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Research article

Fuzzy integral inequalities on coordinates of convex fuzzy interval-valued functions

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Abstract: In this study, we introduce and study new fuzzy-interval integral is known as fuzzy-interval double integral, where the integrand is fuzzy-interval-valued functions (FIVFs). Also, some fundamental properties are also investigated. Moreover, we present a new class of convex fuzzy-interval-valued functions is known as coordinated convex fuzzy-interval-valued functions (coordinated convex FIVFs) through fuzzy order relation (FOR). The FOR $(\preccurlyeq)$ and fuzzy inclusion relation $(\supseteq)$ are two different concepts. With the help of fuzzy-interval double integral and FOR, we have proved that coordinated convex fuzzy-IVF establish a strong relationship between Hermite-Hadamard (HH-) and Hermite-Hadamard-Fejér (HH-Fejér) inequalities. With the support of this relation, we also derive some related HH-inequalities for the product of coordinated convex FIVFs. Some special cases are also discussed. Useful examples that verify the applicability of the theory developed in this study are presented. The concepts and techniques of this paper may be a starting point for further research in this area.

Keywords: fuzzy-interval-valued function; fuzzy double integral; coordinated convex fuzzy-interval-valued function; Hermite-Hadamard inequality; Hermite-Hadamard-Fejér inequality

1. Introduction

It is a familiar fact that integral inequalities have many applications in different mathematical segments as in the theory of differential and integral equation control theory, statistic, among others. Over the past two decades, integral inequalities have attracted good numbers of research to devote themselves because of the importance in different fields. Therefore, several generalizations of classical integral inequalities were obtained in the real, interval and fuzzy spaces. In all of those, HH-inequalities establish strong relationship between different classes of convex functions and have an important place in many areas of mathematics. This double inequality, introduced by Hermite [1] and Hadamard [2], express that if convex function $\mathcal{T}: I \to \mathbb{R}$ on an interval $I = [u, \nu]$ satisfies the following inequality:

$$\mathcal{T}\left(\frac{u+\nu}{2}\right) \leq \frac{1}{\nu-u} \int_u^\nu \mathcal{T}(\omega)d\omega \leq \frac{\mathcal{T}(u) + \mathcal{T}(\nu)}{2} \tag{1}$$

for all $u, \nu \in I.$ If $\mathcal{T}$ is concave, then double inequality (Eq 1) is reversed. A one step forward, Sarikaya et al. provided the fractional version of inequality (Eq 1) in [3]. Moreover, midpoint and trapezoidal inequalities [4,5], which are commonly used in special means and measures errors, are the most well-known results associated with these inequalities. But, the most familiar version of inequality (Eq 1) is known as Hermite-Hadamard-Fejér inequality, was presented by Fejér [6] as follows:

Let $\mathcal{T}: I \to \mathbb{R}$ be a convex function on an interval $I = [u, \nu]$ with $u \leq \nu$, and let $\Omega: I = [u, \nu] \subset \mathbb{R} \to \mathbb{R}$, with $\Omega \geq 0$, be a integrable and symmetric function with respect to $\frac{u+\nu}{2}$. Then, we have the following inequality:

$$\mathcal{T}\left(\frac{u+\nu}{2}\right) \int_u^\nu \Omega(\omega)d\omega \leq \int_u^\nu \mathcal{T}(\omega)\Omega(\omega)d\omega \leq \frac{\mathcal{T}(u) + \mathcal{T}(\nu)}{2} \int_u^\nu \Omega(\omega)d\omega \tag{2}$$

If $\mathcal{T}$ is concave, then double inequality (Eq 2) is reversed. If $\Omega(\omega) = 1$, then we obtain Eq (1) from Eq (2). Many inequalities can be obtained for convex functions using inequality (Eq 2) and the special symmetric function $\Omega(\omega)$. Similarly, many Scholars used fractional integrals to construct new versions of inequality (Eq 2), obtaining new bounds the left- and right-hand of sides of inequality (Eq 2), see [7–9]. What are more, some inequalities for the product of two coordinated convex functions was firstly discussed by Latif and Alomari in [10]. Besides, the most general versions of inequalities were given by Ozdemir et al. [11,12] through product of two coordinated s-convex functions and product of two coordinated h-convex functions. Through fractional integral, Budak and Sarikaya [13] established the strong relationship between new HH-type inequality and the product of two coordinated convex functions.

On the other hand, interval analysis is a well-known method for dealing interval uncertainty; it is an important material that is used in mathematical and computer models. Ramon E. Moore [14], dubbed the "father of Interval analysis" published the first book on the subject in 1966. Thereafter, many authors in the mathematical community have paid close attention to this area of research. In light of this, Sadowska [15] arrived at the following conclusion for an IVF:

Let $\mathcal{T}: [u, \nu] \subset \mathbb{R} \to \mathbb{R}_I^+$ be a convex interval-valued function (convex-IVF) given by $\mathcal{T}(\omega) = [\mathcal{T}_*(\omega), \mathcal{T}^*(\omega)]$ for all $\omega \in [u, \nu]$, where $\mathcal{T}_*(\omega)$ and $\mathcal{T}^*(\omega)$ are convex and concave functions, respectively. If $\mathcal{T}$ is interval Riemann integrable (in sort, IR-integrable), then

$$\mathcal{T}\left(\frac{u+\nu}{2}\right) \supseteq \frac{1}{\nu-u} \ (IR)\int_u^\nu \mathcal{T}(\omega)d\omega \supseteq \frac{\mathcal{T}(u)+\ \mathcal{T}(\nu)}{2} \tag{3}$$

Note that, the inclusion relation Eq (3) is reversed when $\mathcal{T}$ is concave-IVF. Following that, many scholars used inclusion relations and various integral operators to establish a close relationship between inequality and IVFs. Recently, Costa [16] obtained Jensen's type inequality for fuzzy-IVF. Costa and Roman-Flores [17,18] introduced different types of inequalities for fuzzy-IVF and IVF, and discussed their properties. Roman-Flores et al. [19] derived Gronwall for IVFs. Moreover, Chalco-Cano et al. [20,21] presented Ostrowski-type inequalities for IVFs by using the generalized Hukuhara derivative and provided applications in numerical integration in IVF. Nikodem et al. [22], and Matkowski and Nikodem [23] presented the new versions of Jensen inequality for strongly convex and convex functions. Zhao et al. [24,25] derived Chebyshev, Jensen's and HH-type inequalities for IVFs. Recently, Zhang et al. [26] generalized the Jensen inequalities and defined new version of Jensen's inequalities [16] for set-valued and fuzzy-set-valued functions through pseudo order relation. After that, for convex-IVF, Budek [27] established interval-valued fractional Riemann-Liouville HH-inequality by means of inclusion relation. For more useful details, see [28–34] and the references therein.

Recently, Khan et al. [35] introduced the new class of convex fuzzy mappings is known as (h_1, h_2)-convex FIVFs by means of FOR and presented the following new version of HH-type inequality for (h_1, h_2)-convex FIVF involving fuzzy-interval Riemann integrals:

Theorem 1.1. Let $\tilde{\mathcal{T}}: [u, \nu] \to \mathbb{F}_0$ be a (h_1, h_2)-convex FIVF with $h_1, h_2: [0, 1] \to \mathbb{R}^+$ and $h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \neq 0$. Then, from θ-levels, we get the collection of IVFs $\mathcal{T}_\theta: [u, \nu] \subset \mathbb{R} \to \mathbb{R}_I^+$ are given by $\mathcal{T}_\theta(\omega) = [\mathcal{T}_*(\omega, \theta), \mathcal{T}^*(\omega, \theta)]$ for all $\omega \in [u, \nu]$ and for all $\theta \in [0, 1]$. If $\tilde{\mathcal{T}}$ is fuzzy-interval Riemann integrable (in sort, FR-integrable), then

$$\frac{1}{2\, h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right)} \ \tilde{\mathcal{T}}\left(\frac{u+\nu}{2}\right) \preccurlyeq \frac{1}{\nu-u} \ (FR)\int_u^\nu \tilde{\mathcal{T}}(\omega)d\omega \preccurlyeq [\tilde{\mathcal{T}}(u) \ \tilde{+} \ \tilde{\mathcal{T}}(\nu)] \int_0^1 h_1(\tau)\, h_2(1-\tau)\, d\tau \tag{4}$$

If $h_1(\tau) = \tau$ and $h_2(\tau) \equiv 1$, then from Theorem 1.1, we get following the result for convex FIVF:

$$\tilde{\mathcal{T}}\left(\frac{u+\nu}{2}\right) \preccurlyeq \frac{1}{\nu-u} \ (FR)\int_u^\nu \tilde{\mathcal{T}}(\omega)d\omega \preccurlyeq \frac{\tilde{\mathcal{T}}(u) \ \tilde{+} \ \tilde{\mathcal{T}}(\nu)}{2} \tag{5}$$

A one step forward, Khan et al. introduced new classes of convex and generalized convex FIVF, and derived new fractional HH-type and HH-type inequalities for convex FIVF [36], h-convex FIVF [37], (h_1, h_2)-preinvex FIVF [38], log-s-convex FIVFs in the second sense [39], LR-log-h-convex IVFs [40], harmonically convex FIVFs [41] and the references therein. We refer to the readers for further analysis of literature on the applications and properties of fuzzy-interval, and inequalities and generalized convex fuzzy mappings, see [42–51] and the references therein.

2. Preliminaries

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}_I$ be the space of all closed and bounded intervals of $\mathbb{R}$, and $\varpi \in \mathbb{R}_I$ be defined by

$$\varpi = [\varpi_*, \varpi^*] = \{\omega \in \mathbb{R} |\ \varpi_* \leq \omega \leq \varpi^*\},\ (\varpi_*, \varpi^* \in \mathbb{R})$$

If $\varpi_* = \varpi^*$, then ϖ is said to be degenerate. If $\varpi_* \geq 0$, then $[\varpi_*, \varpi^*]$ is called positive interval.

The set of all positive interval is denoted by $\mathbb{R}_I^+$ and defined as $\mathbb{R}_I^+ = \{[\varpi_*, \varpi^*]: [\varpi_*, \varpi^*] \in \mathbb{R}_I \text{ and } \varpi_* \geq 0\}$.

Let $\varrho \in \mathbb{R}$ and $\varrho\varpi$ be defined by

$$\varrho.\varpi = \begin{cases} [\varrho\varpi_*, \varrho\varpi^*] & \text{if } \varrho \geq 0, \\ \{0\} & \text{if } \varrho = 0, \\ [\varrho\varpi^*, \varrho\varpi_*] & \text{if } \varrho < 0. \end{cases} \tag{6}$$

Then the Minkowski difference $\xi - \varpi$, addition $\varpi + \xi$ and $\varpi \times \xi$ for $\varpi, \xi \in \mathbb{R}_I$ are defined by

$$\begin{aligned} [\xi_*, \xi^*] - [\varpi_*, \varpi^*] &= [\xi_* - \varpi_*,\ \xi^* - \varpi^*], \\ [\xi_*, \xi^*] + [\varpi_*, \varpi^*] &= [\xi_* + \varpi_*,\ \xi^* + \varpi^*], \end{aligned} \tag{7}$$

and

$$[\xi_*, \xi^*] \times [\varpi_*, \varpi^*] = [min\{\xi_*\varpi_*, \xi^*\varpi_*, \xi_*\varpi^*, \xi^*\varpi^*\},\ max\{\xi_*\varpi_*, \xi^*\varpi_*, \xi_*\varpi^*, \xi^*\varpi^*\}].$$

The inclusion " $\subseteq$ " means that

$$\xi \subseteq \varpi \text{ if and only if}, [\xi_*, \xi^*] \subseteq [\varpi_*, \varpi^*], \text{ if and only if } \varpi_* \leq \xi_*,\ \xi^* \leq \varpi^*. \tag{8}$$

Remark 2.1. [41] The relation " $\leq_I$ " defined on $\mathbb{R}_I$ by

$$[\xi_*, \xi^*] \leq_I [\varpi_*, \varpi^*] \text{ if and only if } \xi_* \leq \varpi_*,\ \xi^* \leq \varpi^*, \tag{9}$$

for all $[\xi_*, \xi^*]$, $[\varpi_*, \varpi^*] \in \mathbb{R}_I$, it is an order relation. For given $[\xi_*, \xi^*]$, $[\varpi_*, \varpi^*] \in \mathbb{R}_I$, we say that $[\xi_*, \xi^*] \leq_I [\varpi_*, \varpi^*]$ if and only if $\xi_* \leq \varpi_*$, $\xi^* \leq \varpi^*$.

For $[\xi_*, \xi^*]$, $[\varpi_*, \varpi^*] \in \mathbb{R}_I$, the Hausdorff–Pompeiu distance between intervals $[\xi_*, \xi^*]$ and $[\varpi_*, \varpi^*]$ is defined by

$$d([\xi_*, \xi^*], [\varpi_*, \varpi^*]) = max\{|\xi_* - \varpi_*|, |\xi^* - \varpi^*|\} \tag{10}$$

It is familiar fact that $(\mathbb{R}_I, d)$ is a complete metric space.

A fuzzy subset T of $\mathbb{R}$ is characterize by a mapping $\xi: \mathbb{R} \to [0, 1]$ called the membership function, for each fuzzy set and $\theta \in (0, 1]$, then θ-level sets of ξ is denoted and defined as follows $\xi_\theta = \{u \in \mathbb{R} |\ \xi(u) \geq \theta\}$. If $\theta = 0$, then $supp(\xi) = \{\omega \in \mathbb{R} |\ \xi(\omega) > 0\}$ is called support of ξ. By $[\xi]^0$ we define the closure of $supp(\xi)$.

Let $\mathbb{F}(\mathbb{R})$ be the collection of all fuzzy sets and $\xi \in \mathbb{F}(\mathbb{R})$ be a fuzzy set. Then, we define the following:

1) ξ is said to be normal if there exists $\omega \in \mathbb{R}$ and $\xi(\omega) = 1$;
2) ξ is said to be upper semi continuous on $\mathbb{R}$ if for given $\omega \in \mathbb{R}$, there exist $\varepsilon > 0$ there exist $\delta > 0$ such that $\xi(\omega) - \xi(y) < \varepsilon$ for all $y \in \mathbb{R}$ with $|\omega - y| < \delta$;
3) ξ is said to be fuzzy convex if ξ_θ is convex for every $\theta \in [0, 1]$;
4) ξ is compactly supported if $supp(\xi)$ is compact.

A fuzzy set is called a fuzzy number or fuzzy interval if it has properties 1)–4). We denote by $\mathbb{F}_0$ the family of all fuzzy intervals.

Let $\xi \in \mathbb{F}_0$ be a fuzzy-interval, if and only if, θ-levels $[\xi]^\theta$ is a nonempty compact convex set of $\mathbb{R}$. From these definitions, we have

$$[\xi]^\theta = [\xi_*(\theta), \xi^*(\theta)],$$

where

$$\xi_*(\theta) = inf\{\omega \in \mathbb{R}|\ \xi(\omega) \geq \theta\},\ \xi^*(\theta) = sup\{\omega \in \mathbb{R}|\ \xi(\omega) \geq \theta\}.$$

Proposition 2.2. [17] If $\xi, \varpi \in \mathbb{F}_0$, then relation " $\preccurlyeq$ " defined on $\mathbb{F}_0$ by

$$\xi \preccurlyeq \varpi \text{ if and only if, } [\xi]^\theta \leq_I [\varpi]^\theta, \text{ for all } \theta \in [0,1], \tag{11}$$

this relation is known as partial order relation.

For $\xi, \varpi \in \mathbb{F}_0$ and $\varrho \in \mathbb{R}$, the sum $\xi \tilde{+} \varpi$, product $\xi \tilde{\times} \varpi$, scalar product $\varrho . \xi$ and sum with scalar are defined by:

Then, for all $\theta \in [0,1]$, we have

$$[\xi \tilde{+} \varpi]^\theta = [\xi]^\theta + [\varpi]^\theta \tag{12}$$

$$[\xi \tilde{\times} \varpi]^\theta = [\xi]^\theta \times [\ \varpi]^\theta \tag{13}$$

$$[\varrho . \xi]^\theta = \varrho . [\xi]^\theta \tag{14}$$

$$[\varrho \tilde{+}\ \xi]^\theta = \varrho + [\xi]^\theta \tag{15}$$

For $\psi \in \mathbb{F}_0$ such that $\xi = \varpi \tilde{+} \psi$, then by this result we have existence of Hukuhara difference of ξ and ϖ, and we say that ψ is the H-difference of ξ and ϖ, and denoted by $\xi \tilde{-} \varpi$**.** If H-difference exists, then

$$(\psi)^*(\theta) = (\xi \tilde{-} \varpi)^*(\theta) = \xi^*(\theta) - \varpi^*(\theta),\ (\psi)_*(\theta) = (\xi \tilde{-} \varpi)_*(\theta) = \xi_*(\theta) - \varpi_*(\theta) \tag{16}$$

Definition 2.3. [32] The IVF $\mathcal{T}: \Delta = [a,b] \times [u,v] \to \mathbb{R}^+$ is said to be coordinated convex function on Δ if

$$\begin{aligned} &\mathcal{T}(\tau a + (1-\tau)b, su + (1-s)v) \\ &\quad \leq \tau s \mathcal{T}(a,u) + \tau(1-s)\mathcal{T}(a,v) + (1-\tau)s\mathcal{T}(b,u) + (1-\tau)(1-s)\mathcal{T}(b,v) \end{aligned} \tag{17}$$

for all $(a,b),(u,v) \in \Delta, \tau$ and $, \tau, s \in [0,1]$. If inequality (Eq 17) is reversed, then $\mathcal{T}$ is called coordinated concave IVF on Δ.

