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Research article

A mathematical model for the dynamics of SARS-CoV-2 virus using the Caputo-Fabrizio operator

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Abstract: The pandemic of SARS-CoV-2 virus remains a pressing issue with unpredictable characteristics which spread worldwide through human interactions. The current study is focusing on the investigation and analysis of a fractional-order epidemic model that discusses the temporal dynamics of the SARS-CoV-2 virus in a community. It is well known that symptomatic and asymptomatic individuals have a major effect on the dynamics of the SARS-CoV-2 virus therefore, we divide the total population into susceptible, asymptomatic, symptomatic, and recovered groups of the population. Further, we assume that the vaccine confers permanent immunity because multiple vaccinations have commenced across the globe. The new fractional-order model for the transmission dynamics of SARS-CoV-2 virus is formulated via the Caputo-Fabrizio fractional-order approach with the maintenance of dimension during the process of fractionalization. The theory of fixed point will be used to show that the proposed model possesses a unique solution whereas the well-posedness (bounded-ness and positivity) of the fractional-order model solutions are discussed. The steady states of the model are analyzed and the sensitivity analysis of the *basic reproductive number* is explored. Moreover to parameterize the model a real data of SARS-CoV-2 virus reported in the Sultanate of Oman from January 1st, 2021 to May 23rd, 2021 are used. We then perform the large scale numerical findings to show the validity of the analytical work.

Keywords: fractional-order SARS-CoV-2 epidemiological model; existence and uniqueness; bounded-ness and positivity; steady state analysis; sensitivity analysis; numerical simulation

1. Introduction

Infectious diseases are mainly caused by four microorganisms: bacteria, parasites, viruses, and fungi; they can be transmitted in different ways and can cause approximately fifty thousand deaths daily. In the past various epidemics worldwide resulted in millions of deaths. Very recently, a severe outbreak of respiratory illness was begun at the end of 2019 in China, Wuhan; this novel corona-virus was later called named SARS-CoV-2 (COVID-19). It was first detected in January, 2020. Note that the first transmission source of the virus was an animal; however, it rapidly spread through human interaction and such that , by 5th March 2021, a total of 116,216,580 cases were globally reported with 2,581,649 deaths. In every disease controlling, the control scenario, vaccination, etc., play a key role. Vaccines are important weapons in the fight against SARS-CoV-2, and the fact that so many vaccines are being developed and are proving to be effective is extremely promising. Due to the hard work of researchers and scientists vaccines that can save lives and put an end to the pandemic are now available. According to World Health Organization (WHO), safe and reliable vaccinations will be a game-changer. However, the need to wear masks, maintaining social distancing, and avoiding crowds still necessary because being vaccinated does not mean that we can ignore caution and endanger ourselves and others, especially since the extent to which vaccines can protect against not only the disease but also infection and transmission is still unknown.

Mathematical models play an important role in exploring the transmission dynamics of disease and predicting its future spread. Based on these models effective control strategies have been provided for compiling useful guidelines for the health officials and taking various steps towards disease control and eradication. Fractional calculus is a rapidly growing area in the field of mathematics that is used to catch the inherited axioms and memory of different natural and physical phenomena that occur in natural and physical science, technology, and engineering. Numerous classical epidemiological models have proved to be less accurate at predicting the dynamics of a system for the future due to their local nature. On the other hand, models with fractional-order are very usefully subjected to allocating and detaining the missing detail in models of classical case. More specifically, the classical derivative (integer-order) does not explore the dynamics between two distinct points. Various concepts have been developed for overcoming the limitations in case of the classical differentiation (see for more detail [1–4]). Moreover, numerous of researchers used different classical as well as fractional order epidemic models to forecast the long-term behaviors of SARS-CoV-virus and other diseases, and as well suggest some control measures. Particularly the stability analysis of various integer order epidemic models have been reported by different authors, (see for detail [5–10]). Due to the developing of fractional calculus many authors formulated fractional order epidemiological models using various fractional operators, e.g., Atangana proposed a mathematical model by using *ABC*-fractional operator [11]. Another study has been reported to investigate the transmission dynamics of novel corona virus disease [12]. Similarly the dynamics of novel corona virus with control analysis are discussed in [13–16]. In which comparison of the fractional and integer order, epidemiological models show that generally the integer-order models do not explore the dynamics more accurately.

The literature reveals that different dynamical systems are analyzed by different fractional derivative e.g., Riemann and Liouville, Hadamard and Caputo, etc., [17–20]. Many numerical and iterative approaches have been developed to solve the Caputo fractional order epidemic models [21–24], while the complications arise due to a singular kernel. Therefore, a novel idea has been reported by Caputo and Fabrizio to the fractional-order derivative based upon non-singular kernel subject to various important results for the Caputo-Fabrizio fractional integral [25]. Consequently, this operator has a lot of applications in material and thermal sciences [26–28]. Further, the novel idea of fractional Caputo-Fabrizio derivative has been utilized frequently in physical and biological sciences to describe the temporal dynamics of communicable diseases [29–35].

The current pandemic of SARS-CoV-2 and its vaccination remains a challenging issue and therefore requires significant attention. Very few studies have examined to the best of our knowledge that the novel dynamics of SARS-CoV-2 under the effect of vaccination, whereas few of them are the combination of statistical and computational models [36–42]. Very recently a mathematical model has been reported by Rahman et al., to study the dynamics of corona-virus transmission using the aforesaid operator [43]. However many characteristics that can influence the transmission of SARC-CoV-2 have been ignored. Particularly the impact of social behavior, mobility, symptomatic and asymptomatic classification, and vaccination, etc., are not collectively proposed which can influence the dynamics of the human population in ways that are not fully understood [44–47]. Motivated by the fact that fractional-order epidemiological models are more appropriate than the classical order for describing the real-world problem in a true sense as well as to fill the gap in the above-reported study, we propose a model to investigate the dynamics of SARS-COV-2 with vaccination in the frame of Caputo-Fabrizio-Caputo fractional operator by taking into account the vaccination of the susceptible population. We also classify the infected compartment into two groups of symptomatic and asymptomatic according to the SARS-CoV-2 characteristics.

We formulate the new fractional-order epidemiological model by dividing the total population into various population groups of susceptible, asymptomatic, symptomatic, and recovered individuals. Because the asymptomatic and symptomatic population take an important part in the spreading of SARS-CoV-2 virus transmission. Further, we assume that the permanent immunity of the susceptible group by getting vaccinated. We show that the solution of the proposed fractional-order epidemiological model exists and unique with the utilization of the theory of fixed point. We also prove that positivity and boundedness make the problem biologically feasible. The steady-state of the model will be calculated to discuss the stability analysis. We find the *basic reproductive number* and discuss the sensitivity analysis to find the role of the epidemic parameter in the spreading of the disease. We also use the Ordinary Least Square (OLS) method to parameterize the model and estimate the value of model parameters from the real data of SARS-CoV-2 virus reported in Oman. Finally, we present some graphical representations to support our analytical work and to show the effectiveness of the fractional-order epidemiological model.

The organization of the manuscript as follows: In Section 2 we present some basic definitions that will be used in our analysis, while Section 3 is devoted to the formulation of the model. The detailed analysis of the proposed fractional-order epidemiological model in terms of existence and uniqueness, positivity, and bounded-ness are discussed in Section 4. In Section 5, we perform the stability analysis and discuss the sensitivity of the *basic reproductive number*. All the theoretical results are supported with numerical simulation in Section 6 and concluding our work in the last Section.

2. Preliminaries

We give some fundamental concepts that will be helpful to use in the upcoming sections. Let $\phi(t)$ be a function and $\phi \in H^1(0,T), T > 0$, assume that $\alpha > 0$ and $n-1 < \alpha < n$, $n \in \mathbb{N}$, then the fractional order derivative in Caputo and Caputo-Fabrizio-Caputo sense are respectively defined as:

$$ {}^{C}D^{\alpha}_{0,t}\{\phi(t)\} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-x)^{n-\alpha-1}\{\phi^n(x)\}dx, \tag{2.1}$$

and

$$ {}^{CFC}D^{\alpha}_{0,t}\{\phi(t)\} = \frac{M(\alpha)}{(1-\alpha)} \int_0^t \phi^{'}(x) \exp\left(\frac{\alpha(x-t)}{1-\alpha}\right)dx, \tag{2.2}$$

where C and CFC stands for Caputo and Caputo-Fabrizio respectively, while $t > 0$ and $M(\alpha)$ represents the normilization function, such that $M(1) = 0 = M(0)$. We assume that $0 < \alpha < 1$ and $\phi(t)$, varies with t, then the Riemann–Liouville fractional integral of order α is determined by

$$ {}^{RL}J^{\alpha}_{0,t}\{\phi(t)\} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1}\phi(x)dx, \tag{2.3}$$

while the Caputo-Fabrizio-Caputo (CFC) integral having order α takes the form

$$ {}^{CFC}J^{\alpha}_{0,t}\{\phi(t)\} = \frac{2(1-\alpha)\phi(t)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t \phi(x)dx, \tag{2.4}$$

where $t \geq 0$.

