Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/MBE

MBE, 18(4): 3907–3921.
DOI: 10.3934/mbe.2021195
Received: 26 February 2021
Accepted: 20 April 2021
Published: 06 May 2021

Research article

Existence and stability of nonlinear discrete fractional initial value problems with application to vibrating eardrum

George Maria Selvam[1], Jehad Alzabut[2,3], Vignesh Dhakshinamoorthy[1], Jagan Mohan Jonnalagadda[4] and Kamaleldin Abodayeh[2,*]

[1] Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur-635601, Tamil Nadu, India
[2] Department of Mathematics and General Sciences, Prince Sultan University, 11586 Riyadh, Saudi Arabia
[3] Group of Mathematics, Faculty of Engineering, Ostim Technical University, 06374 Ankara, Turkey
[4] Department of Mathematics, Birla Institute of Technology and Science Pilani, Hyderabad-500078, Telangana, India

* **Correspondence:** Email: kamal@psu.edu.sa.

Abstract: It is well known that Newton's second law can be applied in various biological processes including the behavior of vibrating eardrums. In this work, we consider a nonlinear discrete fractional initial value problem as a model describing the dynamic of vibrating eardrum. We establish sufficient conditions for the existence, uniqueness, and Hyers-Ulam stability for the solutions of the proposed model. To examine the validity of our findings, a concrete example of forced eardrum equation along with numerical simulation is analyzed.

Keywords: nonlinear discrete fractional problem; existence and uniqueness; Hyers–Ulam stability; vibrating eardrum

1. Introduction

Understanding the methodology of various physical systems by direct observation consumes time and does not provide clear insight into their working. That is where mathematical descriptions of real-life events play a major role in analyzing their behaviour and nature. Differential equations prove to be ideal tools for modelling phenomena by considering the laws of physics they follow [1, 2]. Qualitative and quantitative studies of natural occurrences by engineers and researchers contribute a lot to developing technology and guidance in the treatment of physical activity.

The derivatives of arbitrary order were initially proposed in the 17^{th} century. It took over three centuries for the expansion in the mathematical theory of fractional calculus. Due to its accuracy in interpreting the system's performance and complexities, arbitrary order calculus proves to be an imperative tool in defining societal problems in various fields of engineering, biotechnology, chemistry, physics, etc., [3–5]. It was in the last few decades when the discrete version of the non-integer order calculus had gained popularity among the researchers [6]. The majority of the research papers on discrete-time calculus of fractional order were dedicated to the existence theory. However, rare results have dealt with the stabilization theory.

The theory of fractional calculus of discrete-time has originated from the ground-works of Atici and Eloe [7–10], Goodrich [11] and Miller and Ross [12]. Many researchers since then have considered further relevant results [13–17]. The stability analysis of nonlinear fractional difference equations is more delicated than the continuous fractional calculus's stability notion. The investigation of asymptotic stability by Chen et al. in [18, 19] has drawn the attention of mathematicians and scientists of various fields. Hyers answers to Ulam's questions have initiated the stability analysis of functional equations [20, 21]. There were significant contributions towards the study of Ulam stability of classical integer order equations and fractional order equations by Niazi et al. [22], Wang et al. [23–25], and Ibrahim [26]. Ulam stability discussion for integrodifferential boundary value problems and Hilfer–Hadamard type coupled system of fractional equations are made in [27, 28]. The study of Ulam stability in [29–32] enriched the qualitative theory of discrete fractional equations.

Oriented by the above work and motivated by the desire to investigate further applications of discrete fractional equations, this paper considers the following nonlinear discrete fractional initial value problem:

$$\begin{cases} \Delta_*^{\rho}[\mathfrak{u}(\varsigma)] + \mathfrak{h}\left(\varsigma+\rho, \mathfrak{u}(\varsigma+\rho), \Delta_*^{\vartheta}[\mathfrak{u}(\varsigma+\rho-\vartheta)]\right) = 0, \\ \varsigma \in [0, \mathcal{T}] \cap \mathbb{N}_{2-\rho},\ 1 < \rho \leq 2,\ 0 < \vartheta \leq 1, \\ \mathfrak{u}(0) = \mathcal{A},\ \Delta(\mathfrak{u}(0)) = \mathcal{B}, \end{cases} \tag{1.1}$$

where Δ_*^{α}, $\alpha \in \{\rho, \vartheta\}$, denotes the α-th order Caputo-like forward difference operator and $\mathfrak{h} : \mathfrak{D} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Here $\mathfrak{D} = [\kappa+2, \kappa+\mathcal{T}] \cap \mathbb{N}_{\kappa+2}$, where $\mathcal{T} \in \mathbb{N}$ and $\mathbb{N}_\kappa = \{\kappa, \kappa+1, \ldots\}, \kappa \in \mathbb{R}$. We establish sufficient conditions on existence, uniqueness and Hyers-Ulam stability of solutions of Eq (1.1). We claim that our results in this paper are new and are even different from the ones reported in the continuous cases [33–35].

The rest of the paper adheres to the following plan: Section 2 presents supporting definitions and lemmas which are essential for subsequent sections. The main results including existence, uniqueness and stability results are conferred in Sections 3–5, respectively. In Section 6, an exciting application on the damped eardrum equation along with numerical simulation is considered and will be analyzed. We end the paper with a conclusion.

2. Basic requisites

This section provides some fundamental mathematical results used throughout this work.

Definition 2.1. [9] ($\vartheta - th$ Fractional Sum) The $\vartheta - th$ fractional sum of $\mathfrak{p}$ is

$$\Delta^{-\vartheta}\mathfrak{p}(\varsigma) = \frac{1}{\Gamma(\vartheta)}\sum_{\ell=\kappa}^{\varsigma-\vartheta}(\varsigma-\ell-1)^{(\vartheta-1)}\mathfrak{p}(\ell), \tag{2.1}$$

where $\vartheta > 0$, $\mathfrak{p}$ is defined for $\ell \equiv \kappa \bmod(1)$ and $\varsigma^{(\vartheta)} = \dfrac{\Gamma(\varsigma+1)}{\Gamma(\varsigma-\vartheta+1)}$.

Definition 2.2. [9] Let $q > 0$ and $j-1 < q < j$, where $j \in \mathbb{N}_1$, $j = \lceil q \rceil$. Set $\vartheta = j - q$. The $q - th$ fractional difference is

$$\begin{aligned}\Delta_*^q\mathfrak{h}(\varsigma) &= \Delta^{-\vartheta}\left(\Delta^j\mathfrak{h}(\varsigma)\right)\\ &= \frac{1}{\Gamma(\vartheta)}\sum_{\ell=\kappa}^{\varsigma-\vartheta}(\varsigma-\ell-1)^{(\vartheta-1)}\left(\Delta^j\mathfrak{h}\right)(\ell), \quad \forall\varsigma \in N_{\kappa+\vartheta},\end{aligned} \tag{2.2}$$

where Δ^j is the $j - th$ order forward difference operator.

Lemma 2.3. *[6] For non-integer $q > 0$, $j = \lceil q \rceil$, $\vartheta = j - q$, we have*

$$\mathfrak{h}(\varsigma) = \sum_{m=0}^{j-1}\frac{(\varsigma-\kappa)^{(m)}}{m!}\Delta^m[\mathfrak{h}(\kappa)] + \frac{1}{\Gamma(q)}\sum_{\ell=\kappa+\vartheta}^{\varsigma-q}(\varsigma-\ell-1)^{(q-1)}\Delta_*^q[\mathfrak{h}(\ell)],$$

where $\mathfrak{h}$ is defined on $\mathbb{N}_\kappa$ with $\kappa \in \mathbb{Z}^+$. If $1 < q < 2$ and $\kappa = 0$, then we have

$$\mathfrak{h}(\varsigma) = \mathfrak{h}(0) + \varsigma\Delta(\mathfrak{h}(0)) + \frac{1}{\Gamma(q)}\sum_{\ell=2-q}^{\varsigma-q}(\varsigma-\ell-1)^{(q-1)}\Delta_*^q[\mathfrak{h}(\ell)], \tag{2.3}$$

where $\mathfrak{h}$ is defined on $\mathbb{N}_1$.