Definition 2.4. [44] The FIVF $\tilde{\mathcal{T}}: [u,v] \to \mathbb{F}_0$ is said to be convex FIVF on $[u,v]$ if

$$\tilde{\mathcal{T}}(\tau x + (1-\tau)\omega) \preccurlyeq \tau \tilde{\mathcal{T}}(x)\ \tilde{+}\ (1-\tau)\tilde{\mathcal{T}}(\omega), \tag{18}$$

for all $x, \omega \in [u,v], \tau \in [0,1]$, where $\mathcal{T}(x) \succcurlyeq \tilde{0}$. If $\tilde{\mathcal{T}}$ is concave FIVF on $[u,v]$, then inequality (Eq 18) is reversed.

Definition 2.5. [35] Let $h_1, h_2: [0,1] \subseteq [u,v] \to \mathbb{R}^+$ such that $h_1, h_2 \not\equiv 0$. Then, FIVF $\tilde{\mathcal{T}}: [u,v] \to \mathbb{F}_0$ is said to be (h_1,h_2)-convex FIVF on $[u,v]$ if

$$\tilde{\mathcal{T}}(\tau x + (1-\tau)\omega\ \ ) \preccurlyeq h_1(\tau)\, h_2(1-\tau)\, \tilde{\mathcal{T}}(x) \tilde{+}\, h_1(1-\tau)\, h_2(\tau)\, \tilde{\mathcal{T}}(\omega) \tag{19}$$

for all $x, \omega \in [u,v], \tau \in [0,1]$, where $\tilde{\mathcal{T}}(x) \succcurlyeq \tilde{0}$. If $\tilde{\mathcal{T}}$ is (h_1,h_2)-concave on $[u,v]$, then inequality (Eq 19) is reversed.

Remark 2.6. [35] If $h_2(\tau) \equiv 1$, then (h_1,h_2)-convex FIVF becomes h_1-convex FIVF, that is

$$\tilde{\mathcal{T}}(\tau x + (1-\tau)\omega) \preccurlyeq h_1(\tau)\, \tilde{\mathcal{T}}(x) \tilde{+}\, h_1(1-\tau)\, \tilde{\mathcal{T}}(\omega),\ \forall\ x, \omega \in [u,v], \tau \in [0,1] \tag{20}$$

If $h_1(\tau) = \tau, h_2(\tau) \equiv 1$, then (h_1,h_2)-convex FIVF becomes convex FIVF, that is

$$\tilde{\mathcal{T}}(\tau x+(1-\tau)\omega) \preccurlyeq \tau\tilde{\mathcal{T}}(x)\tilde{+}(1-\tau)\tilde{\mathcal{T}}(\omega),\ \forall\ x,\omega\in[u,\nu],\tau\in[0,1] \tag{21}$$

If $h_1(\tau)=h_2(\tau)\equiv 1$, then (h_1,h_2)-convex FIVF becomes P-convex FIVF, that is

$$\tilde{\mathcal{T}}(\tau x+(1-\tau)\omega) \preccurlyeq \tilde{\mathcal{T}}(x)\tilde{+}\tilde{\mathcal{T}}(\omega),\ \forall\ x,\omega\in[u,\nu],\tau\in[0,1] \tag{22}$$

Theorem 2.7. [35] Let $\tilde{\mathcal{T}},\tilde{\mathcal{J}}:[u,\nu]\to\mathbb{F}_0$ be two (h_1,h_2)-convex FIVFs with $h_1,h_2:[0,1]\to\mathbb{R}^+$ and $h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\neq 0$. Then, from θ-levels, we get the collection of IVFs $\mathcal{T}_\theta,\ \mathcal{J}_\theta:[u,\nu]\subset\mathbb{R}\to\mathbb{R}_I^+$ are given by $\mathcal{T}_\theta(x)=[\mathcal{T}_*(x,\theta),\mathcal{T}^*(x,\theta)]$ and $\mathcal{J}_\theta(x)=[\mathcal{J}_*(x,\theta),\mathcal{J}^*(x,\theta)]$ for all $x\in[u,\nu]$ and for all $\theta\in[0,1]$. If $\tilde{\mathcal{T}}\tilde{\times}\tilde{\mathcal{J}}$ is fuzzy Riemann integrable, then

$$\frac{1}{\nu-u}\ (FR)\int_u^\nu \tilde{\mathcal{T}}(x)\tilde{\times}\tilde{\mathcal{J}}(x)dx \preccurlyeq \widetilde{\mathcal{M}}(u,\nu)\int_0^1[h_1(\tau)\,h_2(1-\tau)]^2d\tau$$

$$\tilde{+}\widetilde{\mathcal{N}}(u,\nu)\int_0^1 h_1(\tau)\,h_2(\tau)\,h_1(1-\tau)\,h_2(1-\tau)\,d\tau \tag{23}$$

and,

$$\frac{1}{2\left[h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\right]^2}\ \tilde{\mathcal{T}}\left(\frac{u+\nu}{2}\right)\tilde{\mathcal{J}}\left(\frac{u+\nu}{2}\right) \preccurlyeq \frac{1}{\nu-u}\ (FR)\int_u^\nu \tilde{\mathcal{T}}(x)\tilde{\mathcal{J}}(x)dx\ \tilde{+}\ \widetilde{\mathcal{N}}(u,\nu)\int_0^1[h_1(\tau)\,h_2(1-$$

$$\tau)]^2d\tau\,\tilde{+}\widetilde{\mathcal{M}}(u,\nu)\int_0^1 h_1(\tau)\,h_2(\tau)\,h_1(1-\tau)\,h_2(1-\tau)\,d\tau \tag{24}$$

where

$$\widetilde{\mathcal{M}}(u,\nu)=\tilde{\mathcal{T}}(u)\tilde{\times}\tilde{\mathcal{J}}(u)\ \tilde{+}\ \tilde{\mathcal{T}}(\nu)\tilde{\times}\tilde{\mathcal{J}}(\nu),\ \widetilde{\mathcal{N}}(u,\nu)=\tilde{\mathcal{T}}(u)\tilde{\times}\tilde{\mathcal{J}}(\nu)\ \tilde{+}\ \tilde{\mathcal{T}}(\nu)\tilde{\times}\tilde{\mathcal{J}}(u),$$

$$\mathcal{M}_\theta(u,\nu)=\big[\mathcal{M}_*\big((u,\nu),\theta\big),\mathcal{M}^*\big((u,\nu),\theta\big)\big],\ \mathcal{N}_\theta(u,\nu)=\big[\mathcal{N}_*\big((u,\nu),\theta\big),\mathcal{N}^*\big((u,\nu),\theta\big)\big].$$

Remark 2.8. If $h_1(\tau)=\tau$ and $h_2(\tau)\equiv 1$, then Eq (23) reduces to the result for convex FIVF:

$$\frac{1}{\nu-u}\ (FR)\int_u^\nu \tilde{\mathcal{T}}(x)\tilde{\times}\tilde{\mathcal{J}}(x)dx \preccurlyeq \frac{1}{3}\widetilde{\mathcal{M}}(u,\nu)\tilde{+}\ \frac{1}{6}\widetilde{\mathcal{N}}(u,\nu) \tag{25}$$

And if $h_1(\tau)=\tau$ and $h_2(\tau)\equiv 1$, then Eq (24) reduces to the result for convex FIVF:

$$2\ \tilde{\mathcal{T}}\left(\frac{u+\nu}{2}\right)\tilde{\times}\tilde{\mathcal{J}}\left(\frac{u+\nu}{2}\right) \preccurlyeq \frac{1}{\nu-u}\ (FR)\int_u^\nu \tilde{\mathcal{T}}(x)\tilde{\times}\tilde{\mathcal{J}}(x)dx\ \ \tilde{+}\frac{1}{6}\widetilde{\mathcal{M}}(u,\nu)\tilde{+}\ \frac{1}{3}\widetilde{\mathcal{N}}(u,\nu) \tag{26}$$

Theorem 2.9. Let $\tilde{\mathcal{T}}:[u,\nu]\to\mathbb{F}_0$ be a convex FIVF with $u<\nu$. Then, from θ-levels, we get the collection of IVFs $\mathcal{T}_\theta:[u,\nu]\subset\mathbb{R}\to\mathbb{R}_I^+$ are given by $\mathcal{T}_\theta(x)=[\mathcal{T}_*(x,\theta),\mathcal{T}^*(x,\theta)]$ for all $x\in[u,\nu]$ and for all $\theta\in[0,1]$. If $\tilde{\mathcal{T}}\in\ \mathcal{TR}_{([u,\nu],\theta)}$ and $\Omega:[u,\nu]\to\mathbb{R},\Omega(x)\geq 0$, symmetric with respect to $\frac{u+\nu}{2}$, and $\int_u^\nu\Omega(x)dx>0$, then

$$\tilde{\mathcal{T}}\left(\frac{u+\nu}{2}\right)\preccurlyeq\frac{1}{\int_u^\nu\Omega(x)dx}\ (FR)\int_u^\nu\tilde{\mathcal{T}}(x)\Omega(x)dx\preccurlyeq\frac{\tilde{\mathcal{T}}(u)\tilde{+}\tilde{\mathcal{T}}(\nu)}{2} \tag{27}$$

If $\mathcal{T}$ is concave FIVF, then inequality (Eq 27) is reversed.

2.1 Fuzzy-interval double integral and convexity

Firstly, we shall define fuzzy-interval double integrable.

A FIVF $\tilde{\mathcal{T}}: [a, b] \to \mathbb{F}_0$ is said to be continuous at x_0 if for each ϵ there exist a δ such that

$$d\left(\tilde{\mathcal{T}}(x), \tilde{\mathcal{T}}(x_0)\right) < \epsilon,$$

Whenever $|x - x_0| < \delta$. A tagged partition of $[a, b]$ is any finite ordered subset P_1 having the form

$$P_1 = \{a = x_1 < \ x_2 < \ x_3 < x_4 < \ x_5 \ldots \ldots < x_k = b\}.$$

Let $\mathcal{P}(\delta, [a, b])$ be the set of all $P_1 \in \mathcal{P}(\delta, [a, b])$ such that $\Delta(x_i) < \delta$. Then, P_1 is called δ-fine. For each set of numbers $[x_{i-1}, \ x_i]$, where $1 \leq i \leq k$, choose an arbitrary point η_i and taking the sum

$$S(\tilde{\mathcal{T}}, P_1, \delta, [a, b]) = \sum_{i=1}^{k} \tilde{\mathcal{T}}(\eta_i)(x_i - x_{i-1}) \quad (28)$$

where $\tilde{\mathcal{T}}: [a, b] \to \mathbb{F}_0$. We call $S(\tilde{\mathcal{T}}, P_1, \delta[a, b])$ an integral sum of $\tilde{\mathcal{T}}$ corresponding to $P_1 \in \mathcal{P}(\delta, [a, b])$. Let $C([a, b], \mathbb{F}_0)$ be the collection of continuous FIVFs.

If $P_1 \triangleq [x_{i-1}, \ x_i]$ such that $P_1 \in \mathcal{P}(\delta, [a, b])$, where $1 \leq i \leq k$, and $P_2 \triangleq [\omega_{j-1}, \ \omega_j]$ such that $P_2 \in \mathcal{P}(\delta, [u, v])$, where $1 \leq j \leq n$, then rectangles $\Delta_{i,j} = [x_{i-1}, \ x_i] \times [\omega_{j-1}, \ \omega_j]$ partition the rectangle $\Delta = [a, b] \times [u, v]$ and the points $(\eta_i, w_j) \in [x_{i-1}, \ x_i] \times [\omega_{j-1}, \ \omega_j]$. Let $\mathcal{P}(\delta, \Delta)$ be the collection of all δ-fine partition $P = P_1 \times P_2$ of Δ.

Similarly to Eq (28), we have

$$S(\tilde{\mathcal{T}}, P_1, \delta, \Delta) = \sum_{i=1}^{k} \sum_{j=1}^{n} \tilde{\mathcal{T}}(\eta_i, w_j)(x_i - x_{i-1})(\omega_j - \omega_{j-1}) = \sum_{i=1}^{k} \sum_{j=1}^{n} \tilde{\mathcal{T}}(\eta_i, w_j) \Delta A_{i,j} \quad (29)$$

an integral sum of $\tilde{\mathcal{T}}: \Delta = [a, b] \times [u, v] \to \mathbb{F}_0$, where $\Delta A_{i,j}$ be the of rectangle.

Definition 2.1.1. A function $\mathcal{T}: [a, b] \to \mathbb{R}$ is called Riemann integrable (R-integrable) on $[a, b]$ if there exists $B \in \mathbb{R}$ such that, for each ϵ, there exists $\delta > 0$ such that

$$|S(\mathcal{T}, P_1, \delta, [a, b]) - B| < \epsilon,$$

for every Riemann sum of $\mathcal{T}$ corresponding to $P_1 \in \mathcal{P}(\delta, [a, b])$ and for arbitrary choice of $\eta_i \in [x_{i-1}, \ x_i]$ for $1 \leq i \leq k$. Then, we say that B is the R-integral of $\mathcal{T}$ on $[a, b]$ and is denote by $B = (R) \int_a^b \mathcal{T}(x) dx$.

Definition 2.1.2. [25] A function $\mathcal{T}: [a, b] \to \mathbb{R}_I$ is called interval Riemann integrable (IR-integrable) on $[a, b]$ if there exists $B \in \mathbb{R}_I$ such that, for each ϵ, there exists $\delta > 0$ such that

$$d(S(\mathcal{T}, P_1, \delta, [a, b]), B) < \epsilon,$$

for every Riemann sum of $\mathcal{T}$ corresponding to $P_1 \in \mathcal{P}(\delta, [a, b])$ and for arbitrary choice of $\eta_i \in [x_{i-1}, \ x_i]$ for $1 \leq i \leq k$. Then, we say that B is the IR-integral of $\mathcal{T}$ on $[a, b]$ and is denote by $B = (IR) \int_a^b \mathcal{T}(x) dx$.

Definition 2.1.3. A function $\tilde{\mathcal{T}}: [a, b] \to \mathbb{F}_0$ is called fuzzy-interval Riemann integrable

(FR-integrable) on $[a,b]$ if there exists $\tilde{B} \in \mathbb{F}_0$ such that, for each ϵ, there exists $\delta > 0$ such that

$$d\big(S(\tilde{\mathcal{T}}, P_1, \delta, [a,b]), \tilde{B}\big) < \epsilon,$$

for every Riemann sum of $\tilde{\mathcal{T}}$ corresponding to $P_1 \in \mathcal{P}(\delta, [a,b])$ and for arbitrary choice of $\eta_i \in [x_{i-1},\ x_i]$ for $1 \le i \le k$. Then, we say that $\tilde{B}$ is the FR-integral of $\tilde{\mathcal{T}}$ on $[a,b]$ and is denote by $\tilde{B} = (IR)\int_a^b \tilde{\mathcal{T}}(x)dx$.

Definition 2.1.4. [24] A function $\mathcal{T}: \Delta = [a,b] \times [u,v] \to \mathbb{R}_I$ is called interval double integral (ID-integrable) on Δ if there exists $B \in \mathbb{R}_I$ such that, for each ϵ, there exists $\delta > 0$ such that

$$d(S(\mathcal{T}, P, \delta, \Delta), B) < \epsilon,$$

for every Riemann sum of $\mathcal{T}$ corresponding to $P \in \mathcal{P}(\delta, \Delta)$ and for arbitrary choice of $(\eta_i, w_j) \in [x_{i-1},\ x_i] \times [\omega_{j-1},\ \omega_j]$ for $1 \le i \le k$ and $1 \le j \le n$. Then, we say that B is the IR-integral of $\mathcal{T}$ on Δ and is denote by $B = (ID)\int_a^b \int_u^v \mathcal{T}(x,\omega)d\omega dx$ or $B = (ID)\iint_\Delta \mathcal{T}dA$.

