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Research article

Mathematical modeling and dynamic analysis of SIQR model with delay for pandemic COVID-19

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Abstract: On the basis of the SIQR epidemic model, we consider the impact of treatment time on the epidemic situation, and we present a differential equation model with time-delay according to the characteristics of COVID-19. Firstly, we analyze the existence and stability of the equilibria in the modified COVID-19 epidemic model. Secondly, we analyze the existence of Hopf bifurcation, and derive the normal form of Hopf bifurcation by using the multiple time scales method. Then, we determine the direction of Hopf bifurcation and the stability of bifurcating periodic solutions. Finally, we carry out numerical simulations to verify the correctness of theoretical analysis with actual parameters, and show conclusions associated with the critical treatment time and the effect on epidemic for treatment time.

Keywords: COVID-19 model; Time-delay; Hopf bifurcation; Normal form; Multiple time scales method

1. Introduction

Coronavirus disease 2019 (COVID-19) remains an on-going global pandemic at present. The World Health Organization declared COVID-19 as a Public Health Emergency of International Concern (PHEIC) on January 30, 2020. According to data released by Johns Hopkins University, there are 37,213,592 confirmed cases and 1,072,959 deaths in 188 countries and regions around the world by October 11, 2020 [1]. The disease is caused by a novel coronavirus named severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2) [2]. Although most SARS-CoV-2-infected cases have asymptomatic or mild-to-moderate diseases, around 10% of those infected may develop severe pneumonia and other associated organ malfunctions [3].

In recent investigations, many scholars studied different epidemic models of COVID-19. In Zhang's study [4], a new mathematical model (SEIRD) was proposed, which is constructed with five classes including susceptible, exposed, infected, recovered and deaths to describe the possibility of transmission

in a given general population. Bhadauria et al. [5] studied the SIQ model by using the stability theory of nonlinear ordinary differential equations. Li et al. [6] developed a numerical method preserving positivity for a stochastic SIQS epidemic model. In Ref. [7], Higazy proved the existence of a stable solution for the fractional order COVID-19 SIDARTHE model. A $\theta-$SEIHQRD model which is more relevant to COVID-19 was developed by Ramos et al. [8]. Nisar et al. [9] constructed the SIRD model and verified the correctness. Batistela et al. [10] proposed an SIRSi model of COVID-19. Especially, Paré et al. [11] proposed traditional group models, continuous-time and discrete-time versions of the models with non-trivial networks on simple SIR-based models, and supported the need for networked models through presenting a set of simulations.

In the propagation process of COVID-19, we consider that there is a time-delay from infection to recovery. At present, there are some researches on epidemic models with time-delay. Zhu et al. [12] constructed a time delay reaction-diffusion model that is closer to the actual spread of the COVID-19 epidemic with considering the time delay effect of infected persons during the spread of the epidemic. In Ref. [13], the mathematical modeling of COVID-19 fatality trends was constructed by Scheiner et al., which was based on infection-to-death delay rule. Wei et al. [14] considered the time delay from susceptible individuals to infected individuals, thus, proposed a new SVEIR epidemic disease model with time delay, and analyzed the dynamic behavior of the model under pulse vaccination. In Ref. [15], Mukherjee considered an S-I epidemic model with time delay, the time delay is the immune period and the incubation period, and gave an estimate on the length of delays for system which is stable in the absence of delays remains stable.

At the same time, many researchers analyzed the stability of the COVID-19 model. In Ref. [16], Araz dealt with a mathematical model about COVID-19 spread, and analyzed global and local stability for the considered model. Annas et al. [17] carried out the stability analysis and numerical simulation of the SEIR model on the spread of COVID-19 in the research. Besides, some scholars conducted mathematical analysis on the principle of COVID-19 infection. Almocera et al. [18] studied an in-host model, and the stability of a unique positive equilibrium point, with viral load V^*, suggested that the virus may replicate fast enough to overcome T cell response and cause infection. In Ref. [19], Samanta analyzed the stability of the proposed model to control the epidemic.

And some scholars predicted the epidemic situation. León et al. [20] proposed an SEIARD mathematical model and attempted to forecast the evolution of the outbreak. Youssef et al. [21] carried out numerical verification and predictions of the proposed SEIR model, and compared the results with the real data due to the spreading of the COVID-19 in Saudi Arabia.

In addition, some scholars used the models with several compartments to put forward analysis and opinions on epidemic prevention and control. In Ref. [22], Carli et al. proposed a multi-region SIRQTHE model and an optimal control approach, which supported governments in defining the most effective strategies to be adopted during post-lockdown mitigation phases in a multi-region scenario. Giordano et al. [23] established a SIDARTHE model that predicted the course of the epidemic to help plan an effective control strategy. On the basis of the expected utility theory, Odagaki [24] carried out a theoretical framework to find out an optimum strategy for minimizing the maximum number of infeced and for controlling the outbreak of pandemic. In Ref. [25], Saldaña et al. developed a compartmental epidemic model of the COVID-19 epidemic outbreak to evaluate the theoretical impact of plausible control interventions.

In this paper, we establish the COVID-19 epidemic model based on the SIQR model. However,

these large deviations between model predictions and the actually recorded numbers stem from the uncertainty of the underlying SIQR model parameters: they may not be sufficiently well known for the novel COVID-19 pandemic yet. Generally, considering the latent period gives rise to models with the incorporation of delays to solve problems. Therefore, time-delay has important biologic meaning in epidemic models. Differently, by describing the time-delay from the infection period to the recovery period, we propose a new delay SIQR epidemic model with horizontal transmission in our paper, and obtain the critical treatment time through calculation and stability analysis, which can contribute greatly to medicine.

The rest of the paper is organized as follows: In Section 2, considering the treatment time of epidemic, we present a delayed differential equation for COVID-19. In Section 3, we analyze the existence and stability of equilibria and the existence Hopf bifurcation of the system. In Section 4, we derive the normal form of Hopf bifurcation for above model, then determine the direction of Hopf bifurcation and the stability of bifurcating periodic solutions. In Section 5, we present the simulated results to verify the correctness of the theoretical analysis, and show the effect of treatment time on epidemic. Finally, some conclusions are shown in Section 6.

2. Modeling

COVID-19 is similar to other infectious diseases. The susceptible population can be infected by COVID-19 carriers. Some of the infected people are quarantined, and some are killed by COVID-19. Both infected and quarantined people may be cured after treatment. A feature of COVID-19 can be obtained by combining a large number of cases, that is, cured infected people are difficult to be infected again. In Ref. [26], Liu et al. proposed an infectious disease model as follows:

$$\begin{cases} \frac{dS}{dt} = \Lambda - \mu S - \frac{\beta S I}{N}, \\ \frac{dI}{dt} = \frac{\beta S I}{N} - (\mu + \gamma + \delta + \alpha) I, \\ \frac{dQ}{dt} = \delta I - (\mu + \epsilon + \alpha) Q, \\ \frac{dR}{dt} = \gamma I + \epsilon Q - \mu R, \end{cases} \tag{2.1}$$

where S, I, Q, R denote the numbers of susceptible, infective, quarantined and removed, $N = S + I + Q + R$ is the number of total population individuals. The parameter Λ is the recruitment rate of S corresponding to births and immigration; β denotes the average number of adequate contacts; μ is the natural death rate; γ and ϵ denote the recover rates from group I, Q to R, respectively; δ denotes the removal rate from I; α is the disease-caused death rate of I and Q. The parameters involved in system (2.1) are all positive constants.

On the basis of reference [26], we define β as the contact rate, which eliminates the calculation of the total population number N and simplifies the form of the equation. It is a good way to reduce the error of the model results by reducing the parameters. Combined with practical reports, it is known that some quarantined patients will receive treatment, and the mortality of treated patients will be greatly reduced. Therefore, unlike the model in Ref. [26], we believe that the COVID-19 mortality of infected and quarantined person is different. Meanwhile, the uncertainty of the model parameters is also the main reason for the difference between the predicted results and the actual. Here, we consider the time-delay (τ) from infection to recovery process on the basis of the model in Ref. [26], and the critical treatment time obtained by the calculation with time-delay is helpful to the treatment of

COVID-19, which is rarely studied before and should be paid attention to in severe epidemic areas. In particular, the model can change the parameters according to the situation of different regions to get the corresponding critical treatment time, and take reasonable measures to control the epidemic effectively. Finally, we present a new COVID-19 epidemic model, and the relationships between the four populations (susceptible population(S), infected population(I), quarantined population(Q), recovered population(R)) are obtained, as shown in Figure 1.

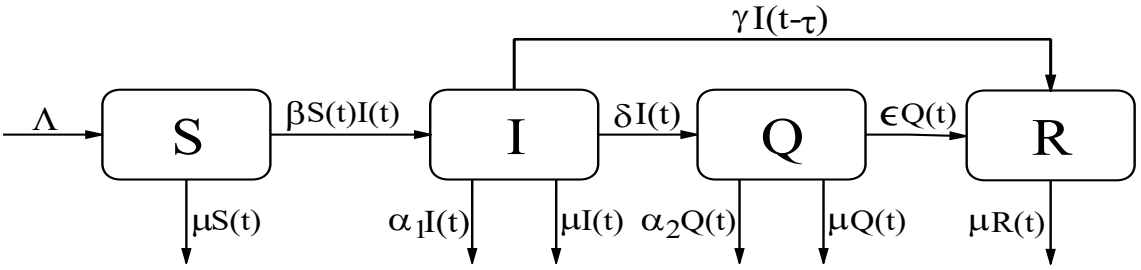

Figure 1. SIQR Model diagram.