Lemma 2.4. $\mathfrak{u} : \mathfrak{D} \to \mathbb{R}$ *is a solution of Eq* (1.1) *if and only if* $\mathfrak{u}$ *solves the fractional Taylor's difference formula given by*

$$\mathfrak{u}(\varsigma) = \mathcal{A} + \varsigma\mathcal{B} + \Delta^{-\rho}\left(-\mathfrak{h}(\varsigma+\rho, \mathfrak{u}(\varsigma+\rho), \Delta_*^\vartheta[\mathfrak{u}(\varsigma+\rho-\vartheta)])\right), \quad \varsigma \in \mathfrak{D}. \tag{2.4}$$

Proof. Let $\mathfrak{u}$ be a solution of Eq (1.1). Then, from Eq (2.3), we have

$$\begin{aligned}\mathfrak{u}(\varsigma) &= [\mathfrak{u}(0) + \varsigma\Delta(\mathfrak{u}(0))] + \frac{1}{\Gamma(\rho)}\sum_{\ell=2-\rho}^{\varsigma-\rho}(\varsigma-\ell-1)^{(\rho-1)}\Delta_*^\rho[\mathfrak{u}(\ell)]\\ &= [\mathcal{A} + \varsigma\mathcal{B}] + \frac{1}{\Gamma(\rho)}\sum_{\ell=2-\rho}^{\varsigma-\rho}(\varsigma-\ell-1)^{(\rho-1)}\left(-\mathfrak{h}(\ell+\rho, \mathfrak{u}(\ell+\rho), \Delta_*^\vartheta[\mathfrak{u}(\ell+\rho-\vartheta)])\right).\end{aligned}$$

Therefore, Eq (2.4) holds.

Conversely, let $\mathfrak{u}$ be a solution of Eq (2.4). The comparison of Eqs (2.3) and (2.4) gives

$$\sum_{\ell=2-\rho}^{\varsigma-\rho}(\varsigma-\ell-1)^{(\rho-1)}\Delta_*^\rho[\mathfrak{u}(\ell)] = \sum_{\ell=2-\rho}^{\varsigma-\rho}(\varsigma-\ell-1)^{(\rho-1)}\left(-\mathfrak{h}(\ell+\rho, \mathfrak{u}(\ell+\rho), \Delta_*^\vartheta[\mathfrak{u}(\ell+\rho-\vartheta)])\right),$$

which takes the form

$$\sum_{\ell=2-\rho}^{\varsigma-\rho}(\varsigma-\ell-1)^{(\rho-1)}[\Delta_*^{\rho}[\mathfrak{u}(\ell)]-\Big(-\mathfrak{h}(\ell+\rho,\mathfrak{u}(\ell+\rho),\Delta_*^{\vartheta}[\mathfrak{u}(\ell+\rho-\vartheta)])\Big)]=0, \tag{2.5}$$

for $\varsigma\in\mathfrak{D}$. Letting $\varsigma=1,2,\cdots$, we have

$$\Delta_*^{\rho}[\mathfrak{u}(\varsigma)]+\mathfrak{h}(\varsigma+\rho,\mathfrak{u}(\varsigma+\rho),\Delta_*^{\vartheta}[\mathfrak{u}(\varsigma+\rho-\vartheta)])=0,\quad \varsigma\in\mathfrak{D}. \tag{2.6}$$

Evidently, $\mathfrak{u}$ is solution of Eq (1.1). □

Lemma 2.5. *The following identity holds*

$$\sum_{\ell=2-\rho}^{\varsigma-\rho}(\varsigma-\ell-1)^{(\rho-1)}=\frac{(\varsigma+\rho-2)^{(\rho)}}{\rho}. \tag{2.7}$$

Proof. Let $r,w\in\mathbb{R}$. For $r>-1, w>-1$ and $r>w$, we have

$$\frac{\Gamma(r+1)}{\Gamma(r-w+1)}=\frac{1}{w+1}\left[\frac{\Gamma(r+2)}{\Gamma(r-w+1)}-\frac{\Gamma(r+1)}{\Gamma(r-w)}\right], \tag{2.8}$$

$$\begin{aligned}\sum_{\ell=2-\rho}^{\varsigma-\rho}(\varsigma-\ell-1)^{(\rho-1)}&=\sum_{\ell=2-\rho}^{\varsigma-\rho}\frac{\Gamma(\varsigma-\ell)}{\Gamma(\varsigma-\ell-\rho+1)}\\&=\sum_{\ell=2-\rho}^{\varsigma-\rho-1}\frac{\Gamma(\varsigma-\ell)}{\Gamma(\varsigma-\ell-\rho+1)}+\Gamma(\rho)\\&=\sum_{\ell=2-\rho}^{\varsigma-\rho-1}\frac{1}{\rho}\left[\frac{\Gamma(\varsigma-\ell+1)}{\Gamma(\varsigma-\ell-\rho+1)}-\frac{\Gamma(\varsigma-\ell)}{\Gamma(\varsigma-\ell-\rho)}\right]+\Gamma(\rho)\\&=\frac{1}{\rho}\left[\frac{\Gamma(\varsigma+\rho-1)}{\Gamma(\varsigma-1)}-\Gamma(\rho+1)\right]+\Gamma(\rho)\\&=\frac{(\varsigma+\rho-2)^{(\rho)}}{\rho}.\end{aligned}$$

This completes the proof. □

3. Existence of solution

First, we state the Krasnoselskii's fixed point theorem.

Theorem 3.1. *[36] Let B be a nonempty closed convex subset of a Banach space $(X,\|\cdot\|)$. Suppose that Λ_1 and Λ_2 map B into X such that*

(I) for any $x, y\in B$, $\Lambda_1x+\Lambda_2y\in B$;
(II) Λ_1 is a contraction;
(III) Λ_2 is continuous and $\Lambda_2(B)$ is contained in a compact set.

Then, there exists $z \in B$ such that $z = \Lambda_1 z + \Lambda_2 z$.

Now, we establish sufficient conditions on the existence of solutions for the initial value problem (1.1) using Krasnoselskii's fixed point theorem.

Define the operators $\mathfrak{T}_1$ and $\mathfrak{T}_2$ by

$$\mathfrak{T}_1\mathfrak{u}(\varsigma) = \Delta^{-\rho}\left(-\mathfrak{h}(\varsigma+\rho, \mathfrak{u}(\varsigma+\rho), \Delta_*^{\vartheta}[\mathfrak{u}(\varsigma+\rho-\vartheta)])\right), \quad \varsigma \in \mathfrak{D}, \tag{3.1}$$

$$\mathfrak{T}_2\mathfrak{u}(\varsigma) = \mathcal{A} + \varsigma\mathcal{B}, \quad \varsigma \in \mathfrak{D}. \tag{3.2}$$

Let $\mathbb{S} = \{\mathbb{C}[\mathfrak{D}, \mathbb{R}]\}$ be the space of all functions $\mathfrak{u}$ with the norm defined by

$$\|\mathfrak{u}\| = \max_{\varsigma\in\mathfrak{D}}\left\{\sup_{\varsigma\in\mathfrak{D}}|\mathfrak{u}(\varsigma)|, \sup_{\varsigma\in\mathfrak{D}}\left|\Delta_*^{\vartheta}\mathfrak{u}(\varsigma)\right|\right\}.$$

Consider the set

$$C = \{u \in \mathbb{S} : \|u\| \le r\}.$$

Then, C is a closed, convex subset of the Banach space $(\mathbb{S}, \|\cdot\|)$. Clearly, $\mathfrak{T}_1, \mathfrak{T}_2 : C \to \mathcal{S}$.