Theorem 2.1.5. [14] If $\mathcal{T}: [u,v] \subset \mathbb{R} \to \mathbb{R}_I$ is an IVF given by (x) $[\mathcal{T}_*(x), \mathcal{T}^*(x)]$, then $\mathcal{T}$ is Riemann integrable over $[u,v]$ if and only if, $\mathcal{T}_*$ and $\mathcal{T}^*$ both are Riemann integrable over $[u,v]$ such that

$$(IR)\int_u^v \mathcal{T}(x)dx = \left[(R)\int_u^v \mathcal{T}_*(x)dx\,, (R)\int_u^v \mathcal{T}^*(x)dx\right] \tag{30}$$

The collection of all Riemann integrable real valued functions and Riemann integrable IVF is denoted by $\mathcal{R}_{[u,v]}$ and $\mathfrak{T}\mathcal{R}_{[u,v]}$, respectively.

Note that, the Theorem 2.1.6 is also true for interval double integrals. The collection of all double integrable IVF is denoted $\mathfrak{T}\mathfrak{D}_\Delta$, respectively.

Theorem 2.1.6. [24] Let $\Delta = [a,b] \times [u,v]$. If $\mathcal{T}: \Delta \to \mathbb{R}_I$ is ID-integrable on Δ, then we have

$$(ID)\int_a^b \int_u^v \mathcal{T}(x,\omega)d\omega dx = (IR)\int_a^b (IR)\int_u^v \mathcal{T}(x,\omega)d\omega dx.$$

Definition 2.1.7. A function $\tilde{\mathcal{T}}$: $\Delta = [a,b] \times [u,v] \to \mathbb{F}_0$ is called fuzzy-interval double integrable (FD-integrable) on Δ if there exists $\tilde{B} \in \mathbb{F}_0$ such that, for each ϵ, there exists $\delta > 0$ such that

$$d\big(S(\tilde{\mathcal{T}}, P, \delta, \Delta), \tilde{B}\big) < \epsilon,$$

for every Riemann sum of $\tilde{\mathcal{T}}$ corresponding to $P \in \mathcal{P}(\delta, \Delta)$ and for arbitrary choice $(\eta_i, w_j) \in [x_{i-1},\ x_i] \times [\omega_{j-1},\ \omega_j]$ for $1 \le i \le k$ and $1 \le j \le n$. Then, we say that $\tilde{B}$ is the FR-integral of $\tilde{\mathcal{T}}$ on Δ and is denote by $\tilde{B} = (FD)\int_a^b \int_u^v \tilde{\mathcal{T}}(x,\omega)d\omega dx$ or $\tilde{B} = (FD)\iint_\Delta \tilde{\mathcal{T}}dA$.

Definition 2.1.8. A fuzzy-interval-valued map $\tilde{\mathcal{T}}: \Delta = [a,b] \times [u,v] \to \mathbb{F}_0$ is called FIVF on coordinates. Then, from θ-levels, we get the collection of IVFs $\mathcal{T}_\theta: \Delta \subset \mathbb{R}^2 \to \mathbb{R}_I$ on coordinates are given by $\mathcal{T}_\theta(x,\omega) = [\mathcal{T}_*((x,\omega),\theta), \mathcal{T}^*((x,\omega),\theta)]$ for all $(x,\omega) \in \Delta$. Here, for each $\theta \in [0,1]$, the end point real valued functions $\mathcal{T}_*(.,\theta), \mathcal{T}^*(.,\theta): (x,\omega) \to \mathbb{R}$ are called lower and upper functions of $\mathcal{T}_\theta$.

Definition 2.1.9. Let $\tilde{\mathcal{T}}: \Delta = [a,b] \times [u,v] \subset \mathbb{R}^2 \to \mathbb{F}_0$ be a coordinate FIVF. Then, $\tilde{\mathcal{T}}(x,\omega)$ is said to be continuous at $(x,\omega) \in \Delta = [a,b] \times [u,v]$, if for each $\theta \in [0,1]$, both end point functions $\mathcal{T}_*((x,\omega),\theta)$ and $\mathcal{T}^*((x,\omega),\theta)$ are continuous at $(x,\omega) \in \Delta$.

Definition 2.1.10. Let $\tilde{\mathcal{T}}: \Delta = [a,b] \times [u,v] \subset \mathbb{R}^2 \to \mathbb{F}_0$ be a FIVF on coordinates. Then, fuzzy double integral of $\tilde{\mathcal{T}}$ over $\Delta = [a,b] \times [u,v]$, denoted by $(FD)\int_a^b \int_u^v \tilde{\mathcal{T}}(x,\omega)d\omega dx$, it is defined level-wise by

$$\left[(FD)\int_a^b \int_u^v \tilde{\mathcal{T}}(x,\omega)d\omega dx\right]^{\boldsymbol{\theta}} = (ID)\int_a^b \int_u^v \mathcal{T}_\theta(x,\omega)d\omega dx$$

$$= (IR)\int_a^b (IR)\int_u^v \mathcal{T}_\theta(x,\omega)d\omega dx, \tag{31}$$

for all $\theta \in [0,1]$, $\tilde{\mathcal{T}}$ is FD-integrable over Δ if $(FD)\int_a^b \int_u^v \tilde{\mathcal{T}}(x,\omega)d\omega dx \in \mathbb{F}_0$. Note that, if end point functions are Lebesgue-integrable, then $\tilde{\mathcal{T}}$ is fuzzy double Aumann-integrable function over Δ.

Theorem 2.1.11. Let $\tilde{\mathcal{T}}: \Delta \subset \mathbb{R}^2 \to \mathbb{F}_0$ be a FIVF on coordinates. Then, from θ-levels, we get the collection of IVFs $\mathcal{T}_\theta: \Delta \subset \mathbb{R}^2 \to \mathbb{R}_I$ are given by $\mathcal{T}_\theta(x,\omega) = [\mathcal{T}_*((x,\omega),\theta), \mathcal{T}^*((x,\omega),\theta)]$ for all $(x,\omega) \in \Delta = [a,b] \times [u,v]$ and for all $\theta \in [0,1]$. Then, $\tilde{\mathcal{T}}$ is FD-integrable over Δ if and only if, $\mathcal{T}_*((x,\omega),\theta)$ and $\mathcal{T}^*((x,\omega),\theta)$ both are D-integrable over Δ. Moreover, if $\tilde{\mathcal{T}}$ is FD-integrable over Δ, then

$$\left[(FD)\int_a^b \int_u^v \tilde{\mathcal{T}}(x,\omega)d\omega dx\right]^\theta = \left[(FR)\int_a^b (FR)\int_u^v \tilde{\mathcal{T}}(x,\omega)d\omega dx\right]^\theta$$

$$= (IR)\int_a^b (IR)\int_u^v \mathcal{T}_\theta(x,\omega)d\omega dx = (ID)\int_a^b \int_u^v \mathcal{T}_\theta(x,\omega)d\omega dx \tag{32}$$

for all $\theta \in [0,1]$.

Proof. The proof of this theorem follows immediately by the Definition 2.1.10 of fuzzy double integral of FIVF.

The family of all FD-integrable FIVFs over coordinates is denoted by $\mathcal{F}\mathfrak{D}_\Delta$ for all $\theta \in [0,1]$.

Theorem 2.1.12. Let $\varrho \in \mathbb{R}$, and $\tilde{\mathcal{T}}, \tilde{\mathcal{J}} \in \mathcal{F}\mathfrak{D}_\Delta$. Then,

1) $\varrho\tilde{\mathcal{T}} \in \mathcal{F}\mathfrak{D}_\Delta$ and

$$(FD)\iint_\Delta \varrho\tilde{\mathcal{T}}dA = \varrho(FD)\iint_\Delta \tilde{\mathcal{T}}dA.$$

2) $\tilde{\mathcal{T}} \tilde{\mp} \tilde{\mathcal{J}} \in \mathcal{F}\mathfrak{D}_\Delta$, and

$$(FD)\iint_\Delta (\tilde{\mathcal{T}} \tilde{\mp} \tilde{\mathcal{J}})dA = (FD)\iint_\Delta \tilde{\mathcal{T}}dA \tilde{\mp} (FD)\iint_\Delta \tilde{\mathcal{J}}dA.$$

3) suppose that Δ_1 and Δ_2 are non-overlapping, then

$$(FD)\iint_{\Delta_1 \cup \Delta_2} \tilde{\mathcal{T}}dA = (FD)\iint_{\Delta_1} \tilde{\mathcal{T}}dA + (FD)\iint_{\Delta_2} \tilde{\mathcal{T}}dA.$$

Proof. The proof of Theorem 2.1.12 is straightforward so it is omitted.

Now we define the new class, namely, coordinated convex FIVF by means of FOR.

Definition 2.1.13. The FIVF $\tilde{\mathcal{T}}: \Delta \to \mathbb{F}_0$ is said to be coordinated convex FIVF on Δ if

$$\tilde{\mathcal{T}}(\tau a + (1-\tau)b, su + (1-s)v)$$
$$\preccurlyeq \tau s\tilde{\mathcal{T}}(a,u) \tilde{+} \tau(1-s)\tilde{\mathcal{T}}(a,v) \tilde{+} (1-\tau)s\tilde{\mathcal{T}}(b,u) \tilde{+} (1-\tau)(1-s)\tilde{\mathcal{T}}(b,v) \tag{33}$$

for all $(a,b),(u,v) \in \Delta$, and $\tau, s \in [0,1]$, where $\tilde{\mathcal{T}}(x) \succcurlyeq \tilde{0}$. If inequality (Eq 33) is reversed, then $\tilde{\mathcal{T}}$ is called coordinate concave FIVF on Δ.

The proof of Lemma 2.1.14 is straightforward will be omitted here.

Lemma 2.1.14. Let $\tilde{\mathcal{T}}: \Delta \to \mathbb{F}_0$ be an coordinated FIVF on Δ. Then, $\tilde{\mathcal{T}}$ is coordinated convex FIVF on Δ, if and only if there exist two coordinated convex FIVFs $\widetilde{\mathcal{T}_x}: [u,v] \to \mathbb{F}_0$, $\tilde{\mathcal{T}}_x(w) = \tilde{\mathcal{T}}(x,w)$ and $\tilde{\mathcal{T}}_\omega: [a,b] \to \mathbb{F}_0$, $\tilde{\mathcal{T}}_\omega(u) = \tilde{\mathcal{T}}(u,\omega)$.

Proof. From the definition of coordinated FIVF, it can be easily proved.

From Lemma 2.1.14, we can easily note each convex FIVF is coordinated convex FIVF. But the converse is not true, see Example 2.1.16.

Theorem 2.1.15. Let $\tilde{\mathcal{T}}: \Delta \to \mathbb{F}_0$ be a FIVF on Δ. Then, from θ-levels, we get the collection of IVFs $\mathcal{T}_\theta: \Delta \to \mathbb{R}_I^+ \subset \mathbb{R}_I$ are given by

$$\mathcal{T}_\theta(x,\omega) = \left[\mathcal{T}_*\big((x,\omega),\theta\big), \mathcal{T}^*\big((x,\omega),\theta\big)\right], \tag{34}$$

for all $(x,\omega) \in \Delta$ and for all $\theta \in [0,1]$. Then, $\tilde{\mathcal{T}}$ is coordinated convex FIVF on Δ, if and only if, for all $\theta \in [0,1]$, $\mathcal{T}_*\big((x,\omega),\theta\big)$ and $\mathcal{T}^*\big((x,\omega),\theta\big)$ are coordinated convex function.

Proof. Assume that for each $\theta \in [0,1]$, $\mathcal{T}_*(x,\theta)$ and $\mathcal{T}^*(x,\theta)$ are coordinated convex on Δ. Then, from Eq (33), for all $(a,b),(u,v) \in \Delta, \tau$ and $s \in [0,1]$ we have

$$\mathcal{T}_*\big((\tau a + (1-\tau)b, su + (1-s)v),\theta\big)$$
$$\leq \tau s\mathcal{T}_*\big((a,u),\theta\big) + t(1-s)\mathcal{T}_*\big((a,v),\theta\big) + s(1-t)\mathcal{T}_*\big((a,u),\theta\big) + (1-\tau)(1-s)\mathcal{T}_*\big((a,v),\theta\big),$$

and

$$\mathcal{T}^*\big((\tau a + (1-\tau)b, su + (1-s)v),\theta\big)$$
$$\leq \tau s\mathcal{T}_*\big((a,u),\theta\big) + t(1-s)\mathcal{T}^*\big((a,v),\theta\big) + s(1-t)\mathcal{T}^*\big((a,u),\theta\big) + (1-\tau)(1-s)\mathcal{T}^*\big((a,v),\theta\big),$$

Then, by Eqs (34), (6) and (7), we obtain

$$\mathcal{T}_\theta\big((\tau a + (1-\tau)b, su + (1-s)v)\big)$$
$$= \left[\mathcal{T}_*\big((\tau a + (1-\tau)b, su + (1-s)v),\theta\big), \mathcal{T}^*\big((\tau a + (1-\tau)b, su + (1-s)v),\theta\big)\right]$$
$$\leq_I \tau s\left[\mathcal{T}_*\big((a,u),\theta\big), \mathcal{T}^*\big((a,u),\theta\big)\right] + t(1-s)\left[\mathcal{T}_*\big((a,v),\theta\big), \mathcal{T}^*\big((a,v),\theta\big)\right]$$
$$+ s(1-\tau)\left[\mathcal{T}_*\big((a,u),\theta\big), \mathcal{T}^*\big((a,u),\theta\big)\right] + (1-\tau)(1-s)\left[\mathcal{T}_*\big((a,v),\theta\big), \mathcal{T}^*\big((a,v),\theta\big)\right]$$

That is

$$\tilde{\mathcal{T}}(\tau a+(1-\tau)b, su+(1-s)\nu)$$
$$\preccurlyeq \tau s\tilde{\mathcal{T}}(a,u)\tilde{+}\tau(1-s)\tilde{\mathcal{T}}(a,\nu)\tilde{+}(1-\tau)s\tilde{\mathcal{T}}(b,u)\tilde{+}(1-\tau)(1-s)\tilde{\mathcal{T}}(b,\nu),$$

hence, $\mathcal{T}$ is coordinated convex FIVF on Δ .
Conversely, let $\mathcal{T}$ be coordinated convex FIVF on Δ. Then, for all $(a,b),(u,\nu)\in\Delta,\tau$ and $s\in[0,1]$, we have

$$\tilde{\mathcal{T}}(\tau a+(1-\tau)b, su+(1-s)\nu)$$
$$\preccurlyeq \tau s\tilde{\mathcal{T}}(a,u)\tilde{+}\tau(1-s)\tilde{\mathcal{T}}(a,\nu)\tilde{+}(1-\tau)s\tilde{\mathcal{T}}(b,u)\tilde{+}(1-\tau)(1-s)\tilde{\mathcal{T}}(b,\nu).$$

Therefore, again from Eq (34), for each $\theta\in[0,1]$, we have

$$\mathcal{T}_\theta\big((\tau a+(1-\tau)b, su+(1-s)\nu)\big)$$

$$=\big[\mathcal{T}_*\big((\tau a+(1-\tau)b, su+(1-s)\nu),\theta\big),\mathcal{T}^*\big((\tau a+(1-\tau)b, su+(1-s)\nu),\theta\big)\big].$$

Again, Eqs (12) and (14), we obtain

$$\tau s\mathcal{T}_\theta(a,u)+\tau(1-s)\mathcal{T}_\theta(a,\nu)+(1-\tau)s\mathcal{T}_\theta(b,u)+(1-\tau)(1-s)\mathcal{T}_\theta(b,\nu)$$

$$=\tau s\big[\mathcal{T}_*\big((a,u),\theta\big),\mathcal{T}^*\big((a,u),\theta\big)\big]+t(1-s)\big[\mathcal{T}_*\big((a,\nu),\theta\big),\mathcal{T}^*\big((a,\nu),\theta\big)\big]$$

$$+s(1-\tau)\big[\mathcal{T}_*\big((a,u),\theta\big),\mathcal{T}^*\big((a,u),\theta\big)\big]+(1-\tau)(1-s)\big[\mathcal{T}_*\big((a,\nu),\theta\big),\mathcal{T}^*\big((a,\nu),\theta\big)\big],$$

for all $x,\omega\in\Delta$ and $\tau\in[0,1]$. Then, by coordinated convexity of $\mathcal{T}$, we have for all $x,\omega\in\Delta$ and $\tau\in[0,1]$ such that

$$\mathcal{T}_*\big((\tau a+(1-\tau)b, su+(1-s)\nu),\theta\big)$$
$$\leq \tau s\mathcal{T}_*(a,u)+\tau(1-s)\mathcal{T}_*(a,\nu)+(1-\tau)s\mathcal{T}_*(b,u)+(1-\tau)(1-s)\mathcal{T}_*(b,\nu),$$

and

$$\mathcal{T}^*\big((\tau a+(1-\tau)b, su+(1-s)\nu),\theta\big)$$
$$\leq \tau s\mathcal{T}^*(a,u)+\tau(1-s)\mathcal{T}^*(a,\nu)+(1-\tau)s\mathcal{T}^*(b,u)+(1-\tau)(1-s)\mathcal{T}^*(b,\nu),$$

for each $\theta\in[0,1]$. Hence, the result follows.
Example 2.1.16. We consider the FIVFs $\tilde{\mathcal{T}}:[0,1]\times[0,1]\to\mathbb{F}_0$ defined by,

$$\mathcal{T}(x)(\sigma)=\begin{cases}\frac{\sigma}{x\omega} & \sigma\in[0,x\omega]\\ \frac{2x\omega-\sigma}{x\omega} & \sigma\in(x\omega,2x\omega]\\ 0 & \text{otherwise}\end{cases}$$

Then, for each $\theta\in[0,1]$, we have $\mathcal{T}_\theta(x)=[\theta x\omega,(2-\theta)x\omega]$. Since end point functions $\mathcal{T}_*\big((x,\omega),\theta\big)$, $\mathcal{T}^*\big((x,\omega),\theta\big)$ are coordinate concave functions for each $\theta\in[0,1]$. Hence $\tilde{\mathcal{T}}(x,\omega)$ is coordinate concave FIVF.
From Example 2.1.16, it can be easily seen that each coordinated convex FIVF is not a convex FIVF.