3. Model formulation

We explore a model using the Caputo-Fabrizio derivative to discuss the SARS-CoV-2 dynamics by categorizing the entire population into susceptible, symptomatic, asymptomatic and recovered groups. These population groups are symbolized respectively with S, A, C, and R. Moreover, rather than direct representation, we must place some constraints by making the following assumptions:

a. The constants, parameters, and variables are taken to be non-negative for the proposed problem.
b. The total population is subdivided into four groups and represented by $N(t)$.
c. The infected groups are assumed to be asymptomatic and symptomatic: as asymptomatic individuals become the major source of disease transmission.
d. Vaccination of susceptible individuals will result in permanent immunity.
e. The transmission of the disease is taken to be probability-based i.e., if p is the probability that the interaction of susceptible with infected may lead to the asymptomatic class, then $(1-p)$ portion of the infected individuals automatically may go to the symptomatic class.
f. It is also noted that some of the individuals in case of SARS-CoV-virus got recovered without showing any symptoms while some may produce complications and causes the symptom of the disease, therefore this has been also proposed.

Based on these assumptions and the disease characteristics, we suggest the below dynamics for the fractional-order epidemic model of SARS-CoV-2 as:

$$\begin{cases} \dot{S}(t) = \Pi - \beta A(t)S(t) - \beta\gamma C(t)S(t) - (\nu + \mu_0)S(t), \\ \dot{A}(t) = p\{\beta A(t)S(t) + \beta\gamma C(t)S(t)\} - (\mu_0 + \mu_1 + \gamma_1)A(t), \\ \dot{C}(t) = (1-p)\{\beta A(t)S(t) + \beta\gamma C(t)S(t)\} + q\gamma_1 A(t) - (\mu_0 + \mu_2 + \gamma_2)C(t), \\ \dot{R}(t) = \gamma_1(1-q)A(t) + \gamma_2 C(t) + \nu S(t) - \mu_0 R(t). \end{cases} \tag{3.1}$$

here Π is the newborn rate and the transmission rate is β for the disease. γ is the reduced transmission rate, and ν is the vaccination rate of the susceptible group. The total natural and disease-related death rates are denoted by μ_0, μ_1, and μ_2 respectively. Similarly, γ_2 and γ_1 are the recovery rates for the asymptomatic and symptomatic groups, respectively. Moreover, p represents the probability of the asymptomatic individuals and q is the probability of these people that recovers directly in the symptomatic stage. We assume some substitutions for the shake of simplicity i.e., $\rho_1 = \nu + \mu_0$, $\rho_2 = \mu_0 + \mu_1 + \gamma_1$ and $\rho_3 = \mu_0 + \mu_2 + \gamma_2$.

We draw up the fractional-order version of the proposed model as reported by Eq (3.1) in the development of fractional calculus using the Caputo-Fabrizio-Caputo (CFC) operator with the maintenance of dimension for each differential equations as given:

$$\begin{aligned} {}^{CFC}D^{\alpha}_{0,t}(S(t)) &= \Pi^{\alpha} - \beta^{\alpha}A(t)S(t) - \gamma^{\alpha}\beta^{\alpha}C(t)S(t) - \rho_1^{\alpha}S(t), \\ {}^{CFC}D^{\alpha}_{0,t}(A(t)) &= (\beta^{\alpha}A(t)S(t) + \gamma^{\alpha}\beta^{\alpha}C(t)S(t))p - \rho_2^{\alpha}A(t), \\ {}^{CFC}D^{\alpha}_{0,t}(C(t)) &= (\beta^{\alpha}A(t)S(t) + \beta^{\alpha}\gamma^{\alpha}C(t)S(t))(1-p) + q\gamma_1^{\alpha}A(t) - \rho_3^{\alpha}C(t), \\ {}^{CFC}D^{\alpha}_{0,t}(R(t)) &= (1-q)\gamma_1^{\alpha}A(t) + \gamma_2^{\alpha}C(t) + \nu^{\alpha}S(t) - \mu_0^{\alpha}R(t), \end{aligned} \tag{3.2}$$

where α is the fractional order parameter.

4. Existence and uniqueness

In this section, we will show that the solution of the fractional-order model reported by Eq (3.2) by analyzing the fixed point theory. We will also prove that the uniqueness of the solution. For this, first the proposed fractional order system can be transformed into an associated integral equation form as:

$$\begin{aligned} S(t) - S(0) &= {}^{CFC}J^{\alpha}_{0,t}\left\{\Pi^{\alpha} - \beta^{\alpha}A(t)S(t) - \gamma^{\alpha}\beta^{\alpha}C(t)S(t) - \rho_1^{\alpha}S(t)\right\}, \\ A(t) - A(0) &= {}^{CFC}J^{\alpha}_{0,t}\left\{(\beta^{\alpha}A(t)S(t) + \gamma^{\alpha}\beta^{\alpha}C(t)S(t))p - \rho_2^{\alpha}A(t)\right\}, \\ C(t) - C(0) &= {}^{CFC}J^{\alpha}_{0,t}\left\{(\beta^{\alpha}A(t)S(t) + \beta^{\alpha}\gamma^{\alpha}C(t)S(t))(1-p) + q\gamma_1^{\alpha}A(t) - \rho_3^{\alpha}C(t)\right\}, \\ R(t) - R(0) &= {}^{CFC}J^{\alpha}_{0,t}\left\{(1-q)\gamma_1^{\alpha}A(t) + \gamma_2^{\alpha}C(t) + \nu^{\alpha}S(t) - \mu_0^{\alpha}R(t)\right\}. \end{aligned}$$

Upon the application of Caputo-Fabrizio-Caputo fractional order integration, one may obtain

$$\begin{aligned}
S(t)-S(0) &= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\{\Pi^{\alpha}-\beta^{\alpha}A(t)S(t)-\gamma^{\alpha}\beta^{\alpha}C(t)S(t)-\rho_1^{\alpha}S(t)\}\\
&+\frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t\{\Pi^{\alpha}-\beta^{\alpha}A(x)S(x)-\gamma^{\alpha}\beta^{\alpha}C(x)S(x)-\rho_1^{\alpha}S(x)\}\,dx,\\
A(t)-A(0) &= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\{p(\beta^{\alpha}A(t)S(t)+\gamma^{\alpha}\beta^{\alpha}C(t)S(t))-\rho_2^{\alpha}A(t)\}\\
&+\frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t\{p(\beta^{\alpha}A(x)S(x)+\gamma^{\alpha}\beta^{\alpha}S(x)C(x))-\rho_2^{\alpha}A(x)\}\,dx,\\
C(t)-C(0) &= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\{(1-p)(\beta^{\alpha}A(t)S(t)+\beta^{\alpha}\gamma^{\alpha}C(t)S(t))+q\gamma_1^{\alpha}A(t)-\rho_3^{\alpha}C(t)\}\\
&+\frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t\{(1-p)(\beta^{\alpha}A(x)S(x)+\beta^{\alpha}\gamma^{\alpha}C(x)S(x))+q\gamma_1^{\alpha}A(x)-\rho_3^{\alpha}C(x)\}\,dx,\\
R(t)-R(0) &= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\{(1-q)\gamma_1^{\alpha}A(t)+\gamma_2^{\alpha}C(t)+\nu^{\alpha}S(t)-\mu_0^{\alpha}R(t)\}\\
&+\frac{2\alpha}{M(\alpha)(2-\alpha)}\int_0^t\{(1-q)\gamma_1^{\alpha}A(x)+\gamma_2^{\alpha}C(x)+\nu^{\alpha}S(x)-\mu_0^{\alpha}R(x)\}\,dx.
\end{aligned}$$

We assume kernels as determined by

$$\begin{aligned}
K_1(S(t),t) &= \Pi^{\alpha}-\beta^{\alpha}S(t)A(t)-\gamma^{\alpha}\beta^{\alpha}C(t)S(t)-\rho_1^{\alpha}S(t),\\
K_2(A(t),t) &= p(\beta^{\alpha}A(t)S(t)+\gamma^{\alpha}\beta^{\alpha}C(t)S(t))-\rho_2^{\alpha}A(t),\\
K_3(C(t),t) &= (\beta^{\alpha}A(x)S(x)+\beta^{\alpha}\gamma^{\alpha}C(t)S(t))(1-p)+q\gamma_1^{\alpha}A(t)-\rho_3^{\alpha}C(t),\\
K_4(R(t),t) &= \gamma_1^{\alpha}(1-q)A(t)+\gamma_2^{\alpha}C(t)+\nu^{\alpha}S(t)-\mu_0^{\alpha}R(t).
\end{aligned}\tag{4.1}$$

