Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/MBE

MBE, 18(4): 3160–3179.
DOI: 10.3934/mbe.2021157
Received: 03 December 2020
Accepted: 25 March 2021
Published: 06 April 2021

Research article

Survival and stationary distribution of a stochastic facultative mutualism model with distributed delays and strong kernels

Ke Qi, Zhijun Liu*, **Lianwen Wang and Qinglong Wang**

Department of Mathematics, Hubei Minzu University, Enshi, Hubei 445000, China

* **Correspondence:** Email: zjliu@hbmzu.edu.cn; Tel: +86(0)7188437732.

Abstract: To investigate the roles of both coupling noises and distributed delays with strong kernels, a novel delayed stochastic two-species facultative mutualism model is established, in where the strong kernels indicate that the maximum influence on the growth rate response at some time is due to population densities at the previous time, and the saturation effect is also incorporated because the facultative capacity of each species is finite and their interspecific mutualism should be upper bounded in real life. We first transfer the two-species stochastic model with strong kernels into an equivalent six-dimensional model through a linear chain technique. Later, sufficient conditions for the extinction exponentially, persistence in the mean, permanent in time average and stationary distribution are respectively obtained. Finally, numerical simulations are supplied to support our theoretical results. Our analytical results show that the coupling noise intensities play an important role in the long-time behaviors while the strong kernels are independent of the above asymptotic properties.

Keywords: facultative mutualism model; noise coupling; distributed delays; persistence and extinction; stationary distribution

1. Introduction

According to the definition of [1], mutualism is the interaction of two/many species that benefits both/each other. As a common occurrence in nature, the mutualism interaction has an important impact and is well documented in many types of communities. Mutualism can be obligate or facultative, more specifically, an obligate mutualist is a species which requires the presence of another species for its survival [2] while a facultative mutualist is one which benefits in the same way from the association with another species but will be survive in its absence [3]. In recent years many mutualism models have been studied intensively and some good results have been obtained, for details to see stability and bifurcation [4–12], persistence and extinction [5, 7, 8, 11, 13–20], periodic solution and almost periodic solution [20–24], optimal control [25, 26], and stationary distribution [27, 28].

Recalling many of the above studies, we can see that the distributed delay does't been taken into account. In fact the evolution of a species may reply on an average over past population or the cumulative effect of the past history, and distributed delays are often incorporated into populations models, for details to see references [30–40]. Particularly, the following Gamma distribution initially given by MacDonald [32]

$$K(t) = \frac{t^n \sigma^{n+1} e^{-\sigma t}}{n!}, \quad \sigma > 0, n = 0, 1, 2, \cdots,$$

is usually used for the delay kernels. It is well known that there exist two types of kernels: weak kernel and strong kernel, which respectively, represented by

$$K(t) = \sigma e^{-\sigma t} \ (n = 0, \text{weak kernel}), \quad K(t) = t\sigma^2 e^{-\sigma t} \ (n = 1, \text{strong kernel}). \tag{1.1}$$

Both weak kernel and strong kernel own different biological meanings: the former implies that the maximum weighted response of the growth rate is due to current population density while past densities have exponentially decreasing influence, and the latter indicate that the maximum influence on the growth rate response at some time is due to population density at the previous time (see [41]).

On the other hand, the deterministic models may be necessary to incorporate the environmental noises into these models. Nisbet and Gurney [42] and May [43] suggested that the growth rates in population systems should own stochasticity and emerge random fluctuation to a certain degree. Thus, some noise sources were incorporated and then corresponding stochastic models were established. However, in many excellent investigations the authors assumed that one noise source only had an effect on the intrinsic growth rate of one species. Obviously, a reasonable idea is to consider that one noise source has influence not only on the intrinsic growth rate of one species but also on that of other species.

Inspired by the above arguments, in the next section we introduce a stochastic facultative mutualism model with distributed delays and strong kernels (see model (2.3)). Survival analysis and stationary distribution will become two topics of our whole research because the survival analysis reveals the persistence or extinction of one or more species in random environment and the stationary distribution is concerned with the stochastic statistical characteristic of the long-term behaviours of the sample trajectories. To the best of our knowledge, there are few published papers concerning model (2.3). The rest of this work is organized as follows. In Section 3, we present the main results including extinction exponentially, persistence in the mean, permanent in time average. In Section 4, we devote to investigating the existence and uniqueness of stationary distribution. In Section 5, numerical simulations are given to support our findings. A brief discussion on the biological meanings is shown in Section 6.