Thus, we construct the following COVID-19 epidemic model:

$$\begin{cases} \dot{S} = \Lambda - \mu S - \beta S I, \\ \dot{I} = \beta S I - \delta I - \mu I - \alpha_1 I - \gamma I(t-\tau), \\ \dot{Q} = \delta I - \epsilon Q - \alpha_2 Q - \mu Q, \\ \dot{R} = \gamma I(t-\tau) + \epsilon Q - \mu R, \end{cases} \tag{2.2}$$

where Λ, μ, β, δ, γ, ϵ, α_1, α_2 are parameters; S, I, Q, R are control variables and τ is the time-delay. The specific definitions are given in the Table 1.

Table 1. Definition of parameters and variables in the model.

Symbol	Definition
S	Number of susceptible people
I	Number of infected people
Q	Number of quarantined people
R	Number of recovered people
Λ	Natural increase of population
β	Transition rate from S to I
δ	Transition rate from I to Q
γ	Transition rate from I to R,the cure rate of infected persons
ϵ	Transition rate from Q to R,the cure rate of quarantined persons
α_1	COVID-19 mortality rate of infected persons
α_2	COVID-19 mortality rate of quarantined persons
μ	Natural death of population
τ	The time-delay from infection to recovery process

3. Stability analysis of equilibrium and existence of Hopf bifurcation

In this section, system (2.2) is considered. Obviously, system (2.2) has two equilibria:

$$E_1 = (S_1^*, I_1^*, Q_1^*, R_1^*),\ E_2 = (S_2^*, I_2^*, Q_2^*, R_2^*),$$

where

$S_1^* = \frac{\Lambda}{\mu}$, $I_1^* = 0$, $Q_1^* = 0$, $R_1^* = 0$, $S_2^* = \frac{\gamma+\alpha_1+\delta+\mu}{\beta}$, $I_2^* = \frac{\Lambda\beta-\mu(\gamma+\alpha_1+\delta+\mu)}{\beta(\gamma+\alpha_1+\delta+\mu)}$,
$Q_2^* = \frac{\delta[\Lambda\beta-\mu(\gamma+\alpha_1+\delta+\mu)]}{\beta(\gamma+\alpha_1+\delta+\mu)(\alpha_2+\epsilon+\mu)}$, $R_2^* = \frac{[\Lambda\beta-\mu(\gamma+\alpha_1+\delta+\mu)][\gamma(\epsilon+\alpha_2+\mu)+\epsilon\delta]}{\beta\mu(\gamma+\alpha_1+\delta+\mu)(\alpha_2+\epsilon+\mu)}$.

Firstly, we consider the first equilibrium $E_1 = (S_1^*, I_1^*, Q_1^*, R_1^*) = (\frac{\Lambda}{\mu}, 0, 0, 0)$. Transferring the equilibrium to the origin and linearizing the system around it, we obtain the characteristic equation of the linearized system as follows:

$$(\lambda+\mu)^2(\lambda+\epsilon+\alpha_2+\mu)(\lambda-\beta S_1^*+\delta+\alpha_1+\mu+\gamma e^{-\lambda\tau}) = 0, \tag{3.1}$$

where $S_1^* = \frac{\Lambda}{\mu}$.

When $\tau = 0$, Eq. (3.1) becomes

$$(\lambda+\mu)^2(\lambda+\epsilon+\alpha_2+\mu)(\lambda-\beta S_1^*+\delta+\alpha_1+\mu+\gamma) = 0. \tag{3.2}$$

Eq. (3.2) has four roots: $\lambda_1{=}\lambda_2{=}-\mu$, $\lambda_3{=}-\epsilon-\alpha_2-\mu$, $\lambda_4 = \beta S_1^*-\delta-\alpha_1-\mu-\gamma$, due to $\mu > 0, \epsilon > 0, \alpha_2 > 0$, so in the actual situation, $\lambda_1 < 0$, $\lambda_2 < 0$, $\lambda_3 < 0$.

We consider the following assumption:

(H1) $\frac{\beta\Lambda}{\mu} < \mu+\delta+\alpha_1+\gamma$.

When (H1) holds, all the roots of Eq. (3.2) have negative real parts, and the equilibrium E_1 is locally asymptotically stable when $\tau = 0$.

When $\tau > 0$, let $\lambda = \mathrm{i}\omega$ ($\omega > 0$) be a root of Eq. (3.1). Due to $\lambda_1 < 0$, $\lambda_2 < 0$, $\lambda_3 < 0$, actually, we only need to discuss the following equation:

$$\lambda-\beta S_1^*+\delta+\alpha_1+\mu+\gamma e^{-\lambda\tau} = 0. \tag{3.3}$$

Substituting $\lambda = \mathrm{i}\omega$ ($\omega > 0$) into Eq. (3.3) and separating the real and imaginary parts, we obtain

$$\begin{cases} \beta S_1^*-(\delta+\alpha_1+\mu) = \gamma\cos(\omega\tau), \\ \omega = \gamma\sin(\omega\tau). \end{cases} \tag{3.4}$$

Eq. (3.4) leads to

$$\begin{cases} \cos(\omega\tau) = \frac{\beta S_1^*-(\delta+\alpha_1+\mu)}{\gamma}, \\ \sin(\omega\tau) = \frac{\omega}{\gamma}. \end{cases} \tag{3.5}$$

Adding the square of two equations of Eq. (3.5), we have

$$h(\omega) = \omega^2+(\beta S_1^*-\delta-\alpha_1-\mu)^2-\gamma^2 = 0. \tag{3.6}$$

Therefore, we give the following assumption:

(H2) $(\beta S_1^* - \delta - \alpha_1 - \mu)^2 - \gamma^2 < 0.$

If (H2) holds, then Eq. (3.6) has one positive root $\omega_0 = \sqrt{\gamma^2 - (\beta S_1^* - \delta - \alpha_1 - \mu)^2}$. Substituting ω_0 into Eq. (3.5), we get

$$\tau_1^{(j)} = \begin{cases} \frac{1}{\omega_0}[\arccos(P_0) + 2j\pi], & Q_0 \geq 0, \\ \frac{1}{\omega_0}[2\pi - \arccos(P_0) + 2j\pi], & Q_0 < 0, \ j = 0, 1, 2, \cdots, \end{cases} \tag{3.7}$$

where

$$Q_0 = \sin(\omega_0 \tau_1^{(j)}) = \frac{\omega_0}{\gamma},$$
$$P_0 = \cos(\omega_0 \tau_1^{(j)}) = \frac{\beta S_1^* - (\delta + \alpha_1 + \mu)}{\gamma}.$$

Lemma 3.1. *If (H2) holds, when $\tau = \tau_1^{(j)}$ ($j = 0, 1, 2, \cdots$), then Eq. (3.1) has a pair of pure imaginary roots $\pm i\omega_0$, and all the other roots of Eq. (3.1) have nonzero real parts.*

Furthermore, let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be the root of Eq. (3.1) satisfying $\alpha(\tau_1^{(j)}) = 0$, $\omega(\tau_1^{(j)}) = \omega_0$ ($j = 0, 1, 2, \cdots$).

Lemma 3.2. *If (H2) holds, we have the following transversality conclusions:*
$\mathrm{Re}(\frac{\mathrm{d}\tau}{\mathrm{d}\lambda})\Big|_{\tau=\tau_1^{(j)}} = \mathrm{Re}(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau})^{-1}\Big|_{\tau=\tau_1^{(j)}} = \frac{1}{\gamma^2} > 0$, *where* $j = 0, 1, 2, \cdots$.

Secondly, for the other equilibrium E_2=$(S_2^*, I_2^*, Q_2^*, R_2^*)$ of the system (2.2), similarly, transferring the equilibrium to the origin and linearizing the system around it, we obtain the characteristic equation of the linearized system as follows:

$$(\lambda + \mu)(\lambda + \epsilon + \alpha_2 + \mu)[(\lambda + \mu + \beta I_2^*)(\lambda - \gamma + \gamma e^{-\lambda\tau}) + \beta^2 I_2^* S_2^*] = 0, \tag{3.8}$$

where $S_2^* = \frac{\gamma+\alpha_1+\delta+\mu}{\beta}$, $I_2^* = \frac{\Lambda\beta-\mu(\gamma+\alpha_1+\delta+\mu)}{\beta(\gamma+\alpha_1+\delta+\mu)}$.

