Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/MBE

MBE, 18(4): 3957–3976.
DOI: 10.3934/mbe.2021198
Received: 07 March 2021
Accepted: 25 April 2021
Published: 07 May 2021

Research article

Initial boundary value problem for a class of p-Laplacian equations with logarithmic nonlinearity

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Abstract: In this paper, we discuss global existence, boundness, blow-up and extinction properties of solutions for the Dirichlet boundary value problem of the p-Laplacian equations with logarithmic nonlinearity $u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \beta|u|^{q-2}u = \lambda|u|^{r-2}u\ln|u|$, where $1 < p < 2$, $1 < q \le 2$, $r > 1$, $\beta, \lambda > 0$. Under some appropriate conditions, we obtain the global existence of solutions by means of the Galerkin approximations, then we prove that weak solution is globally bounded and blows up at positive infinity by virtue of potential well theory and the Nehari manifold. Moreover, we obtain the decay estimate and the extinction of solutions.

Keywords: p-Laplacian; global existence; blow-up; extinction; decay estimate

1. Introduction

In this paper, we consider the following p-Laplacian equations with logarithmic nonlinearity.

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \beta|u|^{q-2}u = \lambda|u|^{r-2}u\ln|u|, & \text{in } \Omega \times (0,T), \\ u(0) = u_0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{1.1}$$

where $1 < p < 2$, $1 < q \le 2$, $r > 1$, $\beta, \lambda > 0$, $T \in (0, +\infty]$, $\Omega \in \mathbb{R}^N$ is a bounded domain with smooth boundary and $u_0(x) \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ is a nonzero non-negative function.

Problem (1.1) is a class of parabolic equation with logarithmic nonlinearity, it is worth pointing out that the interest in studying problem (1.1) relies not only on mathematical purposes, but also on their significance in real models. Among the fields of mathematical physics, biosciences and engineering, problem (1.1) is one of the most important nonlinear evolution equations. For example, in the combustion theory, we can use the function $u(x,t)$ to represent temperature, the $-\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ term to represent thermal diffusion, $\beta|u|^{q-2}u$ to represent absorption, and $\lambda|u|^{r-2}u\ln|u|$ to be the source. In the diffusion theory, we can use $u(x,t)$ to represent the density of a type of population at position x at time t, the $-\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ term represents the diffusion of density, $\lambda|u|^{r-2}u\ln|u|$ and $\beta|u|^{q-2}u$ represents the absorption and the sources, respectively. We refer the reader to [1,2] and the references therein for further details on more practical applications of problem (1.1).

The research with logarithmic nonlinearity terms is the current research hotspot. The literature on the evolution equations with logarithmic nonlinearity term is very interesting, we refer the readers to [3–6] and the references therein. At the same time, the study of p-Laplacian equations has also achieved many important results. The study of p-Laplacian equations can be divided into two cases, namely $1 < p < 2$ and $p > 2$. For the case of $p > 2$, most researchers discuss the global existence and blow-up of solutions of the equations (see [7–10]). For the case of $1 < p < 2$, the extinction and attenuation estimation of solutions are mainly discussed, we refer the readers to [11–13].

In particular, there are also some papers concerning properties such as global existence or extinction for the problem (1.1) for special cases.

In [14], Liu studied a more general form

$$\begin{cases} u_t - \mathrm{div}(|\nabla u|^{p-2}\nabla u) + \beta|u|^q = \lambda|u|^r, & \text{in } \Omega\times(0,T),\\ u(0) = u_0, & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{1.2}$$

where $1 < p < 2$, $q \le 1$, $r > 1$, $\beta, \lambda > 0$, $\Omega \subset \mathbb{R}^N$ $(N > 2)$ is a bounded domain with smooth boundary and $u_0(x) \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ is a nonzero non-negative function. The author gave the extinction properties and attenuation estimates of the solutions by comparison principle and differential inequality.

In [15], Cao and Liu considered the following initial-boundary value problem for a nonlinear evolution equation with logarithmic source

$$u_t - \mathrm{div}(|\nabla u|^{p-2}\nabla u) - k\triangle u_t = |u|^{p-2}u\log|u|, \text{ in } \Omega\times(0,T), \tag{1.3}$$

where $1 < p < 2$, $u_0 \in H_0^1(\Omega)$, $T \in (0,+\infty)$, $k \ge 0$, $\Omega \subset \mathbb{R}^n (n \ge 1)$ is a bounded domain with smooth boundary $\partial\Omega$. They proved the global existence of weak solutions and studied the asymptotic behavior of solutions and gave some decay estimates and growth estimates by constructing a family of potential wells and using the logarithmic Sobolev inequality.

In [16], Pan et al. considered the following pseudo-parabolic equation with p-Laplacian and logarithmic nonlinearity terms

$$u_t - \triangle u_t - \mathrm{div}(|\nabla u|^{p-2}\nabla u) = |u|^{q-2}u\log(|u|), \quad t > 0,\ x \in \Omega, \tag{1.4}$$

in a bounded domain $\Omega \subset \mathbb{R}^n (n \ge 1)$ with smooth boundary, where $u_0 \in W_0^{1,p}(\Omega)\backslash\{0\}$ and the parameters p, q satisfy $2 < p < q < p(1+\frac{2}{n})$. They gave the upper and lower bound estimates of blow-up time and blow-up rate, and established a weak solution with high initial energy.

In [17], Xiang and Yang studied the following initial-boundary value problem for the fractional p-Laplacian Kirchhoff type equations

$$u_t - M(\|u\|^p)(-\triangle)_p^s u = \lambda|u|^{r-2}u - \mu|u|^{q-2}u, \ (x,t) \in \Omega\times(0,\infty), \tag{1.5}$$

where $0 < s < 1 < p < 2$, $1 < q \leq 2$, $r > 1$, $\mu, \lambda > 0$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $M : [0, \infty) \to (0, \infty)$ is a continuous function. By flexible application of differential inequalities, they gave the extinction and the decay estimates of solutions.

Inspired by the above work, we study problem (1.1). Compared with problem (1.2) and (1.5), the focus of our work is partial differential equations with logarithmic nonlinearity. If the nonlinear term $\lambda|u|^{r-2}u\ln|u|$ in problem (1.1) is transformed into $\lambda|u|^{r-2}u$, then the problem (1.1) can be transformed into problem (1.2). For more research results of the logarithmic nonlinear p-Laplacian equations, we refer the readers to [18–21] and the references therein. But as far as we know, no work has dealt with the global existence and extinction properties of solutions for problem (1.1) with both the absorption and source effects as well as the term of logarithmic nonlinearity. To state our main results, we need the following two definitions.

Definition 1.1 (Weak solution). A function $u(x,t)$ is said to be a weak solution of problem (1.1), if $(x,t) \in \Omega \times [0,T)$, $u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap C(0,T;L^\infty(\Omega))$, $u_t \in L^2(0,T;L^2(\Omega))$, $u(x,0) = u_0(x) \in W_0^{1,p}(\Omega)$, for all $v \in W_0^{1,p}(\Omega)$, $t \in (0,T)$, the following equation holds

$$(u_t, v) + (|\nabla u|^{p-2}\nabla u, \nabla v) + (\beta|u|^{q-2}u, v) = (\lambda|u|^{r-2}u\ln|u|, v),$$

where $(\cdot,\cdot)$ means the inner product of $L^2(\Omega)$.

Definition 1.2 (Extinction of solutions). Let $u(t)$ be a weak solution of problem (1.1). We call $u(t)$ is an extinction of solutions if there exists $T > 0$ such that $u(x,t) > 0$ for all $t \in (t,T)$ and $u(x,t) \equiv 0$ for all $t \in [T,+\infty)$.