Theorem 4.1. *The kernels K_1, K_2, K_3 and K_4 satisfies the Lipschitz axioms.*

Proof. Let us assume that S and S_1, A and A_1, C and C_1, R and R_1 are respectively the two functions for the above kernels K_1, K_2, K_3 and K_4, so we establish the following system

$$\begin{aligned}
K_1(S(t),t)-K_1(S_1(t),t) &= \Pi^{\alpha}-(\beta^{\alpha}A(t)+\gamma^{\alpha}\beta^{\alpha}C(t))(S(t)-S_1(t))-\rho_1^{\alpha}(S(t)-S_1(t)),\\
K_2(A(t),t)-K_2(A_1(t),t) &= p(\beta^{\alpha}(A(t)-A_1(t))S(t)+\gamma^{\alpha}\beta^{\alpha}C(t)S(t))-\rho_2^{\alpha}(A(t)-A_1(t)),\\
K_3(C(t),t)-K_3(C_1(t),t) &= (1-p)(\beta^{\alpha}A(x)S(x)+\beta^{\alpha}\gamma^{\alpha}(C(t)-C_1(t))S(t))+q\gamma_1^{\alpha}A(t)\\
&\quad-\rho_3^{\alpha}(C(t)-C_1(t)),\\
K_4(R(t),t)-K_4(R_1(t),t) &= (1-q)\gamma_1^{\alpha}A(t)+\gamma_2^{\alpha}C(t)+\nu^{\alpha}S(t)-\mu_0^{\alpha}(R(t)-R_1(t)).
\end{aligned}$$

With the employment of Cauchy's inequality to the above system, one may obtain

$$\begin{aligned}
\|K_1(S(t),t)-K_1(S_1(t),t)\| &\le \|\Pi^{\alpha}-(\beta^{\alpha}A(t)+\gamma^{\alpha}\beta^{\alpha}C(t))(S(t)-S_1(t))-\rho_1^{\alpha}(S(t)-S_1(t))\|,\\
\|K_2(A(t),t)-K_2(A_1(t),t)\| &\le \|p(\beta^{\alpha}(A(t)-A_1(t))S(t)+\gamma^{\alpha}\beta^{\alpha}C(t)S(t))-\rho_2^{\alpha}(A(t)-A_1(t))\|,\\
\|K_3(C(t),t)-K_3(C_1(t),t)\| &\le \|(1-p)(\beta^{\alpha}A(x)S(x)+\beta^{\alpha}\gamma^{\alpha}(C(t)-C_1(t))S(t))+q\gamma_1^{\alpha}A(t)\\
&\quad-\rho_3^{\alpha}(C(t)-C_1(t))\|,\\
\|K_4(R(t),t)-K_4(R_1(t),t)\| &\le \|(1-q)\gamma_1^{\alpha}A(t)+\gamma_2^{\alpha}C(t)+\nu^{\alpha}S(t)-\mu_0^{\alpha}(R(t)-R_1(t))\|.
\end{aligned}$$

Recursively one may obtain

$$
\begin{aligned}
S(t) &= \frac{2(1-\alpha)K_1(S_{n-1}(t),t)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t K_1(S_{n-1}(x),x)dx,\\
A(t) &= \frac{2(1-\alpha)K_2(A_{n-1}(t),t)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t K_2(A_{n-1}(x),x)dx,\\
C(t) &= \frac{2(1-\alpha)K_3(C_{n-1}(t),t)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t K_3(C_{n-1}(x),x)dx,\\
R(t) &= \frac{2(1-\alpha)K_4(R_{n-1}(t),t)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{M(\alpha)(2-\alpha)}\int_0^t K_4(R_{n-1}(x),x)dx.
\end{aligned}
\tag{4.2}
$$

The application of norm with the concept of majorizing, the difference between successive terms implies

$$
\begin{aligned}
&\|W_n(t)\| = \|S_n(t) - S_{1,n-1}(t)\| \le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\|K_1(S_{n-1}(t),t) - K_1(S_{1,n-2}(t),t)\|\\
&+ \frac{2\alpha}{(2-\alpha)M(\alpha)}\left\|\int_0^t [K_1(S_{n-1}(x),x) - K_1(S_{1,n-2}(x),x)]dx\right\|,\\
&\|X_n(t)\| = \|A_n(t) - A_{1,n-1}(t)\| \le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\|K_2(A_{n-1}(t),t) - K_1(A_{1,n-2}(t),t)\|\\
&+ \frac{2\alpha}{(2-\alpha)M(\alpha)}\left\|\int_0^t [K_1(A_{n-1}(x),x) - K_1(A_{1,n-2}(x),x)]dx\right\|,\\
&\|Y_n(t)\| = \|C_n(t) - C_{1,n-1}(t)\| \le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\|K_3(C_{n-1}(t),t) - K_1(C_{1,n-2}(t),t)\|\\
&+ \frac{2\alpha}{(2-\alpha)M(\alpha)}\left\|\int_0^t [K_1(C_{n-1}(x),x) - K_1(C_{1,n-2}(x),x)]dx\right\|,\\
&\|Z_n(t)\| = \|R_n(t) - R_{1,n-1}(t)\| \le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\|K_4(R_{n-1}(t),t) - K_1(R_{1,n-2}(t),t)\|\\
&+ \frac{2\alpha}{(2-\alpha)M(\alpha)}\left\|\int_0^t [K_1(R_{n-1}(x),x) - K_1(R_{1,n-2}(x),x)]dx\right\|,
\end{aligned}
\tag{4.3}
$$

where

$$
\sum_{i=0}^{\infty} W_i(t) = S_n(t),\ \ \sum_{i=0}^{\infty} X_i(t) = A_n(t),\ \ \sum_{i=0}^{\infty} Y_i(t) = C_n(t),\ \ \sum_{i=0}^{\infty} Z_i(t) = R_n(t). \tag{4.4}
$$

Moreover, the kernels $K_1, \ldots, K_4$ satisfy the Lipschitz property, so one can write

$$
\begin{aligned}
&\|W_n(t)\| = \|S_n(t) - S_{1,n-1}(t)\| \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\eta_1\|S_{n-1}(t) - S_{1,n-2}(t)\| \\
&+ \frac{2\alpha}{M(\alpha)(2-\alpha)}\eta_2 \int_0^t \left\|S_{n-1}(x) - S_{1,n-2}(x)\right\| dx, \\
&\|X_n(t)\| = \|A_n(t) - A_{1,n-1}(t)\| \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\eta_3\|A_{n-1}(t) - A_{1,n-2}(t)\| \\
&+ \frac{2\alpha}{M(\alpha)(2-\alpha)}\eta_4 \int_0^t \left\|A_{n-1}(x) - A_{1,n-2}(x)\right\| dx, \\
&\|Y_n(t)\| = \|C_n(t) - C_{1,n-1}(t)\| \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\eta_5\|C_{n-1}(t) - C_{1,n-2}(t)\| \\
&+ \frac{2\alpha}{M(\alpha)(2-\alpha)}\eta_6 \int_0^t \left\|C_{n-1}(x) - C_{1,n-2}(x)\right\| dx, \\
&\|Z_n(t)\| = \|R_n(t) - R_{1,n-1}(t)\| \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\eta_7\|R_{n-1}(t) - R_{1,n-2}(t)\| \\
&+ \frac{2\alpha}{M(\alpha)(2-\alpha)}\eta_8 \int_0^t \left\|R_{n-1}(x) - R_{1,n-2}(x)\right\| dx.
\end{aligned}
\tag{4.5}
$$

Theorem 4.2. *The solution of the proposed fractional order model (3.2) exists under Caputo-Fabrizio-Caputo operator.*

Proof. The employment of Eq (4.4) and the use of recursive scheme leads to the following system

$$
\begin{aligned}
&\|W_n(t)\| \leq \|S(0)\| + \left\{\left(\frac{2\eta_1(1-\alpha)}{M(\alpha)(2-\alpha)}\right)^n\right\} + \left\{\left(\frac{2\eta_2\alpha t}{M(\alpha)(2-\alpha)}\right)^n\right\}, \\
&\|X_n(t)\| \leq \|A(0)\| + \left\{\left(\frac{2(1-\alpha)\eta_3}{(2-\alpha)M(\alpha)}\right)^n\right\} + \left\{\left(\frac{2\alpha\eta_4 t}{(2-\alpha)M(\alpha)}\right)^n\right\}, \\
&\|Y_n(t)\| \leq \|C(0)\| + \left\{\left(\frac{2\eta_5(1-\alpha)}{M(\alpha)(2-\alpha)}\right)^n\right\} + \left\{\left(\frac{2\eta_6\alpha t}{M(\alpha)(2-\alpha)}\right)^n\right\}, \\
&\|Z_n(t)\| \leq \|R(0)\| + \left\{\left(\frac{2(1-\alpha)\eta_7}{(2-\alpha)M(\alpha)}\right)^n\right\} + \left\{\left(\frac{2\alpha\eta_8 t}{(2-\alpha)M(\alpha)}\right)^n\right\}.
\end{aligned}
\tag{4.6}
$$