Now we consider the following assumptions:

(H_1) $\mathfrak{h} : \mathfrak{D} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H_2) There exists $\mathcal{M} > 0$ such that for all $\mathfrak{u} \in C$ and $\varsigma \in \mathfrak{D}$ the following relation holds:

$$\left|\mathfrak{h}(\varsigma+\rho, \mathfrak{u}(\varsigma+\rho), \Delta_*^{\vartheta}[\mathfrak{u}(\varsigma+\rho-\vartheta)])\right| \le \mathcal{M}\left[|\mathfrak{u}(\varsigma+\rho)| + \left|\Delta_*^{\vartheta}[\mathfrak{u}(\varsigma+\rho-\vartheta)]\right|\right]. \tag{3.3}$$

(H_3) There exists $\mathcal{L} > 0$ such that for all $\mathfrak{u}, \psi, z, v \in C$ and $\varsigma \in \mathfrak{D}$, the following relation holds:

$$|\mathfrak{h}(\varsigma, \mathfrak{u}, z) - \mathfrak{h}(\varsigma, \psi, v)| \le \mathcal{L}\left[|\mathfrak{u}-\psi| + |z-v|\right]. \tag{3.4}$$

Theorem 3.2. *Assume* (H_1)–(H_3) *hold and*

$$\mu = \max\left(\frac{2\mathcal{L}\Gamma(\mathcal{T}+\rho-1)}{\Gamma(\mathcal{T}-1)\Gamma(\rho+1)}, \frac{2\mathcal{L}\Gamma(\mathcal{T}+\rho-\vartheta-1)}{\Gamma(\mathcal{T}-1)\Gamma(\rho-\vartheta+1)}\right) < 1. \tag{3.5}$$

If we choose

$$\max\{\mathfrak{A}, \mathfrak{B}\} \le r, \tag{3.6}$$

where

$$\mathfrak{A} = |\mathcal{A}| + \mathcal{T}|\mathcal{B}| + \frac{2\mathcal{M}r\Gamma(\mathcal{T}+\rho-1)}{\Gamma(\mathcal{T}-1)\Gamma(\rho+1)},$$

and

$$\mathfrak{B} = \frac{|\mathcal{B}|\Gamma(\mathcal{T}+\rho)}{\Gamma(\mathcal{T}+\rho+\vartheta-1)\Gamma(2-\vartheta)} + \frac{2\mathcal{M}r\Gamma(\mathcal{T}+\rho-\vartheta-1)}{\Gamma(\mathcal{T}-1)\Gamma(\rho-\vartheta+1)},$$

then the initial value problem (1.1) *has a solution in* C.

Proof. The proof of the result is divided into three steps:

Step 1: For any $\mathfrak{u}, \mathfrak{v} \in C$, $\mathfrak{T}_1\mathfrak{u} + \mathfrak{T}_2\mathfrak{v} \in C$.

For each $\varsigma \in \mathfrak{D}$, we have

$$|\mathfrak{T}_1\mathfrak{u}(\varsigma) + \mathfrak{T}_2\mathfrak{v}(\varsigma)|$$

$$
\begin{aligned}
&= \left|\mathcal{A} + \varsigma\mathcal{B} + \Delta^{-\rho}\left(-\mathfrak{h}(\varsigma+\rho, \mathfrak{u}(\varsigma+\rho), \Delta_*^{\vartheta}[\mathfrak{u}(\varsigma+\rho-\vartheta)])\right)\right| \\
&\le |\mathcal{A}| + |\varsigma|\,|\mathcal{B}| + \left|\Delta^{-\rho}\left(-\mathfrak{h}(\varsigma+\rho, \mathfrak{u}(\varsigma+\rho), \Delta_*^{\vartheta}[\mathfrak{u}(\varsigma+\rho-\vartheta)])\right)\right| \\
&= |\mathcal{A}| + |\varsigma|\,|\mathcal{B}| \\
&\quad + \left|\frac{1}{\Gamma(\rho)}\sum_{\ell=2-\rho}^{\varsigma-\rho}(\varsigma-\ell-1)^{(\rho-1)}\left(-\mathfrak{h}(\ell+\rho, \mathfrak{u}(\ell+\rho), \Delta_*^{\vartheta}[\mathfrak{u}(\ell+\rho-\vartheta)])\right)\right| \\
&\le |\mathcal{A}| + |\varsigma|\,|\mathcal{B}| \\
&\quad + \frac{1}{\Gamma(\rho)}\sum_{\ell=2-\rho}^{\varsigma-\rho}(\varsigma-\ell-1)^{(\rho-1)}\left|\mathfrak{h}(\ell+\rho, \mathfrak{u}(\ell+\rho), \Delta_*^{\vartheta}[\mathfrak{u}(\ell+\rho-\vartheta)])\right| \\
&\le |\mathcal{A}| + \mathcal{T}\,|\mathcal{B}| + \frac{\mathcal{M}}{\Gamma(\rho)}\sum_{\ell=2-\rho}^{\varsigma-\rho}(\varsigma-\ell-1)^{(\rho-1)}\left[|\mathfrak{u}(\ell+\rho)| + \left|\Delta_*^{\vartheta}[\mathfrak{u}(\ell+\rho-\vartheta)]\right|\right] \\
&\le |\mathcal{A}| + \mathcal{T}\,|\mathcal{B}| + \frac{2\mathcal{M}\|\mathfrak{u}\|}{\Gamma(\rho)}\sum_{\ell=2-\rho}^{\varsigma-\rho}(\varsigma-\ell-1)^{(\rho-1)} \\
&= |\mathcal{A}| + \mathcal{T}\,|\mathcal{B}| + \frac{2\mathcal{M}\|\mathfrak{u}\|}{\Gamma(\rho)}\frac{(\varsigma+\rho-2)^{(\rho)}}{\rho} \\
&\le |\mathcal{A}| + \mathcal{T}\,|\mathcal{B}| + \frac{2\mathcal{M}r}{\Gamma(\rho+1)}(\mathcal{T}+\rho-2)^{(\rho)}.
\end{aligned}
$$

This implies that

$$\sup_{\varsigma\in\mathfrak{D}}|\mathfrak{T}_1\mathfrak{u}(\varsigma) + \mathfrak{T}_2\mathfrak{v}(\varsigma)| \le \mathfrak{A}.$$

Further, we have

$$
\begin{aligned}
&\left|\Delta_*^{\vartheta}\mathfrak{T}_1\mathfrak{u}(\varsigma) + \Delta_*^{\vartheta}\mathfrak{T}_2\mathfrak{v}(\varsigma)\right| \\
&= \left|\mathcal{B}\Delta_*^{\vartheta}\varsigma + \Delta_*^{\vartheta}\Delta^{-\rho}\left(-\mathfrak{h}(\varsigma+\rho, \mathfrak{u}(\varsigma+\rho), \Delta_*^{\vartheta}[\mathfrak{u}(\varsigma+\rho-\vartheta)])\right)\right| \\
&\le |\mathcal{B}|\left|\Delta_*^{\vartheta}\varsigma\right| + \left|\Delta_*^{\vartheta-\rho}\left(-\mathfrak{h}(\varsigma+\rho, \mathfrak{u}(\varsigma+\rho), \Delta_*^{\vartheta}[\mathfrak{u}(\varsigma+\rho-\vartheta)])\right)\right| \\
&= |\mathcal{B}|\frac{\Gamma(2)}{\Gamma(2-\vartheta)}(\varsigma-1+\rho)^{(1-\vartheta)} \\
&\quad + \left|\Delta_*^{\vartheta-\rho}\left(-\mathfrak{h}(\varsigma+\rho, \mathfrak{u}(\varsigma+\rho), \Delta_*^{\vartheta}[\mathfrak{u}(\varsigma+\rho-\vartheta)])\right)\right| \\
&\le |\mathcal{B}|\frac{\Gamma(2)}{\Gamma(2-\vartheta)}(\mathcal{T}-1+\rho)^{(1-\vartheta)} + \frac{2\mathcal{M}\|\mathfrak{u}\|}{\Gamma(\rho-\vartheta+1)}(\mathcal{T}+\rho-\vartheta-2)^{(\rho-\vartheta)},
\end{aligned}
$$

implying that

$$\sup_{\varsigma\in\mathfrak{D}}\left|\Delta_*^{\vartheta}\mathfrak{T}_1\mathfrak{u}(\varsigma) + \Delta_*^{\vartheta}\mathfrak{T}_2\mathfrak{v}(\varsigma)\right| \le \mathfrak{B}.$$

Thus, we have

$$\|\mathfrak{T}_1\mathfrak{u} + \mathfrak{T}_2\mathfrak{v}\| \le \max\{\mathfrak{A}, \mathfrak{B}\} \le r,$$

and hence $\mathfrak{T}_1\mathfrak{u} + \mathfrak{T}_2\mathfrak{v} \in \mathcal{C}$.
Step 2: $\mathfrak{T}_1$ is a contraction.