Further, we respectively define the energy functional $J(u)$ and the Nehari functional $I(u)$ of problem (1.1) as

$$J(u) = \frac{1}{p}\|\nabla u\|_p^p - \lambda\frac{1}{r}\int_\Omega |u|^r \ln|u|dx + \lambda\frac{1}{r^2}\int_\Omega |u|^r dx + \beta\frac{1}{q}\int_\Omega |u|^q dx, \tag{1.6}$$

and

$$I(u) = \|\nabla u\|_p^p - \lambda\int_\Omega |u|^r \ln|u|dx + \beta\int_\Omega |u|^q dx. \tag{1.7}$$

By the subsequent Lemma 2.1 and a simple calculation, we can obtain

$$\int_\Omega |u|^r \ln|u|dx \leq \frac{1}{\sigma e}\|u\|_{r+\sigma}^{r+\sigma} \leq \frac{1}{\sigma e}\|\nabla u\|_2^{r+\sigma},$$

for $0 < \sigma < 2^* - r$. So $J(u)$ and $I(u)$ are well-defined for $u \in W_0^{1,p}(\Omega)$.

Next, the potential well W and its corresponding set V are defined by

$$W := \{u \in W_0^{1,p}(\Omega) \mid I(u) > 0,\ E(u) < d\} \cup \{0\}, \tag{1.8}$$

$$V := \{u \in W_0^{1,p}(\Omega) \mid I(u) < 0,\ E(u) < d\}. \tag{1.9}$$

Let

$$d := \inf_{u\in\mathcal{N}} J(u), \tag{1.10}$$

and define the Nehari manifold

$$\mathcal{N} := \{u \in W_0^{s,p}(\Omega)\backslash\{0\} \mid I(u) = 0\}. \tag{1.11}$$

Moreover, we define

$$\mathcal{N}_+ := \{u \in W_0^{1,p}(\Omega) \mid I(u) > 0\}, \tag{1.12}$$

$$\mathcal{N}_- := \{u \in W_0^{1,p}(\Omega) \mid I(u) < 0\}. \tag{1.13}$$

Next, we state our main results.

Theorem 1.1 (Global existence). Assume that $u_0(x) \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$. Then problem (1.1) admits a global weak solution $u \in L^p(0,T; W_0^{1,p}(\Omega)) \cap C(0,T; L^\infty(\Omega))$ with $u_t \in L^2(0,T; L^2(\Omega))$ for $0 \le t < +\infty$.

Theorem 1.2 (Globally bounded and blow-up). Let $u(x,t)$ be the weak solution of problem (1.1) and $r = p$. If $J(u_0) \le d$, $I(u_0) \ge 0$ and $p = q$, then the weak solution $u(x,t)$ is globally bounded. Moreover, if $J(u_0) < 0$ and $p > q$, the weak solution $u(x,t)$ blows up at $+\infty$.

Theorem 1.3 Assume that $I(u_0) > 0$, $r = p$ and $q = 2$.

(1) If $\lambda < R_0$, then the weak solution of (1.1) satisfies

$$\|u(\cdot,t)\|_2^2 \le \|u_0\|_2^2 e^{-2\beta t} \text{ for all } t \ge 0.$$

(2) If $2N/(N+2s) < p < 2$ and $\lambda < R_0$ or $1 < p \le 2N/(N+2s)$ and $\lambda < R_1$, then the nonnegative solutions of (1.1) vanish in finite time, and

$$\begin{cases} \|u(\cdot,t)\|_2 \le \left[\left(\|u_0\|_2^{2-p} + \frac{C_0}{\beta}\right)e^{(p-2)\beta t} - \frac{C_0}{\beta}\right]^{\frac{1}{2-p}}, & t \in [0, T_1), \\ \|u(\cdot,t)\|_2 \equiv 0, & t \in [T_1, \infty), \end{cases}$$

for $2N/(N+2) < p < 2$, and

$$\begin{cases} \|u(\cdot,t)\|_{l+1} \le \left[\left(\|u_0\|_{l+1}^{2-p} + \frac{C_1}{\beta}\right)e^{(p-2)\beta t} - \frac{C_1}{\beta}\right]^{\frac{1}{2-p}}, & t \in [0, T_2), \\ \|u(\cdot,t)\|_{l+1} \equiv 0, & t \in [T_2, \infty), \end{cases}$$

for $1 < p < 2N/(N+2)$, where

$$R_0 = \frac{\lambda_1}{\lambda_1\Gamma(p,\Omega) + \ln(R)},\ C_0 = C^{-p}|\Omega|^{(\frac{N-p}{Np}-\frac{1}{2})p}\Big(1 - \lambda\Gamma(p,\Omega) - \frac{1}{\lambda_1}\ln(R)\Big),\ l = \frac{2N-(1+N)p}{p},$$

$$R_1 = \frac{\lambda_1 l p^{p-1}}{[\lambda_1\Gamma(p,\Omega) + \ln(R)](p+l-1)^{p-1}}, C_1 = C^{-p}\Big(\frac{lp^p}{(p+l-1)^p} - \lambda p(\frac{\Gamma(p,\Omega)}{p+l-1} + \frac{\ln(R)}{\lambda_1(p+l-1)})\Big)$$

and C is the embedding constant, $\Gamma(p,\Omega)$ will be given in section 3.

Theorem 1.4 Let $r = p$ and $p > q$. If $0 < J(u_0) < R_2$ and $I(u_0) < 0$, then the solution $u(x,t)$ of (1.1) is non-extinct in finite time, where

$$R_2 = \lambda \frac{1}{p^2} (\frac{p^2 e}{\lambda n \mathbb{L}_p})^{\frac{n}{p}}.$$

Theorem 1.5 Assume $I(u_0) > 0$, $r > p$ and $q = 2$, then the nonnegative weak solution of problem (1.1) vanishes in finite time and

$$\begin{cases} \|u(\cdot,t)\|_2 \le \Big[\Big(\|u_0\|_2^{2-p} + \frac{3C_4}{2\beta}\Big) e^{(p-2)\frac{1}{3}\beta t} - \frac{3C_4}{2\beta}\Big]^{\frac{1}{2-p}}, & t \in [0, T_3), \\ \|u(\cdot,t)\|_2 \equiv 0, & t \in [T_3, \infty), \end{cases}$$

for $2N/(N+2) \le p < 2$, $\lambda < R_2$, and

$$\begin{cases} \|u\|_{l+1} \le \Big[\Big(\|u_0\|_{l+1}^{2-p} + \frac{C_8}{C_7}\Big) e^{(p-2)C_7 t} - \frac{C_8}{C_7}\Big]^{\frac{1}{2-p}}, & t \in [0, T_4), \\ \|u\|_{l+1} \equiv 0, & t \in [T_4, \infty), \end{cases}$$

for $1 < p < 2N/(N+2s)$, $\lambda < R_3 = \frac{E_0}{E_1}$, where

$$E_0 = \beta e \sigma |\Omega|^{\frac{l_1 - s_2}{s_2}} C_{p^*}^{(\nu_2 - 1) l_1},$$

$$E_1 = (\frac{l p^p e \sigma}{3(p+l-1)^p} |\Omega|^{\frac{l_1 - s_2}{s_2}} C_{p^*}^{\nu_2 - 1})^{\frac{l_1(\nu_2 - 1)}{p - l_1(1-\nu_2)}},$$

and C_{p^*} is the embedding constant, l_1, ν_2, s_2 will be given in section 3.

The paper is organized as follows. In section 2, we give some necessary Lemmas such as some properties for Nehari functional and known results for ODEs. In section 3, we present the proof of the main theorems.