Now to investigate that the functions in Eq (4.6) are solutions of the model (3.2) we make use of the following substitutions

$$
\begin{aligned}
&S(t) = S_n(t) - \Pi_{1,n}(t),\ A(t) = A_n(t) - \Pi_{2,n}(t),\ C(t) = C_n(t) - \Pi_{3,n}(t), \\
&R(t) = R_n(t) - \Pi_{4,n}(t),
\end{aligned}
\tag{4.7}
$$

where $\Pi_{1,n}(t)$, $\Pi_{2,n}(t)$, $\Pi_{3,n}(t)$, $\Pi_{4,n}(t)$ represent the remainder terms of the series solutions. Thus

$$\begin{aligned}
S(t) - S_{n-1}(t) &= \frac{2(1-\alpha)K_1(S(t) - \Pi_{1,n}(t))}{M(\alpha)(2-\alpha)} + \frac{2\alpha}{M(\alpha)(2-\alpha)}\int_0^t K_1(S(x) - \Pi_{1,n}(x))dx,\\
A(t) - S_{n-1}(t) &= \frac{2K_2(A(t) - \Pi_{2,n}(t))(1-\alpha)}{M(\alpha)(2-\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t K_2(A(x) - \Pi_{2,n}(x))dx,\\
C(t) - S_{n-1}(t) &= \frac{2K_3(C(t) - \Pi_{3,n}(t))(1-\alpha)}{M(\alpha)(2-\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t K_3(C(x) - \Pi_{3,n}(x))dx,\\
R(t) - S_{n-1}(t) &= \frac{2(1-\alpha)K_4(R(t) - \Pi_{4,n}(t))}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t K_4(R(x) - \Pi_{4,n}(x))dx.
\end{aligned} \tag{4.8}$$

Applying the norm on both sides with the application of Lipschitz axiom the above assertion yields

$$\begin{aligned}
&\left\| S(t) - \frac{2(1-\alpha)K_1(S(t),t)}{(2-\alpha)M(\alpha)} - S(0) - \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t K_1(S(x),x)dx \right\|\\
&\le \|\Pi_{1,n}(t)\| \left\{1 + \left(\frac{2(1-\alpha)\eta_1}{(2-\alpha)M(\alpha)} + \frac{2\alpha\eta_2 t}{(2-\alpha)M(\alpha)}\right)\right\},\\
&\left\| A(t) - \frac{2(1-\alpha)K_2(A(t),t)}{(2-\alpha)M(\alpha)} - A(0) - \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t K_2(A(x),x)dx \right\|\\
&\le \|\Pi_{2,n}(t)\| \left\{1 + \left(\frac{2(1-\alpha)\eta_3}{(2-\alpha)M(\alpha)} + \frac{2\alpha\eta_4 t}{(2-\alpha)M(\alpha)}\right)\right\},\\
&\left\| C(t) - \frac{2(1-\alpha)K_3(C(t),t)}{(2-\alpha)M(\alpha)} - C(0) - \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t K_3(C(x),x)dx \right\|\\
&\le \|\Pi_{3,n}(t)\| \left\{1 + \left(\frac{2(1-\alpha)\eta_5}{(2-\alpha)M(\alpha)} + \frac{2\alpha\eta_6 t}{(2-\alpha)M(\alpha)}\right)\right\},\\
&\left\| R(t) - \frac{2(1-\alpha)K_4(R(t),t)}{M(\alpha)(2-\alpha)} - R(0) - \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t K_4(R(x),x)dx \right\|\\
&\le \|\Pi_{4,n}(t)\| \left\{1 + \left(\frac{2(1-\alpha)\eta_7}{(2-\alpha)M(\alpha)} + \frac{2\alpha\eta_8 t}{(2-\alpha)M(\alpha)}\right)\right\}.
\end{aligned} \tag{4.9}$$

Upon the application of lim as t approaches ∞ implies that

$$\begin{aligned}
S(t) &= \frac{2(1-\alpha)K_1(S(t),t)}{M(\alpha)(2-\alpha)} + \frac{2\alpha}{M(\alpha)(2-\alpha)}\int_0^t K_1(S(x),x)dx + S(0),\\
A(t) &= \frac{2(1-\alpha)K_2(A(t),t)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t K_2(A(x),x)dx + A(0),\\
C(t) &= \frac{2(1-\alpha)K_3(C(t),t)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t K_3(C(x),x)dx + C(0),\\
R(t) &= \frac{2(1-\alpha)K_4(R(t),t)}{M(\alpha)(2-\alpha)} + \frac{2\alpha}{M(\alpha)(2-\alpha)}\int_0^t K_4(R(x),x)dx + R(0),
\end{aligned} \tag{4.10}$$

which proves the conclusion i.e., the above are solutions of the model as given by Eq (3.2).

Theorem 4.3. *The fractional order epidemiological model as reported by Eq (3.2) posses a solution which is unique.*

Proof. On the contradiction basis, we assume that $(S^{'}(t), A^{'}(t), C^{'}(t), R^{'}(t))$ is also the solution of the proposed fractional epidemiological model (3.2), thus

$$\begin{aligned}
S(t) - S^{'}(t) &= \frac{2(1-\alpha)\left\{K_1(S(t),t) - K_1(S^{'}(t),t)\right\}}{(2-\alpha)M(\alpha)} \\
&\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t \left\{K_1(S(x),x) - K_1(S^{'}(x),x)\right\} dx, \\
A(t) - A^{'}(t) &= \frac{2(1-\alpha)\left\{K_2(A(t),t) - K_2(A^{'}(t),t)\right\}}{(2-\alpha)M(\alpha)} \\
&\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t \left\{K_2(A(x),x) - K_2(A^{'}(x),x)\right\} dx, \\
C(t) - C^{'}(t) &= \frac{2(1-\alpha)\left\{K_3(C(t),t) - K_3(C^{'}(t),t)\right\}}{(2-\alpha)M(\alpha)} \\
&\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t \left\{K_3(C(x),x) - K_3(C^{'}(x),x)\right\} dx, \\
R(t) - R^{'}(t) &= \frac{2(1-\alpha)\left\{K_4(R(t),t) - K_4(S^{'}(t),t)\right\}}{(2-\alpha)M(\alpha)} \\
&\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t \left\{K_4(R(x),x) - K_4(R^{'}(x),x)\right\} dx.
\end{aligned} \tag{4.11}$$

Upon the property of majorizing we may write the above system as

$$\begin{aligned}
||S(t) - S^{'}(t)|| &= \frac{2(1-\alpha)||K_1(S(t),t) - K_1(S^{'}(t),t)||}{(2-\alpha)M(\alpha)} \\
&\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t ||K_1(S(x),x) - K_1(S^{'}(x),x)||dx, \\
||A(t) - A^{'}(t)|| &= \frac{2(1-\alpha)||K_2(A(t),t) - K_2(A^{'}(t),t)||}{(2-\alpha)M(\alpha)} \\
&\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t ||K_2(A(x),x) - K_2(A^{'}(x),x)||dx, \\
||C(t) - C^{'}(t)|| &= \frac{2(1-\alpha)||K_3(C(t),t) - K_3(C^{'}(t),t)||}{(2-\alpha)M(\alpha)} \\
&\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t ||K_3(C(x),x) - K_3(C^{'}(x),x)||dx, \\
||R(t) - R^{'}(t)|| &= \frac{2(1-\alpha)||K_4(R(t),t) - K_4(S^{'}(t),t)||}{(2-\alpha)M(\alpha)} \\
&\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t ||K_4(R(x),x) - K_4(R^{'}(x),x)||dx.
\end{aligned} \tag{4.12}$$

Using the results derived in Theorems 4.1 and 4.2, we obtain

$$\begin{aligned}
\|S(t)-S^{'}(t)\| &\le \frac{2\eta_1\psi_1(1-\alpha)}{(2-\alpha)M(\alpha)}+\left(\frac{2\eta_2\alpha\psi_2 t}{M(\alpha)(2-\alpha)}\right)^n,\\
\|A(t)-A^{'}(t)\| &\le \frac{2\eta_3(1-\alpha)\psi_3}{M(\alpha)(2-\alpha)}+\left(\frac{2\eta_4\alpha\psi_4 t}{(2-\alpha)M(\alpha)}\right)^n,\\
\|C(t)-C^{'}(t)\| &\le \frac{2(1-\alpha)\eta_5\psi_5}{(2-\alpha)M(\alpha)}+\left(\frac{2\alpha\eta_6\psi_6 t}{(2-\alpha)M(\alpha)}\right)^n,\\
\|R(t)-R^{'}(t)\| &\le \frac{2\eta_7\psi_7(1-\alpha)}{M(\alpha)(2-\alpha)}+\left(\frac{2\alpha\eta_8\psi_8 t}{M(\alpha)(2-\alpha)}\right)^n.
\end{aligned}\tag{4.13}$$

The inequalities as reported in Eq (4.13) holds for every value of n, thus we obtain

$$S(t)=S^{'}(t),\ A(t)=A^{'}(t),\ C(t)=C^{'}(t),\ R(t)=R^{'}(t).\tag{4.14}$$

The positivity and boundedness of the fractional-order model (3.2) will be proved to show the well possed-ness of the problem. Further, we discuss a certain region for the dynamics of the proposed problem which is invariant positively. For this, the *Lemmas* developed has been explored below.

Lemma 4.1. *Let* $(S(t),A(t),C(t),R(t))$ *be the solution of model (3.2) and assume that possessing non-negative initial conditions, then the solutions are non-negative for all* $t\ge 0$.