2. Model and preliminaries

For the final export of the model we will discuss, let us first introduce the following facultative mutualism model with saturation effect which corresponds to a deterministic competitive model proposed by Gopalsamy [29]

$$\begin{cases} dx_1(t) = x_1(t)[a_1 - b_1x_1(t) + \dfrac{c_1x_2(t)}{1 + x_2(t)}]dt, \\ dx_2(t) = x_2(t)[a_2 - b_2x_2(t) + \dfrac{c_2x_1(t)}{1 + x_1(t)}]dt, \end{cases} \tag{2.1}$$

where x_i $(i = 1, 2)$ are the densities of two species, $a_i > 0$ denote the intrinsic growth rates, $b_i > 0$ are the intraspecific competition rates, $c_i > 0$ are the interspecific mutualism rates and the nonlinear term $c_1x_2/(1 + x_2)$ (or $c_2x_1/(1 + x_1)$) reflects a saturation effect for large enough x_2 (or x_1).

With the idea of distributed delays and strong kernels, model (2.1) becomes a delayed version

$$\begin{cases} dx_1(t) = x_1(t)[a_1 - b_1x_1(t) + c_1\displaystyle\int_{-\infty}^{t}(t - s)\sigma_2^2 e^{-\sigma_2(t-s)}\frac{x_2(s)}{1 + x_2(s)}ds]dt, \\ dx_2(t) = x_2(t)[a_2 - b_2x_2(t) + c_2\displaystyle\int_{-\infty}^{t}(t - s)\sigma_1^2 e^{-\sigma_1(t-s)}\frac{x_1(s)}{1 + x_1(s)}ds]dt. \end{cases} \tag{2.2}$$

Similar to [44], we use two coupling noise sources to model the random perturbations and derive a new stochastic model

$$\begin{cases} dx_1(t) = x_1(t)[a_1 - b_1x_1(t) + c_1\displaystyle\int_{-\infty}^{t}(t - s)\sigma_2^2 e^{-\sigma_2(t-s)}\frac{x_2(s)}{1 + x_2(s)}ds]dt + \sum_{i=1}^{2}\alpha_{1i}x_1(t)dB_i(t), \\ dx_2(t) = x_2(t)[a_2 - b_2x_2(t) + c_2\displaystyle\int_{-\infty}^{t}(t - s)\sigma_1^2 e^{-\sigma_1(t-s)}\frac{x_1(s)}{1 + x_1(s)}ds]dt + \sum_{i=1}^{2}\alpha_{2i}x_2(t)dB_i(t), \end{cases} \tag{2.3}$$

with initial values $x_i(s) = \phi_i(s) \geq 0$, $s \in (-\infty, 0]$ and $\phi_i(0) > 0$, where ϕ_i are continuous bounded functions on $(-\infty, 0]$. $B_i(t)$ are standard independent Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. And α_{1i}^2, α_{2i}^2 denote the coupling noise intensities.

Assign

$$\begin{aligned} m_1(t) &= \int_{-\infty}^{t}(t - s)\sigma_1^2 e^{-\sigma_1(t-s)}\frac{x_1(s)}{1 + x_1(s)}ds,\ m_2(t) = \int_{-\infty}^{t}(t - s)\sigma_2^2 e^{-\sigma_2(t-s)}\frac{x_2(s)}{1 + x_2(s)}ds, \\ n_1(t) &= \int_{-\infty}^{t}\sigma_1 e^{-\sigma_1(t-s)}\frac{x_1(s)}{1 + x_1(s)}ds,\ n_2(t) = \int_{-\infty}^{t}\sigma_2 e^{-\sigma_2(t-s)}\frac{x_2(s)}{1 + x_2(s)}ds. \end{aligned} \tag{2.4}$$

With the help of chain techniques, the delayed stochastic facultative mutualism model (2.3) is transformed into an equivalent undelayed stochastic six-dimensional system

$$\begin{cases} dx_1(t) = x_1(t)[a_1 - b_1x_1(t) + c_1m_2(t)]dt + \alpha_{11}x_1(t)dB_1(t) + \alpha_{12}x_1(t)dB_2(t), \\ dx_2(t) = x_2(t)[a_2 - b_2x_2(t) + c_2m_1(t)]dt + \alpha_{21}x_2(t)dB_1(t) + \alpha_{22}x_2(t)dB_2(t), \\ dm_1(t) = \sigma_1(n_1(t) - m_1(t))dt, \\ dm_2(t) = \sigma_2(n_2(t) - m_2(t))dt, \\ dn_1(t) = \sigma_1(\frac{x_1(t)}{1+x_1(t)} - n_1(t))dt, \\ dn_2(t) = \sigma_2(\frac{x_2(t)}{1+x_2(t)} - n_2(t))dt. \end{cases} \tag{2.5}$$

with initial value $(x_1(0), x_2(0), m_1(0), m_2(0), n_1(0), n_2(0))$, where

$$x_i(0) = \phi_i(0),\ m_i(0) = -\int_{-\infty}^{0} s\sigma_i^2 e^{\sigma_i s}\frac{\phi_i(s)}{1+\phi_i(s)}ds,\ n_i(0) = \int_{-\infty}^{0} \sigma_i e^{\sigma_i s}\frac{\phi_i(s)}{1+\phi_i(s)}ds,\ i = 1, 2.$$