Let $\mathfrak{u}, \psi \in C$ and for each $\varsigma \in \mathfrak{D}$, we have

$$
\begin{aligned}
|\mathfrak{T}_1\mathfrak{u} - \mathfrak{T}_1\psi| &= |\mathcal{A} + \varsigma\mathcal{B} + \Delta^{-\rho}\left(-\mathfrak{h}(\varsigma+\rho, \mathfrak{u}(\varsigma+\rho), \Delta_*^{\vartheta}[\mathfrak{u}(\varsigma+\rho-\vartheta)])\right) \\
&- \mathcal{A} - \varsigma\mathcal{B} - \Delta^{-\rho}\left(-\mathfrak{h}(\varsigma+\rho, \psi(\varsigma+\rho), \Delta_*^{\vartheta}[\psi(\varsigma+\rho-\vartheta)])\right)| \\
&\leq \frac{1}{\Gamma(\rho)}\sum_{\ell=2-\rho}^{\varsigma-\rho}(\varsigma-\ell-1)^{(\rho-1)}|\left(\mathfrak{h}(\ell+\rho, \psi(\ell+\rho), \Delta_*^{\vartheta}[\psi(\ell+\rho-\vartheta)])\right) \\
&- \left(\mathfrak{h}(\ell+\rho, \mathfrak{u}(\ell+\rho), \Delta_*^{\vartheta}[\mathfrak{u}(\ell+\rho-\vartheta)])\right)| \\
&\leq \frac{\mathcal{L}\Gamma(\mathcal{T}+\rho-1)}{\Gamma(\mathcal{T}-1)\Gamma(\rho+1)}\left[|\mathfrak{u}-\psi| + \left|\Delta_*^{\vartheta}\mathfrak{u} - \Delta_*^{\vartheta}\psi\right|\right] \\
&\leq \frac{2\mathcal{L}\Gamma(\mathcal{T}+\rho-1)}{\Gamma(\mathcal{T}-1)\Gamma(\rho+1)}\|\mathfrak{u}-\psi\|.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\left|\Delta_*^{\vartheta}\mathfrak{T}_1\mathfrak{u} - \Delta_*^{\vartheta}\mathfrak{T}_1\psi\right| &= |\Delta_*^{\vartheta}\Delta^{-\rho}\left(-\mathfrak{h}(\varsigma+\rho, \mathfrak{u}(\varsigma+\rho), \Delta_*^{\vartheta}[\mathfrak{u}(\varsigma+\rho-\vartheta)])\right) \\
&- \Delta_*^{\vartheta}\Delta^{-\rho}\left(-\mathfrak{h}(\varsigma+\rho, \psi(\varsigma+\rho), \Delta_*^{\vartheta}[\psi(\varsigma+\rho-\vartheta)])\right)| \\
&\leq \Delta_*^{\vartheta-\rho}|\left(\mathfrak{h}(\ell+\rho, \psi(\ell+\rho), \Delta_*^{\vartheta}[\psi(\ell+\rho-\vartheta)])\right) \\
&- \left(\mathfrak{h}(\ell+\rho, \mathfrak{u}(\ell+\rho), \Delta_*^{\vartheta}[\mathfrak{u}(\ell+\rho-\vartheta)])\right)| \\
&\leq \frac{\mathcal{L}\Gamma(\mathcal{T}+\rho-\vartheta-1)}{\Gamma(\mathcal{T}-1)\Gamma(\rho-\vartheta+1)}\left[|\mathfrak{u}-\psi| + \left|\Delta_*^{\vartheta}\mathfrak{u} - \Delta_*^{\vartheta}\psi\right|\right] \\
&\leq \frac{2\mathcal{L}\Gamma(\mathcal{T}+\rho-\vartheta-1)}{\Gamma(\mathcal{T}-1)\Gamma(\rho-\vartheta+1)}\|\mathfrak{u}-\psi\|.
\end{aligned}
$$

Since

$$
\|\mathfrak{T}_1\mathfrak{u} - \mathfrak{T}_1\psi\| = \max_{\varsigma\in\mathfrak{D}}\left\{\sup_{\varsigma\in\mathfrak{D}}|\mathfrak{T}\mathfrak{u} - \mathfrak{T}\psi|, \sup_{\varsigma\in\mathfrak{D}}\left|\Delta_*^{\vartheta}\mathfrak{T}\mathfrak{u} - \Delta_*^{\vartheta}\mathfrak{T}\psi\right|\right\},
$$

we have

$$
\|\mathfrak{T}_1\mathfrak{u} - \mathfrak{T}_1\psi\| \leq \max\left\{\frac{2\mathcal{L}\Gamma(\mathcal{T}+\rho-1)}{\Gamma(\mathcal{T}-1)\Gamma(\rho+1)}, \frac{2\mathcal{L}\Gamma(\mathcal{T}+\rho-\vartheta-1)}{\Gamma(\mathcal{T}-1)\Gamma(\rho-\vartheta+1)}\right\}\|\mathfrak{u}-\psi\|
$$

implying that

$$
\|\mathfrak{T}_1\mathfrak{u} - \mathfrak{T}_1\psi\| \leq \mu\|\mathfrak{u}-\psi\|.
$$

It is clear that $\mathfrak{T}_1$ is a contraction.
Step 3: $\mathfrak{T}_2$ is continuous and $\mathfrak{T}_2(C)$ is contained in a compact set.

For each $\mathfrak{u} \in C$ and $\varsigma \in \mathfrak{D}$, $\mathfrak{T}_2\mathfrak{u}(\varsigma) = \mathcal{A} + \varsigma\mathcal{B}$ is a linear function of ς implying that $\mathfrak{T}_2$ is continuous and $\mathfrak{T}_2(C)$ is contained in a compact set. Hence, by Theorem 3.1, the initial value problem (1.1) has at least one solution in C. □

4. Uniqueness of solution

The following suppositions are needed:

(H_1) : $\mathfrak{h} : \mathfrak{D} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous

(H_4) : There exists $\mathcal{N} > 0$ such that for all $\mathfrak{u}, \psi, z, v \in \mathbb{R}$ and $\varsigma \in \mathfrak{D}$, the following relation hold:

$$|\mathfrak{h}(\varsigma, \mathfrak{u}, z) - \mathfrak{h}(\varsigma, \psi, v)| \leq \mathcal{N}\left[|\mathfrak{u} - \psi| + |z - v|\right]. \tag{4.1}$$

Theorem 4.1. *Assume that the conditions* (H_1) *and* (H_4) *hold. Then, Eq* (1.1) *has a unique solution if*

$$\nu = \max\left(\frac{2\mathcal{N}\Gamma(\mathcal{T}+\rho-1)}{\Gamma(\mathcal{T}-1)\Gamma(\rho+1)}, \frac{2\mathcal{N}\Gamma(\mathcal{T}+\rho-\vartheta-1)}{\Gamma(\mathcal{T}-1)\Gamma(\rho-\vartheta+1)}\right) < 1. \tag{4.2}$$

Proof. Define the operator

$$\mathfrak{T}\mathfrak{u}(\varsigma) = \mathcal{A} + \varsigma\mathcal{B} + \Delta^{-\rho}\left(-\mathfrak{h}(\ell+\rho, \mathfrak{u}(\ell+\rho), \Delta_*^{\vartheta}[\mathfrak{u}(\ell+\rho-\vartheta)])\right), \ \varsigma \in \mathfrak{D}. \tag{4.3}$$