Proof. Let us consider a general fractional-order model of the Eq (3.2) becomes

$$\begin{aligned}
{}^{G}D^{\Omega}_{0,t}(S(t)) &= \Pi^{\Omega}-\beta^{\Omega}A(t)S(t)-\gamma^{\Omega}\beta^{\Omega}C(t)S(t)-\rho^{\Omega}_1 S(t),\\
{}^{G}D^{\Omega}_{0,t}(A(t)) &= p(\beta^{\Omega}A(t)S(t)+\beta^{\Omega}\gamma^{\Omega}C(t)S(t))-\rho^{\Omega}_2 A(t),\\
{}^{G}D^{\Omega}_{0,t}(C(t)) &= (1-p)(\beta^{\Omega}A(t)S(t)+\beta^{\Omega}\gamma^{\Omega}C(t)S(t))+q\gamma^{\Omega}_1 A(t)-\rho^{\Omega}_3 C(t),\\
{}^{G}D^{\Omega}_{0,t}(R(t)) &= (1-q)\gamma^{\Omega}_1 A(t)+\gamma^{\Omega}_2 C(t)+\nu^{\Omega}S(t)-\mu^{\Omega}_0 R(t),
\end{aligned}\tag{4.15}$$

where G is the operator for fractional-order under consideration, while the parameter of fractional order is in Ω. So the above system leads to

$$\begin{aligned}
&{}^{G}D^{\Omega}_{0,t}(S(t))\Big|_{\kappa(S)} = \Pi^{\Omega}>0,\ {}^{G}D^{\Omega}_{0,t}(A(t))\Big|_{\kappa(A)} = (\beta^{\Omega}A(t)S(t)+\gamma^{\Omega}\beta^{\Omega}C(t)S(t))p\ge 0,\\
&{}^{G}D^{\Omega}_{0,t}(C(t))\Big|_{\kappa(C)} = (\beta^{\Omega}A(t)S(t)+\beta^{\Omega}\gamma^{\Omega}C(t)S(t))(1-p)+q\gamma^{\Omega}_1 A(t)\ge 0,\\
&{}^{G}D^{\Omega}_{0,t}(R(t))\Big|_{\kappa(R)} = \gamma^{\Omega}_1(1-q)A(t)+\gamma^{\Omega}_2 C(t)+\nu^{\Omega}S(t)>0,
\end{aligned}\tag{4.16}$$

where $\kappa(\xi)=\{\xi=0$ and S,A,C,R contained in $C(R_+\times R_+)\}$ and $\xi\in\{S,A,C,R\}$. Following the result in [48], it is therefore concluded that any solutions $(S(t),A(t),C(t),R(t))$ of model (4.15) are positive for all non-negative t.

Lemma 4.2. *Let Φ be the region (set) for considering the dynamics of the proposed model (3.2) within it, is invariant positively, then*

$$\Phi = \left\{ (S(t), A(t), C(t), R(t)) \in R_+^4 : S + A + C + R \leq \left(\frac{\Pi}{\rho_1}\right)^{\Omega} \right\}. \tag{4.17}$$

Proof. Since N symbolizes the total population, then $N = S + A + C + R$, which implies that

$${}^{CF}D_{0,t}^{\Omega}(N(t)) + \mu_0^{\Omega} N(t) \leq \Pi^{\Omega}. \tag{4.18}$$

The solution of Eq (4.18) in the Caputo-Fabrizio sense leads to the assertion given by

$$N(t) \leq \left(\frac{\Pi}{\rho_1}\right)^{\Omega} + E_{\Omega}\left(-\mu_0^{\Omega} t^{\Omega}\right)\left(N(0) - \frac{\Pi^{\Omega}}{\mu_0^{\Omega}}\right). \tag{4.19}$$

In Eq (4.19) $E(.)$ represents the Mittag-Leffler function i.e., $E_{\Omega}(Z) = \sum_{n=0}^{\infty} \frac{Z^n}{\Gamma(\Omega i+1)}$. It could be noted from the above Eq (4.19) that whenever time increases with no bound, then $N(t) \to \left(\frac{\Pi}{\rho_1}\right)^{\Omega}$. Hence, if $N(0) \leq \left(\frac{\Pi}{\rho_1}\right)^{\Omega}$, then $N(t) \leq \left(\frac{\Pi}{\rho_1}\right)^{\Omega}$ for all $t > 0$, while whenever $N(0) > \left(\frac{\Pi}{\rho_1}\right)^{\Omega}$, then N goes into the feasible region Φ, and will never leave. So it could be concluded that the fractional order model dynamics can be studied in the feasible region Φ.

5. Steady state analysis

The proposed epidemiological model (3.1) of the SARS-CoV-virus is examined for the equilibria: disease free and endemic states. Let D_1 be the disease free equilibrium point of the model, then for analyzing this point the population under consideration is assumed to be infection free. Thus the system reported by Eq (3.1) has a disease free equilibrium $D_1 = \left(S^0, A^0, C^0, R^0\right)$, where $S^0 = \frac{\Pi}{\rho_1}$, $A^0 = C^0 = 0$ and $R^0 = \frac{\nu\Pi}{\mu_0\rho_1}$. Now to calculate the *basic reproductive number*, we assume $X = (A, C)^T$ then system (3.1) yields

$$\left.\frac{dX}{dt}\right|_{D_1} = F - V, \tag{5.1}$$

where

$$F = \begin{bmatrix} p\beta S^0 & p\beta\gamma S^0 \\ (1-p)p\beta S^0 & (1-p)\beta\gamma S^0 \end{bmatrix}, \; V = \begin{bmatrix} \rho_2 & 0 \\ -\gamma_1 q & \rho_3 \end{bmatrix}. \tag{5.2}$$

Therefore, the *basic reproductive number* is the spectral radius of $\rho(FV^{-1})$, i.e., $R_0 = R_1 + R_2 + R_3$, where

$$R_1 = \frac{p\Pi\beta}{\rho_1\rho_2}, \; R_2 = \frac{\Pi\gamma\gamma_1\beta pq}{\rho_1\rho_2\rho_3}, \; R_3 = \frac{(1-p)\Pi\gamma\beta}{\rho_1\rho_3}. \tag{5.3}$$

In a similar way it is assumed that the endemic state of the model (3.1) is $D_2 = (S^*, A^*, C^*, R^*)$, then

$$\begin{aligned} S^* &= \frac{\rho_2\rho_3}{\beta\left(\rho_3 p + \gamma_1 q\gamma p + \rho_2\gamma(1-p)\right)}, \; A^* = \frac{p\rho_1\rho_2\rho_3\left[R_0 - 1\right]}{\rho_2\beta\left(\rho_3 p + \gamma_1 q\gamma p + \rho_2\gamma(1-p)\right)}, \\ C^* &= \frac{q_1\left(pq\gamma_1 + \rho_2(1-p)\right)\left[R_0 - 1\right]}{\beta\left(\rho_3 p + \gamma_1 q\gamma p + \rho_2\gamma(1-p)\right)}, \; R^* = \frac{1}{\mu_0}\left[(1-q)a^* + \gamma_2 c^* + \nu s^*\right]. \end{aligned} \tag{5.4}$$

The linearizable version of the proposed SARS-CoV-2 virus model (3.1) leads to a matrix given by

$$J = \begin{bmatrix} -\beta A - \gamma\beta C - \rho_1 & -\beta S & -\gamma\beta S & 0 \\ p(\beta A + \gamma\beta C) & p\beta S - \rho_2 & p\gamma\beta S & 0 \\ (1-p)(\beta A + \gamma\beta C) & (1-p)\beta S & (1-p)\gamma\beta S - \rho_3 & 0 \\ \nu & (1-q)\gamma_1 & \gamma_2 & -\mu_0 \end{bmatrix}. \tag{5.5}$$

Two eigenvalues of the Jacobian matrix J around the disease-free state are $-\mu_0$ and $-\rho_1$, while the remaining two are the roots of the quadratic equation is given by

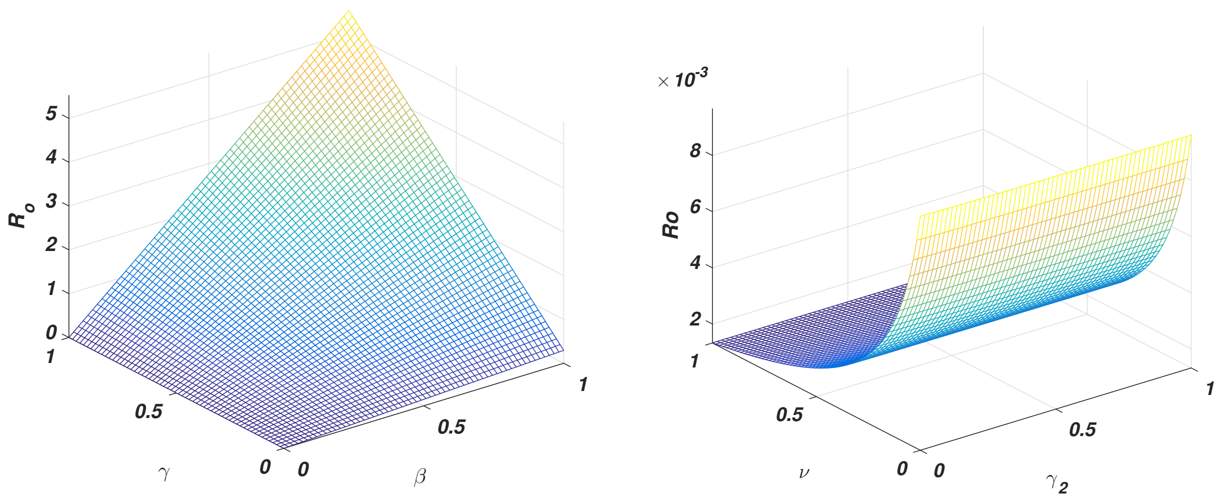
$$\lambda^2 + \{\rho_2(1-R_1) + \rho_3(1-R_3)\}\,\lambda + \rho_2\rho_3\,\{1 - R_1 - R_2\}. \tag{5.6}$$

It could be noted that that the roots of the above Eq (5.6) are negative if $R_1 < 1$, $R_3 < 1$ and $R_1 + R_2 < 1$, which implies that the disease-free state D_1 is stable locally asymptotically whenever $R_0 < 1$. Similarly, it can be shown that the disease endemic state D_2 of the proposed model (3.1) is stable locally asymptotically whenever $R_0 > 1$.