2. Preliminaries and Lemmas

Lemma 2.1 **([22])** Let α be positive number, then

$$t^p \ln(t) \le \frac{1}{e\alpha} t^{p+\alpha}, \quad \text{for all } p, t > 0.$$

Lemma 2.2 Let $u \in W_0^{1,p}(\Omega) \backslash \{0\}$, then

$$\int_\Omega |u|^p \ln |u| dx \le \Gamma(p, \Omega) \|\nabla u\|_p^p + \ln(\|\nabla u\|_p) \|u\|_p^p,$$

where $\Gamma(p,\Omega) := \frac{|\Omega|}{ep} + \frac{1}{e(p^* - p)} C_{p^*}^{p^*}$, C_{p^*} is the best constant of embedding from $W_0^{1,p}(\Omega)$ to $L^{p^*}(\Omega)$.
Proof. For convenience, we provide complete proof. As we know, for $1 < p < 2$,

$$\|u\|_{1,\Omega} = \|u\|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_p. \tag{2.1}$$

Let $\Omega_1 = \{x \in \Omega : |u(x)| \le \|u\|_{1,\Omega}\}$ and $\Omega_2 = \{x \in \Omega : |u(x)| > \|u\|_{1,\Omega}\}$, we can obtain

$$\int_\Omega |u(x)|^p \ln(\frac{|u(x)|}{\|u\|_{1,\Omega}})dx = \int_{\Omega_1} |u(x)|^p \ln(\frac{|u(x)|}{\|u\|_{1,\Omega}})dx + \int_{\Omega_2} |u(x)|^p \ln(\frac{|u(x)|}{\|u\|_{1,\Omega}})dx. \tag{2.2}$$

Using the properties of logarithmic, we have

$$\int_{\Omega_1} \left|u(x)\right|^p \ln(\frac{|u(x)|}{\|u\|_{1,\Omega}})dx \le \|u\|_{1,\Omega} \int_\Omega \left|\frac{|u(x)|}{\|u\|_{1,\Omega}}\right|^p \left|\ln(\frac{|u(x)|}{\|u\|_{1,\Omega}})\right|dx \le \frac{|\Omega|}{pe}\|u\|^p_{1,\Omega}. \tag{2.3}$$

Taking $\sigma = p^* - p$ in Lemma 2.1, and by Sobobev embedding inequality, we obtain

$$\int_{\Omega_2} |u(x)|^p \ln(\frac{|u(x)|}{\|u\|_{1,\Omega}})dx \le \frac{1}{e(p^* - p)\|u\|^{p^*-p}_{1,\Omega}}\|u\|^{p^*}_{p^*} \le \frac{1}{e(p^* - p)}C^{p^*}_{p^*}\|u\|^p_{1,\Omega}. \tag{2.4}$$

By (2.2), (2.3) and (2.4), we get

$$\int_\Omega |u(x)|^p \ln(\frac{|u(x)|}{\|u\|_{1,\Omega}})dx \le \Big(\frac{|\Omega|}{pe} + \frac{1}{e(p^* - p)}C^{p^*}_{p^*}\Big)\|u\|^p_{1,\Omega}.$$

By direct calculation and Eq (2.1), we have

$$\int_\Omega |u|^p \ln|u|dx \le \Gamma(p, \Omega)\|\nabla u\|^p_p + \ln(\|\nabla u\|_p)\|u\|^p_p.$$

The proof is completed. □

Lemma 2.3 **([23])** Let $p > 1$, $\mu > 0$, and $u \in W^{1,p}_0(\Omega)$, then we have

$$p\int_{\mathbb{R}^N} |u(x)|^p \ln\Big(\frac{|u(x)|}{\|u\|_{L^p(\mathbb{R}^N)}}dx\Big) + \frac{n}{p}\ln(\frac{p\mu e}{n\mathbb{L}_p})\int_{\mathbb{R}^N} |u(x)|^p dx \le \mu\int_{\mathbb{R}^N} |\nabla u|^p dx,$$

where

$$\mathbb{L}_p = \frac{p}{n}(\frac{p-1}{e})^{p-1}\pi^{-\frac{p}{2}}\Big[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(n\frac{p-1}{p}+1)}\Big]^{\frac{p}{n}}.$$

Lemma 2.4 Let $u \in W^{1,p}_0\backslash\{0\}$ and $r = p$, then we have

(i) If $0 < \|\nabla u\|_p \le R$, then $I(u) \ge 0$;

(ii) If $I(u) < 0$, then $\|\nabla u\|^p > R$;

where $R = \lambda_1^{\frac{1}{p}}\Big(\frac{p^2 e}{\lambda n\mathbb{L}_p}\Big)^{\frac{n}{p^2}}$ and λ_1 is the first eigenvalue of the following equation

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda|u|^{p-2}u & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Proof. (i) By (1.3) and Lemma 2.3, we can obtain

$$\begin{aligned} I(u) =&\|\nabla u\|^p_p - \lambda\int_\Omega |u|^p \ln|u|dx + \beta\int_\Omega |u|^q dx \\ \ge&\|\nabla u\|^p_p - \lambda\int_\Omega |u|^p \ln|u|dx \\ \ge&(1 - \frac{\lambda\mu}{p})\|\nabla u\|^p_p + \lambda\Big(\ln(\frac{p\mu e}{n\mathbb{L}_p})^{\frac{n}{p^2}} - \ln(\|u\|_p)\Big)\|u\|^p_p. \end{aligned} \tag{2.5}$$

Taking $\mu = \frac{1}{\lambda}p$ in (2.5), we get

$$I(u) \geq \lambda\Big(\ln\big(\frac{p^2 e}{\lambda n \mathbb{L}_p}\big)^{\frac{n}{p^2}} - \ln(\|u\|_p)\Big)\|u\|_p^p. \tag{2.6}$$

If $0 < \|\nabla u\|_p \leq R$, then $\|u\|_p \leq \lambda_1^{-\frac{1}{p}}\|\nabla u\|_p \leq \big(\frac{p^2 e}{\lambda n \mathbb{L}_p}\big)^{\frac{n}{p^2}}$, we have $I(u) \geq 0$.

(ii) If $I(u) < 0$, by (2.6), we have

$$\ln\big(\frac{p^2 e}{\lambda n \mathbb{L}_p}\big)^{\frac{n}{p^2}} < \ln(\|u\|_p),$$

namely

$$R = \lambda_1^{\frac{1}{p}}\Big(\frac{p^2 e}{\lambda n \mathbb{L}_p}\Big)^{\frac{n}{p^2}} < \lambda_1^{\frac{1}{p}}\|u\|_p \leq \|\nabla u\|_p.$$

The proof is completed. □

Lemma 2.5 **([24])** If $1 \leq p_0 < p_\theta < p_1 \leq \infty$, then we have

$$\|u\|_{p_\theta} \leq \|u\|_{p_0}^{1-\theta}\|u\|_{p_1}^{\theta},$$

for all $u \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$ with $\theta \in (0,1)$ defined by $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Lemma 2.6 **([25])** Let $y(t)$ be a non-negative absolutely continuous function on $[T_0, +\infty)$ satisfying

$$\frac{dy}{dt} + \alpha y^k + \beta y \leq 0, \quad t \geq 0, \quad y(0) \geq 0,$$

where $\alpha, \beta > 0$ are constants and $k \in (0,1)$. Then

$$\begin{cases} y(t) \leq \Big[(y^{1-k}(T_0) + \frac{\alpha}{\beta})e^{(k-1)\beta(t-T_0)} - \frac{\alpha}{\beta}\Big]^{\frac{1}{1-k}}, & t \in [T_0, T_*), \\ y(t) \equiv 0, & t \in [T_*, +\infty), \end{cases}$$

where $T_* = \frac{1}{(1-k)\beta}\ln(1 + \frac{\beta}{\alpha}y^{1-k}(T_0))$.

Lemma 2.7 Assume that $J(u_0) \leq d$, then the sets $\mathcal{N}_+$ and $\mathcal{N}_-$ are both invariant for $u(t)$, i.e, if $u_0 \in \mathcal{N}_-$ (resp. $u_0 \in \mathcal{N}_+$), then $u(t) \in \mathcal{N}_-$ (resp. $u(t) \in \mathcal{N}_+$) for all $t \in [0, T)$.

Proof. We only prove the case of $u(t) \in \mathcal{N}_-$, and the proof of $u(t) \in \mathcal{N}_+$ is similar.