When $\tau = 0$, Eq. (3.8) becomes

$$(\lambda + \mu)(\lambda + \epsilon + \alpha_2 + \mu)[\lambda^2 + (\mu + \beta I_2^*)\lambda + \beta^2 I_2^* S_2^*] = 0. \tag{3.9}$$

Eq. (3.9) has four roots: $\lambda_1 = -\mu$, $\lambda_2 = -\epsilon - \alpha_2 - \mu$, $\lambda_3 = \frac{-\mu-\beta S_2^* + \sqrt{(\mu+\beta S_2^*)^2 - 4\beta^2 S_2^* I_2^*}}{2}4$, $\lambda_4 = \frac{-\mu-\beta S_2^* - \sqrt{(\mu+\beta S_2^*)^2 - 4\beta^2 S_2^* I_2^*}}{2}$. Due to $\mu > 0$, $\epsilon > 0$, $\alpha_2 > 0$, in the actual situation, $\lambda_1 < 0$, $\lambda_2 < 0$.

Note that $\beta^2 S_2^* I_2^* > 0$, we consider the following assumption:

(H3) $\frac{\beta\Lambda}{\mu} > \mu + \delta + \alpha_1 + \gamma$.

Therefore, under the assumption (H3), all the roots of Eq. (3.9) have negative real parts, and the equilibrium $E_2 = (S_2^*, I_2^*, Q_2^*, R_2^*)$ is locally asymptotically stable when $\tau = 0$.

When $\tau > 0$, let $\lambda = i\omega$ ($\omega > 0$) be a root of Eq. (3.8). Due to $\mu > 0$, $\epsilon > 0$, $\alpha_2 > 0$, actually, we only need to consider the equation:

$$(\lambda + \mu + \beta I_2^*)(\lambda - \gamma + \gamma e^{-\lambda\tau}) + \beta^2 I_2^* S_2^* = 0. \tag{3.10}$$

Substituting $\lambda = \mathrm{i}\omega$ ($\omega > 0$) into Eq. (3.10) and separating the real and imaginary parts, we have

$$\begin{cases} \omega^2 + \gamma\mu + \gamma\beta I_2^* - \beta^2 S_2^* I_2^* = (\gamma\mu + \gamma\beta I_2^*)\cos\omega\tau + \gamma\omega\sin\omega\tau, \\ \omega(\mu - \gamma + \beta I_2^*) = -\gamma\omega\cos\omega\tau + (\gamma\mu + \gamma\beta I_2^*)\sin\omega\tau. \end{cases} \tag{3.11}$$

Eq. (3.11) leads to

$$\begin{cases} \cos\omega\tau = \frac{(\omega^2+\gamma\mu+\gamma\beta I_2^* - \beta^2 S_2^* I_2^*)(\gamma\mu+\gamma\beta I_2^*) - \gamma\omega^2(\mu-\gamma+\beta I_2^*)}{(\gamma\mu+\gamma\beta I_2^*)^2+\gamma^2\omega^2}, \\ \sin\omega\tau = \frac{\gamma\omega(\mu+\beta I_2^*)(\mu-\gamma+\beta I_2^*) + \gamma\omega(\omega^2+\gamma\mu+\gamma\beta I_2^* - \beta^2 S_2^* I_2^*)}{(\gamma\mu+\gamma\beta I_2^*)^2+\gamma^2\omega^2}. \end{cases} \tag{3.12}$$

Adding the square of two equations of Eq. (3.12), let $z = \omega^2$, then

$$h(z) = z^2 + c_1 z + c_0 = 0, \tag{3.13}$$

where $c_1 = (\mu + \beta I_2^*)^2 - 2\beta^2 S_2^* I_2^*$, $c_0 = (\gamma\mu + \gamma\beta I_2^* - \beta^2 S_2^* I_2^*)^2 - (\gamma\mu + \gamma\beta I_2^*)^2$. Therefore, we give the following assumptions:

(H4) $c_0 < 0$.

(H5) $c_1^2 - 4c_0 > 0, c_1 < 0, c_0 > 0$.

If (H4) holds, then Eq. (3.13) has only one positive real root z_1. If (H5) holds, then Eq. (3.13) has two positive real roots z_2 and z_3. Substituting $\omega_k = \sqrt{z_k}$ ($k = 1, 2, 3$) into Eq. (3.12), we get

$$\tau_{2,k}^{(j)} = \begin{cases} \frac{1}{\omega_k}[\arccos(P_k) + 2j\pi], & Q_k \geq 0, \\ \frac{1}{\omega_k}[2\pi - \arccos(P_k) + 2j\pi], & Q_k < 0, \ k = 1, 2, 3, \ j = 0, 1, 2, \cdots, \end{cases} \tag{3.14}$$

where

$$Q_k = \sin(\omega_k \tau_{2,k}^{(j)}) = \frac{\gamma\omega_k(\mu + \beta I_2^*)(\mu - \gamma + \beta I_2^*) + \gamma\omega_k(\omega_k^2 + \gamma\mu + \gamma\beta I_2^* - \beta^2 S_2^* I_2^*)}{(\gamma\mu + \gamma\beta I_2^*)^2 + \gamma^2\omega_k^2},$$

$$P_k = \cos(\omega_k \tau_{2,k}^{(j)}) = \frac{(\omega_k^2 + \gamma\mu + \gamma\beta I_2^* - \beta^2 S_2^* I_2^*)(\gamma\mu + \gamma\beta I_2^*) - \gamma\omega_k^2(\mu - \gamma + \beta I_2^*)}{(\gamma\mu + \gamma\beta I_2^*)^2 + \gamma^2\omega_k^2}.$$

Lemma 3.3. *If (H4) or (H5) holds, when $\tau = \tau_{2,k}^{(j)}$ ($k = 1, 2, 3$; $j = 0, 1, 2, \cdots$), then Eq. (3.8) has a pair of pure imaginary roots $\pm\mathrm{i}\omega_k$, and all the other roots of Eq. (3.8) have nonzero real parts.*

Furthermore, let $\lambda(\tau) = \alpha(\tau) + \mathrm{i}\omega(\tau)$ be the root of Eq. (3.8) satisfying $\alpha(\tau_{2,k}^{(j)}) = 0$, $\omega(\tau_{2,k}^{(j)}) = \omega_k$ ($k = 1, 2, 3$; $j = 0, 1, 2, \cdots$).

Lemma 3.4. *If (H4) or (H5) holds, and $z_k = \omega_k^2$, $h'(z_k) \neq 0$, then we have the following transversality conclusions:*

$$\mathrm{Re}\left(\frac{\mathrm{d}\tau}{\mathrm{d}\lambda}\right)\Big|_{\tau=\tau_{2,k}^{(j)}} = \mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\Big|_{\tau=\tau_{2,k}^{(j)}} = \frac{h'(z_k)}{\gamma^2[(\mu+\beta I_2^*)^2+\omega_k^2]} \neq 0, \ k = 1, 2, 3, \ j = 0, 1, 2, \cdots.$$

Theorem 3.1. *We show the conclusion associated with two equilibria of the system (2.2).*

(1) If the assumptions (H1) and (H2) hold, the equilibrium E_1 of the system (2.2) undergoes Hopf bifurcation at $\tau = \tau_1^{(j)}$ ($j = 0, 1, 2, \cdots$), where $\tau_1^{(j)}$ is given by Eq. (3.7), and we have: when $\tau \in [0, \tau_1^{(0)})$, the equilibrium E_1 is locally asymptotically stable, and the equilibrium E_1 is unstable when $\tau > \tau_1^{(0)}$.

(2) If the assumptions (H4) or (H5) holds, the equilibrium E_2 of the system (2.2) undergoes Hopf bifurcation at $\tau = \tau_{2,k}^{(j)}$ ($k = 1, 2, 3$; $j = 0, 1, 2, \cdots$), where $\tau_{2,k}^{(j)}$ is given by Eq. (3.14), and

(a) If the assumptions (H3) and (H4) hold, $h(z)$ has one positive root z_1, then when $\tau \in [0, \tau_{2,1}^{(0)})$, the equilibrium E_2 is locally asymptotically stable, and the equilibrium E_2 is unstable when $\tau > \tau_{2,1}^{(0)}$.

(b) If the assumptions (H3) and (H5) hold, $h(z)$ has two positive roots z_2 and z_3, we suppose $z_2 < z_3$, then $h'(z_2) < 0, h'(z_3) > 0$, note that $\tau_{2,2}^{(0)} > \tau_{2,3}^{(0)}$. Then $\exists\, m \in N$ makes $0 < \tau_{2,3}^{(0)} < \tau_{2,2}^{(0)} < \tau_{2,3}^{(1)} < \tau_{2,2}^{(1)} < \cdots < \tau_{2,2}^{(m-1)} < \tau_{2,3}^{(m)} < \tau_{2,3}^{(m+1)}$. When $\tau \in [0, \tau_{2,3}^{(0)}) \cup \bigcup_{l=1}^{m} (\tau_{2,2}^{(l-1)}, \tau_{2,3}^{(l)})$, the equilibrium E_2 of the system (2.2) is locally asymptotically stable, and when $\tau \in \bigcup_{l=0}^{m-1} (\tau_{2,3}^{(l)}, \tau_{2,2}^{(l)}) \cup (\tau_{2,3}^{(m)}, +\infty)$, the equilibrium E_2 of the system (2.2) is unstable.