To discuss the sensitivity of the *basic reproductive number* (R_0) to each epidemic parameters of the model (3.1) we perform the following

$$\begin{aligned} \frac{\partial R_0}{\partial \beta} &= \frac{p\Pi}{\rho_1\rho_2} + \frac{\Pi\gamma\gamma_1 pq}{\rho_1\rho_2\rho_3} + \frac{(1-p)\,\Pi\gamma_0}{\rho_1\rho_3} > 0, \quad \frac{\partial R_0}{\partial \gamma} = \frac{\Pi\gamma_1\beta pq}{\rho_1\rho_2\rho_3} + \frac{(1-p)\,\Pi\beta}{\rho_1\rho_3} > 0, \\ \frac{\partial R_0}{\partial \nu} &= -\frac{p\Pi\beta}{\rho_1^2\rho_2} - \frac{\Pi\gamma\gamma_1\beta pq}{\rho_1^2\rho_2\rho_3} < 0, \quad \frac{\partial R_0}{\partial \gamma_2} = -\frac{pq\Pi\beta\gamma\gamma_1}{\rho_1\rho_2\rho_3^2} - \frac{(1-p)\Pi\gamma\beta}{\rho_1\rho_3^2\rho_3} < 0. \end{aligned} \tag{5.7}$$

Eq (5.7) describes that the *basic reproductive number* rises with the increasing of β and γ, as depicted by Figure 1a, while reduces whenever the value of ν and γ_2 rises, as shown in Figure 1b.



(a) R_0 verses γ and β **(b)** R_0 verses ν and γ_2

Figure 1. The graph visualizing the sensitivity analysis of the *basic reproductive number* verses the epidemic parameters γ, β, ν and γ_2.

6. Numerical simulation

In this section, the numerical simulations are carried out to understand the temporal dynamical behavior corresponding with the SARS-CoV-virus fractional-order epidemiological model (3.2). We parameterize the proposed model against the real data of SARS-CoV-2 virus reported in the Sultanate of Oman from 1st January 2021 to 23rd May 2021. Based on reported data we estimate the epidemic parameters and then simulate the model for the long run. For this, the Ordinary Least Square (OLS) method is used. Using OLS to minimize the error terms for daily reported cases and the simulated data in Eq (6.1), and the associated relative error is used in the goodness of fit

$$\min\left\{\sum_{i=1}^{n}\frac{C_i-\hat{c}_i}{C_i^2}\right\}. \tag{6.1}$$

In the above Eq (6.1), C_i and $\hat{C}_i$ are the cumulative number of reported cases and the cumulative number of simulated cases. It could be noted that the crude birth rate (per 1000 population) is 25.2 and the total population of the Sultanate of Oman is 4.975 million, and our unit of time is the day, therefore the value of Π is calculated as $\Pi = (4975000 \times 25.2)/(1000 \times 365)$. Moreover, the data fitting verse the proposed model is depicted in Figure 2. Thus the parameter's value is estimated and presented in Table 1. It is also is very important to show the feasibility of the reported work and investigate its validity

Table 1. The estimated value of the model parameters fitted by Ordinary Least Square (OLS) method, while some of the parameters value are assumed with biological feasibility.

Parameter	Value	Source	Parameter	Value	Source
Π	343.0	assumed	β	0.440	fitted
ν	0.010	fitted	γ	0.457	fitted
γ_1	0.059	fitted	p	0.260	fitted
γ_2	0.0081	fitted	q	0.59	assumed
μ_1	0.080	assumed	μ_2	0.033	assumed
μ_0	0.01	assumed			

of using large-scale numerical simulations. Unlike the traditional numerical analysis, there are not as many options to choose schemes for the numerical analysis of the fractional-order epidemiological model's simulations [49, 50]. Thus, there is a need for extensive research to develop new schemes and techniques that are both convergent and robust in the field of fractional calculus. By following the numerical schemes as reported in [51–53], we assume $[0, t]$ interval of simulation and $h = 10^{-3}$ is the time step for integration, and $n = \frac{T}{h}$, $n \in \mathbb{N}$, and $u = 0, 1, 2, \ldots, n$. So the scheme may take the

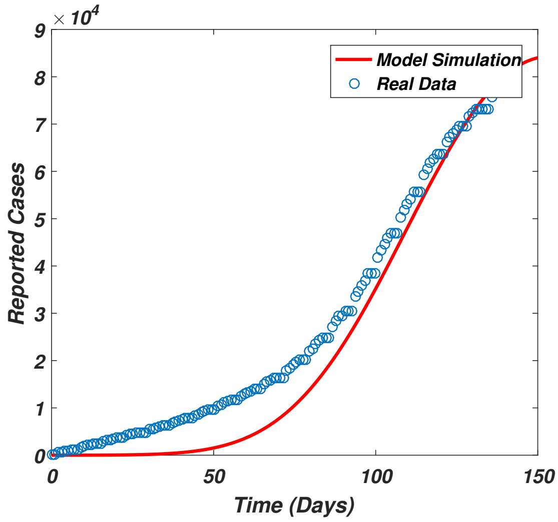


Figure 2. The trajectories visualizes the real data of SARS-CoV virus vs model fitting.

following structure:

$$
\begin{aligned}
{}^{CFC}S_{u+1} &= S(0) + (1-\alpha)\{\Pi^{\alpha} - \beta^{\alpha}A(t)S(t) - \gamma^{\alpha}\beta^{\alpha}C(t)S(t) - \rho_1^{\alpha}S(t)\} \\
&\quad + \alpha h \sum_{k=0}^{u}\{\Pi^{\alpha} - \beta^{\alpha}A(t)S(t) - \gamma^{\alpha}\beta^{\alpha}C(t)S(t) - \rho_1^{\alpha}S(t)\}, \\
{}^{CFC}A_{u+1} &= A(0) + (1-\alpha)\{p(\beta^{\alpha}A(t)S(t) + \gamma^{\alpha}\beta^{\alpha}C(t)S(t)) - \rho_2^{\alpha}A(t)\} \\
&\quad + \alpha h \sum_{k=0}^{u}\{p(\beta^{\alpha}A(t)S(t) + \gamma^{\alpha}\beta^{\alpha}C(t)S(t)) - \rho_2^{\alpha}A(t)\}, \\
{}^{CFC}C_{u+1} &= C(0) + (1-\alpha)\{(1-p)(\beta^{\alpha}A(t)S(t) + \beta^{\alpha}\gamma^{\alpha}C(t)S(t)) + q\gamma_1^{\alpha}A(t) \\
&\quad - \rho_3^{\alpha}C(t)\} + \alpha h \sum_{k=0}^{u}((1-p)\{\beta^{\alpha}S(t)A(t) + \beta^{\alpha}\gamma^{\alpha}C(t)S(t)\} + q\gamma_1^{\alpha}A(t) \\
&\quad - \rho_3^{\alpha}C(t)), \\
{}^{CFC}R_{u+1} &= (1-\alpha)\{(1-q)\gamma_1^{\alpha}A(t) + \gamma_2^{\alpha}C(t) + \nu^{\alpha}S(t) - \mu_0^{\alpha}R(t)\} \\
&\quad + \alpha h \sum_{k=0}^{u}\{(1-q)\gamma_1^{\alpha}A(t) + \gamma_2^{\alpha}C(t) + \nu^{\alpha}S(t) - \mu_0^{\alpha}R(t)\} + R(0).
\end{aligned}
\tag{6.2}
$$

This part of the study is specified to present the graphical illustrations of the proposed model. For this purpose, the value of the model parameters is considered from the Table 1, while moving on the same way the various order of α are taken to be 0.6, 0.7, 0.8, 0.9 and 1.0 to demonstrate the difference between integer-order and fractional-order and its effect on the disease transmission. Consequently, the results are depicted in Figures 3–6, which represent the dynamical behaviors of the compartmental population of the Caputo-Fabrizio-Caputo fractional-order model (3.2). More precisely, different trajectories of Figure 3 visualize the dynamical behaviors of the susceptible population for the different values of α. Similarly, the various trajectories of Figures 4–6 demonstrate the dynamical behaviors of the symptomatic, asymptomatic, and recovered population against the different values of the fractional-order parameter (α). It is observed that the fractional-order parameter has a great influence on disease transmission. It can be also seen that there is an inverse relationship between the fractional-order parameter (α) and the dynamics of the susceptible, asymptomatic, and symptomatic population i.e., increasing the value of α decreases the density of S, A and C are decreases as shown in Figures 3–5 respectively. On the other, a direct relation has been observed in the case of the dynamics of recovered population, and so increases the density of recovered population whenever the value of fractional parameter increasing as reported in Figure 6. This indicates that the Caputo-Fabrizio-Caputo, fractional-order reveals more valuable outputs regarding the model behavior which are usually could not be obtained in the case of the integer-order model.