Step 1: $J(u_0) < d$. If $u(t) \notin \mathcal{N}_-$, there exists a $t_0 > 0$, such that

$$I(u(t_0)) = 0, \text{ and } I(u(t)) < 0 \text{ for all } t \in [0, t_0).$$

By (2.1), we have

$$\begin{aligned} I(u) =& \|u\|_{1,\Omega}^p - \lambda\int_\Omega |u|^r \ln|u| dx + \beta\int_\Omega |u|^q dx \\ \geq& \|u\|_{1,\Omega}^p - \lambda\frac{1}{\sigma}\|u\|_{r+\sigma}^{r+\sigma} \\ \geq& (1 - \lambda\frac{1}{\sigma}S^{r+\sigma}\|u\|_{1,\Omega}^{r+\sigma-p})\|u\|_{1,\Omega}^p, \end{aligned} \tag{2.7}$$

where S is the embedding constant, $0 < \sigma \leq p^* - r$. Note that $I(u) \leq 0$, we have $\|u\|_{1,\Omega} = \|\nabla u\|_p > (\frac{\sigma}{\lambda S_{r+\sigma}^{r+\sigma}})^{\frac{1}{r+\sigma-p}} > 0$, which implies $\|u(t_0)\|_{1,\Omega} = \|\nabla u(t_0)\|_p > (\frac{\sigma}{\lambda S_{r+\sigma}^{r+\sigma}})^{\frac{1}{r+\sigma-p}} > 0$, so we get $u(t_0) \in \mathcal{N}$. Choosing $v = u_t$ in Definition 1.1 and integrating with respect to time 0 to t, we can obtain

$$\int_0^t \|u_\tau\|_2^2 dx + J(u(t)) = J(u_0) < d, \text{ for all } t \in [0, T), \tag{2.8}$$

namely

$$J(u(t)) = J(u_0) < d, \text{ for all } t \in [0, T).$$

While by the definition of d in (1.10), we get $J(u(t_0)) \geq d$, which gives a contradiction.

Step 2: $J(u_0) = d$. Similarly, we assume that the conclusion is incorrect, then it exists a $t_1 > 0$, such that $I(u(t_1)) = 0$, and $I(u(t)) < 0$ for all $t \in [0, t_1)$. By calculation of (2.7), we get $\|\nabla u\|_p > 0$, which implies $u(t_1) \in \mathcal{N}$. Since $\frac{1}{2}\frac{d}{dt}\|u\|_2^2 = -I(u(t))$ for all $t \in [0, t_1)$, combining with boundary conditions, we obtain $u_t \neq 0$. By (2.8), we have

$$J(u(t_1)) \leq J(u_0) - \int_0^{t_1} \|u_\tau\|_2^2 dx < d, \tag{2.9}$$

which gives a contradiction with the definition of d. □

3. Proof of main results

In this section, we prove that the main results of problem (1.1).

Proof of Theorem 1.1

First we let $\{\omega_j(x)\}$ be the basis function of $W_0^{1,p}(\Omega)$. Next, we construct the following approximate solutions $u_m(t)$ of problem (1.1) as follows:

$$u_m = \sum_{j=1}^{m} g_{jm}(t)\omega_j(x), \quad j = 1, 2, \cdots,$$

which satisfy

$$(u_{mt}, \omega_j) + (|\nabla u|^{p-2}\nabla u_m, \omega_j) + (\beta|u_m|^{q-2}u_m, \omega_j) = (\lambda|u_m|^{r-2}u_m \ln|u_m|, \omega_j), \tag{3.1}$$

and

$$u_m(x, 0) = \sum_{j=1}^{m} \xi_{jm}\omega_j(x) \to u_0, \text{ in } W_0^{1,p}(\Omega), \tag{3.2}$$

where $j = 1, 2, ..., m$, and $\xi_{jm} = (u_m(0), \omega_j)$ are given constants. We use $(\cdot, \cdot)$ to represent the inner product in $L^2(\Omega)$. The standard theory of ODEs, e.g. Peano's theorem, yields that $g_{jm}(t) \in C^1([0, \infty); W_0^{1,p}(\Omega))$ and $g_{jm}(0) = \xi_{jm}$, thus $u_m \in C^1([0, \infty); W_0^{1,p}(\Omega))$.

Next, we try to get a priori estimates of the approximate solutions u_m. Multiplying (3.1) by $g'_{jm}(t)$, summing for j from 1 to m and integrating with respect to time from 0 to t, we can obtain

$$\begin{aligned}&\int_0^t \|u_{m\tau}\|_2^2 d\tau + \frac{1}{p}\|\nabla u_m\|_p^p - \lambda\frac{1}{r}\int_\Omega |u_m|^r \ln|u_m| dx + \lambda\frac{1}{r^2}\|u_m\|_r^r + \beta\frac{1}{q}\|u_m\|_q^q \\ &= \frac{1}{p}\|\nabla u_m(0)\|_p^p - \lambda\frac{1}{r}\int_\Omega |u_m(0)|^r \ln|u_m(0)| dx + \lambda\frac{1}{r^2}\|u_m(0)\|_r^r + \beta\frac{1}{q}\|u_m(0)\|_q^q.\end{aligned} \tag{3.3}$$

For sufficiently large m and by (3.3), we have

$$\frac{1}{p}\|\nabla u_m(0)\|_p^p - \lambda\frac{1}{r}\int_\Omega |u_m(0)|^r \ln|u_m(0)|dx + \lambda\frac{1}{r^2}\|u_m(0)\|_r^r + \beta\frac{1}{q}\|u_m(0)\|_q^q \le \Phi(u_0), \tag{3.4}$$

where

$$\Phi(u_0) = \|u_0\|_p^p - \lambda\frac{1}{r}\int_\Omega |u_0|^r \ln|u_0|dx + \lambda\frac{1}{r^2}\|u_0\|_r^r + \beta\frac{1}{q}\|u_0\|_q^q + 1.$$

Multiplying (3.1) by $g_{jm}(t)$, then summing j from 0 to m, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_m\|_2^2 + \|\nabla u_m\|_p^p + \beta\|u_m\|_q^q = \lambda\int |u_m|^r \ln|u_m|dx.$$

By Lemma 2.1, we have

$$\lambda\int |u_m|^r \ln|u_m|dx \le \lambda\frac{1}{e\sigma}(\int_\Omega |u_m|^2 dx)^{\frac{r+\sigma}{2}}. \tag{3.5}$$

For $\sigma \in [1, 2-p)$, namely

$$\frac{1}{2}\frac{d}{dt}\|u_m\|_2^2 \le \frac{1}{e\sigma}(\int_\Omega |u_m|^2 dx)^{\frac{r+\sigma}{2}},$$

which implies

$$\|u_m\|_2^2 \le (\frac{2-(r+\sigma)}{e\sigma}t + \|u_{m0}\|_2^{2-(r+\sigma)})^{\frac{2}{2-(r+\sigma)}}. \tag{3.6}$$

By (3.3), (3.4) and (3.6), we can obtain

$$\begin{aligned}&\int_0^t \|u_{m\tau}\|_2^2 d\tau + \frac{1}{p}\|\nabla u_m\|_p^p + \lambda\frac{1}{r^2}\|u_m\|_r^r + \beta\frac{1}{q}\|u_m\|_q^q\\ &\le \Phi(u_0) + \Psi(u_0, t),\end{aligned} \tag{3.7}$$

where

$$\Psi(u_0, t) = \lambda\frac{1}{e\sigma}\Big(\frac{2-(r+\sigma)}{e\sigma}t + \|u_{m0}\|_2^{2-(r+\sigma)}\Big)^{\frac{r+\sigma}{2-(r+\sigma)}}.$$

Therefore, by (3.6) and (3.7), there is a function $u \in L^p(0, T; W_0^{1,p}(\Omega)))$ with $u_t \in L^2(0, T; L^2(\Omega))$, and a subsequence of $\{u_m\}_{m=1}^\infty$ (still denoted by $\{u_m\}_{m=1}^\infty$) such that for $t \in (0, \infty)$, as $m \to \infty$, we obtain

$$u_{mt} \rightharpoonup u_t \text{ weakly in } L^2(0, \infty; L^2(\Omega)), \tag{3.8}$$

$$u_m \rightharpoonup u \text{ weakly star in } L^\infty(0, \infty; W_0^{1,p}(\Omega)), \tag{3.9}$$

$$|\nabla u_m|^{p-2}\nabla u_m \to \chi(t) \text{ weakly star in } L^\infty(0, \infty; L^{\frac{p}{p-1}(\Omega)}). \tag{3.10}$$

Since $W_0^{1,p} \hookrightarrow L^p(\Omega)$ is compact, by (3.8), (3.9) and using Aubin-Lions compactness theorem, we have

$$u_m \to u \text{ strongly in } C(0, \infty; L^2(\Omega)), \tag{3.11}$$

which implies $u_m \to u$ a.e. in $\Omega \times (0, \infty)$, and then $|u_m|^{p-2}u_m \ln|u_m| \to |u|^{p-2}u \ln|u|$ a.e. in $\Omega \times (0, \infty)$.