4. Direction of Hopf bifurcation and stability of periodic solution

In this section, we discuss the normal form of Hopf bifurcation for the system (2.2) by using the multiple time scales method. Combining with actual situation, we concern about the impact of treatment time on epidemic control. Therefore, we consider the time-delay τ as a bifurcation parameter, let $\tau{=}\tau_c + \varepsilon\tau_\varepsilon$, where τ_c is the critical value of Hopf bifurcation given in Eq. (3.7) or Eq. (3.14) respectively, τ_ε is the disturbance parameter, and ε is the dimensionless scale parameter. We suppose the characteristic Eq. (3.1) and Eq. (3.8) have eigenvalue $\lambda = \mathrm{i}\omega^{(k)}$ ($k = 1, 2$), where $\omega^{(1)} = \omega_0$, $\omega^{(2)} = \omega_1$, ω_2 or ω_3, at which system (2.2) undergoes a Hopf bifurcation at equilibrium $E_k = (S_k^*, I_k^*, Q_k^*, R_k^*)$, $k = 1, 2$, respectively.

Then system (2.2) can be written as

$$\dot{X}(t) = AX(t) + BX(t-\tau) + F[X(t), X(t-\tau)], \tag{4.1}$$

where $X(t) = (S_k, I_k, Q_k, R_k)^{\mathrm{T}}$, $X(t-\tau) = (S_k(t-\tau), I_k(t-\tau), Q_k(t-\tau), R_k(t-\tau))^{\mathrm{T}}$,

$$A = \begin{pmatrix} -\mu - \beta I_k^* & -\beta S_k^* & 0 & 0 \\ \beta I_k^* & \beta S_k^* - (\delta + \mu + \alpha_1) & 0 & 0 \\ 0 & \delta & -(\epsilon + \alpha_2 + \mu) & 0 \\ 0 & 0 & -\epsilon & -\mu \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\gamma & 0 & 0 \end{pmatrix},$$

$$F(X(t), X(t-\tau)) = \begin{pmatrix} F_S \\ F_I \\ F_Q \\ F_R \end{pmatrix} = \begin{pmatrix} \Lambda - \mu S_k^* - \beta S_k^* I_k^* - \beta S_k I_k \\ \beta S_k I_k + \beta S_k^* I_k^* - (\delta + \mu + \alpha_1 + \gamma) I_k^* \\ \delta I_k^* - (\epsilon + \alpha_2 + \mu) Q_k^* \\ \gamma I_k^* - \epsilon Q_k^* - \mu R_k^* \end{pmatrix}.$$

We suppose $h_k, h_k^* (k = 1, 2)$ are the eigenvector of the corresponding eigenvalue $\lambda = \mathrm{i}\omega^{(k)}, \lambda = -\mathrm{i}\omega^{(k)}$ respectively of Eq. (4.1) for equilibrium E_k, and satisfies $\left\langle h_k^*, h_k \right\rangle = \overline{h_k^*}^{\mathrm{T}} h_k = 1$. By simple calculation, we can get:

$$\begin{aligned} h_k &= (h_{k1}, h_{k2}, h_{k3}, h_{k4})^{\mathrm{T}} = (1, -\tfrac{\lambda+\mu+\beta I_k^*}{\beta S_k^*}, -\tfrac{\delta(\lambda+\mu+\beta I_k^*)}{(\lambda+\epsilon+\alpha_2+\mu)\beta S_k^*}, \tfrac{\delta\epsilon(\lambda+\mu+\beta I_k^*) - \gamma(\lambda+\mu+\beta I_k^*)(\lambda+\epsilon+\alpha_2+\mu)\mathrm{e}^{-\lambda\tau}}{(\lambda+\mu)(\lambda+\epsilon+\alpha_2+\mu)\beta S_k^*})^{\mathrm{T}}, \\ h_k^* &= (h_{k1}^*, h_{k2}^*, h_{k3}^*, h_{k4}^*)^{\mathrm{T}} = d_k(1, \tfrac{-\lambda+\mu+\beta I_k^*}{\beta I_k^*}, 0, 0)^{\mathrm{T}}, \end{aligned} \tag{4.2}$$

where $d_k = \frac{\beta^2 S_k^* I_k^*}{\beta^2 S_k^* I_k^* - (-\lambda+\mu+\beta I_k^*)^2}$, $k = 1, 2$.

We suppose the solution of Eq. (4.1) as follows:

$$X(t) = X(T_0, T_1, T_2, \cdots) = \sum_{k=1}^{\infty} \varepsilon^k X_k(T_0, T_1, T_2, \cdots), \tag{4.3}$$

where $X(T_0, T_1, T_2, \cdots) = [S(T_0, T_1, T_2, \cdots), I(T_0, T_1, T_2, \cdots), Q(T_0, T_1, T_2, \cdots), R(T_0, T_1, T_2, \cdots)]^{\mathrm{T}}$, $X_k(T_0, T_1, T_2, \cdots) = [S_k(T_0, T_1, T_2, \cdots), I_k(T_0, T_1, T_2, \cdots), Q_k(T_0, T_1, T_2, \cdots), R_k(T_0, T_1, T_2, \cdots)]^{\mathrm{T}}$.

The derivative with respect to t is transformed into:

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^2 \frac{\partial}{\partial T_2} + \cdots = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots,$$

where $D_i = \frac{\partial}{\partial T_i}$, $i = 0, 1, 2, \cdots$.

We make $X_j = (S_j, I_j, Q_j, R_j)^{\mathrm{T}} = X_j(T_0, T_1, T_2, \cdots)$, $X_{j,\tau_c} = (S_{j,\tau_c}, I_{j,\tau_c}, Q_{j,\tau_c}, R_{j,\tau_c})^{\mathrm{T}} = X_j(T_0 - \tau_c, T_1, T_2, \cdots)$, $j = 1, 2, 3, \cdots$.

From Eq. (4.3) we can get:

$$\dot{X}(t) = \varepsilon D_0 X_1 + \varepsilon^2 D_1 X_1 + \varepsilon^3 D_2 X_1 + \varepsilon^2 D_0 X_2 + \varepsilon^3 D_0 X_3 + \cdots. \tag{4.4}$$

The Taylor expansion of $X(t - \tau)$ is carried out:

$$\begin{aligned} X(t-\tau) =& \varepsilon X_{1,\tau_c} + \varepsilon^2 X_{2,\tau_c} + \varepsilon^3 X_{3,\tau_c} - \varepsilon^2 \tau_\varepsilon D_0 X_{1,\tau_c} - \varepsilon^3 \tau_\varepsilon D_0 X_{2,\tau_c} - \varepsilon^2 \tau_c D_1 X_{1,\tau_c} \\ & - \varepsilon^3 \tau_\varepsilon D_1 X_{1,\tau_c} - \varepsilon^3 \tau_c D_2 X_{1,\tau_c} - \varepsilon^3 \tau_c D_1 X_{2,\tau_c} + \cdots, \end{aligned} \tag{4.5}$$

where $X_{j,\tau_c} = X_j(T_0 - \tau_c, T_1, T_2, \cdots)$, $j = 1, 2, 3, \cdots$.

Substituting Eqs. (4.3) ~ (4.5) into Eq. (4.1), and balancing the coefficients before ε on both sides of the equation, the following expression is obtained:

$$\begin{aligned} & D_0 S_{k1} + \mu S_{k1} + \beta I_k^* S_{k1} + \beta S_k^* I_{k1} = 0, \\ & D_0 I_{k1} - \beta I_k^* S_{k1} - \beta S_k^* I_{k1} + (\delta + \mu + \alpha_1) I_{k1} + \gamma I_{k1,\tau_c} = 0, \\ & D_0 Q_{k1} - \delta I_{k1} + (\epsilon + \alpha_2 + \mu) Q_{k1} = 0, \\ & D_0 R_{k1} - \gamma I_{k1,\tau_c} + \epsilon Q_{k1} + \mu R_{k1} = 0, \ k = 1, 2. \end{aligned} \tag{4.6}$$

Thus Eq. (4.6) has the following solution form:

$$X_1(T_1, T_2, T_3, \cdots) = G(T_1, T_2, T_3, \cdots) \mathrm{e}^{\mathrm{i}\omega^{(k)} T_0} h_k + \bar{G}(T_1, T_2, T_3, \cdots) \mathrm{e}^{-\mathrm{i}\omega^{(k)} T_0} \bar{h}_k, \ k = 1, 2. \tag{4.7}$$

The expression of the coefficient before ε^2 is as follows:

$$\begin{aligned} & D_0 S_{k2} + \mu S_{k2} + \beta I_k^* S_{k2} + \beta S_k^* I_{k2} = -D_1 S_{k1} - \beta S_{k1} I_{k1}, \\ & D_0 I_{k2} - \beta I_k^* S_{k2} - \beta S_k^* I_{k2} + (\delta + \mu + \alpha_1) I_{k2} + \gamma I_{k2,\tau_c} = -D_1 I_{k1} + \beta S_{k1} I_{k1} + \\ & \qquad \gamma(\tau_\varepsilon D_0 I_{k1,\tau_c} + \tau_c D_1 I_{k1,\tau_c}), \\ & D_0 Q_{k2} - \delta I_{k2} + (\epsilon + \alpha_2 + \mu) Q_{k2} = -D_1 Q_{k1}, \\ & D_0 R_{k2} - \gamma I_{k2,\tau_c} + \epsilon Q_{k2} + \mu R_{k2} = -D_1 R_{k1} - \gamma(\tau_\varepsilon D_0 I_{k1,\tau_c} + \tau_c D_1 I_{k1,\tau_c}), \ k = 1, 2. \end{aligned} \tag{4.8}$$

Substituting Eq. (4.7) into the right hand side of Eq. (4.8), and the coefficient vector of $\mathrm{e}^{\mathrm{i}\omega^{(k)} T_0}$ is denoted by m_1. According to the solvable condition $\left\langle h_k^*, m_1 \right\rangle = 0$, the expression of $\frac{\partial G}{\partial T_1}$ can be obtained as follows:

$$\frac{\partial G}{\partial T_1} = N_k \tau_\varepsilon G, \tag{4.9}$$

where $N_k = \frac{(\mathrm{i}\omega^{(k)}+\mu+\beta I_k^*)^2\gamma\cdot\mathrm{i}\omega^{(k)}\mathrm{e}^{-\mathrm{i}\omega^{(k)}\tau_c}}{(\mathrm{i}\omega^{(k)}+\mu+\beta I_k^*)^2-\beta^2 S_k^* I_k^*-(\mathrm{i}\omega^{(k)}+\mu+\beta I_k^*)^2\gamma\tau_c\mathrm{e}^{-\mathrm{i}\omega^{(k)}\tau_c}}$, $k = 1, 2$.