We show that the operator $\mathfrak{T} : \mathbb{S} \to \mathbb{S}$ has a unique solution. Let $\mathfrak{u}, \psi \in \mathbb{S}$. Then, for each $\varsigma \in \mathfrak{D}$, we have

$$\begin{aligned}
|\mathfrak{T}\mathfrak{u} - \mathfrak{T}\psi| &= |\mathcal{A} + \varsigma\mathcal{B} + \Delta^{-\rho}\left(-\mathfrak{h}(\ell+\rho, \mathfrak{u}(\ell+\rho), \Delta_*^{\vartheta}[\mathfrak{u}(\ell+\rho-\vartheta)])\right) \\
&\quad - \mathcal{A} - \varsigma\mathcal{B} - \Delta^{-\rho}\left(-\mathfrak{h}(\ell+\rho, \psi(\ell+\rho), \Delta_*^{\vartheta}[\psi(\ell+\rho-\vartheta)])\right)| \\
&\leq \frac{1}{\Gamma(\rho)} \sum_{\ell=2-\rho}^{\varsigma-\rho} (\varsigma - \ell - 1)^{(\rho-1)} |\left(\mathfrak{h}(\ell+\rho, \psi(\ell+\rho), \Delta_*^{\vartheta}[\psi(\ell+\rho-\vartheta)])\right) \\
&\quad - \left(\mathfrak{h}(\ell+\rho, \mathfrak{u}(\ell+\rho), \Delta_*^{\vartheta}[\mathfrak{u}(\ell+\rho-\vartheta)])\right)| \\
&\leq \frac{\mathcal{N}\Gamma(\mathcal{T}+\rho-1)}{\Gamma(\mathcal{T}-1)\Gamma(\rho+1)}\left[|\mathfrak{u}-\psi| + \left|\Delta_*^{\vartheta}\mathfrak{u} - \Delta_*^{\vartheta}\psi\right|\right] \\
&\leq \frac{2\mathcal{N}\Gamma(\mathcal{T}+\rho-1)}{\Gamma(\mathcal{T}-1)\Gamma(\rho+1)}\|\mathfrak{u}-\psi\|.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\left|\Delta_*^{\vartheta}\mathfrak{T}\mathfrak{u} - \Delta_*^{\vartheta}\mathfrak{T}\psi\right| &= |\Delta_*^{\vartheta}\Delta^{-\rho}\left(-\mathfrak{h}(\ell+\rho, \mathfrak{u}(\ell+\rho), \Delta_*^{\vartheta}[\mathfrak{u}(\ell+\rho-\vartheta)])\right) \\
&\quad - \Delta_*^{\vartheta}\Delta^{-\rho}\left(-\mathfrak{h}(\ell+\rho, \psi(\ell+\rho), \Delta_*^{\vartheta}[\psi(\ell+\rho-\vartheta)])\right)| \\
&\leq \Delta_*^{\vartheta-\rho}|\left(\mathfrak{h}(\ell+\rho, \psi(\ell+\rho), \Delta_*^{\vartheta}[\psi(\ell+\rho-\vartheta)])\right) \\
&\quad - \left(\mathfrak{h}(\ell+\rho, \mathfrak{u}(\ell+\rho), \Delta_*^{\vartheta}[\mathfrak{u}(\ell+\rho-\vartheta)])\right)| \\
&\leq \frac{\mathcal{N}\Gamma(\mathcal{T}+\rho-\vartheta-1)}{\Gamma(\mathcal{T}-1)\Gamma(\rho-\vartheta+1)}\left[|\mathfrak{u}-\psi| + \left|\Delta_*^{\vartheta}\mathfrak{u} - \Delta_*^{\vartheta}\psi\right|\right] \\
&\leq \frac{2\mathcal{N}\Gamma(\mathcal{T}+\rho-\vartheta-1)}{\Gamma(\mathcal{T}-1)\Gamma(\rho-\vartheta+1)}\|\mathfrak{u}-\psi\|.
\end{aligned}$$

Since

$$\|\mathfrak{T}\mathfrak{u} - \mathfrak{T}\psi\| = \max_{\varsigma\in\mathfrak{D}}\left\{\sup_{\varsigma\in\mathfrak{D}}|\mathfrak{T}\mathfrak{u} - \mathfrak{T}\psi|, \sup_{\varsigma\in\mathfrak{D}}\left|\Delta_*^{\vartheta}\mathfrak{T}\mathfrak{u} - \Delta_*^{\vartheta}\mathfrak{T}\psi\right|\right\},$$

we see that,

$$\|\mathfrak{T}\mathfrak{u} - \mathfrak{T}\psi\| \leq \max\left\{\frac{2\mathcal{N}\Gamma(\mathcal{T}+\rho-1)}{\Gamma(\mathcal{T}-1)\Gamma(\rho+1)}, \frac{2\mathcal{N}\Gamma(\mathcal{T}+\rho-\vartheta-1)}{\Gamma(\mathcal{T}-1)\Gamma(\rho-\vartheta+1)}\right\}\|\mathfrak{u}-\psi\|.$$

Therefore,

$$\|\mathfrak{T}\mathfrak{u} - \mathfrak{T}\psi\| \le \nu \|\mathfrak{u} - \psi\| .$$

It is clear that $\mathfrak{T}$ is a contraction. Therefore, Banach fixed point theorem guarantees that the unique fixed point of $\mathfrak{T}$ is the unique solution of Eq (1.1). □

5. Hyers-Ulam stability

In this section, we study the stability results of Eq (1.1).

Definition 5.1. [30] The discrete time fractional order initial value problem (1.1) is Hyers-Ulam Stable, if a constant $\mathbb{U} > 0$ exists such that for every $\epsilon > 0$, $\psi(\varsigma) \in \mathbb{R}$ satisfy

$$\left|\Delta_*^{\rho}[\psi(\varsigma)] + \mathfrak{h}(\varsigma+\rho, \psi(\varsigma+\rho), \Delta_*^{\vartheta}[\psi(\varsigma+\rho-\vartheta)])\right| \le \epsilon \quad \varsigma \in \mathfrak{D} \tag{5.1}$$

with $\psi(0) = \mathcal{A}, \Delta(\psi(0)) = \mathcal{B}$ then the solution $\mathfrak{u}(\varsigma)$ of Eq (1.1) exists such that

$$|\psi(\varsigma) - \mathfrak{u}(\varsigma)| \le \mathbb{U}\epsilon.$$

Remark 5.2. $\psi(\varsigma) \in \mathbb{R}$ solves Eq (5.1) if and only if there exists a function $\mathfrak{g} : \mathfrak{D} \times \mathbb{R} \to \mathbb{R}$ such that

A1 : $|\mathfrak{g}(\varsigma+\rho, \psi(\varsigma+\rho))| \le \epsilon \qquad \varsigma \in \mathfrak{D}$.
A2 : $\Delta_*^{\rho}[\psi(\varsigma)] + \mathfrak{h}(\varsigma+\rho, \psi(\varsigma+\rho), \Delta_*^{\vartheta}[\psi(\varsigma+\rho-\vartheta)]) = \mathfrak{g}(\varsigma+\rho, \psi(\varsigma+\rho))$.

Lemma 5.3. *If $\psi(\varsigma)$ solves Eq (5.1), then*

$$\left|\psi(\varsigma) - \mathcal{A} - \varsigma\mathcal{B} + \Delta^{-\rho}\left(\mathfrak{h}(\ell+\rho, \psi(\ell+\rho), \Delta_*^{\vartheta}[\psi(\ell+\rho-\vartheta)])\right)\right| \le \epsilon\frac{(\mathcal{T}+\rho-2)^{(\rho)}}{\Gamma(\rho+1)}, \tag{5.2}$$

for $\varsigma \in \mathfrak{D}$.