Fixing j in (3.1) and letting $m \to \infty$, we get

$$(u_t, \omega_j) + (\chi(t), \nabla\omega_j) + (\beta|u|^{q-2}u, \omega_j) = (\lambda|u|^{r-2}u \ln|u|, \omega_j),$$

which implies

$$(u_t, v) + (\chi(t), \nabla v) + (\beta|u|^{q-2}u, v) = (\lambda|u|^{r-2}u \ln|u|, v), \tag{3.12}$$

for all $v \in W_0^{1,p}(\Omega)$. The next work is to prove that $\chi(t) = |\nabla u|^{p-2}\nabla u$, that is to say, we should change Eq (3.12) into the following equation

$$(u_t, v) + (|\nabla u|^{p-2}\nabla u, \nabla v) + (\beta|u|^{q-2}u, v) = (\lambda|u|^{r-2}u \ln|u|, v).$$

The remainder of the proof is the same as that in [15]. □

Proof of Theorem 1.2

We first consider the case $0 < J(u_0) < d$ and $I(u_0) > 0$. Choosing $v = u$ in Definition 1.1, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \|\nabla u\|_p^p + \beta\|u\|_q^q = \lambda\int_\Omega |u|^r \ln|u| dx, \tag{3.13}$$

which implies

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + I(u) = 0. \tag{3.14}$$

Taking $v = u_t$ in Definition 1.1 and integrating with respect to time 0 to t, we get

$$\int_0^t \|u_\tau\|_2^2 dx + J(u(t)) = J(u_0) < d, \text{ for } t > 0. \tag{3.15}$$

Next, we proof that $u(x,t) \in W$ for any $t > 0$. If there exists a $t_0 > 0$, such that $u(x,t_0) \in \partial W$, namely

$$I(u(x,t_0)) = 0 \text{ or } J(u(x,t_0)) = d.$$

By (3.15), we get $J(u(x,t_0)) = d$ is not true. If $u(x,t_0) \in \mathcal{N}$, then by the definition of d in (1.10), we get $J(u(x,t_0)) \geq d$, which also contradict with (3.15). So we have $u(x,t) \in W$.

By (1.6), (1.7) and (3.15), we have

$$\frac{1}{p}I(u) + \lambda\frac{1}{p^2}\|u\|_p^p = J(u) < d.$$

Note that $I(u) > 0$, we obtain

$$\lambda\|u\|_p^p \leq p^2 d. \tag{3.16}$$

Through (3.14) and a simple calculation, we have the following inequality

$$\|u\|_2^2 \leq \|u_0\|_2^2. \tag{3.17}$$

By Lemma 2.4, we have

$$\|\nabla u\|_p^p \leq \lambda_1^{\frac{1}{p}}\Big(\frac{p^2 e}{\lambda n \mathbb{L}_p}\Big)^{\frac{n}{p^2}}. \tag{3.18}$$

Thus, combining with the above inequality, we know that the weak solution of problem (1.1) is globally bounded.

Now we consider $J(u_0) = d$. Take a function ρ_m which satisfies $\rho_m > 0$ and $\lim_{n\to+\infty} \rho_m = 1$. Let $u(x,0) = u_{0m}(x) = \rho_m u_0(x)$, $x \in \Omega$, for the following equations

$$\begin{cases} u_t - \mathrm{div}(|\nabla u|^{p-2}\nabla u) + \beta|u|^{q-2}u = \lambda|u|^{r-2}u\ln|u|, & \text{in } \Omega\times(0,T), \\ u(x,0) = u_{0m}(x) = \rho_m u_0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega\times(0,T). \end{cases} \tag{3.19}$$

Since $u_0(x) \in L^\infty(\Omega)\cap W_0^{1,p}(\Omega)$ is a nonzero non-negative function, $\rho_m \in (0,1)$ and $I(u_0) \geq 0$, then we have

$$\|\nabla u_0\|_p^p + \beta\|u\|_q^q \geq \int_\Omega |u|^r \ln|u|dx, \tag{3.20}$$

and

$$\begin{aligned} I(\rho_m u_0) &= \rho_m^p\|\nabla u_0\|_p^p + \beta\rho_m^q\|u_0\|_q^q - \rho_m^r\int_\Omega |u|^r\ln|u_0|dx - \rho^r\ln(\rho_m)\int_\Omega |u_0|^r dx \\ &> \rho_m^p\|\nabla u_0\|_p^p + \beta\rho_m^q\|u_0\|_q^q - \rho_m^r\int_\Omega |u|^r\ln|u_0|dx \\ &= \rho_m^p\Big(\|\nabla u_0\|_p^p + \beta\rho_m^{q-p}\|u_0\|_q^q - \rho_m^{r-p}\int_\Omega |u_0|\ln|u|dx\Big). \end{aligned} \tag{3.21}$$

Note that $r = p$ and $p = q$, we have

$$I(\rho_m u_0) = \rho_m^p\Big(\|\nabla u_0\|_p^p + \beta\|u_0\|_p^p - \int_\Omega |u_0|\ln|u|dx\Big). \tag{3.22}$$

If $\int_\Omega |u_0|\ln|u|dx < 0$, by (3.22), we can obtain

$$I(\rho_m u_0) > \rho_m^p\Big(\|\nabla u_0\|_p^p + \beta\|u_0\|_p^q\Big) > 0. \tag{3.23}$$

If $\int_\Omega |u_0|\ln|u|dx \geq 0$, by (3.20) and (3.22), we can obtain

$$I(\rho_m u_0) = \rho_m^p\Big(\|\nabla u_0\|_p^p + \beta\|u_0\|_p^p - \int_\Omega |u_0|\ln|u|dx\Big) > 0. \tag{3.24}$$

Moreover, by a simple calculation, we have

$$\begin{aligned} J'(\rho_m u_0) &= \rho_m^{p-1}\|\nabla u_0\|_p^p + \beta\rho_m^{p-1}\|u_0\|_p^p - \rho_m^{p-1}\int_\Omega |u_0|^p\ln|u_0|dx - \rho_m^{p-1}\ln(\rho_m)\int_\Omega |u_0|dx \\ &= \frac{1}{\rho_m}\Big(\rho_m^p\|\nabla u_0\|_p^p + \beta\rho_m^p\|u_0\|_p^p - \rho_m^p\int_\Omega |u_0|^p\ln|u_0|dx - \rho_m^p\ln(\rho_m)\int_\Omega |u_0|dx\Big) \\ &= \frac{1}{\rho_m}I(\rho_m u_0), \end{aligned} \tag{3.25}$$

thus, we get

$$J'(\rho_m u_0) = \frac{1}{\rho_m}I(\rho_m u_0) = \frac{1}{\rho_m}I(u_{0m}) > 0,$$

which implies that $J(\rho_m u_0)$ is strictly increasing with respect to ρ_m. Then we have

$$J(u_{0m}) = J(\rho_m u_0) < J(u_0) = d.$$

From the results above, we can derive that the weak solution of Eq (3.19) is globally bounded.