Since τ_ε is a disturbance parameter, we only consider its effect on the linear part. It has little effect on the high order, so it can be ignored. Therefore, we ignore the part containing τ_ε in the higher order. We suppose:

$$\begin{aligned}
S_{k2} &= g_{k1}\mathrm{e}^{2\mathrm{i}\omega^{(k)}T_0}G^2 + \bar{g}_{k1}\mathrm{e}^{-2\mathrm{i}\omega^{(k)}T_0}\bar{G}^2 + l_{k1}G\bar{G},\\
I_{k2} &= g_{k2}\mathrm{e}^{2\mathrm{i}\omega^{(k)}T_0}G^2 + \bar{g}_{k2}\mathrm{e}^{-2\mathrm{i}\omega^{(k)}T_0}\bar{G}^2 + l_{k2}G\bar{G},\\
Q_{k2} &= g_{k3}\mathrm{e}^{2\mathrm{i}\omega^{(k)}T_0}G^2 + \bar{g}_{k3}\mathrm{e}^{-2\mathrm{i}\omega^{(k)}T_0}\bar{G}^2 + l_{k3}G\bar{G},\\
R_{k2} &= g_{k4}\mathrm{e}^{2\mathrm{i}\omega^{(k)}T_0}G^2 + \bar{g}_{k4}\mathrm{e}^{-2\mathrm{i}\omega^{(k)}T_0}\bar{G}^2 + l_{k4}G\bar{G},
\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}
&g_{k1} = \tfrac{-\beta h_{k2}(2\mathrm{i}\omega^{(k)}+\delta+\mu+\alpha_1+\gamma\mathrm{e}^{-2\mathrm{i}\omega^{(k)}\tau_c})}{J},\ g_{k2} = \tfrac{(2\mathrm{i}\omega^{(k)}+\mu)\beta h_{k2}}{J},\\
&g_{k3} = \tfrac{(2\mathrm{i}\omega^{(k)}+\mu)\delta\beta h_{k2}}{(2\mathrm{i}\omega^{(k)}+\mu+\alpha_2+\epsilon)J},\ g_{k4} = \tfrac{(2\mathrm{i}\omega^{(k)}+\mu)\beta h_{k2}[\epsilon(2\mathrm{i}\omega^{(k)}+\mu+\alpha_2+\epsilon)-\delta\epsilon]}{(2\mathrm{i}\omega^{(k)}+\mu+\alpha_2+\epsilon)(2\mathrm{i}\omega^{(k)}+\mu)J},\\
&l_{k1} = \tfrac{-\beta(h_{k2}+\bar{h}_{k2})(\delta+\mu+\alpha_1+\gamma)}{V},\ l_{k2} = \tfrac{\beta\mu(h_{k2}+\bar{h}_{k2})}{V},\ l_{k3} = \tfrac{\delta\beta\mu(h_{k2}+\bar{h}_{k2})}{(\mu+\alpha_2+\epsilon)V},\\
&l_{k4} = \tfrac{\gamma\beta\mu(h_{k2}+\bar{h}_{k2})(\mu+\alpha_2+\epsilon)-\epsilon\delta\beta\mu(h_{k2}+\bar{h}_{k2})}{\mu(\mu+\alpha_2+\epsilon)V},\\
&J = (2\mathrm{i}\omega^{(k)}+\mu+\beta I_k^*)(2\mathrm{i}\omega^{(k)}+\delta+\mu+\alpha_1+\gamma\mathrm{e}^{-2\mathrm{i}\omega^{(k)}\tau_c}) - \beta S_k^*(2\mathrm{i}\omega^{(k)}+\mu),\\
&V = (\mu+\beta I_k^*)(\delta+\mu+\alpha_1+\gamma)+\beta\mu S_k^*,\\
&h_{k2} = -\tfrac{\mathrm{i}\omega^{(k)}+\mu+\beta I_k^*}{\beta S_k^*},\ k = 1, 2.
\end{aligned} \tag{4.11}$$

The expression of the coefficient before ε^3 is:

$$\begin{aligned}
&D_0S_{k3}+\mu S_{k3}+\beta I_k^*S_{k3}+\beta S_k^*I_{k3} = -D_2S_{k1}-D_1S_{k2}-\beta S_{k2}I_{k1}-\beta S_{k1}I_{k2},\\
&D_0I_{k3}-\beta I_k^*S_{k3}-\beta S_k^*I_{k3}+(\delta+\mu+\alpha_1)I_{k3}+\gamma I_{k3,\tau_c} = -D_2I_{k1}-D_1I_{k2}+\beta S_{k2}I_{k1}+\beta S_{k1}I_{k2}\\
&\qquad\qquad +\gamma(\tau_cD_1I_{k2,\tau_c}+\tau_cD_2I_{k1,\tau_c}),\\
&D_0Q_{k3}-\delta I_{k3}+(\epsilon+\alpha_2+\mu)Q_{k3} = -D_2Q_{k1}-D_1Q_{k2},\\
&D_0R_{k3}-\gamma I_{k3,\tau_c}+\epsilon Q_{k3}+\mu R_{k3} = -D_2R_{k1}-D_1R_{k2}-\gamma(\tau_cD_1I_{k2,\tau_c}+\tau_cD_2I_{k1,\tau_c}),\ k = 1, 2.
\end{aligned} \tag{4.12}$$

Substituting Eq. (4.7), Eq. (4.10) and Eq. (4.11) into the right hand side of Eq. (4.12), and the coefficient vector of $\mathrm{e}^{\mathrm{i}\omega^{(k)}T_0}$ is denoted by m_2. According to the solvable condition $\left\langle h_k^*, m_2\right\rangle = 0$, the expression of $\frac{\partial G}{\partial T_2}$ can be obtained as follows:

$$\frac{\partial G}{\partial T_2} = \xi_k G^2\bar{G}, \tag{4.13}$$

where

$$\xi_k = \frac{\beta(l_{k2}+g_{k2}+l_{k1}h_{k2}+g_{k1}\bar{h}_{k2})(\mathrm{i}\omega^{(k)}+\mu)}{\beta I_k^*-(\mathrm{i}\omega^{(k)}+\mu+\beta I_k^*)(\gamma\tau_c\mathrm{e}^{-\mathrm{i}\omega^{(k)}\tau_c}-1)h_{k2}},\ k = 1, 2.$$

Let $G \to G/\varepsilon$, then the deduce to third-order normal form of Hopf bifurcation of system (2.2) is:

$$\dot{G} = N_k\tau_\varepsilon G + \xi_k G^2\bar{G}, \tag{4.14}$$

where N_k is given in Eq. (4.9), and ξ_k is given in Eq. (4.13), $k = 1, 2$.

By substituting $G = re^{i\theta}$ into Eq. (4.14), the following normal form of Hopf bifurcation in polar coordinates can be obtained:

$$\begin{cases} \dot{r} = \mathrm{Re}(N_k)\tau_\varepsilon r + \mathrm{Re}(\xi_k)r^3, \\ \dot{\theta} = \mathrm{Im}(N_k)\tau_\varepsilon + \mathrm{Im}(\xi_k)r^2, \end{cases} \tag{4.15}$$

where N_k is given in Eq. (4.9), and ξ_k is given in Eq. (4.13), $k = 1, 2$.