Proof. If ψ solves Eq (5.1), using Remark (5.2) and Eq (2.3), then the solution of $(A2)$ satisfies

$$\psi(\varsigma) = \mathcal{A} + \varsigma\mathcal{B} + \Delta^{-\rho}\left(\mathfrak{g}(\ell+\rho, \psi(\ell+\rho)) - \mathfrak{h}(\ell+\rho, \psi(\ell+\rho), \Delta_*^{\vartheta}[\psi(\ell+\rho-\vartheta)])\right),$$

for $\varsigma \in \mathfrak{D}$. Hence,

$$\begin{aligned}
&|\psi(\varsigma) - \mathcal{A} - \varsigma\mathcal{B} - \Delta^{-\rho}\left(-\mathfrak{h}(\varsigma+\rho, \psi(\varsigma+\rho), \Delta_*^{\vartheta}[\psi(\varsigma+\rho-\vartheta)])\right)| \\
&\quad = \left|\Delta^{-\rho}\mathfrak{g}(\varsigma+\rho, \psi(\varsigma+\rho))\right| \\
&\quad \le \frac{1}{\Gamma(\rho)}\sum_{\ell=2-\rho}^{\varsigma-\rho}(\varsigma-\ell-1)^{(\rho-1)}\,|\mathfrak{g}(\ell+\rho, \psi(\ell+\rho))| \\
&\quad \le \epsilon\frac{1}{\Gamma(\rho)}\sum_{\ell=2-\rho}^{\varsigma-\rho}(\varsigma-\ell-1)^{(\rho-1)} \\
&\quad \le \epsilon\frac{(\mathcal{T}+\rho-2)^{(\rho)}}{\Gamma(\rho+1)}.
\end{aligned}$$

This completes the proof. □

Remark 5.4. As a direct outcome of Lemma Eq (5.3), we get

$$\begin{aligned}
&|\Delta_*^{\vartheta}\left[\psi(\varsigma)-\mathcal{A}-\varsigma\mathcal{B}-\Delta^{-\rho}\left(-\mathfrak{h}(\varsigma+\rho,\psi(\varsigma+\rho),\Delta_*^{\vartheta}[\psi(\varsigma+\rho-\vartheta)])\right)\right]| \\
&\quad\le\epsilon\frac{(\mathcal{T}+\rho-\vartheta-2)^{(\rho-\vartheta)}}{\Gamma(\rho-\vartheta+1)}.
\end{aligned}\tag{5.3}$$

Theorem 5.5. *Assume that* (H_4) *and Eq* (4.2) *hold. Let* $\psi(\varsigma)\in\mathbb{R}$ *solve Eq* (5.1) *for some* $\epsilon>0$ *and let* $\mathfrak{u}(\varsigma)\in\mathbb{R}$ *be the solution of*

$$\begin{cases}\Delta_*^{\rho}[\mathfrak{u}(\varsigma)]+\mathfrak{h}(\varsigma+\rho,\mathfrak{u}(\varsigma+\rho),\Delta_*^{\vartheta}[\mathfrak{u}(\varsigma+\rho-\vartheta)])=0,\\ \mathfrak{u}(0)=\psi(0),\ \Delta(\mathfrak{u}(0))=\Delta(\psi(0)),\end{cases}\tag{5.4}$$

for $\varsigma\in[0,\mathcal{T}]\cap\mathbb{N}_{2-\rho}$, $1<\rho\le 2$, $0<\vartheta\le 1$. *Then Eq* (1.1) *is Hyers Ulam Stable.*

Proof. In view of Lemma (2.4), the solution $\mathfrak{u}$ of Eq (5.4) satisfies

$$\mathfrak{u}(\varsigma)=\psi(0)+\varsigma\Delta\psi(0)+\Delta^{-\rho}\left(-\mathfrak{h}(\ell+\rho,\mathfrak{u}(\ell+\rho),\Delta_*^{\vartheta}[\mathfrak{u}(\ell+\rho-\vartheta)])\right),\ \varsigma\in\mathfrak{D}.$$

Therefore,

$$\begin{aligned}
&|\psi(\varsigma)-\mathfrak{u}(\varsigma)|\\
&=\left|\psi(\varsigma)-\psi(0)-\varsigma\Delta\psi(0)-\Delta^{-\rho}\left(-\mathfrak{h}(\varsigma+\rho,\mathfrak{u}(\varsigma+\rho),\Delta_*^{\vartheta}[\mathfrak{u}(\varsigma+\rho-\vartheta)])\right)\right|\\
&=|\psi(\varsigma)-\psi(0)-\varsigma\Delta\psi(0)-\Delta^{-\rho}\left(-\mathfrak{h}(\varsigma+\rho,\psi(\varsigma+\rho),\Delta_*^{\vartheta}[\psi(\varsigma+\rho-\vartheta)])\right)\\
&-\Delta^{-\rho}\left(-\mathfrak{h}(\varsigma+\rho,\mathfrak{u}(\varsigma+\rho),\Delta_*^{\vartheta}[\mathfrak{u}(\varsigma+\rho-\vartheta)])\right)\\
&+\Delta^{-\rho}\left(-\mathfrak{h}(\varsigma+\rho,\psi(\varsigma+\rho),\Delta_*^{\vartheta}[\psi(\varsigma+\rho-\vartheta)])\right)|\\
&\le\epsilon\frac{(\mathcal{T}+\rho-2)^{(\rho)}}{\Gamma(\rho+1)}+\frac{\mathcal{N}\Gamma(\mathcal{T}+\rho-1)}{\Gamma(\mathcal{T}-1)\Gamma(\rho+1)}\left\|\psi(\varsigma)-\mathfrak{u}(\varsigma)\right\|.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\left|\Delta_*^{\vartheta}\psi(\varsigma)-\Delta_*^{\vartheta}\mathfrak{u}(\varsigma)\right|\\
&=|\Delta_*^{\vartheta}\psi(\varsigma)+\Delta_*^{\vartheta}[\Delta^{-\rho}\left(\mathfrak{h}(\varsigma+\rho,\mathfrak{u}(\varsigma+\rho),\Delta_*^{\vartheta}[\mathfrak{u}(\varsigma+\rho-\vartheta)])\right)\\
&-\psi(0)-\varsigma\Delta\psi(0)]|\\
&=|\Delta_*^{\vartheta}[\psi(\varsigma)-\Delta^{-\rho}\left(-\mathfrak{h}(\varsigma+\rho,\psi(\varsigma+\rho),\Delta_*^{\vartheta}[\psi(\varsigma+\rho-\vartheta)])\right)\\
&-\psi(0)-\varsigma\Delta\psi(0)]-\Delta_*^{\vartheta-\rho}\left(-\mathfrak{h}(\varsigma+\rho,\mathfrak{u}(\varsigma+\rho),\Delta_*^{\vartheta}[\mathfrak{u}(\varsigma+\rho-\vartheta)])\right)\\
&+\Delta_*^{\vartheta}\Delta^{-\rho}\left(-\mathfrak{h}(\varsigma+\rho,\psi(\varsigma+\rho),\Delta_*^{\vartheta}[\psi(\varsigma+\rho-\vartheta)])\right)|\\
&\le\epsilon\frac{(\mathcal{T}+\rho-\vartheta-2)^{(\rho-\vartheta)}}{\Gamma(\rho-\vartheta+1)}+\frac{\mathcal{N}\Gamma(\mathcal{T}+\rho-\vartheta-1)}{\Gamma(\mathcal{T}-1)\Gamma(\rho-\vartheta+1)}\left\|\psi(\varsigma)-\mathfrak{u}(\varsigma)\right\|.
\end{aligned}$$

Therefore,

$$\left\|\psi(\varsigma)-\mathfrak{u}(\varsigma)\right\|\le\nu\left\|\psi(\varsigma)-\mathfrak{u}(\varsigma)\right\|+\mathbb{V}\epsilon,$$

where ν is defined in Eq (4.2) and

$$\mathbb{V} = \max\left\{\frac{(\mathcal{T}+\rho-2)^{(\rho)}}{\Gamma(\rho+1)}, \frac{(\mathcal{T}+\rho-\vartheta-2)^{(\rho-\vartheta)}}{\Gamma(\rho-\vartheta+1)}\right\}.$$

Thus, Eq (1.1) is Hyers Ulam stable with $\mathbb{U} = \dfrac{\mathbb{V}}{(1-\mu)}$. This completes the proof. □

6. A numerical example

The eardrum is a vibrating membrane inside the ear that converts the vibrations caused by waves hitting them into nerve signals that are sent to the brain. The frequency response produced by the vibrations produced when hit by sound waves is the hearing range. The hearing ranges gets lowered with age, and the normal range of hearing frequency ranges between 20–20,000 Hz. Most of the biological models like the vibrating eardrum follow the 2^{nd} Newton's law [37]. In the eardrum, the tympanic membrane with one-dimensional vibration is considered a mechanical system, and the sound wave that enters with different pressure is its forcing factor. Transmission of waves of different frequencies can be modeled with nonlinearity in order to understand frequency responses. It is necessary to analyze the qualitative behavior of the vibrating eardrum. This section establishes the stability of a nonlinear vibrating eardrum equation with a driving force.