Then, we discuss that weak solution blows up at infinity. Let $M(t) = \frac{1}{2}\int_\Omega |u(x,t)|^2 dx$, then we have

$$M'(t) = \int_\Omega u_t u dx = -I(u). \tag{3.26}$$

By (3.26) and the following equation

$$J(u) = \frac{1}{p}I(u) + \lambda\frac{1}{p^2}\|u\|_p^p + \beta\frac{p-q}{qp}\|u\|_q^q,$$

which implies

$$M'(t) = -pJ(u) + \beta\frac{p-q}{q}\|u\|_q^q + \lambda\frac{1}{p}\|u\|_p^p. \tag{3.27}$$

Making $v = u_t$ in Definition 1.1, we get

$$\int_\Omega u_t u_t dx = -(|\nabla u|^{p-2}\nabla u, \nabla u_t) + \lambda\int_\Omega u_t|u|^{p-2}u\ln|u|dx - \beta\int_\Omega u_t|u|^{q-2}udx.$$

By a simple calculation, we get

$$\frac{d}{dt}E(u) = \frac{1}{p}\frac{d}{dt}\|\nabla u\|_p^p - \lambda\int |u|^{r-2}uu_t\ln|u|dx + \beta\int_\Omega |u|^{q-2}uu_t dx.$$

Thus, we can obtain

$$J(u) = J(u_0) - \int_0^t \|u_\tau\|_2^2 d\tau. \tag{3.28}$$

By (3.27) and (3.28), we have

$$M'(t) = -pJ(u_0) + p\int_0^t \|u_\tau\|_2^2 d\tau + \beta\frac{p-q}{q}\|u\|_q^q + \lambda\frac{1}{p}\|u\|_p^p,$$

namely

$$M'(t) \geq -pJ(u_0) > 0.$$

Therefore, the following inequality holds

$$\|u\|_2^2 \geq -2pE(u_0)t + 2\|u_0\|_2^2, \text{ for all } t > 0.$$

The proof is completed. □

Proof of Theorem 1.3

(1) Choosing $v = u$ in Definition 1.1, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \|\nabla u\|_p^p + \beta\int_\Omega u^2 dx = \lambda\int_\Omega |u|^p\ln|u|dx. \tag{3.29}$$

By Lemma 2.2 and (3.29), we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \|\nabla u\|_p^p + \beta\|u\|_2^2 \leq \lambda\Gamma(p,\Omega)\|\nabla u\|_p^p + \lambda\ln(\|\nabla u\|_p)\|u\|_p^p. \tag{3.30}$$

Combining (3.30) and Lemma 2.4, we can obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + (1 - \frac{1}{\lambda_1}\lambda(\lambda_1\Gamma(p,\Omega) + \ln(R))\|\nabla u\|_p^p + \beta\|u\|_2^2 \leq 0.$$

Note that $\lambda < R_0$, it follows that

$$\frac{d}{dt}\|u\|_2^2 + 2\beta\|u\|_2^2 \leq 0,$$

so we have

$$\|u(\cdot,t)\|_2^2 \leq \|u_0\|_2^2 e^{-2\beta t}.$$

Therefore, we conclude that $\|u(\cdot,t)\|_2 \to 0$ as $t \to \infty$.

(2) We consider first the case $2N/(N+2) < p < 2$ with $\lambda < R_0$. Multiplying (1.1) by u and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \|\nabla u\|_p^p + \beta\|u\|_2^2 = \lambda\int_\Omega |u|^p \ln|u| dx. \tag{3.31}$$

By the first eigenvalue λ_1 and Lemma 2.4, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + (1 - \frac{1}{\lambda_1}\lambda(\lambda_1\Gamma(p,\Omega) + \ln(R))\|\nabla u\|_p^p + \beta\int_\Omega u^2 dx \leq 0. \tag{3.32}$$

By virtue of Holder's inequality and the embedding theorem, we obtain

$$\|u\|_2 \leq |\Omega|^{\frac{1}{2}-\frac{N-p}{Np}}\|u\|_{\frac{Np}{N-p}} \leq C|\Omega|^{\frac{1}{2}-\frac{N-p}{Np}}\|\nabla u\|_p, \tag{3.33}$$

where C is the embedding constant.

By (3.32) and (3.33), we get the following differential inequality

$$\frac{d}{dt}\|u\|_2^2 + 2C_0\|\nabla u\|_2^p + 2\beta\|u\|_2^2 \leq 0, \tag{3.34}$$

where

$$C_0 = C|\Omega|^{(\frac{N-p}{Np}-\frac{1}{2})p}\Big(1 - \lambda\Gamma(p,\Omega) - \frac{1}{\lambda_1}\ln(R)\Big) > 0. \tag{3.35}$$

Setting $y(t) = \|u(\cdot,t)\|_2^2$, $y(0) = \|u_0(\cdot)\|_2^2$, by Lemma 2.2, we obtain

$$\begin{cases} \|u(\cdot,t)\|_2 \leq \Big[\Big(\|u_0\|_2^{2-p} + \frac{C_0}{\beta}\Big)e^{(p-2)\beta t} - \frac{C_0}{\beta}\Big]^{\frac{1}{2-p}}, & t \in [0,T_1), \\ \|u(\cdot,t)\|_2 \equiv 0, & t \in [T_1,\infty), \end{cases}$$

where

$$T_1 = \frac{1}{(2-p)\beta}\ln(1 + \frac{\beta}{C_0}\|u_0\|_2^{2-p}). \tag{3.36}$$

We now turn to the case $1 < p \leq 2N/(N+2)$ and $\lambda < R_1$.

Multiplying (1.1) by u^l ($l = \frac{2N-(1+N)p}{p} \geq 1$) and integrating over Ω, we can obtain

$$\frac{1}{l+1}\frac{d}{dt}\|u\|_{l+1}^{l+1} + \Big(\frac{lp^p}{(p+l-1)^p} - \frac{\lambda p\Gamma(p,\Omega)}{p+l-1} - \frac{\lambda p\ln(R)}{\lambda_1(p+l-1)}\Big)\|\nabla u^{\frac{p+l-1}{p}}\|^p + \beta\|u\|_{l+1}^{l+1} \leq 0, \tag{3.37}$$

By the embedding theorem and the specific choice of l, we have

$$\|u\|_{l+1}^{\frac{p+l-1}{p}} = \Big(\int_\Omega u^{\frac{p+l-1}{p}\cdot\frac{Np}{N-p}}dx\Big)^{\frac{N-p}{Np}} \leq C\|\nabla u^{\frac{p+l-1}{p}}\|, \tag{3.38}$$

where C is the embedding constant. Thus (3.37) becomes

$$\frac{1}{l+1}\frac{d}{dt}\|u\|_{l+1}^{l+1} + C_1\|u\|_{l+1}^{p+l-1} + \beta\|u\|_{l+1}^{l+1} \leq 0, \tag{3.39}$$

where

$$C_1 = C\Big(\frac{lp^p}{(p+l-1)^p} - \lambda p(\frac{\Gamma(p,\Omega)}{p+l-1} + \frac{\ln(R)}{\lambda_1(p+l-1)})\Big) > 0.$$

Setting $y(t) = \|u(\cdot,t)\|_{l+1}$, $y(0) = \|u_0(\cdot)\|_{l+1}$, by Lemma 2.6, we can obtain

$$\begin{cases} \|u(\cdot,t)\|_{l+1} \leq \Big[\Big(\|u_0\|_{l+1}^{2-p} + \frac{C_1}{\beta}\Big)e^{(p-2)\beta t} - \frac{C_1}{\beta}\Big]^{\frac{1}{2-p}}, & t \in [0, T_2), \\ \|u(\cdot,t)\|_{l+1} \equiv 0, & t \in [T_2, \infty), \end{cases}$$

where

$$T_2 = \frac{1}{(2-p)\beta}\ln\big(1 + \frac{\beta}{C_1}\|u_0\|_{l+1}^{2-p}\big). \tag{3.40}$$

The proof is completed. □

Proof of Theorem 1.4

We first define

$$M(t) := \int_0^t \|u\|_2^2 dr,$$

then, we obtain

$$M'(t) = \|u\|_2^2,$$

and

$$\begin{aligned} M''(t) &= -2I(u) \\ &= -2pJ(u) + 2p\beta(\frac{1}{q} - \frac{1}{p})\|u\|_q^q + 2\lambda\frac{1}{p}\|u\|_p^p. \end{aligned} \tag{3.41}$$