According to the normal form of Hopf bifurcation in polar coordinates, we just need to consider the first equation in system (4.15). Thus, there is the following theorem:

Theorem 4.1. *For the system (4.15), when* $\frac{\mathrm{Re}(N_k)\tau_\varepsilon}{\mathrm{Re}(\xi_k)} < 0$ ($k = 1, 2$)*, there is a nontrivial fixed point* $r = \sqrt{-\frac{\mathrm{Re}(N_k)\tau_\varepsilon}{\mathrm{Re}(\xi_k)}}$*, and system (2.2) has periodic solution:*

(1) If $\mathrm{Re}(N_k)\tau_\varepsilon < 0$*, then the periodic solution reduced on the center manifold is unstable.*

(2) If $\mathrm{Re}(N_k)\tau_\varepsilon > 0$*, then the periodic solution reduced on the center manifold is stable.*

5. Numerical simulations

In this section, according to the data presented in Refs. [27] and [28], we let $\Lambda = 10.48, \beta = 0.0004$, $\delta = 0.7$, $\alpha_1 = 0.0484$, $\alpha_2 = 0.022778$, $\gamma = 0.42386$, $\epsilon = 0.475$, $\mu = 0.00714$. Then, system (2.2) only has one nonnegative equilibrium $E_1 = (S_1^*, I_1^*, Q_1^*, R_1^*) \approx (1467.8, 0, 0, 0)$. Obviously, the assumption (H1) holds, thus the equilibrium E_1 is locally asymptotically stable when $\tau = 0$.

Substituding these parameter values into the Eqs. (3.5)~(3.7), using MATLAB, we can obtain $\omega_0 \approx 0.3900$, $Q_0 \approx 0.9201$, $P_0 \approx -0.4284$, $\tau_1^{(0)} \approx 5.1629$. According to the Theorem 3.1, the equilibrium E_1 is locally asymptotically stable at $\tau \in [0, \tau_1^{(0)})$, and when $\tau = \tau_1^{(0)}$, Hopf bifurcation occurs near the equilibrium E_1. Then, we obtain $\mathrm{Re}(N_1) > 0$, $\mathrm{Re}(\xi_1) > 0$ from Eqs. (4.9)~(4.13), thus according to the Theorem 4.1, the system (2.2) has backward periodic solution and the periodic solution is unstable when $\tau_\varepsilon < 0$.

When $\tau = 0$, we choose the initial value (1473, 1, 2, 1) and the corresponding locally asymptotically stable equilibrium is shown in Figure 2.

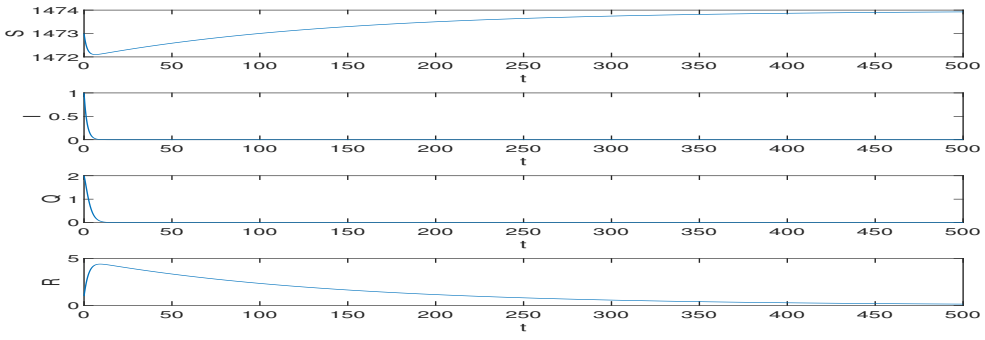

Figure 2. When $\tau = 0$, the equilibrium E_1 of system (2.2) is locally asymptotically stable.

When $\tau = 4.2 \in [0, \tau_1^{(0)})$, we choose the initial value (1473, 1, 2, 1), the equilibrium E_1 is also locally asymptotically stable (see Figure 3). Thus, the epidemic will be controlled eventually.

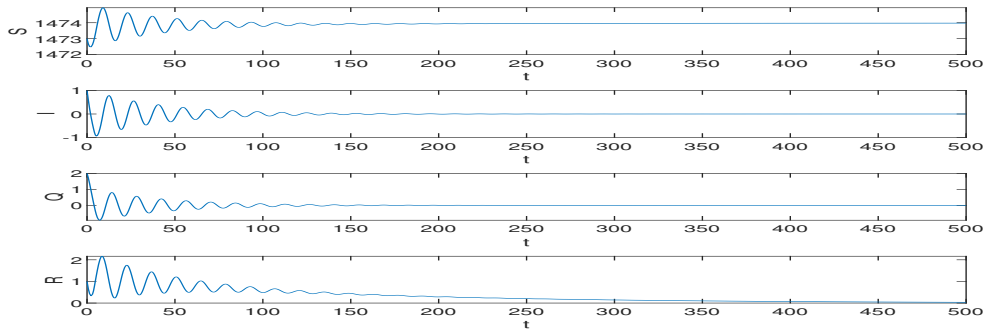

Figure 3. When $\tau = 4.2$, the system (2.2) has locally asymptotically stable equilibrium.

When $\tau = 5.5 \in (\tau_1^{(0)}, +\infty)$, we choose the initial value $(1473, 1, 2, 1)$, and the equilibrium E_1 is unstable shown in Figure 4.

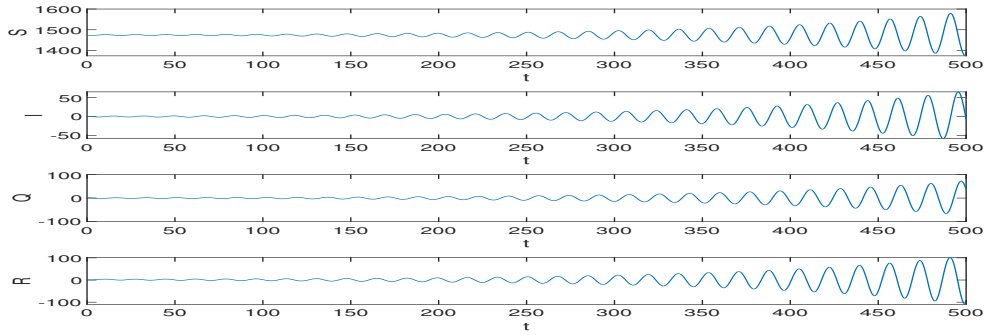

Figure 4. When $\tau = 5.5$, the system (2.2) has unstable equilibrium.

It can be seen from Figure 2∼ Figure 4, when $\tau \in [0, \tau_1^{(0)})$, the equilibrium E_1 of system (2.2) is locally asymptotically stable (see Fig. 2 and Fig. 3). When $\tau \in (\tau_1^{(0)}, +\infty)$, the equilibrium E_1 of system (2.2) is unstable, and accompanied by the fluctuation with increased amplitude (see Fig. 4). There is no periodic solution, so we believe the theoretical analysis is correct.

Remark 1: Through numerical simulations, it can be found that when the treatment time $\tau < \tau_1^{(0)}$, the shorter treatment time is, the faster the epidemic tends to be stable. If $\tau > \tau_1^{(0)}$, the epidemic cannot be controlled effectively, and accompanied by the fluctuation with increased amplitude with time. Thus, we obtain the critical treatment time $\tau_1^{(0)}$ for the epidemic.

According to the data presented in Refs. [27] and [28], we choose another set of parameters, $\Lambda = 2.48$, $\beta = 0.86834$, $\delta = 0.3$, $\alpha_1 = 0.0484$, $\alpha_2 = 0.022778$, $\gamma = 0.42386$, $\epsilon = 0.475$, $\mu = 0.00714$. We obtain $E_1 = (S_1^*, I_1^*, Q_1^*, R_1^*) \approx (347.3, 0, 0, 0)$, $E_2 = (S_2^*, I_2^*, Q_2^*, R_2^*) \approx (0.8976, 3.1737, 1.8857, 313.8526)$. Apparently, the assumption (H1) does not hold, but the assumption (H3) holds, therefore, the equilibrium E_1 is unstable and the equilibrium E_2 is locally asymptotically stable when $\tau = 0$.

Substituding these parameter values into the Eq. (3.13), we get $c_1 \approx 3.3383$, $c_0 \approx -0.4174$. It satisfies the assumption (H4), therefore, Eq. (3.13) has only one positive root $z_1 \approx 0.1204$. Substituting these parameter values into the Eqs. (3.12)∼(3.14), using MATLAB, we can obtain $\omega_1 \approx 0.3474$,

$Q_1 \approx 0.5926$, $P_1 \approx -0.8055$, $\tau_{2,1}^{(0)} \approx 7.2175$. According to the Theorem 3.1, the equilibrium E_2 is locally asymptotically stable at $\tau \in [0, \tau_{2,1}^{(0)})$, and when $\tau = \tau_{2,1}^{(0)}$, Hopf bifurcation occurs near the equilibrium E_2. Then we obtain $\mathrm{Re}(N_2) > 0$, $\mathrm{Re}(\xi_2) < 0$ by Eqs. (4.9)~(4.13), according to the Theorem 4.1, the system (2.2) has stable periodic solution near E_2, and the periodic solution reduced on the center manifold is stable.

When τ= 0, we choose the initial value $(0.85, 3, 1.9, 300)$ and the corresponding locally asymptotically stable equilibrium is shown in Figure 5.