Example 6.1. Consider the discrete fractional damped eardrum equation

$$\begin{cases}\Delta_*^{1.9}[\mathfrak{u}(\varsigma)] + \lambda\Delta_*^{0.7}\mathfrak{u}(\varsigma+1.2) + a\mathfrak{u}(\varsigma+1.9) + (b)\,(\mathfrak{u}(\varsigma+1.9))^2 \quad = c\sin(0.1(\varsigma+1.9)),\\ \mathfrak{u}(0)=0, \Delta(\mathfrak{u}(0))=0\end{cases} \tag{6.1}$$

where $\varsigma \in [0,8] \cap N_{0.1}$, and λ is the damping coefficient. We shall now prove the stability of Eq (6.1). Let the parameters take the values $\lambda = \dfrac{0.1}{3}, a = 0.02, b = 0.03, c = 0.01$.
By checking, we find that

$$\begin{aligned}&\mathfrak{h}(\varsigma+1.9, \mathfrak{u}(\varsigma+1.9), \Delta_*^{0.7}\mathfrak{u}(\varsigma+1.2))\\ &\qquad = c\sin(0.1(\varsigma+1.9)) - \lambda\Delta_*^{0.7}\mathfrak{u}(\varsigma+1.2) - a\mathfrak{u}(\varsigma+1.9) - (b)\,(\mathfrak{u}(\varsigma+1.9))^2\end{aligned}$$

satisfies (H_2) with $\max_{\varsigma\in\mathfrak{D}} |\mathfrak{u}(\varsigma)| = 0.1$. Moreover, we get

$$\begin{aligned}&|\mathfrak{h}(\varsigma+1.9,\mathfrak{u}(\varsigma+1.9), \Delta_*^{0.7}\mathfrak{u}(\varsigma+1.2)) - \mathfrak{h}(\varsigma+1.9, \psi(\varsigma+1.9), \Delta_*^{0.7}\psi(\varsigma+1.2))|\\ &\quad = |c\sin(0.1(\varsigma+1.9)) - \lambda\Delta_*^{0.7}\mathfrak{u}(\varsigma+1.2) - a\mathfrak{u}(\varsigma+1.9) - (b)\,(\mathfrak{u}(\varsigma+1.9))^2\\ &\quad - c\sin(0.1(\varsigma+1.9)) + \lambda\Delta_*^{0.7}\psi(\varsigma+1.2) + a\psi(\varsigma+1.9) + (b)\,(\psi(\varsigma+1.9))^2\,|\\ &\quad \le 0.033\left[|\mathfrak{u}-\psi| + \left|\Delta_*^{0.7}\mathfrak{u} - \Delta_*^{0.7}\psi\right|\right],\end{aligned}$$

where $\varsigma \in [2,8] \cap N_2$ and $\mathcal{L} = 0.033$. Thus, (H_3) holds and $\mathfrak{h}$ is Lipschitz continuous. Further, it is clear that $\mu = \max\{0.8169, 0.3169\} < 1$. By the conclusion of Theorem (3.2), the initial value problem (6.1) has a solution.

Let $\epsilon = 0.82$ and $\mathfrak{u}(\varsigma) = \dfrac{\varsigma^{(2)}}{20}, \varsigma \in [2,8] \cap N_2$. We make sure that the inequality (5.1) holds.

$$
\begin{aligned}
&|\Delta_*^{1.9}[\mathfrak{u}(\varsigma)] + \lambda\Delta_*^{0.7}\mathfrak{u}(\varsigma + 1.2) + a\mathfrak{u}(\varsigma + 1.9) + (b)\,(\mathfrak{u}(\varsigma + 1.9))^2 - c\sin(0.1(\varsigma + 1.9))| \\
&\quad = |\frac{2}{20}\frac{\Gamma(\varsigma + 0.1)}{\Gamma(1.1)\Gamma(\varsigma)} + a\frac{(\varsigma + 1.9)^{(2)}}{20} + b\left(\frac{(\varsigma + 1.9)^{(2)}}{20}\right)^2 - c\sin(0.1(\varsigma + 1.9)) \\
&\quad + \frac{3\lambda}{20}\frac{\Gamma(2)}{\Gamma(2.3)}(\varsigma + 1.2)^{(1.3)}| \\
&\quad \leq 0.8138 < \epsilon.
\end{aligned}
$$

Under this, we conclude that Theorem (5.5) confirms the Hyers-Ulam stability of Eq (6.1) with constant $\mathbb{U}$.

Table 1. Illustration of μ and ρ.

ρ	1.0900	1.1900	1.2900	1.3900	1.4900	1.5900	1.6900	1.7900	1.8900	1.9900
μ	0.2661	0.3099	0.3593	0.4149	0.4773	0.5469	0.6246	0.7110	0.8068	0.9128

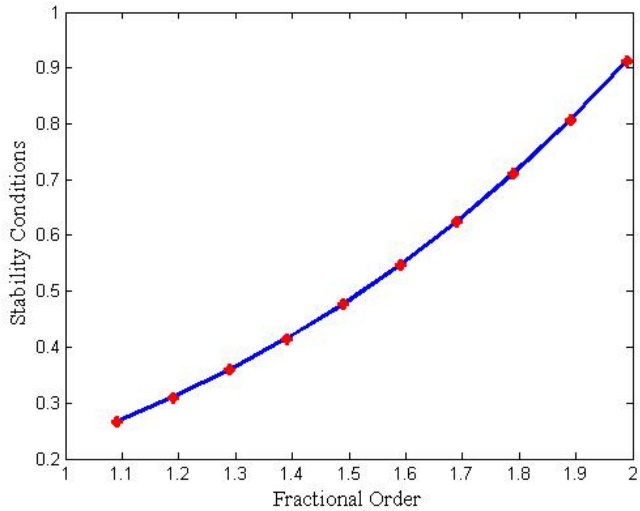

Figure 1. μ versus ρ.

The values of μ given in Theorem (4.1) for different fractional orders are tabulated in Table 1 and are plotted in Figure 1.

7. Conclusions

In this paper, we prove the existence, uniqueness, and stability of the nonlinear discrete fractional initial value problem using the technique of fixed point hypothesis. The stability results are presented in the sense of Hyers-Ulam for the proposed initial value problem. As an application, the vibrating eardrum model is considered for illustrating the efficiency of the stability results. The conditions for the stability for different order of the equations are numerically calculated and are presented in a table and a graph. We claim that the proposed results of this paper are new and are different from the results given in the literature such as the results reported in [33–35].

Results reported in this paper have verified that the descretization of some physical models provides an efficient and practical tool to study their qualitative behavior. We believe that descretizing and investigating some different models can be a promising topic in the future.

Acknowledgments

J. Alzabut and K. Abodayeh would like to thank Prince Sultan University for supporting this work.

Conflict of interest

The authors declare that they have no competing interests.