To show the novelty of our work, we explicate the following two facts:

(I) Zuo et al. [36] recently investigated the following stochastic two-species cooperative model with distributed delays and weak kernels

$$\begin{cases} dx_1(t) = x_1(t)[a_1 - b_1x_1(t) + c_1\int_{-\infty}^{t}\sigma_2 e^{-\sigma_2(t-s)}x_2(s)ds]dt + \alpha_{11}x_1(t)dB_1(t), \\ dx_2(t) = x_2(t)[a_2 - b_2x_2(t) + c_2\int_{-\infty}^{t}\sigma_1 e^{-\sigma_1(t-s)}x_1(s)ds]dt + \alpha_{22}x_2(t)dB_2(t). \end{cases} \quad (2.6)$$

Obviously, there exists a limitation in model (2.6): with the increase of one cooperator's density, its cooperative capacity will increase and tend to infinity (see $c_i\int_{-\infty}^{t}\sigma_j e^{-\sigma_j(t-s)}x_j(s)ds$). But in real life, this interaction between different species should be upper-bounded (a similar argument can be seen in [45]). In our model (2.3), the interspecific mutualism terms $c_i\int_{-\infty}^{t}(t-s)\sigma_i^2 e^{-\sigma_i(t-s)}\frac{x_j(s)}{1+x_j(s)}ds$ show saturation effects. Also, by the chain techniques, model (2.6) can be transformed into an equivalent four-dimensional system (see model (2.2) in [36]) whose dimension is lower than the above six-dimensional system (2.5). Finally, we must point out that there are two noise sources in model (2.6), but one noise source only has an effect on one species. It is easy to see that one noise source affects two species at the same time in model (2.3).

(II) In a recent investigation, Ning et al. [40] discussed a stochastic competitive model with distributed delays and weak kernels

$$\begin{cases} dx_1(t) = x_1(t)[a_1 - b_1x_1(t) - c_1\int_{-\infty}^{t}\sigma_2 e^{-\sigma_2(t-s)}\frac{x_2(s)}{1+x_2(s)}ds]dt + \sum_{i=1}^{2}\alpha_{1i}x_1(t)dB_i(t), \\ dx_2(t) = x_2(t)[a_2 - b_2x_2(t) - c_2\int_{-\infty}^{t}\sigma_1 e^{-\sigma_1(t-s)}\frac{x_1(s)}{1+x_1(s)}ds]dt + \sum_{i=1}^{2}\alpha_{2i}x_2(t)dB_i(t). \end{cases} \quad (2.7)$$

Clearly, there exists a remarkably different mechanism of action between model (2.3) and model (2.7) because the former is interspecific mutualism (see the positive feedback parameters c_i) and the latter is interspecific competition (see the negative feedback parameters $-c_i$). The strong kernel functions of our model (2.3) differs distinctly from the weak kernel functions of the above model (2.7) (also see Eq (1.1)). By the chain techniques, an equivalent undelayed six-dimensional system to model (2.3) is also different from that of the undelayed four-dimensional system to model (2.7) (see model (3) in [40]).

As a continuation of previous work [40], our main purpose of this contribution is to investigate the effects of both coupling noise sources on the long-time behaviors of facultative mutualism model (2.3) with distributed delays and strong kernels by analyzing its equivalent system (2.5). For the convenience of the subsequent analysis, we list the following two definitions.

Definition 2.1. *Signals and abbreviations are defined in Table 1.*

Table 1. Signal abbreviation.

Signal	$\langle x(t)\rangle$	x_*	x^*
Expression	$t^{-1}\int_0^t x(s)ds$	$\liminf_{t\to+\infty} x(t)$	$\limsup_{t\to+\infty} x(t)$

Definition 2.2. *Survival results of species are defined in Table 2.*

Table 2. Survival of species.