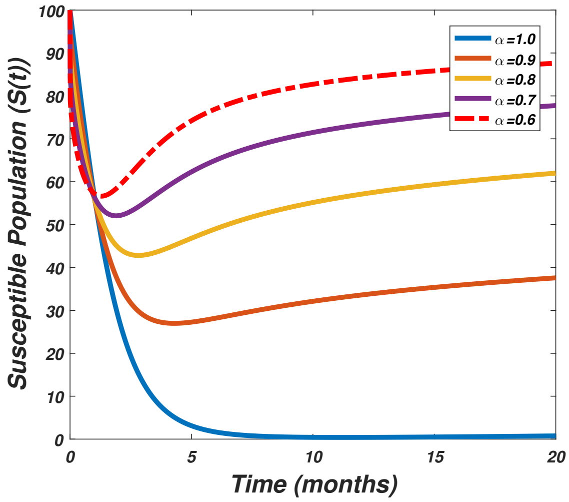


Figure 3. The various trajectories visualizing the large-scale dynamics of the proposed fractional order Caputo-Fabrizio-Caputo model (3.2) for the susceptible individuals ($S(t)$) against different instances of fractional order parameter (α), where the other epidemic parameters values are reported in Table 1, and initial sizes of the population are $(100, 90, 80, 70)$.

7. Conclusions

The research work carried out in this analysis consists of a new fractional-order epidemiological model related to the SARS-CoV-2 virus disease by using the CFC operator. The proposed Caputo-Fabrizio model has been balanced dimensionally in respect of involved parameters. Upon the application of the theory of fixed point, it has been rigorously proved that the solution of the model under the CFC operator exists and is unique. We also discussed the well possed-ness of the problem and showed that the solutions are bounded as well as positive. Steady-state analysis with sensitivity is also examined. Real data of SARS-CoV-2 virus are used and parameterized the proposed model. Finally, both the classical and fractional order Caputo-Fabrizio-Caputo model is simulated numerically and showed the feasibility and advantage of the obtained result. Thus the major findings of this work investigate that CFC fractional-order operator is the best choice instead of classical order, where the long run of the models show that the SARS-CoV-2 virus infection decreasing

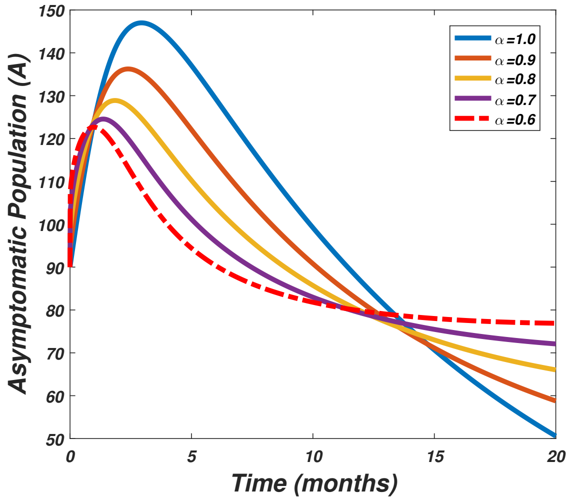


Figure 4. The various trajectories visualizing the large-scale dynamics of the Caputo-Fabrizio-Caputo model (3.2) for the asymptomatic individuals ($A(t)$) against different instances of fractional order parameter (α), where the other epidemic parameters values are reported in Table 1, and initial sizes of the population are $(100, 90, 80, 70)$.

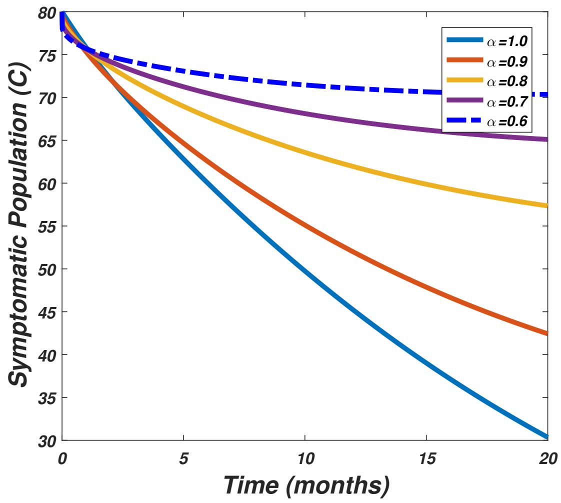


Figure 5. The trajectories visualizing the large-scale dynamics of the Caputo-Fabrizio-Caputo model (3.2) for the symptomatic individuals ($C(t)$) against different instances of fractional order parameter (α), where the values of other epidemic parameters are reported in Table 1, and the initial sizes of the population are $(100, 90, 80, 70)$.

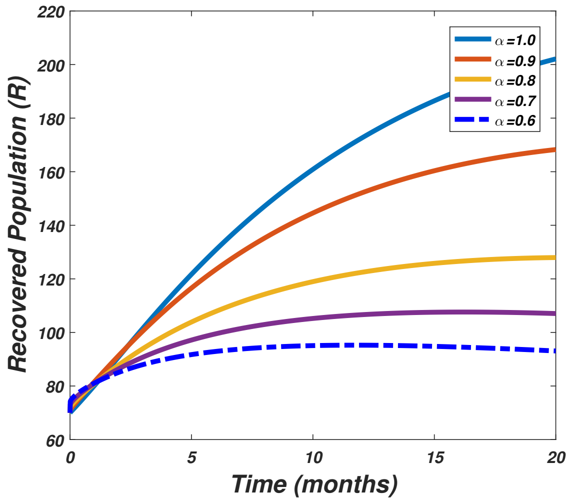


Figure 6. The trajectories visualizing the large-scale dynamics of the Caputo-Fabrizio-Caputo model (3.2) for the recovered individuals ($R(t)$) against different instances of fractional order parameter (α), where the value of other epidemic parameters are reported in Table 1, and the initial sizes of the population are $(100, 90, 80, 70)$.

asymptotically.

Nonetheless, the CFC operator yielded interesting output in the reported work however there are some other operators are introduced very recently such as Atangana–Baleanu–Caputo, Atangana–Gomez, Atangana bi-order, Atangana–Koca, variable order, distributed orders and the fractal-fractional operator to capture much more information and complexities occurs in the real world problems, therefore in a near future, these operators are to be considered to analyze the epidemiological models of SARS-CoV-virus infection and other diseases.

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Conflict of interest

The authors declare that there is no conflict of interest.

References

1. S. G. Samko, *Fractional Integrals and Derivatives, Theory and Applications*, Minsk, Nauka I Tekhnika, 1987.

2. I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Elsevier, 1999.
3. D. Baleanu, Z. B. Güvenç, J. T. Machado, *New Trends in Nanotechnology and Fractional Calculus Applications*, Springer, New York, 2010.
4. D. Baleanu, J. A. T. Machado, A. C. Luo, *Fractional Dynamics and Control*, Springer Science and Business Media, 2011.
5. G. Zaman, Y. H. Kang, I. H. Jung, Stability analysis and optimal vaccination of an SIR epidemic model, *BioSystems*, **93** (2008), 240-249.
6. Y. Wang, J. Cao, Global dynamics of a network epidemic model for waterborne diseases spread, *Appl. Math. Comput.*, **237** (2014), 474–488.
7. H. Abboubakar, J. C. Kamgang, D. Tieudjo, Backward bifurcation and control in transmission dynamics of arboviral diseases, *Math. Biosci.*, **278** (2016), 100-129.
8. T. Khan, G. Zaman, M. I. Chohan, The transmission dynamic and optimal control of acute and chronic hepatitis B, *J. Biol. Dyn.*, **11** (2017), 172-189.
9. Y. Tu, S. Gao, Y. Liu, D. Chen, Y. Xu, Transmission dynamics and optimal control of stage-structured HLB model, *J. Biol. Dyn.*, **16** (2019), 5180.
10. A. Akgül, S. H. Khoshnaw, A. S. Abdalrahman, Mathematical modeling for enzyme inhibitors with slow and fast subsystems, *Arab J. Basic Appl. Sci.*, **27** (2020), 442–449.
11. A. Atangana, Modelling the spread of COVID-19 with new fractal-fractional operators: can the lockdown save mankind before vaccination, *Chaos, Solitons Fractals*, **136** (2020), 109860.
12. B. Ghanbari, A. Atangana, A new application of fractional Atangana–Baleanu derivatives: designing ABC-fractional masks in image processing, *Physica A: Stat. Mech. Appl.*, **542** (2020), 123516.
13. A. G. M. Selvam, J. Alzabut, D. A. Vianny, M. Jacintha, F. B. Yousef, Modeling and stability analysis of the spread of novel coronavirus disease COVID-19, *Int. J. Biomath.*, **2021** (2021), 2150035.
14. F. Bozkurt, A. Yousef, D. Baleanu, J. Alzabut, A mathematical model of the evolution and spread of pathogenic coronaviruses from natural host to human host, *Chaos, Solitons Fractals*, **138** (2020), 109931.
15. K. M. Owolabi, A. Atangana, A. Akgul, Modelling and analysis of fractal-fractional partial differential equations: application to reaction-diffusion model, *Alexandria Eng. J.*, **59** (2020), 2477–2490.
16. H. Mohammadi, S. Rezapour, A. Jajarmi, On the fractional SIRD mathematical model and control for the transmission of COVID-19: the first and the second waves of the disease in Iran and Japan, *ISA Trans.*, 2021.
17. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, **204** (2006).