Theorem 2.1.17. Let Δ be a coordinated convex set, and let $\tilde{\mathcal{T}}: \Delta \rightarrow \mathbb{F}_0$ be a FIVF. Then, from θ-levels, we obtain the collection of IVFs $\mathcal{T}_\theta: \Delta \rightarrow \mathbb{R}_I^+ \subset \mathbb{R}_I$ are given by

$$\mathcal{T}_\theta(x, \omega) = \big[\mathcal{T}_*\big((x, \omega), \theta\big), \mathcal{T}^*\big((x, \omega), \theta\big)\big] \tag{35}$$

for all $(x, \omega) \in \Delta$ and for all $\theta \in [0, 1]$. Then, $\tilde{\mathcal{T}}$ is coordinated convex FIVF on Δ, if and only if, for all $\theta \in [0, 1]$, $\mathcal{T}_*\big((x, \omega), \theta\big)$ and $\mathcal{T}^*\big((x, \omega), \theta\big)$ are coordinated convex function.

Proof. The demonstration of proof of Theorem 2.1.17 is similar to the demonstration proof of Theorem 2.1.15.

Theorem 2.1.18. We consider the FIVFs $\tilde{\mathcal{T}}: [0, 1] \times [0, 1] \rightarrow \mathbb{F}_0$ defined by,

$$\tilde{\mathcal{T}}(x)(\sigma) = \begin{cases} \frac{\sigma}{2(6-e^x)(6-e^\omega)}, & \sigma \in [0, 2(6-e^x)(6-e^\omega)] \\ \frac{4(6-e^x)(6-e^\omega)-\sigma}{2(6-e^x)(6-e^\omega)}, & \sigma \in (2(6-e^x)(6-e^\omega), 4(6-e^x)(6-e^\omega)] \\ 0, & \text{otherwise} \end{cases}$$

Then, for each $\theta \in [0, 1]$, we have $\mathcal{T}_\theta(x) = [2\theta(6-e^x)(6-e^\omega), (4-2\theta)(6-e^x)(6-e^\omega)\ ]$. Since end point functions $\mathcal{T}_*\big((x, \omega), \theta\big)$, $\mathcal{T}^*\big((x, \omega), \theta\big)$ are coordinate concave functions for each $\theta \in [0, 1]$. Hence $\tilde{\mathcal{T}}(x, \omega)$ is coordinate concave FIVF.

In the next results, to avoid confusion, we will not include the symbols (R), (IR), (FR), (ID), and (FD) before the integral sign.

3. Fuzzy-interval Hermite-Hadamard inequalities

In this section, we propose HH- and HH-Fejér inequalities for coordinated convex FIVFs, and verify with the help of some nontrivial example.

Theorem 3.1. Let $\tilde{\mathcal{T}}: \Delta \rightarrow \mathbb{F}_0$ be a coordinate convex FIVF on Δ. Then, from θ-levels, we get the collection of IVFs $\mathcal{T}_\theta: \Delta \rightarrow \mathbb{R}_I^+$ are given by $\mathcal{T}_\theta(x, \omega) = \big[\mathcal{T}_*\big((x, \omega), \theta\big), \mathcal{T}^*\big((x, \omega), \theta\big)\big]$ for all $(x, \omega) \in \Delta$ and for all $\theta \in [0, 1]$. Then, following inequality holds:

$$\begin{aligned} \tilde{\mathcal{T}}\left(\frac{a+b}{2}, \frac{u+\nu}{2}\right) &\preccurlyeq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b \tilde{\mathcal{T}}\left(x, \frac{u+\nu}{2}\right) dx \tilde{+} \frac{1}{\nu-u}\int_u^\nu \tilde{\mathcal{T}}\left(\frac{a+b}{2}, \omega\right) d\omega\right] \\ &\preccurlyeq \frac{1}{(b-a)(\nu-u)} \int_a^b \int_u^\nu \tilde{\mathcal{T}}(x, \omega) d\omega dx \\ &\preccurlyeq \frac{1}{4(b-a)}\left[\int_a^b \tilde{\mathcal{T}}(x, u)\, dx \tilde{+} \int_a^b \tilde{\mathcal{T}}(x, \nu) dx\right] \\ &\quad + \frac{1}{4(\nu-u)}\left[\int_u^\nu \tilde{\mathcal{T}}(a, \omega) d\omega \tilde{+} \int_u^\nu \tilde{\mathcal{T}}(b, \omega) d\omega\right] \\ &\preccurlyeq \frac{\tilde{\mathcal{T}}(a,u) \tilde{+} \tilde{\mathcal{T}}(b,u) \tilde{+} \tilde{\mathcal{T}}(a,\nu) \tilde{+} \tilde{\mathcal{T}}(b,\nu)}{4} \end{aligned} \tag{36}$$

If $\mathcal{T}(x)$ concave FIVF then,

$$\tilde{\mathcal{T}}\left(\frac{a+b}{2},\frac{u+\nu}{2}\right) \succcurlyeq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b \tilde{\mathcal{T}}\left(x,\frac{u+\nu}{2}\right)dx \tilde{+} \frac{1}{\nu-u}\int_u^\nu \tilde{\mathcal{T}}\left(\frac{a+b}{2},\omega\right)d\omega\right]$$

$$\succcurlyeq \frac{1}{(b-a)(\nu-u)} \int_a^b\int_u^\nu \tilde{\mathcal{T}}(x,\omega)d\omega dx$$

$$\succcurlyeq \frac{1}{4(b-a)}\left[\int_a^b \tilde{\mathcal{T}}(x,u)\,dx \tilde{+} \int_a^b \tilde{\mathcal{T}}(x,\nu)dx\right] + \frac{1}{4(\nu-u)}\left[\int_u^\nu \tilde{\mathcal{T}}(a,\omega)\,d\omega \tilde{+} \int_u^\nu \tilde{\mathcal{T}}(b,\omega)d\omega\right]$$

$$\succcurlyeq \frac{\tilde{\mathcal{T}}(a,u)\tilde{+}\tilde{\mathcal{T}}(b,u)\tilde{+}\tilde{\mathcal{T}}(a,\nu)\tilde{+}\tilde{\mathcal{T}}(b,\nu)}{4} \quad (37)$$

Proof. Let $\tilde{\mathcal{T}}: [a,b] \to \mathbb{F}_0$ be a coordinated convex FIVF. Then, by hypothesis, we have

$$4\tilde{\mathcal{T}}\left(\frac{a+b}{2},\frac{u+\nu}{2}\right) \preccurlyeq \tilde{\mathcal{T}}(\tau a+(1-\tau)b, \tau u+(1-\tau)\nu)\tilde{+}\tilde{\mathcal{T}}\big((1-\tau)a+\tau b,(1-\tau)u+\tau\nu\big).$$

By using Theorem 3.9, for every $\theta \in [0,1]$, we have

$$4\mathcal{T}_*\left(\left(\frac{a+b}{2},\frac{u+\nu}{2}\right),\theta\right) \le \mathcal{T}_*\big((\tau a+(1-\tau)b,\tau u+(1-\tau)\nu),\theta\big)$$

$$4\mathcal{T}^*\left(\left(\frac{a+b}{2},\frac{u+\nu}{2}\right),\theta\right) \le \mathcal{T}^*\big((\tau a+(1-\tau)b,\tau u+(1-\tau)\nu),\theta\big)$$

By using Lemma 2.1.14, we have

$$\begin{aligned} 2\mathcal{T}_*\left(\left(x,\frac{u+\nu}{2}\right),\theta\right) &\le \mathcal{T}_*\big((x,\tau u+(1-\tau)\nu),\theta\big)+\mathcal{T}_*\big((x,(1-\tau)u+\tau\nu),\theta\big) \\ 2\mathcal{T}^*\left(\left(x,\frac{u+\nu}{2}\right),\theta\right) &\le \mathcal{T}^*\big((x,\tau u+(1-\tau)\nu),\theta\big)+\mathcal{T}^*\big((x,(1-\tau)u+\tau\nu),\theta\big) \end{aligned} \quad (38)$$

and

$$\begin{aligned} 2\mathcal{T}_*\left(\left(\frac{a+b}{2},\omega\right),\theta\right) &\le \mathcal{T}_*\big((\tau a+(1-\tau)b,\omega),\theta\big)+\mathcal{T}_*\big(((1-\tau)a+tb,\omega),\theta\big) \\ 2\mathcal{T}^*\left(\left(\frac{a+b}{2},\omega\right),\theta\right) &\le \mathcal{T}^*\big((\tau a+(1-\tau)b,\omega),\theta\big)+\mathcal{T}^*\big(((1-\tau)a+tb,\omega),\theta\big) \end{aligned} \quad (39)$$

From Eqs (38) and (39), we have

$$2\left[\mathcal{T}_*\left(\left(x,\frac{u+\nu}{2}\right),\theta\right),\mathcal{T}^*\left(\left(x,\frac{u+\nu}{2}\right),\theta\right)\right] \le_I \big[\mathcal{T}_*\big((x,\tau u+(1-\tau)\nu),\theta\big),\mathcal{T}^*\big((x,\tau u+$$

$$(1-\tau)\nu),\theta\big)\big]+\big[\mathcal{T}_*\big((x,(1-\tau)u+\tau\nu),\theta\big),\mathcal{T}^*\big((x,(1-\tau)u+\tau\nu),\theta\big)\big]$$

and

$$2\left[\mathcal{T}_*\left(\left(\frac{a+b}{2},\omega\right),\theta\right),\mathcal{T}^*\left(\left(\frac{a+b}{2},\omega\right),\theta\right)\right] \le_I \big[\mathcal{T}_*\big((\tau a+(1-\tau)b,\omega),\theta\big),\mathcal{T}^*\big((\tau a+(1-\tau)b,\omega),\theta\big)\big]$$

$$+\big[\mathcal{T}_*\big((\tau a+(1-\tau)b,\omega),\theta\big),\mathcal{T}^*\big((\tau a+(1-\tau)b,\omega),\theta\big)\big]$$

It follows that

$$\mathcal{T}_\theta\left(x,\frac{u+\nu}{2}\right) \le_I \mathcal{T}_\theta(x,\tau u+(1-\tau)\nu)+\mathcal{T}_\theta(x,(1-\tau)u+\tau\nu) \quad (40)$$

and

$$\mathcal{T}_\theta\left(\frac{a+b}{2}, \omega\right) \leq_I \mathcal{T}_\theta(\tau a + (1-\tau)b, \omega) + \mathcal{T}_\theta(\tau a + (1-\tau)b, \omega) \tag{41}$$

Since $\mathcal{T}_\theta(x, .)$ and $\mathcal{T}_\theta(., \omega)$, both are coordinated convex-IVFs, then from inequality (Eq 5), for every $\theta \in [0, 1]$, inequality (Eqs 40 and 41) we have

$$\mathcal{T}_\theta\left(x, \frac{u+\nu}{2}\right) \leq_I \frac{1}{\nu-u} \int_u^\nu \mathcal{T}_\theta(x, \omega) d\omega \leq_I \frac{\mathcal{T}_\theta(x,u) + \mathcal{T}_\theta(x,\nu)}{2} \tag{42}$$

and

$$\mathcal{T}_\theta\left(\frac{a+b}{2}, \omega\right) \leq_I \frac{1}{b-a} \int_a^b \mathcal{T}_\theta(x, \omega) dx \leq_I \frac{\mathcal{T}_\theta(a,\omega) + \mathcal{T}_\theta(b,\omega)}{2} \tag{43}$$

Dividing double inequality (Eq 42) by $(b-a)$, and integrating with respect to x over $[a, b]$, we have

$$\frac{1}{b-a} \int_a^b \mathcal{T}_\theta\left(x, \frac{u+\nu}{2}\right) dx \leq_I \frac{1}{(b-a)(\nu-u)} \int_a^b \int_u^\nu \mathcal{T}_\theta(x, \omega) d\omega dx$$

$$\leq_I \frac{1}{2(b-a)}\left[\int_a^b \mathcal{T}_\theta(x, u) dx + \int_a^b \mathcal{T}_\theta(x, \nu) dx\right] \tag{44}$$

Similarly, dividing double inequality (Eq 43) by $(\nu-u)$, and integrating with respect to x over $[u, \nu]$, we have

$$\frac{1}{\nu-u} \int_u^\nu \mathcal{T}_\theta\left(\frac{a+b}{2}, \omega\right) d\omega \leq_I \frac{1}{(b-a)(\nu-u)} \int_a^b \int_u^\nu \mathcal{T}_\theta(x, \omega) d\omega dx$$

$$\leq_I \frac{1}{2(\nu-u)}\left[\int_u^\nu \mathcal{T}_\theta(a, \omega) d\omega + \int_u^\nu \mathcal{T}_\theta(b, \omega) d\omega\right] \tag{45}$$

By adding Eqs (44) and (45), we have

$$\frac{1}{2}\left[\frac{1}{b-a} \int_a^b \mathcal{T}_\theta\left(x, \frac{u+\nu}{2}\right) dx + \frac{1}{\nu-u} \int_u^\nu \mathcal{T}_\theta\left(\frac{a+b}{2}, \omega\right) d\omega\right]$$

$$\leq_I \frac{1}{(b-a)(\nu-u)} \int_a^b \int_u^\nu \mathcal{T}_\theta(x, \omega) d\omega dx$$

$$\leq_I \frac{1}{4(b-a)}\left[\int_a^b \mathcal{T}_\theta(x, u) dx + \int_a^b \mathcal{T}_\theta(x, \nu) dx\right] + \frac{1}{4(\nu-u)}\left[\int_u^\nu \mathcal{T}_\theta(a, \omega) d\omega + \int_u^d \mathcal{T}_\theta(b, \omega) d\omega\right] \tag{46}$$

Since $\mathcal{T}$ is FIVF, then inequality (Eq 46), we have

$$\frac{1}{2}\left[\frac{1}{b-a} \int_a^b \tilde{\mathcal{T}}\left(x, \frac{u+\nu}{2}\right) dx \,\tilde{+}\, \frac{1}{\nu-u} \int_u^\nu \tilde{\mathcal{T}}\left(\frac{a+b}{2}, \omega\right) d\omega\right] \preccurlyeq \frac{1}{(b-a)(\nu-u)} \int_a^b \int_u^\nu \tilde{\mathcal{T}}(x, \omega) d\omega dx$$

$$\preccurlyeq \frac{1}{4(b-a)}\left[\int_a^b \tilde{\mathcal{T}}(x, u) dx \,\tilde{+}\, \int_a^b \tilde{\mathcal{T}}(x, \nu) dx\right] \tilde{+} \frac{1}{4(\nu-u)}\left[\int_u^\nu \tilde{\mathcal{T}}(a, \omega) d\omega \,\tilde{+}\, \int_u^\nu \tilde{\mathcal{T}}(b, \omega) d\omega\right] \tag{47}$$

From the left side of inequality (Eq 5), for each $\theta \in [0, 1]$, we have

$$\mathcal{T}_\theta\left(\frac{a+b}{2}, \frac{u+\nu}{2}\right) \leq_I \frac{1}{b-a} \int_a^b \mathcal{T}_\theta\left(x, \frac{u+\nu}{2}\right) dx \tag{48}$$

$$\mathcal{T}_\theta\left(\frac{a+b}{2}, \frac{u+\nu}{2}\right) \leq_I \frac{1}{\nu-u} \int_u^\nu \mathcal{T}_\theta\left(\frac{a+b}{2}, \omega\right) d\omega \tag{49}$$

Taking addition of inequality (Eq 48) with inequality (Eq 49), we have

$$\mathcal{T}_\theta\left(\frac{a+b}{2},\frac{u+\nu}{2}\right) \leq_I \frac{1}{2}\left[\frac{1}{b-a}\int_a^b \mathcal{T}_\theta\left(x,\frac{u+\nu}{2}\right)dx + \frac{1}{\nu-u}\int_u^\nu \mathcal{T}_\theta\left(\frac{a+b}{2},\omega\right)d\omega\right].$$