Cases	Conditions
x-Extinct exponentially (EE)	$\limsup_{t\to+\infty} t^{-1}\ln x(t) < -\varpi_1$ a.s. ($\varpi_1 > 0$)
x-Persistence in the mean (PM)	$\lim_{t\to+\infty}\langle x(t)\rangle = \varpi_2$ a.s. ($\varpi_2 > 0$)
x-Permanent in time average (PTA)	$\varpi_3 \le \langle x(t)\rangle_* \le \langle x(t)\rangle^* \le \varpi_4$ a.s. ($\varpi_3,\ \varpi_4 > 0$)

3. Survival analysis

This section is determined to analyze the survival of system (2.5). For the convenience of the subsequent discussion, we first estimate $m_i(t)$ and $n_i(t)$.

Lemma 3.1. *$m_i(t)$, $n_i(t) \le 1$ and $\lim_{t\to+\infty} m_i(t)/t = \lim_{t\to+\infty} n_i(t)/t = 0$, $i = 1, 2$.*

Proof. We first consider the upper bound of $m_i(t)$. It follows from Eq (2.4) that

$$m_i(t) = \int_{-\infty}^{t} (t-s)\sigma_i^2 e^{-\sigma_i(t-s)} \frac{x_i(s)}{1+x_i(s)} ds \le \int_{-\infty}^{t} (t-s)\sigma_i^2 e^{-\sigma_i(t-s)} ds = 1,\ i = 1, 2. \tag{3.1}$$

Also, we obtain from Eq (2.4) that

$$n_i(t) = \int_{-\infty}^{t} \sigma_i e^{-\sigma_i(t-s)} \frac{x_i(s)}{1+x_i(s)} ds \le \int_{-\infty}^{t} \sigma_i e^{-\sigma_i(t-s)} ds = 1,\ i = 1, 2. \tag{3.2}$$

Obviously, Eqs (3.1) and (3.2) imply that $\lim_{t\to+\infty} m_i(t)/t = 0$, $\lim_{t\to+\infty} n_i(t)/t = 0$, $i = 1, 2$. □

We continue to give the following fundamental lemma on the global existence and uniqueness of positive solution to system (2.5).

Lemma 3.2. *For any initial value $X(0) = (x_1(0), x_2(0), m_1(0), m_2(0), n_1(0), n_2(0)) > 0$, system (2.5) admits a unique global solution $X(t) = (x_1(t), x_2(t), m_1(t), m_2(t), n_1(t), n_2(t)) > 0$ for $t \ge 0$ a.s.*

Proof. Using Itô's formula, we obtain from system (2.5) that

$$\begin{cases} d\ln x_1(t) = [a_1 - b_1 x_1(t) + c_1 m_2(t) - (\alpha_{11}^2 + \alpha_{12}^2)/2]dt + \alpha_{11}dB_1(t) + \alpha_{12}dB_2(t), \\ d\ln x_2(t) = [a_2 - b_2 x_2(t) + c_2 m_1(t) - (\alpha_{21}^2 + \alpha_{22}^2)/2]dt + \alpha_{21}dB_1(t) + \alpha_{22}dB_2(t). \end{cases} \tag{3.3}$$

Define a C^2-function $U(X(t))$ by

$$U(X(t)) = \sum_{i=1}^{2} (x_i - 1 - \ln x_i + m_i - 1 - \ln m_i + n_i - 1 - \ln n_i). \tag{3.4}$$

Applying Itô's formula to Eq (3.4) leads to

$$dU(X(t)) = LU(X(t)) + \sum_{i=1}^{2}\Big(\alpha_{1i}x_1(t)dB_i(t) - \alpha_{1i}dB_i(t) + \alpha_{2i}x_2(t)dB_i(t) - \alpha_{2i}dB_i(t)\Big),$$

where

$$\begin{aligned} LU(X(t)) = &-b_1x_1^2 - b_2x_2^2 + c_1m_2x_1 + c_2m_1x_2 + a_1x_1 + a_2x_2 + b_1x_1 + b_2x_2 + \sigma_1\tfrac{x_1}{1+x_1} \\ &+ \sigma_2\tfrac{x_2}{1+x_2} + (\alpha_{11}^2 + \alpha_{12}^2)/2 + (\alpha_{21}^2 + \alpha_{22}^2)/2 - \sigma_1\tfrac{1}{n_1}\tfrac{x_1}{1+x_1} - \sigma_2\tfrac{1}{n_2}\tfrac{x_2}{1+x_2} \\ &+ 2\sigma_1 + 2\sigma_2 - a_1 - a_2 - c_1m_2 - c_2m_1 - \sigma_1m_1 - \sigma_2m_2 - \sigma_1\tfrac{n_1}{m_1} - \sigma_2\tfrac{n_2}{m_2} \\ \le\ & K + 3\sigma_1 + 3\sigma_2 + (\alpha_{11}^2 + \alpha_{12}^2)/2 + (\alpha_{21}^2 + \alpha_{22}^2)/2, \end{aligned}$$

and

$$K = -b_1x_1^2 - b_2x_2^2 + c_1m_2x_1 + c_2m_1x_2 + a_1x_1 + a_2x_2 + b_1x_1 + b_2x_2.$$