18. H. M. Srivastava, K. M. Saad, J. F. Gómez-Aguilar, A. A. Almadiy, Some new mathematical models of the fractional-order system of human immune against IAV infection, *Math. Biosci. Eng.*, **17** (2020), 4942–4969.

19. N. H. Tuan, H. Mohammadi, S. Rezapour, A mathematical model for COVID-19 transmission by using the Caputo fractional derivative, *Chaos, Solitons Fractals*, **140** (2020), 110107.

20. L. F. Ávalos-Ruiz, J. F. Gomez-Aguilar, A. Atangana, K. M. Owolabi, On the dynamics of fractional maps with power-law, exponential decay and Mittag–Leffler memory, *Chaos, Solitons Fractals*, **127** (2019), 364–388.

21. K. M. Owolabi, A. Atangana, Numerical approximation of nonlinear fractional parabolic differential equations with Caputo–Fabrizio derivative in Riemann–Liouville sense, *Chaos, Solitons Fractals*, **99** (2017), 171–179.

22. O. A. Arqub, Application of residual power series method for the solution of time-fractional Schrödinger equations in one-dimensional space, *Fundam. Informaticae*, **166** (2019), 87–110.

23. O. A. Arqub, Numerical algorithm for the solutions of fractional order systems of Dirichlet function types with comparative analysis, *Fundam. Informaticae*, **166** (2019), 111–137.

24. S. Rezapour, H. Mohammadi, M. E. Samei, SEIR epidemic model for COVID-19 transmission by Caputo derivative of fractional order. Advances in difference equations, *Adv. Differ. Equations*, **1** (2020), 1–9.

25. M. Caputo, Michele, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.*, **1** (2015), 1–13.

26. H. A. A. El-Saka, The fractional-order SIS epidemic model with variable population size, *J. Egypt. Math. Soc.*, **22** (2014), 50–54.

27. R. Toledo-Hernandez, V. Rico-Ramirez, G. A. Iglesias-Silva, U. M. Diwekar, A fractional calculus approach to the dynamic optimization of biological reactive systems. Part I: Fractional models for biological reactions, *Chem. Eng. Sci.*, **117** (2014), 217–228.

28. K. M. Owolabi, Computational analysis of different Pseudoplatystoma species patterns the Caputo-Fabrizio derivative, *Chaos, Solitons Fractals*, **144** (2021), 110675.

29. D. Baleanu, A. Mousalou, S. Rezapour, A new method for investigating approximate solutions of some fractional integro-differential equations involving the Caputo-Fabrizio derivative, *Adv. Differ. Equations*, **2017** (2017), 1–12.

30. M. S. Aydogan, D. Baleanu, A. Mousalou, S. Rezapour, On high order fractional integro-differential equations including the Caputo-Fabrizio derivative, *Boundary Value Probl.*, **2018** (2018), 1–15.

31. D. Baleanu, S. Rezapour, Z. Saberpour, On fractional integro-differential inclusions via the extended fractional Caputo-Fabrizio derivation, *Boundary Value Probl.*, **2019** (2019), 1–17.

32. D. Baleanu, A. Jajarmi, H. Mohammadi, S. Rezapour, A new study on the mathematical modelling of human liver with Caputo–Fabrizio fractional derivative, *Chaos, Solitons Fractals*, **134** (2020), 109705.

33. S. Rezapour, H. Mohammadi, A study on the AH1N1/09 influenza transmission model with the fractional Caputo-Fabrizio derivative, *Adv. Differ. Equations*, **2020** (2020), 1–15.

34. S. Rezapour, H. Mohammadi, A. Jajarmi, A new mathematical model for Zika virus transmission, *Adv. Differ. Equations*, **2020** (2020), 1–15.

35. D. Baleanu, H. Mohammadi, S. Rezapour, A mathematical theoretical study of a particular system of Caputo-Fabrizio fractional differential equations for the Rubella disease model, *Adv. Differ. Equations*, **2020** (2020), 1–19.

36. N. M. Ferguson, D. Laydon, G. Nedjati-Gilani, N. Imai, K. Ainslie, M. Baguelin, et al., Impact of non-pharmaceutical interventions (NPIs) to reduce COVID-19 mortality and healthcare demand. Imperial College COVID-19 Response Team, *Imp. Coll. COVID-19 Response Team*, 2020.

37. A. J. Kucharski, T. W. Russell, C. Diamond, Y. Liu, J. Edmunds, S. Funk, et al., Early dynamics of transmission and control of COVID-19: a mathematical modelling study, *Lancet Infect. Dis.*, **20** (2020), 553–558.

38. T. Kuniya, Prediction of the epidemic peak of coronavirus disease in Japan, 2020, *J. Clin. Med.*, **9** (2020), 789.

39. R. O. Stutt, R. Retkute, M. Bradley, C. A. Gilligan, J. Colvin, A modelling framework to assess the likely effectiveness of facemasks in combination with 'lock-down'in managing the COVID-19 pandemic, *Proc. R. Soc. A*, **476** (2020), 2238.

40. Z. Tang, X. Li, H. Li, Prediction of new coronavirus infection based on a modified SEIR model, *medRxiv*, 2020.

41. Z. Tang, X. Li, H. Li, New applications related to Covid-19, *Results Phys.*, **20** (2021), 103663.

42. A. Atangana, S. İ. Araz, Nonlinear equations with global differential and integral operators: existence, uniqueness with application to epidemiology, *Results Phys.*, **20** (2021), 103593.

43. M. Rahman, S. Ahmad, R. T. Matoog, N. A. Alshehri, T. Khan, Study on the mathematical modelling of COVID-19 with Caputo-Fabrizio operator, *Chaos, Solitons Fractals*, **2021** (2021), 111121.

44. H. M. Dobrovolny, Modeling the role of asymptomatics in infection spread with application to SARS-CoV-2, *Plos one*, **15** (2020), e0236976.

45. H. M. Dobrovolny, Quantifying the effect of remdesivir in rhesus macaques infected with SARS-CoV-2, *Virology*, **550** (2020), 61–69.

46. S. Mandal, T. Bhatnagar, N. Arinaminpathy, A. Agarwal, A. Chowdhury, M. Murhekar, et al., Prudent public health intervention strategies to control the coronavirus disease 2019 transmission in India: A mathematical model-based approach, *Indian J. Med. Res.*, **151** (2020), 2–3.

47. R. F. Reis, B. de Melo Quintela, J. de Oliveira Campos, J. M. Gomes, B. M. Rocha, M. Lobosco, et al., Characterization of the COVID-19 pandemic and the impact of uncertainties, mitigation strategies, and underreporting of cases in South Korea, Italy, and Brazil, *Chaos, Solitons Fractals*, **136** (2020), 109888.

48. X. Yang, L. Chen, J. Chen, Permanence and positive periodic solution for the single-species nonautonomous delay diffusive models, *Comput. Math. Appl.*, **32** (1996), 109–116.

49. H. Ramos, Z. Kalogiratou, T. Monovasilis, T. E. Simos, An optimized two-step hybrid block method for solving general second order initial-value problems, *Numer. Algorithms*, **72** (2016), 1089–1102.

50. J. Vigo-Aguiar, H. Ramos, C. Clavero, A first approach in solving initial-value problems in ODEs by elliptic fitting methods, *J. Comput. Appl. Math.*, **318** (2017), 599–603.

51. C. Li, F. Zeng, *Numerical Methods for Fractional Calculus*, Chapman and Hall/CRC, 2019.

52. A. Jajarmi, D. Baleanu, A new fractional analysis on the interaction of HIV with CD4+ T-cells, *Chaos, Solitons Fractals*, **113** (2018), 221–229.

53. A. Jajarmi, D. Baleanu, On the nonlinear dynamical systems within the generalized fractional derivatives with Mittag-Leffler kernel, *Nonlinear Dyn.*, **94** (2018), 397–414.

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