Since $\tilde{\mathcal{T}}$ is a FIVF, then it follows that

$$\tilde{\mathcal{T}}\left(\frac{a+b}{2},\frac{u+\nu}{2}\right) \preccurlyeq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b \tilde{\mathcal{T}}\left(x,\frac{u+\nu}{2}\right)dx \;\tilde{+}\; \frac{1}{\nu-u}\int_u^\nu \tilde{\mathcal{T}}\left(\frac{a+b}{2},\omega\right)d\omega\right] \tag{50}$$

Now from right side of inequality (Eq 5), for every $\theta \in [0,1]$, we have

$$\frac{1}{b-a}\int_a^b \mathcal{T}_\theta(x,u)dx \leq_I \frac{\mathcal{T}_\theta(a,u)+\mathcal{T}_\theta(b,u)}{2} \tag{51}$$

$$\frac{1}{b-a}\int_a^b \mathcal{T}_\theta(x,\nu)dx \leq_I \frac{\mathcal{T}_\theta(a,\nu)+\mathcal{T}_\theta(b,\nu)}{2} \tag{52}$$

$$\frac{1}{\nu-u}\int_u^\nu \mathcal{T}_\theta(a,\omega)d\omega \leq_I \frac{\mathcal{T}_\theta(a,\nu)+\mathcal{T}_\theta(a,u)}{2} \tag{53}$$

$$\frac{1}{\nu-u}\int_u^\nu \mathcal{T}_\theta(b,\omega)d\omega \leq_I \frac{\mathcal{T}_\theta(b,\nu)+\mathcal{T}_\theta(b,u)}{2} \tag{54}$$

By adding inequalities (Eqs 51–54), we have

$$\frac{1}{4(b-a)}\left[\int_a^b \mathcal{T}_\theta(x,u)\,dx + \int_a^b \mathcal{T}_\theta(x,\nu)dx\right] + \frac{1}{4(\nu-u)}\left[\int_u^\nu \mathcal{T}_\theta(a,\omega)d\omega + \int_u^\nu \mathcal{T}_\theta(b,\omega)d\omega\right]$$
$$\leq_I \frac{\mathcal{T}_\theta(a,u)+\mathcal{T}_\theta(b,u)+\mathcal{T}_\theta(a,\nu)+\mathcal{T}_\theta(b,\nu)}{4}$$

Since $\mathcal{T}$ is a FIVF, then it follows that

$$\frac{1}{4(b-a)}\left[\int_a^b \tilde{\mathcal{T}}(x,u)dx \;\tilde{+}\; \int_a^b \tilde{\mathcal{T}}(x,\nu)dx\right] + \frac{1}{4(\nu-u)}\left[\int_u^\nu \tilde{\mathcal{T}}(a,\omega)\,d\omega \;\tilde{+}\; \int_u^\nu \tilde{\mathcal{T}}(b,\omega)d\omega\right]$$
$$\preccurlyeq \frac{\tilde{\mathcal{T}}(a,u)\tilde{+}\tilde{\mathcal{T}}(b,u)\tilde{+}\tilde{\mathcal{T}}(a,\nu)\tilde{+}\tilde{\mathcal{T}}(b,\nu)}{4} \tag{55}$$

By combining inequalities Eqs (47), (50) and (55), we get the desired result.

Example 3.2. We consider the FIVFs $\tilde{\mathcal{T}}: [0,1]\times[0,1] \to \mathbb{F}_0$ defined by,

$$\mathcal{T}(x)(\sigma) = \begin{cases} \frac{\sigma}{2(6+e^x)(6+e^\omega)}, & \sigma \in [0, 2(6+e^x)(6+e^\omega)] \\ \frac{4(6+e^x)(6+e^\omega)-\sigma}{2(6+e^x)(6+e^\omega)}, & \sigma \in (2(6+e^x)(6+e^\omega), 4(6+e^x)(6+e^\omega)] \\ 0, & \text{otherwise} \end{cases}$$

Then, for each $\theta \in [0,1]$, we have $\mathcal{T}_\theta(x) = [2\theta(6+e^x)(6+e^\omega), (4+2\theta)(6+e^x)(6+e^\omega)\ ]$. Since end point functions $\mathcal{T}_*((x,\omega),\theta)$, $\mathcal{T}^*((x,\omega),\theta)$ are coordinate concave functions for each $\theta \in [0,1]$. Hence $\tilde{\mathcal{T}}(x,\omega)$ is coordinate concave FIVF.

$$\mathcal{T}_\theta\left(\frac{a+b}{2},\frac{u+\nu}{2}\right) = \left[2\theta\left(5+e^{\frac{1}{2}}\right)^2, 2(2+\theta)\left(6+e^{\frac{1}{2}}\right)^2\ \right]$$

$$\frac{1}{2}\left[\frac{1}{b-a}\int_a^b \mathcal{T}_\theta\left(x,\frac{u+\nu}{2}\right)dx+\frac{1}{\nu-u}\int_u^\nu \mathcal{T}_\theta\left(\frac{a+b}{2},\omega\right)d\omega\right]=\left[4\theta\left(6+e^{\frac{1}{2}}\right)(5+e),4(2+\theta)\left(6+e^{\frac{1}{2}}\right)(5+e)\right]$$

$$\frac{1}{(b-a)(\nu-u)}\int_a^b\int_u^\nu \mathcal{T}_\theta(x,\omega)d\omega dx=[2\theta(5+e)^2,2(2+\theta)(5+e)^2]$$

$$\frac{1}{4(b-a)}\left[\int_a^b \mathcal{T}_\theta(x,u)dx+\int_a^b \mathcal{T}_\theta(x,\nu)dx\right]+\frac{1}{4(\nu-u)}\left[\int_u^\nu \mathcal{T}_\theta(a,\omega)d\omega+\int_u^\nu \mathcal{T}_\theta(b,\omega)d\omega\right]$$

$$=[\theta(5+e)(13+e),(2+\theta)(5+e)(13+e)]$$

$$\frac{\mathcal{T}_\theta(a,u)+\mathcal{T}_\theta(b,u)+\mathcal{T}_\theta(a,\nu)+\mathcal{T}_\theta(b,\nu)}{4}=\left[\theta\frac{(6+e)(20+e)+49}{2},2(2+\theta)\frac{(6+e)(20+e)+49}{2}\right]$$

That is

$$\left[2\theta\left(5+e^{\frac{1}{2}}\right)^2,2(2+\theta)\left(6+e^{\frac{1}{2}}\right)^2\right]\leq_I\left[4\theta\left(6+e^{\frac{1}{2}}\right)(5+e),4(2+\theta)\left(6+e^{\frac{1}{2}}\right)(5+e)\right]$$
$$\leq_I[2\theta(5+e)^2,2(2+\theta)(5+e)^2]\leq_I[\theta(5+e)(13+e),(2+\theta)(5+e)(13+e)]$$
$$\leq_I\left[\theta\frac{(6+e)(20+e)+49}{2},2(2+\theta)\frac{(6+e)(20+e)+49}{2}\right].$$

Hence, Theorem 3.1 has been verified.

We now give HH-Fejér inequality for coordinated convex FIVFs by means of FOR in the following result.

Theorem 3.3. Let $\tilde{\mathcal{T}}:\Delta=[a,b]\times[u,\nu]\to\mathbb{F}_0$ be a coordinated convex FIVF with $a<b$ and $u<\nu$. Then, from θ-levels, we get the collection of IVFs $\mathcal{T}_\theta:\Delta\to\mathbb{R}_I^+$ are given by $\mathcal{T}_\theta(x,\omega)=[\mathcal{T}_*((x,\omega),\theta),\mathcal{T}^*((x,\omega),\theta)]$ for all $(x,\omega)\in\Delta$ and for all $\theta\in[0,1]$. Let $\Omega:[a,b]\to\mathbb{R}$ with $\Omega(x)\geq 0$, $\int_a^b\Omega(x)dx>0$ and $\mathcal{W}:[u,\nu]\to\mathbb{R}$ with $\mathcal{W}(\omega)\geq 0$, $\int_u^\nu\mathcal{W}(\omega)d\omega>0$, be two symmetric functions with respect to $\frac{a+b}{2}$ and $\frac{u+\nu}{2}$ respectively. Then, following inequality holds:

$$\tilde{\mathcal{T}}\left(\frac{a+b}{2},\frac{u+\nu}{2}\right)\preccurlyeq\frac{1}{2}\left[\frac{1}{\int_a^b\Omega(x)dx}\int_a^b\tilde{\mathcal{T}}\left(x,\frac{u+\nu}{2}\right)\Omega(x)dx\,\tilde{+}\,\frac{1}{\int_a^b\mathcal{W}(\omega)d\omega}\int_a^b\tilde{\mathcal{T}}\left(\frac{a+b}{2},\omega\right)\mathcal{W}(\omega)d\omega\right]$$
$$\preccurlyeq\frac{1}{\int_a^b\Omega(x)dx\int_a^b\mathcal{W}(\omega)d\omega}\int_a^b\int_u^\nu\tilde{\mathcal{T}}(x,\omega)\Omega(x)\mathcal{W}(\omega)d\omega dx$$
$$\preccurlyeq\frac{1}{4\int_a^b\Omega(x)dx}\left[\int_a^b\tilde{\mathcal{T}}(x,u)\,dx\,\tilde{+}\int_a^b\tilde{\mathcal{T}}(x,\nu)dx\right]$$
$$\tilde{+}\,\frac{1}{4\int_a^b\mathcal{W}(\omega)d\omega}\left[\int_u^\nu\tilde{\mathcal{T}}(a,\omega)d\omega\,\tilde{+}\int_u^\nu\tilde{\mathcal{T}}(b,\omega)d\omega\right]$$
$$\preccurlyeq\frac{\tilde{\mathcal{T}}(a,u)\tilde{+}\tilde{\mathcal{T}}(b,u)\tilde{+}\tilde{\mathcal{T}}(a,\nu)\tilde{+}\tilde{\mathcal{T}}(b,\nu)}{4}.\tag{56}$$

Proof. Since $\tilde{\mathcal{T}}$ both is a coordinated convex FIVF on Δ, it follows that functions, then by Lemma 2.1.14, there exist

$$\tilde{\mathcal{T}}_x:[u,\nu]\to\mathbb{F}_0,\ \tilde{\mathcal{T}}_x(\omega)=\tilde{\mathcal{T}}(x,\omega),\qquad \tilde{\mathcal{T}}_\omega:[a,b]\to\mathbb{F}_0,\ \tilde{\mathcal{T}}_\omega(x)=\tilde{\mathcal{T}}(x,\omega).$$

Thus from inequality (Eq 27), for each $\theta \in [0,1]$, we have

$$\mathcal{T}_{\theta_x}\left(\frac{u+v}{2}\right) \leq_I \frac{1}{\int_u^v \mathcal{W}(\omega)d\omega} \int_u^v \mathcal{T}_{\theta_x}(\omega)\mathcal{W}(\omega)d\omega \leq_I \frac{\mathcal{T}_{\theta_x}(u)+\mathcal{T}_{\theta_x}(v)}{2}$$

and

$$\mathcal{T}_{\theta_\omega}\left(\frac{a+b}{2}\right) \leq_I \frac{1}{\int_a^b \Omega(x)dx} \int_a^b \mathcal{T}_{\theta_\omega}(x)\Omega(x)dx \leq_I \frac{\mathcal{T}_{\theta_\omega}(a)+\mathcal{T}_{\theta_\omega}(b)}{2}$$

The above inequalities can be written as

$$\mathcal{T}_{\theta}\left(x,\frac{u+v}{2}\right) \leq_I \frac{1}{\int_u^v \mathcal{W}(\omega)d\omega} \int_u^v \mathcal{T}_{\theta}(x,\omega)\mathcal{W}(\omega)d\omega \leq_I \frac{\mathcal{T}_{\theta}(x,u)+\mathcal{T}_{\theta}(x,v)}{2} \tag{57}$$

and

$$\mathcal{T}_{\theta}\left(\frac{a+b}{2},\omega\right) \leq_I \frac{1}{\int_a^b \Omega(x)dx} \int_a^b \mathcal{T}_{\theta}(x,\omega)\Omega(x)dx \leq_I \frac{\mathcal{T}_{\theta}(a,\omega)+\mathcal{T}_{\theta}(b,\omega)}{2} \tag{58}$$

Multiplying Eq (57) by $\Omega(x)$ and then integrating the resultant with respect to x over $[a,b]$, we have

$$\int_a^b \mathcal{T}_{\theta}\left(x,\frac{u+v}{2}\right)\Omega(x)dx \leq_I \frac{1}{\int_u^v \mathcal{W}(\omega)d\omega} \int_a^b \int_u^v \mathcal{T}_{\theta}(x,\omega)\Omega(x)\mathcal{W}(\omega)d\omega dx \leq_I \int_a^b \frac{\mathcal{T}_{\theta}(x,u)+\mathcal{T}_{\theta}(x,v)}{2}\Omega(x)dx. \tag{59}$$

Now, multiplying Eq (58) by $\mathcal{W}(\omega)$ and then integrating the resultant with respect to ω over $[u,v]$, we have

$$\int_u^v \mathcal{T}_{\theta}\left(\frac{a+b}{2},\omega\right)\mathcal{W}(\omega)d\omega \leq_I \frac{1}{\int_a^b \Omega(x)dx} \int_a^b \int_u^v \mathcal{T}_{\theta}(x,\omega)\Omega(x)\mathcal{W}(\omega)dxd\omega \leq_I \int_a^b \frac{\mathcal{T}_{\theta}(a,\omega)+\mathcal{T}_{\theta}(b,\omega)}{2}\mathcal{W}(\omega)d\omega \tag{60}$$

Since $\int_a^b \Omega(x)dx > 0$ and $\int_a^b \mathcal{W}(\omega)d\omega > 0$, then dividing Eqs (59) and (60) by $\int_a^b \Omega(x)dx > 0$ and $\int_a^b \mathcal{W}(\omega)d\omega > 0$, respectively, we get

$$\frac{1}{2}\left[\frac{1}{\int_a^b \Omega(x)dx} \int_a^b \mathcal{T}_{\theta}\left(x,\frac{u+v}{2}\right)\Omega(x)dx + \frac{1}{\int_a^b \mathcal{W}(\omega)d\omega} \int_a^b \mathcal{T}_{\theta}\left(\frac{a+b}{2},\omega\right)\mathcal{W}(\omega)d\omega\right]$$

$$\leq_I \frac{1}{\int_a^b \Omega(x)dx \int_a^b \mathcal{W}(\omega)d\omega} \int_a^b \int_u^v \mathcal{T}_{\theta}(x,\omega)\Omega(x)\mathcal{W}(\omega)d\omega dx.$$

$$\leq_I \left[\frac{1}{\int_a^b \Omega(x)dx} \int_a^b \frac{\mathcal{T}_{\theta}(x,u)+\mathcal{T}_{\theta}(x,v)}{4}\Omega(x)dx + \frac{1}{\int_u^v \mathcal{W}(\omega)d\omega} \int_a^b \frac{\mathcal{T}_{\theta}(a,\omega)+\mathcal{T}_{\theta}(b,\omega)}{4}\mathcal{W}(\omega)d\omega\right] \tag{61}$$

Now, from the left part of double inequalities (Eqs 57 and 58), we obtain

$$\mathcal{T}_{\theta}\left(\frac{a+b}{2},\frac{u+v}{2}\right) \leq_I \frac{1}{\int_u^v \mathcal{W}(\omega)d\omega} \int_u^v \mathcal{T}_{\theta}\left(\frac{a+b}{2},\omega\right)\mathcal{W}(\omega)d\omega \tag{62}$$

and

$$\mathcal{T}_{\theta}\left(\frac{a+b}{2},\frac{u+v}{2}\right) \leq_I \frac{1}{\int_a^b \Omega(x)dx} \int_a^b \mathcal{T}_{\theta}\left(x,\frac{u+v}{2}\right)\Omega(x)dx \tag{63}$$

Summing the inequalities (Eqs 62 and 63), we get

$$\mathcal{T}_\theta\left(\frac{a+b}{2},\frac{u+v}{2}\right) \leq_I \frac{1}{2}\left[\frac{1}{\int_a^b \Omega(x)dx}\int_a^b \mathcal{T}_\theta\left(x,\frac{u+v}{2}\right)\Omega(x)dx + \frac{1}{\int_u^v \mathcal{W}(\omega)d\omega}\int_u^v \mathcal{T}_\theta\left(\frac{a+b}{2},\omega\right)\mathcal{W}(\omega)d\omega\right] \quad (64)$$