It follows from Lemma 3.1 that $m_1 \le 1$ and $m_2 \le 1$, and then K is bounded when $x_1, x_2 \in (0, +\infty)$. As a consequence, $LU(X(t))$ is bounded. The rest proof is similar to that of Theorem 2.1 in [46], and hence we omit it. □

Assign $\xi_1 = 0.5(\alpha_{11}^2 + \alpha_{12}^2)$, $\xi_2 = 0.5(\alpha_{21}^2 + \alpha_{22}^2)$. The following Theorems 3.1–3.4 focus on the survival results of both species.

Theorem 3.1. *Both species are extinct exponentially if $a_1 + c_1 < \xi_1$ and $a_2 + c_2 < \xi_2$.*

Proof. An integration of Eq (3.3) over $[0, t]$ leads to

$$\ln\frac{x_i(t)}{x_i(0)} = (a_i - \xi_i)t - b_i\int_0^t x_i(s)ds + c_i\int_0^t m_j(s)ds + \alpha_{i1}B_1(t) + \alpha_{i2}B_2(t). \tag{3.5}$$

Dividing by t and using Lemma 3.1, one has

$$t^{-1}\ln x_i(t) - t^{-1}\ln x_i(0) \le a_i + c_i - \xi_i + t^{-1}(\alpha_{i1}B_1(t) + \alpha_{i2}B_2(t)),\ i = 1, 2. \tag{3.6}$$

The strong law of local martingales [47] states $\lim_{t\to+\infty} t^{-1}B_i(t) = 0$, and moreover, we derive from Eq (3.6) that

$$\limsup_{t\to+\infty} t^{-1}\ln x_i(t) \le a_i + c_i - \xi_i,\ i = 1, 2.$$

Thus, both species are extinct exponentially by the assumptions of Theorem 3.1. □

Theorem 3.2. *Assume that $a_1 + c_1 < \xi_1$ and $a_2 > \xi_2$, then species x_1 goes to exponential extinction while species x_2 is persistent in the mean and* $\lim_{t\to+\infty}\langle x_2(t)\rangle = (a_2 - \xi_2)/b_2$ *a.s.*

Proof. If $a_1 + c_1 < \xi_1$, then we obtain from Theorem 3.1 that species x_1 will be extinct exponentially and $\limsup_{t\to+\infty} t^{-1}\ln x_1(t) < 0$. Thus, we further derive that

$$\lim_{t\to+\infty} x_1(t) = 0\ \ a.s. \tag{3.7}$$

An integration of the last four equations of system (2.5) on both sides results in

$$\begin{aligned} m_i(t) - m_i(0) &= \sigma_i(\textstyle\int_0^t n_i(s)ds - \int_0^t m_i(s)ds), \\ n_i(t) - n_i(0) &= \sigma_i(\textstyle\int_0^t \frac{x_i(s)}{1+x_i(s)}ds - \int_0^t n_i(s)ds),\ i = 1, 2. \end{aligned}$$

Consequently, we have

$$\lim_{t\to+\infty} \frac{m_i(t)-m_i(0)}{t} = \sigma_i \lim_{t\to+\infty}(\langle n_i\rangle - \langle m_i\rangle), \quad \lim_{t\to+\infty} \frac{n_i(t)-n_i(0)}{t} = \sigma_i \lim_{t\to+\infty}(\langle \frac{x_i}{1+x_i}\rangle - \langle n_i\rangle),$$

furthermore, it follows from Lemma 3.1 that $\lim_{t\to+\infty} m_i(t)/t = 0$ and $\lim_{t\to+\infty} n_i(t)/t = 0$. And $\lim_{t\to+\infty} m_i(0)/t = 0$, $\lim_{t\to+\infty} n_i(0)/t = 0$. So we have

$$\lim_{t\to+\infty}\langle m_i\rangle = \lim_{t\to+\infty}\langle n_i\rangle = \lim_{t\to+\infty}\langle \frac{x_i}{1+x_i}\rangle, \ i = 1, 2. \tag{3.8}$$