References

1. V. A. Dobrushkin, *Applied Differential Equations with Boundary Value Problems*, CRC Press, Boca Raton, 2017.
2. A. Mondol, R. Gupta, S. Das, T. Dutta, An insight into Newton's cooling law using fractional calculus, *J. Appl. Phys.*, **123** (2018), 064901.
3. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Elsevier, 2006.
4. A. Pratap, R. Raja, J. Alzabut, J. Cao, G. Rajachakit, C. Huang, Mittag-Leffler stability and adaptive impulsive synchronization of fractional order neural networks in quaternion field, *Math. Meth. Appl. Sci.*, **43** (2020), 6223–6253.
5. J. Dianavinnarasi, R. Raja, J. Alzabut, M. Niezabitowski, G. Selvam, O. O. Bagdasar, An LMI approach based mathematical model to control Aedes Aegypti mosquitoes population via biological control, *Math. Probl. Eng.*, **2021** (2021), 5565949.
6. G. A. Anastassiou, Discrete fractional calculus and inequalities, preprint, arXiv:0911.3370v1.
7. F. M. Atici, S. Seng*ü*l, Modeling with fractional difference equations, *J. Math. Anal. Appl.*, **369** (2010), 1–9.
8. F. M. Atici, P. W. Eloe, Two-point boundary value problems for finite fractional difference equations, *J. Differ. Equ. Appl.*, **17** (2011), 445–456.
9. F. M. Atici, P. W. Eloe, Initial value problems in discrete fractional calculus, *Proc. Am. Math. Soc.*, **137** (2009), 981–989.
10. F. M. Atici, P. W. Eloe, Discrete fractional calculus with the nabla operator, *Electron. J. Qual. Theory Differ. Equ., Spec. Ed. I*, **3** (2009), 1–12.
11. C. S. Goodrich, A. C. Peterson, Discrete fractional calculus, Springer, Cham, 2015.
12. K. S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Difference Equations*, Wiley, New York, 1993.

13. J. Alzabut, T. Abdeljawad, D. Baleanu, Nonlinear delay fractional difference equations with applications on discrete fractional Lotka–Volterra competition model, *J. Comput. Anal. Appl.*, **25** (2018), 889–898.

14. S. Kang, H. Chen, J. Guo, Existence of positive solutions for a system of Caputo fractional difference equations depending on parameters, *Adv. Differ. Equ.*, **2015** (2015), 138.

15. R. Dahal, D. Duncan, C. S. Goodrich, Systems of semipositone discrete fractional boundary value problems, *J. Differ. Equ. Appl.*, **20** (2014), 473–491.

16. J. Alzabut, T. Abdeljawad, A generalized discrete fractional Gronwall's inequality and its application on the uniqueness of solutions for nonlinear delay fractional difference system, *Appl. Anal. Discret. Math.*, **12** (2018), 036–048.

17. S. Djennoune, M. Bettayeb, U. Muhsen Al-Saggaf, Synchronization of fractional–order discrete–time chaotic systems by an exact delayed state reconstructor: Application to secure communication, *Int. J. Appl. Math. Comput. Sci.*, **29** (2019), 179–194.

18. F. Chen, Fixed points and asymptotic stability of nonlinear fractional difference equations, *Electron. J. Qual. Theory Differ. Equ.*, **39** (2011), 1–18.

19. F. Chen, Z. Liu, Asymptotic stability results for nonlinear fractional difference equations, *J. Appl. Math.*, **2012** (2012), 879657.

20. D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. U S A.*, **1941** (1941), 222–224.

21. S. Ulam, Problems in Modern Mathematics, New York: Science Editions John Wiley & Sons, Inc., 1964.

22. A. U. K Niazi, J. Wei, M. U. Rehman, P. Denghao, Ulam-Hyers-Mittag-Leffler stability of nonlinear fractional neutral differential equations, *Mat. Sb.*, **209** (2018), 1337–1350.

23. J. R. Wang, Y. Zhang, Ulam-Hyers-Mittag-Leffler stability of fractional-order delay differential equations, *Optimization*, **63** (2014), 1181–1190.

24. J. R. Wang, M. Feckan, Y. Zhou, Ulam's type stability of impulsive ordinary differential equations, *J. Math. Anal. Appl.*, **395** (2012), 258–264.

25. J. R. Wang, Y. Zhou, M. Feckan, Nonlinear impulsive problems for fractional differential equations and Ulam stability, *Comput.Math.Appl.*, **64** (2012), 3389–3405.

26. R. W. Ibrahim, Generalized Ulam-Hyers stability for fractional differential equations, *Internat. J. Math.*, **23** (2012), 1250056.

27. M. Ahmad, A. Zada, J. Alzabut, Hyers-Ulam stability of a coupled system of fractional differential equations of Hilfer-Hadamard type, *Demonstr. Math.*, **52** (2019), 283–295.

28. A. Zada, J. Alzabut, H. Waheed , I. L. Popa, Ulam-Hyers stability of impulsive integrodifferential equations with Riemann–Liouville boundary conditions, *Adv. Differ. Equ.*, **2020** (2020), 64.

29. S. S. Haider, M. ur Rehman, Ulam-Hyers-Rassias stability and existence of solutions to nonlinear fractional difference equations with multipoint summation boundary condition, *Acta Math. Sci.*, **40** (2020), 589–602.

30. C. Chen, M. Bohner, B. Jia, Ulam-Hyers stability of Caputo fractional difference equations, *Math. Methods Appl. Sci.*, **42** (2019), 7461–7470.

31. F. Chen, Y. Zhou, Existence of Ulam stability of solutions for discrete fractional boundary value problem, *Discrete Dyn. Nat. Soc.*, **2013** (2013), 459161.

32. A. G. M. Selvam, J. Alzabut, R. Dhineshbabu, M. Rehman, S. Rashid, Discrete fractional order two point boundary value problems with some relevant physical applications, *J. Inequal. Appl.*, **221** (2020).

33. Y. Guo, X. B. Shu, Y. Li, F. Xu, The existence and Hyers-Ulam stability of solution for an impulsive Riemann–Liouville fractional neutral functional stochastic differential equation with infinite delay of order $1 < \beta < 2$, *Bound. Value Probl.*, **59** (2019).

34. S. Li, L. Shu, X. B. Shu, F. Xu, Existence and Hyers-Ulam stability of random impulsive stochastic functional differential equations with finite delays, *Stochastics*, **91** (2019), 857–872.

35. A. Zada, W. Ali, S. Farina, Hyers-Ulam stability of nonlinear differential equations with fractional integrable impulses, *Math. Methods Appl. Sci.*, **40** (2017), 5502–5514.

36. M. A. Krasnosel'skiĭ, Two remarks on the method of successive approximation, *Usp. Mat. Nauk.*, **10** (1955), 123–127.

37. R. H. Enns, G. C. Mcguire, *Nonlinear Physics with Mathematica for Scientists and Engineers*, Birkhauser, Boston, 2001.

38. M. S. Abdo, T. Abdeljawad, K. Shah, S. M. Ali, On nonlinear coupled evolution system with nonlocal subsidiary conditions under fractal-fractional order derivative, *Math. Methods Appl. Sci.*, (2021), 1–20.

39. A. Ali, I. Mahariq, K. Shah, T. Abdeljawad, B.Al-Sheikh, Stability analysis of initial value problem of pantograph-type implicit fractional differential equations with impulsive conditions, *Adv. Differ. Equ.*, **2021** (2021).

40. M. Arfan, K. Shah, A. Ullah, S. Salahshour, A. Ahmadian, M. Ferrara, A novel semi-analytical method for solutions of two dimensional fuzzy fractional wave equation using natural transform, *Discrete & Continuous Dyn. Syst.-S*, 2021.

41. H. Alrabaiah, A. Zeb, E. Alzahrani, K. Shah, Dynamical analysis of fractional-order tobacco smoking model containing snuffing class, *Alex. Eng. J.*, **60** (2021), 3669–3678.

AIMS Press