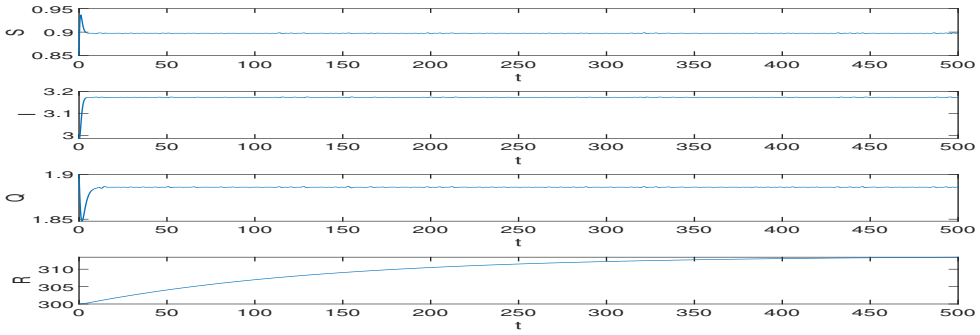

Figure 5. When $\tau = 0$, the equilibrium E_2 of system (2.2) is locally asymptotically stable.

When $\tau = 4 \in [0, \tau_{2,1}^{(0)})$, we choose the initial value $(0.85, 3, 1.9, 300)$, and the periodic solution of this model is locally asymptotically stable shown in Figure 6.

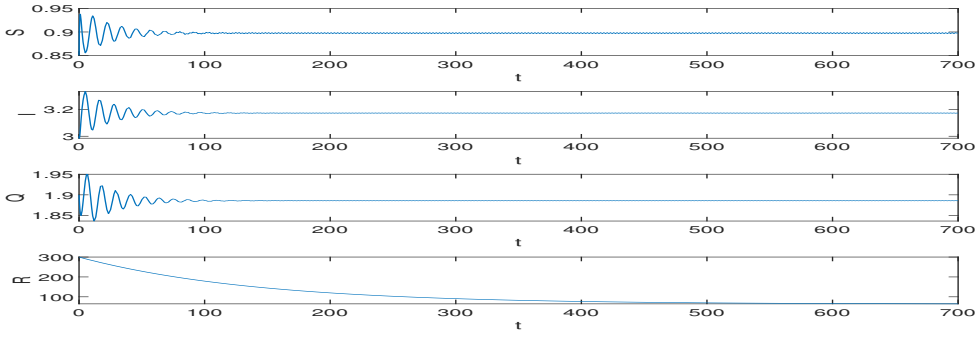

Figure 6. When $\tau = 4$, the equilibrium E_2 of system (2.2) is locally asymptotically stable.

When $\tau = 5 \in [0, \tau_{2,1}^{(0)})$, we choose the initial value $(0.85, 3, 1.9, 300)$, the equilibrium E_2 has a stable equilibrium is shown in Figure 7.

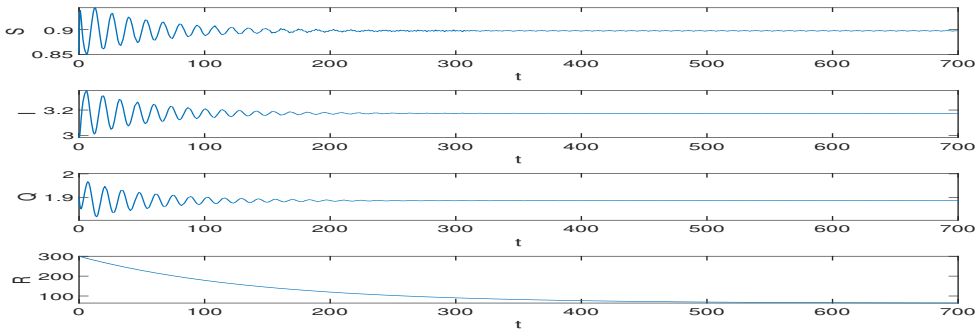

Figure 7. When $\tau = 5$, the equilibrium E_2 of system (2.2) is locally asymptotically stable.

It can be seen from Figure 5~Figure 7, when $\tau \in [0, \tau_{2,1}^{(0)})$, the equilibrium E_2 of system (2.2) is locally asymptotically stable. Especially, if τ is larger, the time required for the system (2.2) to be controlled stable is longer.

When $\tau = 7.23 > \tau_{2,1}^{(0)} = 7.2175$, we choose the initial value $(0.85, 3, 1.9, 300)$, the model has stable periodic solution shown in Figure 8.

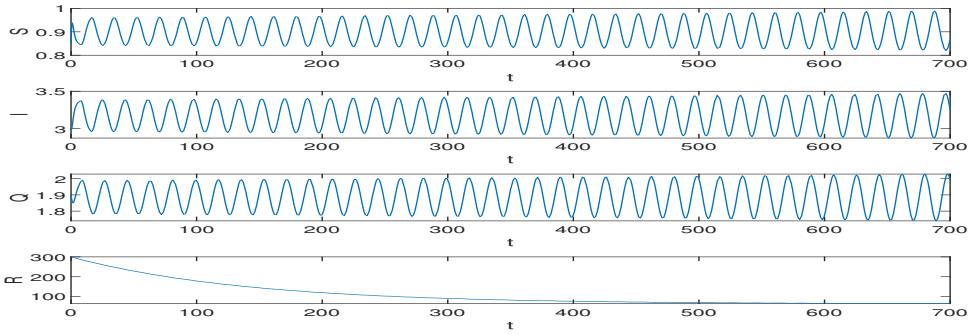

Figure 8. When $\tau = 7.23$, system (2.2) has stable periodic solution near E_2.

When $\tau = 7.5 > 7.23$, we choose the initial value $(0.85, 3, 1.9, 300)$, and the equilibrium E_2 is unstable shown in Figure 9.

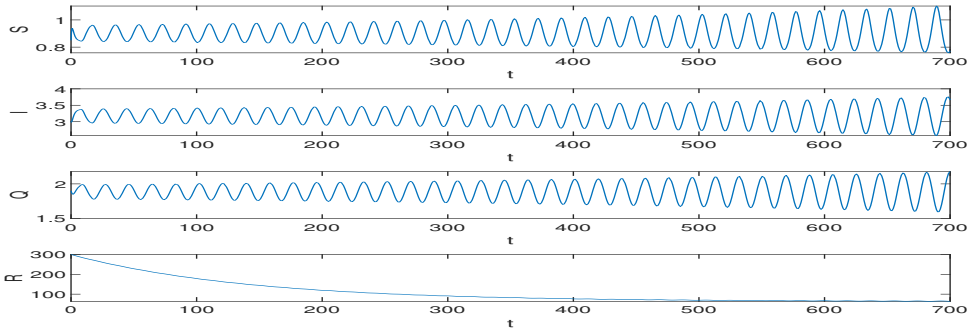

Figure 9. When $\tau = 7.5$, system (2.2) is unstable near E_2.

When $\tau = 7.9 > 7.5$, we choose the initial value $(0.85, 3, 1.9, 300)$, and the equilibrium E_2 is unstable shown in Figure 10.

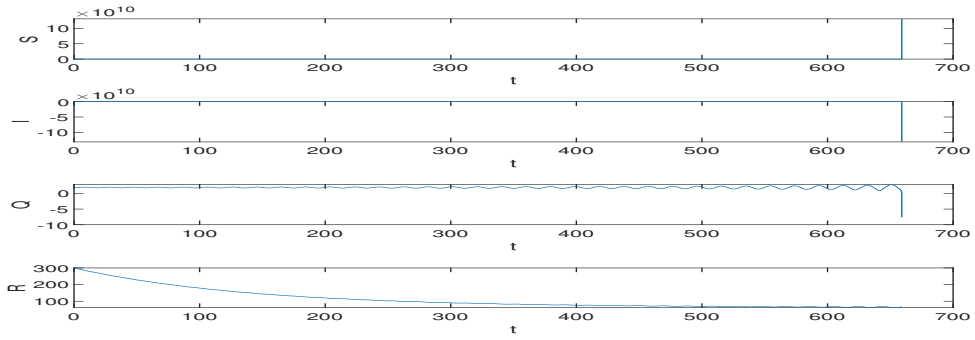

Figure 10. When $\tau = 7.9$, the equilibrium E_2 of system (2.2) is unstable.

It can be seen from the Figure 8∼Figure 10, when $\tau \in (\tau_{2,1}^{(0)}, +\infty)$, the equilibrium E_2 of system (2.2) is unstable. When τ approaches the critical value $\tau_{2,1}^{(0)}$, the equilibrium E_2 of system (2.2) exhibits periodic fluctuation and bifurcates stable periodic solutions (see Fig. 8). As τ increasing, the fluctuation phenomenon lasts for a period of time, and shows increasing volatility trends (see Fig. 9) and disappears eventually (see Fig. 10). According to the Theorem 3.1 and Theorem 4.1, we know that the theoretical analysis is correct.

Remark 2: Through numerical simulations, it can be found that when the treatment time $\tau < \tau_{2,1}^{(0)}$, the epidemic can be controlled effectively, and the smaller the time-delay τ is the faster the epidemic is controlled. When the treatment time $\tau \in (\tau_{2,1}^{(0)}, +\infty)$ is close to $\tau_{2,1}^{(0)}$, the epidemic will repeatedly occur periodically, otherwise, the epidemic cannot be controlled. Since the small variation in treatment time will lead to the epidemic polarization, therefore, whether to grasp the treatment time is essential to control the epidemic.