Next, by Eq (3.7) one gets for arbitrarily small $\varepsilon > 0$, there is $T > 0$ such that for $t \geq T$,

$$0 < \langle \frac{x_1}{1+x_1}\rangle < \varepsilon/(2c_2),$$

which together with Eq (3.8) leads to

$$0 < \langle m_1(t)\rangle < \varepsilon/(2c_2). \tag{3.9}$$

Let $-\varepsilon < t^{-1}\ln x_2(0) < \varepsilon/2$ for $t \geq T$. We obtain from Eqs (3.5) and (3.9) that for $t \geq T$,

$$\begin{aligned}\ln x_2(t) &\leq (a_2 - \xi_2 + \varepsilon)t - b_2\int_0^t x_2(s)ds + \alpha_{21}B_1(t) + \alpha_{22}B_2(t),\\ \ln x_2(t) &\geq (a_2 - \xi_2 - \varepsilon)t - b_2\int_0^t x_2(s)ds + \alpha_{21}B_1(t) + \alpha_{22}B_2(t).\end{aligned}$$

An application of Lemma 4 in [48] to the above two inequalities gives that

$$\langle x_2(t)\rangle^* \leq (a_2 - \xi_2 + \varepsilon)/b_2, \ \langle x_2(t)\rangle_* \geq (a_2 - \xi_2 - \varepsilon)/b_2 \ a.s.$$

Thus

$$\lim_{t\to+\infty}\langle x_2(t)\rangle = (a_2 - \xi_2)/b_2$$

is acquired by the arbitrariness of ε. □

Theorem 3.3. *Suppose that $a_1 > \xi_1$ and $a_2 + c_2 < \xi_2$, then species x_2 goes to exponent extinction while species x_1 is persistent in the mean and* $\lim_{t\to+\infty}\langle x_1(t)\rangle = (a_1 - \xi_1)/b_1$ *a.s.*

Proof. The proof is similar to that of Theorem 3.2, and hence we omit it. □

Theorem 3.4. *Assume that $a_1 > \xi_1$ and $a_2 > \xi_2$, then both species will be permanent in time average and $(a_1-\xi_1)/b_1 \leq \langle x_1\rangle_* \leq \langle x_1\rangle^* \leq (a_1-\xi_1+c_1)/b_1$, $(a_2-\xi_2)/b_2 \leq \langle x_2\rangle_* \leq \langle x_2\rangle^* \leq (a_2-\xi_2+c_2)/b_2$ a.s.*

Proof. Recalling Eq (3.5) and Lemma 3.1, we have

$$\begin{aligned}\ln x_i(t) &\leq \ln x_i(0) + (a_i - \xi_i + c_i)t - b_i\int_0^t x_i(s)ds + \alpha_{i1}B_1(t) + \alpha_{i2}B_2(t),\\ \ln x_i(t) &\geq \ln x_i(0) + (a_i - \xi_i)t - b_i\int_0^t x_i(s)ds + \alpha_{i1}B_1(t) + \alpha_{i2}B_2(t), \ i = 1, 2.\end{aligned}$$

It follows from Lemma 4 in [48] that

$$\langle x_i(t)\rangle^* \leq (a_i - \xi_i + c_i)/b_i, \quad \langle x_i(t)\rangle_* \geq (a_i - \xi_i)/b_i \ a.s.$$

So the desired conclusion is obtained. □

4. Stationary distribution

In this section, to discuss the stationary distribution of system (2.5) we make some preliminaries. Consider the following integral equation

$$X(t) = X(t_0) + \int_{t_0}^{t} g(s, X(s))ds + \sum_{l=1}^{k} \int_{t_0}^{t} \varsigma_l(s, X(s))dB_l(s). \tag{4.1}$$

Lemma 4.1. *[49]. Assume that the coefficients of Eq* (4.1) *are independent of t and satisfy the following conditions for a constant κ:*

$$|g(s, x_1) - g(s, x_2)| + \sum_{l=1}^{k} |\varsigma_l(s, x_1) - \varsigma_l(s, x_2)| \leq \kappa|x_1 - x_2|,\ |g(s, x)| + \sum_{l=1}^{k} |\varsigma_l(s, x)| \leq \kappa(1 + |x|)$$

in $O_{\mathbb{R}} \subset \mathbb{R}_+^d$ and there exists a nonnegative C^2-function $V(x)$ in $\mathbb{R}_+^d$ satisfying $LV(x) \leq -1$ outside some compact set. Then Eq (4.1) *exists a solution which has a stationary distribution.*

Remark 4.1. *[36]. The condition in Lemma 4.1 may be replaced by the global existence of the solution to Eq* (4.1) *according to Remark 5 in Xu et al. [50].*

We first give Lemma 4.2 which is important for the subsequent discussions.