Similarly, from the right part of Eqs (57) and (58), we can obtain

$$\frac{1}{\int_u^v \mathcal{W}(\omega)d\omega}\int_u^v \mathcal{T}_\theta(a,\omega)\mathcal{W}(\omega)d\omega \leq_I \frac{\mathcal{T}_\theta(a,u)+\mathcal{T}_\theta(a,v)}{2} \quad (65)$$

$$\frac{1}{\int_u^v \mathcal{W}(\omega)d\omega}\int_u^v \mathcal{T}_\theta(b,\omega)\mathcal{W}(\omega)d\omega \leq_I \frac{\mathcal{T}_\theta(b,u)+\mathcal{T}_\theta(b,v)}{2} \quad (66)$$

and

$$\frac{1}{\int_a^b \Omega(x)dx}\int_a^b \mathcal{T}_\theta(x,u)\Omega(x)dx \leq_I \frac{\mathcal{T}_\theta(a,u)+\mathcal{T}_\theta(b,u)}{2} \quad (67)$$

$$\frac{1}{\int_a^b \Omega(x)dx}\int_a^b \mathcal{T}_\theta(x,v)\Omega(x)dx \leq_I \frac{\mathcal{T}_\theta(a,v)+\mathcal{T}_\theta(b,v)}{2} \quad (68)$$

Adding Eqs (65)–(68) and dividing by 4, we get

$$\frac{1}{4\int_u^v \mathcal{W}(\omega)d\omega}\left[\int_u^v \mathcal{T}_\theta(a,\omega)\mathcal{W}(\omega)d\omega + \int_u^v \mathcal{T}_\theta(b,\omega)\mathcal{W}(\omega)d\omega\right] + \frac{1}{4\int_a^b \Omega(x)dx}\left[\int_a^b \mathcal{T}_\theta(x,u)\Omega(x)dx + \int_a^b \mathcal{T}_\theta(x,v)\Omega(x)dx\right] \leq_I \frac{\mathcal{T}_\theta(a,u)+\mathcal{T}_\theta(a,v)+\mathcal{T}_\theta(b,u)+\mathcal{T}_\theta(b,v)}{4} \quad (69)$$

Combing inequalities Eqs (61), (64) and (69), we obtain

$$\begin{aligned}\mathcal{T}_\theta\left(\frac{a+b}{2},\frac{u+v}{2}\right) &\leq_I \frac{1}{2}\left[\frac{1}{\int_a^b \Omega(x)dx}\int_a^b \mathcal{T}_\theta\left(x,\frac{u+v}{2}\right)\Omega(x)dx + \frac{1}{\int_u^v \mathcal{W}(\omega)d\omega}\int_u^v \mathcal{T}_\theta\left(\frac{a+b}{2},\omega\right)\mathcal{W}(\omega)d\omega\right] \\ &\leq_I \frac{1}{\int_a^b \Omega(x)dx\int_a^b \mathcal{W}(\omega)d\omega}\int_a^b\int_u^v \mathcal{T}(x,\omega)\Omega(x)\mathcal{W}(\omega)d\omega dx. \\ &\leq_I \frac{1}{\mathbf{4}\int_u^v \mathcal{W}(\omega)d\omega}\left[\int_u^v \mathcal{T}_\theta(a,\omega)\mathcal{W}(\omega)d\omega + \int_u^v \mathcal{T}_\theta(b,\omega)\mathcal{W}(\omega)d\omega\right] \\ &+ \frac{1}{\mathbf{4}\int_a^b \Omega(x)dx}\left[\int_a^b \mathcal{T}_\theta(x,u)\Omega(x)dx + \int_a^b \mathcal{T}_\theta(x,v)\Omega(x)dx\right] \\ &\leq_I \frac{\mathcal{T}_\theta(a,u)+\mathcal{T}_\theta(a,v)}{2} + \frac{\mathcal{T}_\theta(b,u)+\mathcal{T}_\theta(b,v)}{2} + \frac{\mathcal{T}_\theta(a,u)+\mathcal{T}_\theta(b,u)}{2} + \frac{\mathcal{T}_\theta(a,v)+\mathcal{T}_\theta(b,v)}{2}\end{aligned}$$

That is

$$\begin{aligned}\tilde{\mathcal{T}}\left(\frac{a+b}{2},\frac{u+v}{2}\right) &\preccurlyeq \frac{1}{2}\left[\frac{1}{\int_a^b \Omega(x)dx}\int_a^b \tilde{\mathcal{T}}\left(x,\frac{u+v}{2}\right)\Omega(x)dx \,\tilde{+}\, \frac{1}{\int_a^b \mathcal{W}(\omega)d\omega}\int_a^b \tilde{\mathcal{T}}\left(\frac{a+b}{2},\omega\right)\mathcal{W}(\omega)d\omega\right] \\ &\preccurlyeq \frac{1}{\int_a^b \Omega(x)dx\int_a^b \mathcal{W}(\omega)d\omega}\int_a^b\int_u^v \tilde{\mathcal{T}}(x,\omega)\Omega(x)\mathcal{W}(\omega)d\omega dx \\ &\preccurlyeq \frac{1}{4\int_a^b \Omega(x)dx}\left[\int_a^b \tilde{\mathcal{T}}(x,u)\,dx \,\tilde{+} \int_a^b \tilde{\mathcal{T}}(x,v)dx\right]\end{aligned}$$

$$\widetilde{+}\frac{1}{4\int_a^b \mathcal{W}(\omega)d\omega}\left[\int_u^v \tilde{\mathcal{T}}(a,\omega)d\omega \,\widetilde{+}\, \int_u^v \tilde{\mathcal{T}}(b,\omega)d\omega\right]$$

$$\preccurlyeq \frac{\tilde{\mathcal{T}}(a,u)\widetilde{+}\tilde{\mathcal{T}}(b,u)\widetilde{+}\tilde{\mathcal{T}}(a,v)\widetilde{+}\tilde{\mathcal{T}}(b,v)}{4}$$

Hence, this concludes the proof.

We now obtain some HH-inequalities for the product of coordinated convex FIVFs. These inequalities are refinements of some known inequalities, see [11,13].

Theorem 3.4. Let $\tilde{\mathcal{T}}, \tilde{\mathcal{J}} : \Delta = [a,b]\times[u,v] \subset \mathbb{R}^2 \to \mathbb{F}_0$ be two coordinated convex FIVFs on Δ, whose θ-levels $\mathcal{T}_\theta,\ \mathcal{J}_\theta: [a,b]\times[u,v] \to \mathbb{R}_I^+$ are defined by $\mathcal{T}_\theta(x,\omega) = \left[\mathcal{T}_*\big((x,\omega),\theta\big), \mathcal{T}^*\big((x,\omega),\theta\big)\right]$ and $\mathcal{J}_\theta(x,\omega) = \left[\mathcal{J}_*\big((x,\omega),\theta\big), \mathcal{J}^*\big((x,\omega),\theta\big)\right]$ for all $(x,\omega)\in\Delta$ and for all $\theta\in[0,1]$. Then, following inequality holds:

$$\frac{1}{(b-a)(v-u)}\int_a^b\int_u^v \tilde{\mathcal{T}}(x,\omega)\,\tilde{\times}\,\tilde{\mathcal{J}}(x,\omega)d\omega dx \preccurlyeq \frac{1}{9}\tilde{P}(a,b,u,v)\widetilde{+}\frac{1}{18}\widetilde{\mathcal{M}}(a,b,u,v)\widetilde{+}\frac{1}{36}\widetilde{\mathcal{N}}(a,b,u,v)$$

where

$$\tilde{P}(a,b,u,v) = \tilde{\mathcal{T}}(a,u)\,\tilde{\times}\,\tilde{\mathcal{J}}(a,u)\ \widetilde{+}\ \tilde{\mathcal{T}}(a,v)\,\tilde{\times}\,\tilde{\mathcal{J}}(a,v)\widetilde{+}\tilde{\mathcal{T}}(b,u)\,\tilde{\times}\,\tilde{\mathcal{J}}(b,u)\ \widetilde{+}\ \tilde{\mathcal{T}}(b,v)\,\tilde{\times}\,\tilde{\mathcal{J}}(b,v)$$

$$\widetilde{\mathcal{M}}(a,b,u,v) = \tilde{\mathcal{T}}(a,u)\,\tilde{\times}\,\tilde{\mathcal{J}}(a,v)\ \widetilde{+}\ \tilde{\mathcal{T}}(a,v)\,\tilde{\times}\,\tilde{\mathcal{J}}(a,u)\widetilde{+}\tilde{\mathcal{T}}(b,u)\,\tilde{\times}\,\tilde{\mathcal{J}}(b,v)\ \widetilde{+}\ \tilde{\mathcal{T}}(b,v)\,\tilde{\times}\,\tilde{\mathcal{J}}(b,u)$$
$$\widetilde{+}\tilde{\mathcal{T}}(a,u)\,\tilde{\times}\,\tilde{\mathcal{J}}(b,u)\ \widetilde{+}\ \tilde{\mathcal{T}}(b,v)\,\tilde{\times}\,\tilde{\mathcal{J}}(a,v)\widetilde{+}\tilde{\mathcal{T}}(b,u)\,\tilde{\times}\,\tilde{\mathcal{J}}(a,u)\ \widetilde{+}\ \tilde{\mathcal{T}}(a,v)\,\tilde{\times}\,\tilde{\mathcal{J}}(b,v)$$

$$\widetilde{\mathcal{N}}(a,b,u,v) = \tilde{\mathcal{T}}(a,u)\,\tilde{\times}\,\tilde{\mathcal{J}}(b,v)\ \widetilde{+}\ \tilde{\mathcal{T}}(b,u)\,\tilde{\times}\,\tilde{\mathcal{J}}(a,v)\widetilde{+}\ \tilde{\mathcal{T}}(b,v)\,\tilde{\times}\,\tilde{\mathcal{J}}(a,u)\widetilde{+}\tilde{\mathcal{T}}(b,u)\,\tilde{\times}\,\tilde{\mathcal{J}}(a,v)$$

and for each $\theta\in[0,1]$, $\tilde{P}(a,b,u,v)$, $\widetilde{\mathcal{M}}(a,b,u,v)$ and $\widetilde{\mathcal{N}}(a,b,u,v)$ are defined as follows:

$$P_\theta(a,b,u,v) = \left[P_*\big((a,b,u,v),\theta\big), P^*\big((a,b,u,v),\theta\big)\right],$$

$$\mathcal{M}_\theta(a,b,u,v) = \left[\mathcal{M}_*\big((a,b,u,v),\theta\big), \mathcal{M}^*\big((a,b,u,v),\theta\big)\right],$$

$$\mathcal{N}_\theta(a,b,u,v) = \left[\mathcal{N}_*\big((a,b,u,v),\theta\big), \mathcal{N}^*\big((a,b,u,v),\theta\big)\right].$$

Proof. Let $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{J}}$ both are coordinated convex FIVFs on $[a,b]\times[u,v]$. Then

$$\tilde{\mathcal{T}}(\tau a+(1-\tau)b, su+(1-s)v)$$
$$\preccurlyeq \tau s\tilde{\mathcal{T}}(a,u)\widetilde{+}\tau(1-s)\tilde{\mathcal{T}}(a,v)\widetilde{+}(1-\tau)s\tilde{\mathcal{T}}(b,u)\widetilde{+}(1-\tau)(1-s)\tilde{\mathcal{T}}(b,v)$$

and

$$\tilde{\mathcal{J}}(\tau a+(1-\tau)b, su+(1-s)v)$$
$$\preccurlyeq \tau s\tilde{\mathcal{J}}(a,u)\widetilde{+}\tau(1-s)\tilde{\mathcal{J}}(a,v)\widetilde{+}(1-\tau)s\tilde{\mathcal{J}}(b,u)\widetilde{+}(1-\tau)(1-s)\tilde{\mathcal{J}}(b,v).$$

Since $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{J}}$ both are coordinated convex FIVFs, then by Lemma 2.1.14, there exist

$$\tilde{\mathcal{T}}_x: [u,v]\to\mathbb{F}_0,\ \tilde{\mathcal{T}}_x(\omega) = \tilde{\mathcal{T}}(x,\omega), \qquad \tilde{\mathcal{J}}_x: [u,v]\to\mathbb{F}_0,\ \tilde{\mathcal{J}}_x(\omega) = \tilde{\mathcal{J}}(x,\omega),$$

and

$$\tilde{\mathcal{T}}_\omega: [a,b]\to\mathbb{F}_0,\ \tilde{\mathcal{T}}_\omega(x) = \tilde{\mathcal{T}}(x,\omega), \qquad \tilde{\mathcal{J}}_\omega: [a,b]\to\mathbb{F}_0,\ \tilde{\mathcal{J}}_\omega(x) = \tilde{\mathcal{J}}(x,\omega).$$

Since $\tilde{\mathcal{T}}_x$, $\tilde{\mathcal{J}}_x$, $\tilde{\mathcal{T}}_\omega$ and $\tilde{\mathcal{J}}_\omega$ are FIVFs, then by inequality (Eq 25), we have

$$\frac{1}{b-a}\int_a^b \tilde{\mathcal{T}}_\omega(x)\times\tilde{\mathcal{J}}_\omega(x)dx \preccurlyeq \frac{1}{3}[\tilde{\mathcal{T}}_\omega(a)\times\tilde{\mathcal{J}}_\omega(a)+\tilde{\mathcal{T}}_\omega(b)\times\tilde{\mathcal{J}}_\omega(b)]+\frac{1}{6}[\tilde{\mathcal{T}}_\omega(a)\times\tilde{\mathcal{J}}_\omega(b)+\tilde{\mathcal{T}}_\omega(b)\times\tilde{\mathcal{J}}_\omega(a)]$$

and

$$\frac{1}{\nu-u}\int_u^\nu \tilde{\mathcal{T}}_x(\omega)\times\tilde{\mathcal{J}}_x(\omega)d\omega \preccurlyeq \frac{1}{3}[\tilde{\mathcal{T}}_x(u)\times\tilde{\mathcal{J}}_x(u)+\tilde{\mathcal{T}}_x(\nu)\times\tilde{\mathcal{J}}_x(\nu)]+\frac{1}{6}[\tilde{\mathcal{T}}_x(u)\times\tilde{\mathcal{J}}_x(\nu)+\tilde{\mathcal{T}}_x(u)\times\tilde{\mathcal{J}}_x(\nu)]$$

For each $\theta\in[0,1]$, we have

$$\frac{1}{b-a}\int_a^b \mathcal{T}_{\theta_\omega}(x)\times\mathcal{J}_{\theta_\omega}(x)dx \leq_I \frac{1}{3}\left[\mathcal{T}_{\theta_\omega}(a)\times\mathcal{J}_{\theta_\omega}(a)+\mathcal{T}_{\theta_\omega}(b)\times\mathcal{J}_{\theta_\omega}(b)\right]+\frac{1}{6}\left[\mathcal{T}_{\theta_\omega}(a)\times\mathcal{J}_{\theta_\omega}(b)+\mathcal{T}_{\theta_\omega}(b)\times\mathcal{J}_{\theta_\omega}(a)\right]$$

and

$$\frac{1}{\nu-u}\int_u^\nu \mathcal{T}_{\theta_x}(\omega)\times\mathcal{J}_{\theta_x}(\omega)d\omega \leq_I \frac{1}{3}\left[\mathcal{T}_{\theta_x}(u)\times\mathcal{J}_{\theta_x}(u)+\mathcal{T}_{\theta_x}(\nu)\times\mathcal{J}_{\theta_x}(\nu)\right]+\frac{1}{6}\left[\mathcal{T}_{\theta_x}(u)\times\mathcal{J}_{\theta_x}(\nu)+\mathcal{T}_{\theta_x}(u)\times\mathcal{J}_{\theta_x}(\nu)\right]$$

The above inequalities can be written as

$$\frac{1}{b-a}\int_a^b \mathcal{T}_\theta(x,\omega)\times\mathcal{J}_\theta(x,\omega)dx \leq_I \frac{1}{3}[\mathcal{T}_\theta(a,\omega)\times\mathcal{J}_\theta(a,\omega)+\mathcal{T}_\theta(b,\omega)\times\mathcal{J}_\theta(b,\omega)]+\frac{1}{6}[\mathcal{T}_\theta(a,\omega)\times\mathcal{J}_\theta(b,\omega)+\mathcal{T}_\theta(b,\omega)\times\mathcal{J}_\theta(a,\omega)] \quad (70)$$

and

$$\frac{1}{\nu-u}\int_u^\nu \mathcal{T}_\theta(x,\omega)\times\mathcal{J}_\theta(x,\omega)d\omega \leq_I \frac{1}{3}[\mathcal{T}_\theta(x,u)\times\mathcal{J}_\theta(x,u)+\mathcal{T}_\theta(x,\nu)\times\mathcal{J}_\theta(x,\nu)]+\frac{1}{6}[\mathcal{T}_\theta(x,u)\times\mathcal{J}_\theta(x,u)+\mathcal{T}_\theta(x,\nu)\times\mathcal{J}_\theta(x,\nu)] \quad (71)$$