Remark 3: In Ref. [26], Liu et al. used the Milstein's Higher Order Method mentioned to illustrate their main results. And they obtained that the disease is persistent. Similarly, in our simulation, the epidemic is persistent and tends to stable with time.

In Ref. [29], the study carried out numerical simulation. First of all, they changed the value of parameters to get the "without control" and "with control" graphs, from which we can see that different populations tend to be stable in the end. At the same time, through comparison, it is concluded that the control technique is useful in controlling the disease. Our simulation results show that the system is locally asymptotically stable in the critical treatment time, and the disease will be effectively controlled. Then they fit real data with the infected class of their model and found that the number of cases grows exponentially with time, the disease would be serious without applying the proper optimal control strategies. And we conclude that if not treat within the critical time, that is to say, without timely and effective treatment, the epidemic will be uncontrollable.

In a word, although the model studied is different from Ref. [29], and we focus on the impact of time delay on the epidemic situation, but the overall idea and conclusion are consistent, so we can still verify the correctness of our simulation results. Moreover, we emphasize the time-delay of treatment, so we also obtain the critical treatment time, which can take more targeted measures to control the epidemic. To effectively control the epidemic situation, we need to take measures not only in the

external environment, but also in the internal medical care, and now there is little research on the treatment direction. Therefore, the critical treatment time we get is helpful for the alleviation of the disease.

6. Conclusion

In this paper, according to the propagation characteristics of COVID-19, we have constructed the SIRQ epidemic model with time delay for the COVID-19. We have also analyzed the stability of the equilibria and the existence of Hopf bifurcation associated with both equilibria. Then, we have used the multiple time scales method to derive the normal form of Hopf bifurcation for above COVID-19 epidemic model. Finally, We have chosen two groups of parameter values according to the data presented in Refs. [27] and [28] for numerical simulations to verify the correctness of the theoretical analysis. Compared with the numerical simulation results of Refs. [26] and [29], we have got the conclusion that our numerical simulation results were accuracy.

The numerical simulations showed that the small change of time-delay τ leads to the epidemic from stable to uncontrollable: the smaller τ is the better the epidemic controlled, but with τ increasing, the epidemic will occur repeatedly and outbreak eventually. Therefore, it is significant that find the critical treatment time to control the epidemic. In addition, we would predict the critical treatment time of epidemic in different regions according to the relevant parameters, so as to effectively control the epidemic in treatment, and realize the major breakthrough of medicine in the epidemic situation.

As for the part of numerical simulation, according to the opinions of reviewers, we will use more accurate data to get more valuable conclusions in further research.

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Conflict of interest

The Authors declare that this work has no conflict of interest.

References

1. Z. Liao, P. Lan, Z. Liao, Y. Zhang, S. Liu, TW-SIR: Time-window based SIR for COVID-19 forecasts, *Sci. Rep.*, **10** (2020), 22454.
2. C. Yang, J. Wang, Modeling the transmission of COVID-19 in the US–A case study, *Infect. Dis. Model.*, **6** (2021), 195–211.

3. G. Xu, F. Qi, H. Li, Q. Yang, H. Wang, X. Wang, et al., The differential immune responses to COVID-19 in peripheral and lung revealed by single-cell RNA sequencing, *Cell Discov.*, **6** (2020), 1–14.
4. Z. Zhang, A novel covid-19 mathematical model with fractional derivatives: Singular and nonsingular kernels, *Chaos Soliton. Fract.*, **139** (2020), 110060.
5. A. S. Bhadauria, R. Pathak, M. Chaudhary, A SIQ mathematical model on COVID-19 investigating the lockdown effect, *Infect. Dis. Model.*, **6** (2021), 244–257.
6. Y. Li, Q. Zhang, The balanced implicit method of preserving positivity for the stochastic SIQS epidemic model, *Physica A*, **538** (2020), 122972.
7. M. Higazy, Novel fractional order SIDARTHE mathematical model of COVID-19 pandemic, *Chaos Soliton. Fract.*, **138** (2020), 110007.
8. A. M. Ramos, M. R. Ferrández, M. Vela-Pérez, A. B. Kubik, B. Ivorra, A simple but complex enough θ-SIR type model to be used with COVID-19 real data. Application to the case of Italy, *Physica D*, **421** (2021), 132839.
9. K. S. Nisar, S. Ahmad, A. Ullah, K. Shah, H. Alrabaiah, M. Arfan, Mathematical analysis of SIRD model of COVID-19 with Caputo fractional derivative based on real data, *Results Phys.*, **21** (2021), 103772.
10. C. M. Batistela, D. P. F. Correa, Á. M. Bueno, J. R. C. Piqueira, SIRSi compartmental model for COVID-19 pandemic with immunity loss, *Chaos Soliton. Fract.*, **142** (2021), 110388.
11. P. E. Paré, C. L. Beck, T. Başar, Modeling, estimation, and analysis of epidemics over networks: An overview, *Annu. Rev. Control*, **50** (2020), 345–360.
12. C.-C. Zhu, J. Zhu, Dynamic analysis of a delayed COVID-19 epidemic with home quarantine in temporal-spatial heterogeneous via global exponential attractor method, *Chaos Soliton. Fract.*, **143** (2021), 110546.
13. S. Scheiner, N. Ukaj, C. Hellmich, Mathematical modeling of COVID-19 fatality trends: Death kinetics law versus infection-to-death delay rule, *Chaos Soliton. Fract.*, **136** (2020), 109891.
14. H. Wei, Y. Jiang, X. Song, G. H. Su, S. Z. Qiu, Global attractivity and permanence of a SVEIR epidemic model with pulse vaccination and time delay, *J. Comput. Appl. Math.*, **229** (2009), 302–312.
15. D. Mukherjee, Stability analysis of an S-I epidemic model with time delay, *Math. Comput. Model.*, **24** (1996), 63–68.
16. S. İğret Araz, Analysis of a Covid-19 model: Optimal control, stability and simulations, *Alex. Eng. J.*, **60** (2021), 647–658.
17. S. Annas, Muh. Isbar Pratama, Muh. Rifandi, W. Sanusi, S. Side, Stability analysis and numerical simulation of SEIR model for pandemic COVID-19 spread in Indonesia, *Chaos Soliton. Fract.*, **139** (2020), 110072.
18. A. E. S. Almocera, G. Quiroz, E. A. Hernandez-Vargas, Stability analysis in COVID-19 within-host model with immune response, *Commun. Nonlinear Sci. Numer. Simul.*, **95** (2021), 105584.
19. G. P. Samanta, Permanence and extinction of a nonautonomous HIV/AIDS epidemic model with distributed time delay, *Nonlinear Anal.-Real World Appl.*, **12** (2011), 1163–1177.

20. U. Avila-Ponce de León, Á. G. C. Pérez, E. Avila-Vales, An SEIARD epidemic model for COVID-19 in Mexico: Mathematical analysis and state-level forecast, *Chaos Soliton. Fract.*, **140** (2020), 110165.

21. H. M. Youssef, N. A. Alghamdi, M. A. Ezzat, A. A. El-Bary, A. M. Shawky, A new dynamical modeling SEIR with global analysis applied to the real data of spreading COVID-19 in Saudi Arabia, *Math. Biosci. Eng.*, **17** (2020), 7018–7044.

22. R. Carli, G. Cavone, N. Epicoco, P. Scarabaggio, M. Dotoli, Model predictive control to mitigate the COVID-19 outbreak in a multi-region scenario, *Annu. Rev. Control*, **50** (2020), 373–393.

23. G. Giordano, F. Blanchini, R. Bruno, P. Colaneri, A. D. Filippo, A. D. Matteo, et al., Modelling the COVID-19 epidemic and implementation of population-wide interventions in Italy, *Nature Med.*, **26** (2020), 855—860.

24. T. Odagaki, Exact properties of SIQR model for COVID-19, *Physica A*, **564** (2021), 125564.

25. F. Saldaña, H. Flores-Arguedas, J. A. Camacho-Gutiérrez, I. Barradas, Modeling the transmission dynamics and the impact of the control interventions for the COVID-19 epidemic outbreak, *Math. Biosci. Eng.*, **17** (2020), 4165—4183.

26. Q. Liu, D. Jiang, N. Shi, Threshold behavior in a stochastic SIQR epidemic model with standard incidence and regime switching, *Appl. Math. Comput.*, **316** (2018), 310–325.

27. M. Chen, M. Li, Y. Hao, Z. Liu, L. Hu, L. Wang, The introduction of population migration to SEIAR for COVID-19 epidemic modeling with an efficient intervention strategy, *Inf. Fusion*, **64** (2020), 252–258.

28. A. B. Gumel, E. A. Iboi, C. N. Ngonghala, E. H. Elbasha, A primer on using mathematics to understand COVID-19 dynamics: Modeling, analysis and simulations, *Infect. Dis. Model.*, **6** (2021), 148–168.

29. A. Khan, R. Zarin, G. Hussain, N. A. Ahmad, M. H. Mohd, A. Yusuf, Stability analysis and optimal control of covid-19 with convex incidence rate in Khyber Pakhtunkhawa (Pakistan), *Results Phys.*, **20** (2021), 103703.

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