Lemma 4.2. *Assume that $X(t)$ is a solution to system* (2.5) *with initial value $X(0) > 0$. Then there is a positive constant Q_q such that for $q > 0$,*

$$\mathbb{E}[x_i^q] \leq Q_q,\ \mathbb{E}[m_i^q] \leq Q_q,\ \mathbb{E}[n_i^q] \leq Q_q,\ i = 1, 2.$$

Proof. Let

$$V(X(t)) = \sum_{i=1}^{2} (\frac{1}{q} x_i^q + \frac{b_i}{2\sigma_i} m_i^{q+1} + \frac{b_i}{2\sigma_i} n_i^{q+1}). \tag{4.2}$$

Applying Itô's formula to Eq (4.2), one can derive that

$$dV(X(t)) = LV(X(t))dt + \sum_{i=1}^{2} (\alpha_{i1} dB_1(t) + \alpha_{i2} dB_2(t)) x_i^q,$$

in which

$$\begin{aligned} LV(X(t)) = {} & (a_1 - b_1 x_1 + c_1 m_2) x_1^q + (a_2 - b_2 x_2 + c_2 m_1) x_2^q + \textstyle\sum_{i=1}^2 \frac{1}{2}(q-1)(\alpha_{i1}^2 + \alpha_{i2}^2) x_i^q \\ & + \textstyle\sum_{i=1}^2 [\frac{b_i}{2}(q+1)(n_i m_i^q - m_i^{q+1}) + \frac{b_i}{2}(q+1)(\frac{x_i}{1+x_i} n_i^q - n_i^{q+1})]. \end{aligned} \tag{4.3}$$

Obviously, we get, by Young's inequality, that

$$\begin{aligned} &\tfrac{b_i}{2}(q+1)(n_i m_i^q - m_i^{q+1}) \leq \tfrac{b_i}{2}(q+1)[\tfrac{1}{q+1} n_i^{q+1} - \tfrac{1}{q+1} m_i^{q+1}] = \tfrac{b_i}{2}(n_i^{q+1} - m_i^{q+1}), \\ &\tfrac{b_i}{2}(q+1)(\tfrac{x_i}{1+x_i} n_i^q - n_i^{q+1}) \leq \tfrac{b_i}{2}(q+1)[\tfrac{1}{q+1}(\tfrac{x_i}{1+x_i})^{q+1} - \tfrac{1}{q+1} {n_i}^{q+1}] \leq \tfrac{b_i}{2}(x_i^{q+1} - {n_i}^{q+1}). \end{aligned}$$

It follows from Lemma 3.1 that $m_1 \leq 1$ and $m_2 \leq 1$. By Eq (4.3) we have

$$LV(X(t)) \leq \sum_{i=1}^{2} \{-\frac{b_i}{2} x_i^{q+1} + [a_i + c_i + \frac{1}{2}(q-1)(\alpha_{i1}^2 + \alpha_{i2}^2)] x_i^q - \frac{b_i}{2} m_i^{q+1}\}.$$

For a constant $\eta > 0$, we have

$$\begin{aligned} L(e^{\eta t}V(X(t))) &= \eta e^{\eta t}V(X(t)) + e^{\eta t}LV(X(t)) \\ &\leq e^{\eta t}\sum_{i=1}^{2}\{-\tfrac{b_i}{2}x_i^{q+1} + [a_i + c_i + \tfrac{1}{2}(q-1)(\alpha_{i1}^2 + \alpha_{i2}^2) + \tfrac{\eta}{q}]x_i^q + (\tfrac{b_i\eta}{2\sigma_i} - \tfrac{b_i}{2})m_i^{q+1} + \tfrac{b_i\eta}{2\sigma_i}n_i^{q+1}\}. \end{aligned}$$

Choosing the above constant η small enough such that $b_i\eta/(2\sigma_i) - b_i/2 < 0$, and noting that $b_i\eta/(2\sigma_i)n_i^{q+1} \leq b_i\eta/(2\sigma_i)$ (see Lemma 3.1, $n_i \leq 1$), we further obtain

$$L(e^{\eta t}V(X(t))) \leq G_1 e^{\eta t}, \tag{4.4}$$

where

$$G_1 = \max_{x_1,x_2\in(0,+\infty)}\sum_{i=1}^{2}\{-\frac{b_i}{2}x_i^{q+1} + [a_i + c_i + \frac{1}{2}(q-1)(\alpha_{i1}^2 + \alpha_{i2}^2) + \frac{\eta}{q}]x_i^q + \frac{b_i\eta}{2\sigma_i}\}.$$