Firstly we solve inequality (Eq 70), taking integration on the both sides of inequality with respect to ω over interval $[u,\nu]$ and dividing both sides by $\nu-u$, we have

$$\frac{1}{(b-a)(\nu-u)}\int_a^b\int_u^\nu \mathcal{T}_\theta(x,\omega)\times\mathcal{J}_\theta(x,\omega)d\omega dx \leq_I \frac{1}{3(\nu-u)}\int_u^\nu[\mathcal{T}_\theta(a,\omega)\times\mathcal{J}_\theta(a,\omega)+\mathcal{T}_\theta(b,\omega)\times\mathcal{J}_\theta(b,\omega)]d\omega+\frac{1}{6(\nu-u)}\int_u^\nu[\mathcal{T}_\theta(a,\omega)\times\mathcal{J}_\theta(b,\omega)+\mathcal{T}_\theta(b,\omega)\times\mathcal{J}_\theta(a,\omega)]\,d\omega \quad (72)$$

Now again by inequality (Eq 25), for each $\theta\in[0,1]$, we have

$$\frac{1}{(\nu-u)}\int_u^\nu \mathcal{T}_\theta(a,\omega)\times\mathcal{J}_\theta(a,\omega)d\omega \leq_I \frac{1}{3}\int_u^\nu[\mathcal{T}_\theta(a,u)\times\mathcal{J}_\theta(a,u)+\mathcal{T}_\theta(a,\nu)\times\mathcal{J}_\theta(a,\nu)]d\omega$$

$$+\frac{1}{6}\int_u^\nu[\mathcal{T}_\theta(a,u)\times\mathcal{J}_\theta(a,\nu)+\ \mathcal{T}_\theta(a,u)\times\mathcal{J}_\theta(a,\nu)]\,d\omega \tag{73}$$

$$\frac{1}{(\nu-u)}\int_u^\nu\mathcal{T}_\theta(b,\omega)\times\mathcal{J}_\theta(b,\omega)d\omega\leq_I\frac{1}{3}\int_u^\nu[\mathcal{T}_\theta(b,u)\times\mathcal{J}_\theta(b,u)+\ \mathcal{T}_\theta(b,\nu)\times\mathcal{J}_\theta(b,\nu)]d\omega$$

$$+\frac{1}{6}\int_u^\nu[\mathcal{T}_\theta(b,u)\times\mathcal{J}_\theta(b,\nu)+\ \mathcal{T}_\theta(b,u)\times\mathcal{J}_\theta(a,\nu)]\,d\omega \tag{74}$$

$$\frac{1}{(\nu-u)}\int_u^\nu\mathcal{T}_\theta(a,\omega)\times\mathcal{J}_\theta(b,\omega)\,d\omega\ \ \leq_I\frac{1}{3}\int_u^\nu[\mathcal{T}_\theta(a,u)\times\mathcal{J}_\theta(b,u)+\ \mathcal{T}_\theta(a,\nu)\times\mathcal{J}_\theta(b,\nu)]d\omega$$

$$+\frac{1}{6}\int_u^\nu[\mathcal{T}_\theta(a,u)\times\mathcal{J}_\theta(b,\nu)+\ \mathcal{T}_\theta(a,\nu)\times\mathcal{J}_\theta(b,u)]d\omega \tag{75}$$

$$\frac{1}{(\nu-u)}\int_u^\nu\mathcal{T}_\theta(b,\omega)\times\mathcal{J}_\theta(a,\omega)\,d\omega\leq_I\frac{1}{3}\int_u^\nu[\mathcal{T}_\theta(b,u)\times\mathcal{J}_\theta(a,u)+\ \mathcal{T}_\theta(b,\nu)\times\mathcal{J}_\theta(a,\nu)]d\omega$$

$$+\frac{1}{6}\int_u^\nu[\mathcal{T}_\theta(b,u)\times\mathcal{J}_\theta(a,\nu)+\ \mathcal{T}_\theta(b,\nu)\times\mathcal{J}_\theta(a,u)]\,d\omega \tag{76}$$

From Eqs (73)–(76), inequality (Eq 72) we have

$$\frac{1}{(b-a)(\nu-u)}\ \int_a^b\int_u^\nu\mathcal{T}_\theta(x,\omega)\times\mathcal{J}_\theta(x,\omega)d\omega dx\leq_I\frac{1}{9}P_\theta(a,b,u,\nu)+\frac{1}{18}\mathcal{M}_\theta(a,b,u,\nu)+$$
$$\frac{1}{36}\mathcal{N}_\theta(a,b,u,\nu)$$

That is

$$\frac{1}{(b-a)(\nu-u)}\ \int_a^b\int_u^\nu\tilde{\mathcal{T}}(x,\omega)\,\tilde{\times}\,\tilde{\mathcal{J}}(x,\omega)d\omega dx\preccurlyeq\frac{1}{9}\tilde{P}(a,b,u,\nu)\tilde{+}\frac{1}{18}\widetilde{\mathcal{M}}(a,b,u,\nu)\tilde{+}\frac{1}{36}\widetilde{\mathcal{N}}(a,b,u,\nu)$$

Hence, this concludes the proof of theorem.

Theorem 3.5. Let $\tilde{\mathcal{T}},\tilde{\mathcal{J}}:\Delta=[a,b]\times[u,\nu]\subset\mathbb{R}^2\to\mathbb{F}_0$ be two convex FIVFs. Then, from θ-levels, we get the collection of IVFs $\mathcal{T}_\theta,\ \mathcal{J}_\theta:\Delta\subset\mathbb{R}^2\to\mathbb{R}_I^+$ are given by $\mathcal{T}_\theta(x)=[\mathcal{T}_*((x,\omega),\theta),\mathcal{T}^*((x,\omega),\theta)]$ and $\mathcal{J}_\theta(x)=[\mathcal{J}_*((x,\omega),\theta),\mathcal{J}^*((x,\omega),\theta)]$ for all $(x,\omega)\in\Delta$ and for all $\theta\in[0,1]$. Then, following inequality holds:

$$4\ \tilde{\mathcal{T}}\left(\frac{a+b}{2},\frac{u+\nu}{2}\right)\tilde{\times}\,\tilde{\mathcal{J}}\left(\frac{a+b}{2},\frac{u+\nu}{2}\right)\quad\preccurlyeq$$
$$\frac{1}{(b-a)(\nu-u)}\ \int_a^b\int_u^\nu\tilde{\mathcal{T}}(x,\omega)\,\tilde{\times}\,\tilde{\mathcal{J}}(x,\omega)d\omega dx\,\tilde{+}\ \frac{5}{36}\tilde{P}(a,b,u,\nu)\tilde{+}\frac{7}{36}\widetilde{\mathcal{M}}(a,b,u,\nu)\tilde{+}\frac{2}{9}\widetilde{\mathcal{N}}(a,b,u,\nu)$$

where $\tilde{P}(a,b,u,\nu)$, $\widetilde{\mathcal{M}}(a,b,u,\nu)$ and $\widetilde{\mathcal{N}}(a,b,u,\nu)$ are given in Theorem 3.4.

Proof. Since $\tilde{\mathcal{T}},\tilde{\mathcal{J}}:\Delta\to\mathbb{F}_0$ be two convex FIVFs, then from inequality (Eq 26) and for each $\theta\in[0,1]$, we have

$$2\mathcal{T}_\theta\left(\frac{a+b}{2},\frac{u+\nu}{2}\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},\frac{u+\nu}{2}\right)\leq_I\frac{1}{b-a}\ \int_a^b\mathcal{T}_\theta\left(x,\frac{u+\nu}{2}\right)\times\mathcal{J}_\theta\left(x,\frac{u+\nu}{2}\right)dx$$

$$+\frac{1}{6}\left[\mathcal{T}_\theta\left(a,\frac{u+\nu}{2}\right)\times\mathcal{J}_\theta\left(a,\frac{u+\nu}{2}\right)+\ \mathcal{T}_\theta\left(b,\frac{u+\nu}{2}\right)\times\mathcal{J}_\theta\left(b,\frac{u+\nu}{2}\right)\right]$$

$$+\frac{1}{3}\left[\mathcal{T}_\theta\left(a,\frac{u+\nu}{2}\right)\times\mathcal{J}_\theta\left(b,\frac{u+\nu}{2}\right)+\ \mathcal{T}_\theta\left(b,\frac{u+\nu}{2}\right)\times\mathcal{J}_\theta\left(a,\frac{u+\nu}{2}\right)\right] \tag{77}$$

and

$$2\mathcal{T}_\theta\left(\frac{a+b}{2},\frac{u+\nu}{2}\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},\frac{u+\nu}{2}\right)\leq_I\frac{1}{\nu-u}\ \int_u^\nu\mathcal{T}_\theta\left(\frac{a+b}{2},\omega\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},\omega\right)d\omega$$

$$+\frac{1}{6}\left[\mathcal{T}_\theta\left(\frac{a+b}{2},u\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},u\right)+\ \mathcal{T}_\theta\left(\frac{a+b}{2},\nu\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},\nu\right)\right]$$

$$+\frac{1}{3}\left[\mathcal{T}_\theta\left(\frac{a+b}{2},u\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},\nu\right)+\ \mathcal{T}_\theta\left(\frac{a+b}{2},\nu\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},u\right)\right] \tag{78}$$

Summing the inequalities (Eqs 77 and 78), then taking the multiplication of the resultant one by 2, we obtain

$$8\mathcal{T}_\theta\left(\frac{a+b}{2},\frac{u+\nu}{2}\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},\frac{u+\nu}{2}\right)\leq_I\frac{2}{b-a}\ \int_a^b\mathcal{T}_\theta\left(x,\frac{u+\nu}{2}\right)\times\mathcal{J}_\theta\left(x,\frac{u+\nu}{2}\right)dx\ +\frac{2}{\nu-u}\ \int_u^\nu\mathcal{T}_\theta\left(\frac{a+b}{2},\omega\right)\times$$

$$\mathcal{J}_\theta\left(\frac{a+b}{2},\omega\right)dx+\frac{1}{6}\left[2\mathcal{T}_\theta\left(a,\frac{u+\nu}{2}\right)\times\mathcal{J}_\theta\left(a,\frac{u+\nu}{2}\right)+2\mathcal{T}_\theta\left(b,\frac{u+\nu}{2}\right)\times\mathcal{J}_\theta\left(b,\frac{u+\nu}{2}\right)\right]+\frac{1}{6}\left[2\mathcal{T}_\theta\left(\frac{a+b}{2},u\right)\times\right.$$

$$\left.\mathcal{J}_\theta\left(\frac{a+b}{2},u\right)+2\mathcal{T}_\theta\left(\frac{a+b}{2},\nu\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},\nu\right)\right]+\frac{1}{3}\left[2\mathcal{T}_\theta\left(a,\frac{u+\nu}{2}\right)\times\mathcal{J}_\theta\left(b,\frac{u+\nu}{2}\right)+2\mathcal{T}_\theta\left(b,\frac{u+\nu}{2}\right)\times\right.$$

$$\left.\mathcal{J}_\theta\left(a,\frac{u+\nu}{2}\right)\right]+\frac{1}{3}\left[2\mathcal{T}_\theta\left(\frac{a+b}{2},u\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},\nu\right)+2\mathcal{T}_\theta\left(\frac{a+b}{2},\nu\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},u\right)\right] \tag{79}$$

Now, with the help of integral inequality (Eq 26) for each integral on the right-hand side of Eq (79), we have

$$2\mathcal{T}_\theta\left(a,\frac{u+\nu}{2}\right)\times\mathcal{J}_\theta\left(a,\frac{u+\nu}{2}\right)$$

$$\leq_I\frac{1}{\nu-u}\ \int_u^\nu\mathcal{T}_\theta(a,\omega)\times\mathcal{J}_\theta(a,\omega)d\omega\ +\frac{1}{6}[\mathcal{T}_\theta(a,u)\times\mathcal{J}_\theta(a,u)+\ \mathcal{T}_\theta(a,\nu)\times\mathcal{J}_\theta(a,\nu)]$$

$$+\frac{1}{3}[\mathcal{T}_\theta(a,u)\times\mathcal{J}_\theta(a,\nu)+\ \mathcal{T}_\theta(a,\nu)\times\mathcal{J}_\theta(a,u)] \tag{80}$$

$$2\mathcal{T}_\theta\left(b,\frac{u+\nu}{2}\right)\times\mathcal{J}_\theta\left(b,\frac{u+\nu}{2}\right)$$

$$\leq_I\frac{1}{\nu-u}\ \int_u^\nu\mathcal{T}_\theta(b,\omega)\times\mathcal{J}_\theta(b,\omega)d\omega\ +\frac{1}{6}[\mathcal{T}_\theta(b,u)\times\mathcal{J}_\theta(b,u)+\ \mathcal{T}_\theta(b,\nu)\times\mathcal{J}_\theta(b,\nu)]$$

$$+\frac{1}{3}[\mathcal{T}_\theta(b,u)\times\mathcal{J}_\theta(b,\nu)+\ \mathcal{T}_\theta(b,\nu)\times\mathcal{J}_\theta(b,u)] \tag{81}$$

$$2\mathcal{T}_\theta\left(a,\frac{u+\nu}{2}\right)\times\mathcal{J}_\theta\left(b,\frac{u+\nu}{2}\right)$$

$$\leq_I\frac{1}{\nu-u}\ \int_u^\nu\mathcal{T}_\theta(a,\omega)\times\mathcal{J}_\theta(b,\omega)d\omega\ +\frac{1}{6}[\mathcal{T}_\theta(a,u)\times\mathcal{J}_\theta(b,u)+\ \mathcal{T}_\theta(a,\nu)\times\mathcal{J}_\theta(b,\nu)]$$

$$+\frac{1}{3}[\mathcal{T}_\theta(a,u)\times\mathcal{J}_\theta(b,\nu)+\ \mathcal{T}_\theta(a,\nu)\times\mathcal{J}_\theta(b,u)] \tag{82}$$

$$2\mathcal{T}_\theta\left(b,\frac{u+\nu}{2}\right)\times\mathcal{J}_\theta\left(a,\frac{u+\nu}{2}\right)$$
$$\leq_I \frac{1}{\nu-u}\int_u^\nu \mathcal{T}_\theta(b,\omega)\times\mathcal{J}_\theta(a,\omega)d\omega+\frac{1}{6}[\mathcal{T}_\theta(b,u)\times\mathcal{J}_\theta(a,u)+\mathcal{T}_\theta(b,\nu)\times\mathcal{J}_\theta(a,\nu)]$$
$$+\frac{1}{3}[\mathcal{T}_\theta(b,u)\times\mathcal{J}_\theta(a,\nu)+\mathcal{T}_\theta(b,\nu)\times\mathcal{J}_\theta(a,u)] \tag{83}$$

$$2\mathcal{T}_\theta\left(\frac{a+b}{2},u\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},u\right)$$
$$\leq_I \frac{1}{b-a}\int_a^b \mathcal{T}_\theta(x,u)\times\mathcal{J}_\theta(x,u)dx+\frac{1}{6}[\mathcal{T}_\theta(a,u)\times\mathcal{J}_\theta(a,u)+\mathcal{T}_\theta(b,u)\times\mathcal{J}_\theta(b,u)]$$
$$+\frac{1}{3}\left[\mathcal{T}_\theta\left(\frac{a+b}{2},u\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},u\right)+\mathcal{T}_\theta\left(\frac{a+b}{2},u\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},u\right)\right] \tag{84}$$

$$2\mathcal{T}_\theta\left(\frac{a+b}{2},\nu\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},\nu\right)$$
$$\leq_I \frac{1}{b-a}\int_a^b \mathcal{T}_\theta(x,\nu)\times\mathcal{J}_\theta(x,\nu)dx+\frac{1}{6}[\mathcal{T}_\theta(a,\nu)\times\mathcal{J}_\theta(a,\nu)+\mathcal{T}_\theta(b,\nu)\times\mathcal{J}_\theta(b,\nu)]$$
$$+\frac{1}{3}\left[\mathcal{T}_\theta\left(\frac{a+b}{2},\nu\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},\nu\right)+\mathcal{T}_\theta\left(\frac{a+b}{2},\nu\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},\nu\right)\right] \tag{85}$$

$$2\mathcal{T}_\theta\left(\frac{a+b}{2},u\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},\nu\right)$$
$$\leq_I \frac{1}{b-a}\int_a^b \mathcal{T}_\theta(x,u)\times\mathcal{J}_\theta(x,\nu)dx+\frac{1}{6}[\mathcal{T}_\theta(a,u)\times\mathcal{J}_\theta(a,\nu)+\mathcal{T}_\theta(b,u)\times\mathcal{J}_\theta(b,\nu)]$$
$$+\frac{1}{3}\left[\mathcal{T}_\theta\left(\frac{a+b}{2},u\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},\nu\right)+\mathcal{T}_\theta\left(\frac{a+b}{2},u\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},\nu\right)\right] \tag{86}$$

$$2\mathcal{T}_\theta\left(\frac{a+b}{2},\nu\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},u\right)$$
$$\leq_I \frac{1}{b-a}\int_a^b \mathcal{T}_\theta(x,\nu)\times\mathcal{J}_\theta(x,u)dx+\frac{1}{6}[\mathcal{T}_\theta(a,\nu)\times\mathcal{J}_\theta(a,u)+\mathcal{T}_\theta(b,\nu)\times\mathcal{J}_\theta(b,u)]$$
$$+\frac{1}{3}\left[\mathcal{T}_\theta\left(\frac{a+b}{2},\nu\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},u\right)+\mathcal{T}_\theta\left(\frac{a+b}{2},\nu\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},u\right)\right] \tag{87}$$