By (3.28) and (3.41), we can obtain

$$M''(t) \geq -2pJ(u_0) + 2p\int_0^t \|u_r\|dr + 2\lambda\frac{1}{p}\|u\|_p^p. \tag{3.42}$$

Not that

$$\begin{aligned} I(u) \geq& \|\nabla u\|_p^p - \lambda\int_\Omega |u|^p\ln|u|dx \\ \geq& (1 - \frac{\lambda\mu}{p})\|\nabla u\|_p^p + \lambda\Big(\ln\big(\frac{p\mu e}{n\mathbb{L}_p}\big)^{\frac{n}{p^2}} - \ln(\|u\|_p)\Big)\|u\|_p^p. \end{aligned} \tag{3.43}$$

Choosing $\mu = \frac{1}{\lambda}p$ in (3.43), we have

$$I(u) \geq \lambda\Big(\ln\big(\frac{p^2 e}{\lambda n \mathbb{L}_p}\big)^{\frac{n}{p^2}} - \ln(\|u\|_p)\Big)\|u\|_p^p.$$

Since $I(u_0) < 0$, we get

$$\big(\frac{p^2 e}{\lambda n \mathbb{L}_p}\big)^{\frac{n}{p^2}} \leq \|u\|_p. \tag{3.44}$$

By (3.42), (3.44) and $p > q$, we obtain

$$\begin{aligned} M''(t) &\geq -2pJ(u_0) + 2p\int_0^t \|u_r\|dr + 2\lambda\frac{1}{p}\big(\frac{p^2 e}{\lambda n \mathbb{L}_p}\big)^{\frac{n}{p}} \\ &= 2p(R_2 - J(u_0)) + 2p\int_0^t \|u_r\|dr, \end{aligned} \tag{3.45}$$

where $R_2 = \lambda\frac{1}{p^2}\big(\frac{p^2 e}{\lambda n \mathbb{L}_p}\big)^{\frac{n}{p}}$. Multiplying both sides by $M(t)$ in inequality (3.45), we get

$$M''(t)M(t) \geq 2p(R_2 - J(u_0))M(t) + 2p\int_0^t \|u_r\|dr\int_0^t \|u(r)\|dr. \tag{3.46}$$

Since

$$\frac{1}{4}(M'(t)) \leq \Big(\int_0^t\int_\Omega u_r(r)u(r)dxdr\Big)^2 \leq \int_0^t \|u(r)\|_2^2 ds\int_0^t \|u_r(r)\|_2^2 dr,$$

thus we have

$$M''(t)M(t) \geq 2p(R_2 - J(u_0))M(t) + \frac{p}{2}(M'(t))^2.$$

namely

$$M''(t)M(t) - \frac{p}{2}(M'(t))^2 \geq 2p(R_2 - J(u_0))M(t) > 0.$$

So, there exists a finite time $T_0 > 0$ such that

$$\lim_{t\to T_0^-} M(t) = +\infty,$$

which implies

$$\lim_{t\to T_0^-} \|u\|_2^2 = +\infty.$$

The proof is completed. □

Proof of Theorem 1.5

We consider first the case $p < r < 2$ and $2N/(N+2) < p < 2$. Choosing $v = u$ in Definition 1.1, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \|u\|_{1,\Omega}^p + \beta\int_\Omega u^2 dx = \int_\Omega |u|^r \ln|u|dx. \tag{3.47}$$

By Lemma 2.2, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} u^2 dx + \|u\|_{1,\Omega}^p + \beta\int_{\Omega} u^2 dx \le \lambda\Gamma(p,\Omega)\|u\|_{1,\Omega}^r + \lambda\ln(\|u\|_{1,\Omega})\int_{\Omega} u^r dx. \tag{3.48}$$

Since $I(u_0) > 0$, combining Lemma 2.4 and Lemma 2.7, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} u^2 dx + (1-\lambda\Gamma(p,\Omega)R^{r-p})\|u\|_{1,\Omega}^p + \beta\int_{\Omega} u^2 dx \le \lambda\ln(R)\|u\|_r^r. \tag{3.49}$$

To facilitate discussion, we let

$$s_1 = \frac{v_1}{2} + \frac{1-v_1}{p^*},\ v_1 = \frac{2(r-p)}{r(2-p)}.$$

Note that $s_1 > r$ and $v_1 \in (0,1)$, by the Hölder's inequality, the Sobolev embedding theorem and Lemma 2.5, we have

$$\|u\|_r^r \le |\Omega|^{\frac{s_1-r}{s_1}}\|u\|_{s_1}^r \le |\Omega|^{\frac{s_1-r}{s_1}} C_{p^*}^{r(1-\vartheta_1)}\|u\|_2^{rv_1}\|u\|_{1,\Omega}^{r(1-v_1)}.$$

Further, by the Young inequality for any $\varepsilon > 0$, we get

$$\|u\|_r^r \le |\Omega|^{\frac{s_1-r}{s_1}} C_{p^*}^{r(1-\vartheta_1)}(\varepsilon\|u\|_{1,\Omega}^p + \varepsilon^{\frac{r(v_1-1)}{p-r(1-v_1)}}\|u\|_2^2). \tag{3.50}$$

By (3.49) and (3.50), we can obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + (1-\lambda\Gamma(p,\Omega)R^{r-p})\|u\|_{1,\Omega}^p + \beta\|u\|_2^2 \le \lambda\ln(R)|\Omega|^{\frac{s_1-r}{s_1}} C_{p^*}^{r(1-\vartheta_1)}(\varepsilon\|u\|_{1,\Omega}^p + \varepsilon^{\frac{r(v_1-1)}{p-r(1-v_1)}}\|u\|_2^2),$$

namely

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + C_2\|u\|_{1,\Omega}^p + C_3\|u\|_2^2 \le 0, \tag{3.51}$$

where

$$C_2 = 1-\lambda\Gamma(p,\Omega)R^{r-p} - \varepsilon\lambda\ln(R)|\Omega|^{\frac{s_1-r}{s_1}} C_{p^*}^{r(1-\vartheta_1)},\ C_3 = \beta - \lambda\ln(R)|\Omega|^{\frac{s_1-r}{s_1}} C_{p^*}^{r(1-\vartheta_1)}\varepsilon^{\frac{r(v_1-1)}{p-r(1-v_1)}}.$$

Taking $\varepsilon = \Big(\frac{\beta}{3\lambda\ln(R)|\Omega| C_{p^*}^{r(1-v_1)}}\Big)^{\frac{p-r(1-v_1)}{r(v_1-1)}}$ in (3.51), we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + C_4\|u\|_{1,\Omega}^p + \frac{2}{3}\beta\|u\|_2^2 \le 0, \tag{3.52}$$

where

$$C_4 = 1-\lambda\Gamma(p,\Omega)R^{r-p} - \lambda\ln(R)|\Omega|^{\frac{s_1-r}{s_1}} C_{p^*}^{r(1-\vartheta_1)}\Big(\frac{\beta}{3\lambda\ln(R)|\Omega| C_{p^*}^{r(1-v_1)}}\Big)^{\frac{p-r(1-v_1)}{r(v_1-1)}}.$$

Since $\lambda < R_2$, thus $C_4 > 0$. By Lemma 2.6, we can obtain

$$\begin{cases} \|u(\cdot,t)\|_2 \le \Big[\Big(\|u_0\|_2^{2-p} + \frac{3C_4}{2\beta}\Big)e^{(p-2)\frac{1}{3}\beta t} - \frac{3C_4}{2\beta}\Big]^{\frac{1}{2-p}}, & t\in[0,T_3), \\ \|u(\cdot,t)\|_2 \equiv 0, & t\in[T_3,\infty), \end{cases}$$

where

$$T_3 = \frac{3}{(2-p)\beta}\ln(1+\frac{2\beta}{3C_4}\|u_0\|_2^{2-p}). \tag{3.53}$$