From Eqs (80)–(87), we have

$$8\mathcal{T}_\theta\left(\frac{a+b}{2},\frac{u+\nu}{2}\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},\frac{u+\nu}{2}\right)$$
$$\leq_I \frac{2}{b-a}\int_a^b \mathcal{T}_\theta\left(x,\frac{u+\nu}{2}\right)\times\mathcal{J}_\theta\left(x,\frac{u+\nu}{2}\right)dx+\frac{2}{\nu-u}\int_u^\nu \mathcal{T}_\theta\left(\frac{a+b}{2},\omega\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},\omega\right)dx$$
$$+\frac{1}{6(\nu-u)}\int_u^\nu \mathcal{T}_\theta(a,\omega)\times\mathcal{J}_\theta(a,\omega)d\omega+\frac{1}{6(\nu-u)}\int_u^\nu \mathcal{T}_\theta(b,\omega)\times\mathcal{J}_\theta(b,\omega)d\omega$$
$$+\frac{1}{6(b-a)}\int_a^b \mathcal{T}_\theta(x,u)\times\mathcal{J}_\theta(x,u)dx+\frac{1}{6(b-a)}\int_a^b \mathcal{T}_\theta(x,\nu)\times\mathcal{J}_\theta(x,\nu)dx$$

$$+\frac{1}{3(\nu-u)}\int_u^\nu \mathcal{T}_\theta(a,\omega)\times\mathcal{J}_\theta(b,\omega)d\omega \ +\frac{1}{3(\nu-u)}\int_u^\nu \mathcal{T}_\theta(b,\omega)\times\mathcal{J}_\theta(a,\omega)d\omega$$

$$+\frac{1}{3(b-a)}\ \int_a^b \mathcal{T}_\theta(x,u)\times\mathcal{J}_\theta(x,\nu)dx+\frac{1}{3(b-a)}\ \int_a^b \mathcal{T}_\theta(x,\nu)\times\mathcal{J}_\theta(x,u)dx$$

$$+\ \frac{1}{18}P_\theta(a,b,u,\nu)+\frac{1}{9}\mathcal{M}_\theta(a,b,u,\nu)+\frac{2}{9}\mathcal{N}_\theta(a,b,u,\nu) \qquad (88)$$

Now, again with the help of integral inequality (Eq 26) for first two integrals on the right-hand side of Eq (88), we have the following relation

$$\frac{2}{b-a}\ \int_a^b \mathcal{T}_\theta\left(x,\frac{u+\nu}{2}\right)\times\mathcal{J}_\theta\left(x,\frac{u+\nu}{2}\right)dx$$

$$\leq_I \frac{1}{(b-a)(\nu-u)}\ \int_a^b\int_u^\nu \mathcal{T}_\theta(x,\omega)\times\mathcal{J}_\theta(x,\omega)d\omega dx$$

$$+\frac{1}{3(b-a)}\int_a^b[\mathcal{T}_\theta(x,u)\times\mathcal{J}_\theta(x,u)+\ \mathcal{T}_\theta(x,\nu)\times\mathcal{J}_\theta(x,\nu)]dx$$

$$+\frac{1}{6(b-a)}\int_a^b[\mathcal{T}_\theta(u,x)\times\mathcal{J}_\theta(x,\nu)+\ \mathcal{T}_\theta(x,\nu)\times\mathcal{J}_\theta(x,u)]\,dx, \qquad (89)$$

$$\frac{2}{\nu-u}\ \int_u^\nu \mathcal{T}_\theta\left(\frac{a+b}{2},\omega\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},\omega\right)dx$$

$$\leq_I \frac{1}{(b-a)(\nu-u)}\ \int_a^b\int_u^\nu \mathcal{T}_\theta(x,\omega)\times\mathcal{J}_\theta(x,\omega)d\omega dx$$

$$+\frac{1}{3(\nu-u)}\int_u^\nu[\mathcal{T}_\theta(a,\omega)\times\mathcal{J}_\theta(a,\omega)+\ \mathcal{T}_\theta(b,\omega)\times\mathcal{J}_\theta(b,\omega)]d\omega$$

$$+\frac{1}{6(\nu-u)}\int_u^\nu[\mathcal{T}_\theta(a,\omega)\times\mathcal{J}_\theta(b,\omega)+\ \mathcal{T}_\theta(b,\omega)\times\mathcal{J}_\theta(a,\omega)]\,d\omega \qquad (90)$$

From Eqs (89) and (90), we have

$$8\mathcal{T}_\theta\left(\frac{a+b}{2},\frac{u+\nu}{2}\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},\frac{u+\nu}{2}\right)\leq_I \frac{1}{(b-a)(\nu-u)}\ \int_a^b\int_u^\nu \mathcal{T}_\theta(x,\omega)\times\mathcal{J}_\theta(x,\omega)d\omega dx+$$

$$\frac{1}{3(b-a)}\int_a^b[\mathcal{T}_\theta(x,u)\times\mathcal{J}_\theta(x,u)+\ \mathcal{T}_\theta(x,\nu)\times\mathcal{J}_\theta(x,\nu)]dx+\frac{1}{6(b-a)}\int_a^b[\mathcal{T}_\theta(x,u)\times\mathcal{J}_\theta(x,\nu)+$$

$$\mathcal{T}_\theta(x,\nu)\times\mathcal{J}_\theta(x,u)]\,dx+\frac{1}{(b-a)(\nu-u)}\ \int_a^b\int_u^\nu \mathcal{T}_\theta(x,\omega)\times\mathcal{J}_\theta(x,\omega)d\omega dx+\frac{1}{3(\nu-u)}\int_u^\nu[\mathcal{T}_\theta(a,\omega)\times$$

$$\mathcal{J}_\theta(a,\omega)+\ \mathcal{T}_\theta(b,\omega)\times\mathcal{J}_\theta(b,\omega)]d\omega+\frac{1}{6(\nu-u)}\int_u^\nu[\mathcal{T}_\theta(a,\omega)\times\mathcal{J}_\theta(b,\omega)+\ \mathcal{T}_\theta(b,\omega)\times$$

$$\mathcal{J}_\theta(a,\omega)]\,d\omega+\frac{1}{6(\nu-u)}\ \int_u^\nu \mathcal{T}_\theta(a,\omega)\times\mathcal{J}_\theta(a,\omega)d\omega\ +\frac{1}{6(\nu-u)}\ \int_u^\nu \mathcal{T}_\theta(b,\omega)\times\mathcal{J}_\theta(b,\omega)d\omega\ +$$

$$\frac{1}{6(b-a)}\ \int_a^b \mathcal{T}_\theta(x,u)\times\mathcal{J}_\theta(x,u)dx+\frac{1}{6(b-a)}\ \int_a^b \mathcal{T}_\theta(x,\nu)\times\mathcal{J}_\theta(x,\nu)dx\ \ +\frac{1}{3(\nu-u)}\int_u^\nu \mathcal{T}_\theta(a,\omega)\times$$

$$\mathcal{J}_\theta(b,\omega)d\omega\ \ +\frac{1}{3(\nu-u)}\int_u^\nu \mathcal{T}_\theta(b,\omega)\times\mathcal{J}_\theta(a,\omega)d\omega+\frac{1}{3(b-a)}\ \int_a^b \mathcal{T}_\theta(x,u)\times\mathcal{J}_\theta(x,\nu)dx+$$

$$\frac{1}{3(b-a)}\ \int_a^b \mathcal{T}_\theta(x,\nu)\times\mathcal{J}_\theta(x,u)dx\ \ +\ \ \frac{1}{18}P_\theta(a,b,u,\nu)+\frac{1}{9}\mathcal{M}_\theta(a,b,u,\nu)+\frac{2}{9}\mathcal{N}_\theta(a,b,u,\nu)$$

It follows that

$$8\mathcal{T}_\theta\left(\frac{a+b}{2},\frac{u+v}{2}\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},\frac{u+v}{2}\right)$$

$$\leq_I \frac{2}{(b-a)(v-u)}\int_a^b\int_u^v \mathcal{T}_\theta(x,\omega)\times\mathcal{J}_\theta(x,\omega)d\omega dx$$

$$+\frac{2}{3(b-a)}\int_a^b[\mathcal{T}_\theta(x,u)\times\mathcal{J}_\theta(x,u)+\ \mathcal{T}_\theta(x,v)\times\mathcal{J}_\theta(x,v)]dx$$

$$+\frac{1}{3(b-a)}\int_a^b[\mathcal{T}_\theta(x,u)\times\mathcal{J}_\theta(x,v)+\ \mathcal{T}_\theta(x,v)\times\mathcal{J}_\theta(x,u)]\,dx$$

$$+\frac{2}{3(v-u)}\int_u^v[\mathcal{T}_\theta(a,\omega)\times\mathcal{J}_\theta(a,\omega)+\ \mathcal{T}_\theta(b,\omega)\times\mathcal{J}_\theta(b,\omega)]d\omega$$

$$+\frac{1}{3(v-u)}\int_u^v[\mathcal{T}_\theta(a,\omega)\times\mathcal{J}_\theta(b,\omega)+\ \mathcal{T}_\theta(b,\omega)\times\mathcal{J}_\theta(a,\omega)]\,d\omega$$

$$+\ \frac{1}{18}P_\theta(a,b,u,v)+\frac{1}{9}\mathcal{M}_\theta(a,b,u,v)+\frac{2}{9}\mathcal{N}_\theta(a,b,u,v) \tag{91}$$

Now, using integral inequality (Eq 25) for integrals on the right-hand side of Eq (91), we have the following relation

$$\frac{1}{b-a}\int_a^b \mathcal{T}_\theta(x,u)\times\mathcal{J}_\theta(x,u)dx \leq_I \frac{1}{3}[\mathcal{T}_\theta(a,u)\times\mathcal{J}_\theta(a,u)+\ \mathcal{T}_\theta(b,u)\times\mathcal{J}_\theta(b,u)]+\frac{1}{6}[\mathcal{T}_\theta(a,u)\times \mathcal{J}_\theta(b,u)+\ \mathcal{T}_\theta(b,u)\times\mathcal{J}_\theta(a,u)] \tag{92}$$

$$\frac{1}{b-a}\int_a^b \mathcal{T}_\theta(x,v)\times\mathcal{J}_\theta(x,v)dx \leq_I \frac{1}{3}[\mathcal{T}_\theta(a,v)\times\mathcal{J}_\theta(a,v)+\ \mathcal{T}_\theta(b,v)\times\mathcal{J}_\theta(b,v)]+\frac{1}{6}[\mathcal{T}_\theta(a,v)\times \mathcal{J}_\theta(b,v)+\ \mathcal{T}_\theta(b,v)\times\mathcal{J}_\theta(a,v)] \tag{93}$$

$$\frac{1}{b-a}\int_a^b \mathcal{T}_\theta(x,u)\times\mathcal{J}_\theta(x,v)dx \leq_I \frac{1}{3}[\mathcal{T}_\theta(a,u)\times\mathcal{J}_\theta(a,v)+\ \mathcal{T}_\theta(b,u)\times\mathcal{J}_\theta(b,v)]+\frac{1}{6}[\mathcal{T}_\theta(a,u)\times \mathcal{J}_\theta(b,v)+\ \mathcal{T}_\theta(b,u)\times\mathcal{J}_\theta(a,v)] \tag{94}$$

$$\frac{1}{b-a}\int_a^b \mathcal{T}_\theta(x,v)\times\mathcal{J}_\theta(x,u)dx \leq_I \frac{1}{3}[\mathcal{T}_\theta(a,v)\times\mathcal{J}_\theta(a,u)+\ \mathcal{T}_\theta(b,v)\times\mathcal{J}_\theta(b,u)]+\frac{1}{6}[\mathcal{T}_\theta(a,v)\times \mathcal{J}_\theta(b,u)+\ \mathcal{T}_\theta(b,v)\times\mathcal{J}_\theta(a,u)] \tag{95}$$

$$\frac{1}{v-u}\int_u^v \mathcal{T}_\theta(a,\omega)\times\mathcal{J}_\theta(a,\omega)d\omega \leq_I \frac{1}{3}[\mathcal{T}_\theta(a,u)\times\mathcal{J}_\theta(a,u)+\ \mathcal{T}_\theta(a,v)\times\mathcal{J}_\theta(a,v)]+\frac{1}{6}[\mathcal{T}_\theta(a,u)\times \mathcal{J}_\theta(a,v)+\ \mathcal{T}_\theta(a,v)\times\mathcal{J}_\theta(a,u)] \tag{96}$$

$$\frac{1}{v-u}\int_u^v \mathcal{T}_\theta(b,\omega)\times\mathcal{J}_\theta(b,\omega)d\omega \leq_I \frac{1}{3}[\mathcal{T}_\theta(b,u)\times\mathcal{J}_\theta(b,u)+\ \mathcal{T}_\theta(b,v)\times\mathcal{J}_\theta(b,v)]+\frac{1}{6}[\mathcal{T}_\theta(b,u)\times \mathcal{J}_\theta(b,v)+\ \mathcal{T}_\theta(b,v)\times\mathcal{J}_\theta(b,u)] \tag{97}$$

$$\frac{1}{v-u}\int_u^v \mathcal{T}_\theta(a,\omega)\times\mathcal{J}_\theta(b,\omega)d\omega \leq_I \frac{1}{3}[\mathcal{T}_\theta(a,u)\times\mathcal{J}_\theta(b,u)+\ \mathcal{T}_\theta(a,v)\times\mathcal{J}_\theta(b,v)]+\frac{1}{6}[\mathcal{T}_\theta(a,u)\times \mathcal{J}_\theta(b,v)+\ \mathcal{T}_\theta(a,v)\times\mathcal{J}_\theta(b,u)] \tag{98}$$

$$\frac{1}{\nu-u}\int_u^\nu \mathcal{T}_\theta(b,\omega)\times\mathcal{J}_\theta(a,\omega)d\omega \leq_I \frac{1}{3}[\mathcal{T}_\theta(b,u)\times\mathcal{J}_\theta(a,u)+\mathcal{T}_\theta(b,\nu)\times\mathcal{J}_\theta(a,\nu)]+\frac{1}{6}[\mathcal{T}_\theta(b,u)\times \mathcal{J}_\theta(a,\nu)+\mathcal{T}_\theta(b,\nu)\times\mathcal{J}_\theta(a,u)] \quad (99)$$

From Eqs (92)–(99), inequality (Eq 91) we have

$$4\ \mathcal{T}_\theta\left(\frac{a+b}{2},\frac{u+\nu}{2}\right)\times\mathcal{J}_\theta\left(\frac{a+b}{2},\frac{u+\nu}{2}\right)\leq_I \frac{1}{(b-a)(\nu-u)}\int_a^b\int_u^\nu \mathcal{T}_\theta(x,\omega)\times\mathcal{J}_\theta(x,\omega)d\omega dx + \frac{5}{36}P_\theta(a,b,u,\nu)+\frac{7}{36}\mathcal{M}_\theta(a,b,u,\nu)+\frac{2}{9}\mathcal{N}_\theta(a,b,u,\nu)$$

That is

$$4\ \tilde{\mathcal{T}}\left(\frac{a+b}{2},\frac{u+\nu}{2}\right)\tilde{\times}\tilde{\mathcal{J}}\left(\frac{a+b}{2},\frac{u+\nu}{2}\right)\preccurlyeq \frac{1}{(b-a)(\nu-u)}\int_a^b\int_u^\nu \tilde{\mathcal{T}}(x,\omega)\tilde{\times}\tilde{\mathcal{J}}(x,\omega)d\omega dx\,\tilde{+}\,\frac{5}{36}\tilde{P}(a,b,u,\nu)\tilde{+}\frac{7}{36}\tilde{\mathcal{M}}(a,b,u,\nu)\tilde{+}\frac{2}{9}\tilde{\mathcal{N}}(a,b,u,\nu).$$

4. Conclusions and future plan

In this study, firstly we introduced the notion of double integrals where the integrand is FIVFs. Secondly, we have presented the new class of convex FIVFs is known as coordinated convex FIVFs by means of FOR. Then, we established a strong relationship between HH-inequalities and coordinated convex FIVFs through FOR and fuzzy double integral. In future, we shall try to explore this concept for generalized coordinated FIVF, and with the help of fuzzy fractional integral operators; we shall derive some new versions of fuzzy-interval HH-type inequalities by means of FOR. We hope that this concept will be helpful for other authors to contribute their roles in different fields of sciences.

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Conflict of interest

The authors declare that they have no competing interests.

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