When $1 < p \le 2N/(N+2)$, $p < r \le 2$ and $\lambda < R_3$. Taking $v = u^l$ ($l = \frac{2N-(1+N)p}{p} \ge 1$) in Definition 1.1, we obtain

$$\frac{1}{l+1}\frac{d}{dt}\|u\|_{l+1}^{l+1} + \frac{lp^p}{(p+l-1)^p}\|\nabla u^{\frac{l+p-1}{p}}\|_p^p + \beta\|u\|_{l+1}^{l+1} = \int_\Omega |u|^{r+l-1}\ln|u|dx. \tag{3.54}$$

By Lemma 2.1, we have

$$\frac{1}{l+1}\frac{d}{dt}\|u\|_{l+1}^{l+1} + \frac{lp^p}{(p+l-1)^p}\|\nabla u^{\frac{l+p-1}{p}}\|_p^p + \beta\|u\|_{l+1}^{l+1} \le \lambda\frac{1}{e\sigma}\|u\|_{p_1}^{p_1}, \tag{3.55}$$

where $p_1 = l + p + \sigma - 1$. Let

$$\frac{1}{s_2} = \frac{v_2}{l_0} + \frac{1-v_2}{p^*},\ l_0 = \frac{p(l+1)}{l+p-1},\ v_2 = \frac{l_0(l_1-p)}{l_1(l_0-p)},\ l_1 = \frac{pp_1}{l+p-1}.$$

Thus we have $l_0 > l_1 > p$, $v_1 \in (0,1)$ and

$$\frac{1}{l+1}\frac{d}{dt}\|u^{\frac{l+p-1}{p}}\|_{l_0}^{l_0} + \frac{lp^p}{(p+l-1)^p}\|\nabla u^{\frac{l+p-1}{p}}\|_p^p + \beta\|u^{\frac{l+p-1}{p}}\|_{l_0}^{l_0} \le \lambda\frac{1}{e\sigma}\|u^{\frac{l+p-1}{p}}\|_{l_1}^{l_1}. \tag{3.56}$$

By the Hölder's inequality, the following inequality holds

$$\|u^{\frac{l+p-1}{p}}\|_{l_1}^{l_1} \le |\Omega|^{\frac{s_2-l_1}{s_2}}\|u^{\frac{l+p-1}{p}}\|_{s_2}^{l_1} \le |\Omega|^{\frac{s_2-l_1}{s_2}}C^{(1-v_2)l_1}\|u^{\frac{l+p-1}{p}}\|_{l_0}^{l_1v_2}\|u^{\frac{l+p-1}{p}}\|_{1,\Omega}^{(1-v_2)l_1}.$$

Further, by the Young inequality for any $\varepsilon > 0$, we get

$$\|u^{\frac{l+p-1}{p}}\|_{l_1}^{l_1} \le |\Omega|^{\frac{s_2-l_1}{s_2}}C_{p^*}^{(1-v_2)l_1}(\varepsilon\|u^{\frac{l+p-1}{p}}\|_{1,\Omega}^p + \varepsilon^{\frac{l_1(v_2-1)}{p-l_1(1-v_2)}}\|u^{\frac{l+p-1}{p}}\|_{l_0}^{l_0}). \tag{3.57}$$

Substituting (3.57) into (3.56), we get

$$\frac{1}{l+1}\frac{d}{dt}\|u^{\frac{l+p-1}{p}}\|_{l_0}^{l_0} + C_5\|\nabla u^{\frac{l+p-1}{p}}\|_p^p + C_6\|u^{\frac{l+p-1}{p}}\|_{l_0}^{l_0} \le 0, \tag{3.58}$$

where

$$C_5 = \frac{lp^p}{(p+l-1)^p} - \lambda\frac{1}{e\sigma}|\Omega|^{\frac{s_2-l_1}{s_2}}C_{p^*}^{(1-v_2)l_1},\ C_6 = \beta - |\Omega|^{\frac{s_2-l_1}{s_2}}C_{p^*}^{(1-V_2)l_1}\lambda\frac{1}{e\sigma}\varepsilon^{\frac{l_1(v_2-1)}{p-l_1(1-v_2)}}.$$

Taking $\varepsilon = \frac{1}{3}\frac{lp^pe\sigma}{(p+l-1)^p}|\Omega|^{\frac{l_1-s_2}{s_2}}C_{p^*}^{v_2-1}$ in (3.58), we can obtain

$$\frac{1}{l+1}\frac{d}{dt}\|u^{\frac{l+p-1}{p}}\|_{l_0}^{l_0} + \frac{1}{3}\frac{lp^p}{(p+l-1)^p}\|\nabla u^{\frac{l+p-1}{p}}\|_p^p + C_7\|u^{\frac{l+p-1}{p}}\|_{l_0}^{l_0} \le 0, \tag{3.59}$$

where

$$C_7 = \beta - |\Omega|^{\frac{s_2-l_1}{s_2}}C_{p^*}^{(1-v_2)l_1}\lambda\frac{1}{e\sigma}\Big(\frac{1}{3}\frac{lp^pe\sigma}{(p+l-1)^p}|\Omega|^{\frac{l_1-s_2}{s_2}}C_{p^*}^{v_2-1}\Big)^{\frac{l_1(v_2-1)}{p-l_1(1-v_2)}}.$$

Since $l_0 < p^*$, by the Sobolev embedding theorem and the Hölder's inequality, we get

$$\|u^{\frac{l+p-1}{p}}\|_{l_0}^{p} \leq |\Omega|^{(\frac{1}{l_0}-\frac{1}{p^*})p} C_{p^*}^{p} \|u^{\frac{l+p-1}{p}}\|_{1,\Omega}^{p} = |\Omega|^{(\frac{1}{l_0}-\frac{1}{p^*})p} C_{p^*}^{p} \|\nabla u^{\frac{l+p-1}{p}}\|_{p}^{p}. \tag{3.60}$$

Combining (3.59) with (3.60), we obtain

$$\frac{1}{l+1}\frac{d}{dt}\|u^{\frac{l+p-1}{p}}\|_{l_0}^{l_0} + C_8\|u^{\frac{l+p-1}{p}}\|_{l_0}^{p} + C_7\|u^{\frac{l+p-1}{p}}\|_{l_0}^{l_0} \leq 0, \tag{3.61}$$

where

$$C_8 = \frac{1}{3}\frac{lp^p}{(p+l-1)^p}|\Omega|^{(\frac{1}{p^*}-\frac{1}{l_0})p} C_{p^*}^{-p} > 0.$$

Since $\lambda < R_3$, thus $C_7 > 0$. By Lemma 2.6, we can obtain

$$\begin{cases} \|u\|_{l+1} \leq \left[\left(\|u_0\|_{l+1}^{2-p} + \frac{C_8}{C_7}\right)e^{(p-2)C_7 t} - \frac{C_8}{C_7}\right]^{\frac{1}{2-p}}, & t \in [0, T_4), \\ \|u\|_{l+1} \equiv 0, & t \in [T_4, \infty), \end{cases}$$

where

$$T_4 = \frac{1}{(2-p)C_7}\ln(\frac{C_7}{C_8}\|u_0\|_{l+1}^{2-p} + 1). \tag{3.62}$$

The proof is completed. □

4. Conclusions

In this work, we study the initial boundary value problem for a class of p-Laplacian diffusion equations with logarithmic nonlinearity. Compared with the research in literature [26], we further discussed the integer-order Laplacian equations when $1 < p < 2$, and proved the global existence of the solution of problem (1.1) by the Galerkin approximation method. Compared with problems (1.2) and (1.5), we give the extinction and attenuation estimates of the weak solution of problem (1.1) by using potential well theory and Nehari manifold. In addition, we also prove that the weak solution of problem (1.1) is globally bounded and blows up at infinity. In the next work, we will further discuss the properties of the solution of Eq (1.1) when $r \neq p$ and $q \neq 2$, and study the diffusion $p(x)$-Laplacian with logarithmic nonlinearity.

Acknowledgments

This research was supported by the project of Guizhou province science and technology plan under (No.Qiankehe foundation-ZK [2021] YIBAN317), and by the project of Guizhou Minzu University under (No.GZMU [2019] YB04).

Conflict of interest

The authors declare no conflict of interest.

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