Integrating Eq (4.4) from 0 to t and then taking the expectation, one has

$$\mathbb{E}[V(X(t))] \leq e^{-\eta t}V(X(0)) + G_1/\eta, \quad t \geq 0,$$

which together with the continuity of $V(X(t))$ and the boundedness of $e^{-\eta t}V(X(0))$ and G_1/η, implies that there exists a constant $G_2 > 0$ such that for all $t \geq 0$

$$\mathbb{E}[V(X(t))] \leq G_2.$$

We further obtain from Eq (4.2) that $\mathbb{E}[x_i^q/q] \leq \mathbb{E}[V(X(t))] \leq G_2$, and hence

$$\mathbb{E}[x_i^q] \leq qG_2, \; i = 1, 2.$$

Also, it follows from Eq (4.2) that $\mathbb{E}[m_i^{q+1}] \leq 2\sigma_i G_2/b_i$. And by the Young's inequality, there exist $A_{1i} > 0$ such that

$$\mathbb{E}[m_i^q] \leq A_{1i}\mathbb{E}[m_i^{q+1}]^{\frac{q}{q+1}} \leq A_{1i}(2\sigma_i G_2/b_i)^{\frac{q}{q+1}}, \; i = 1, 2.$$

Similarly, we can prove there exist A_{2i} such that

$$\mathbb{E}[n_i^q] \leq A_{2i}(2\sigma_i G_2/b_i)^{\frac{q}{q+1}}, \; i = 1, 2.$$

Let

$$Q_q = \max\{qG_2, A_{1i}(2\sigma_i G_2/b_i)^{\frac{q}{q+1}}, A_{2i}(2\sigma_i G_2/b_i)^{\frac{q}{q+1}}, \; i = 1, 2\},$$

then for $q > 0$,

$$\mathbb{E}[x_i^q] \leq Q_q, \;\; \mathbb{E}[m_i^q] \leq Q_q, \;\; \mathbb{E}[n_i^q] \leq Q_q. \tag{4.5}$$

The proof is complete. □

Lemma 4.3. *Suppose that $X(t)$ is a solution to system* (2.5) *with $X(0) > 0$, then almost every path of $X(t)$ to system* (2.5) *will be uniformly continuous.*

Proof. First let us consider $x_1(t)$. For any $0 \le t_1 \le t_2$, an integration of the first equation of system (2.5) yields

$$x_1(t_2) - x_1(t_1) = \int_{t_1}^{t_2} x_1(s)(a_1 - b_1 x_1(s) + c_1 m_2(s))ds + \sum_{i=1}^{2} \alpha_{1i} \int_{t_1}^{t_2} x_1(s)dB_i(s). \tag{4.6}$$

Let $p > 2$, by the elementary inequality $|a + b + c|^p \le 3^{p-1}(|a^p + b^p + c^p|)$, one has

$$\begin{aligned}
&\mathbb{E}[|x_1(t_2) - x_1(t_1)|^p] \\
&= \mathbb{E}[|\int_{t_1}^{t_2} x_1(s)(a_1 - b_1 x_1(s) + c_1 m_2(s))ds + \sum_{i=1}^{2} \alpha_{1i} \int_{t_1}^{t_2} x_1(s)dB_i(s)|^p] \\
&\le 3^{p-1}\{\mathbb{E}[|\int_{t_1}^{t_2} x_1(s)(a_1 - b_1 x_1(s) + c_1 m_2(s))ds|^p] + \sum_{i=1}^{2} \mathbb{E}[|\int_{t_1}^{t_2} \alpha_{1i} x_1(s)dB_i(s)|^p]\}.
\end{aligned} \tag{4.7}$$

Recalling Lemma 4.2 (see Eq (4.5)) and using the Hölder inequality result in

$$\begin{aligned}
&\mathbb{E}[|\int_{t_1}^{t_2} x_1(s)(a_1 - b_1 x_1(s) + c_1 m_2(s))ds|^p] \\
&\le \mathbb{E}[|(\int_{t_1}^{t_2} 1^{\frac{p}{p-1}} ds)^{\frac{p-1}{p}} (\int_{t_1}^{t_2} x_1(s)^p (a_1 - b_1 x_1(s) + c_1 m_2(s))^p ds)^{\frac{1}{p}}|^p] \\
&\le (t_2 - t_1)^{p-1} \mathbb{E}[\int_{t_1}^{t_2} |x_1(s)(a_1 - b_1 x_1(s) + c_1 m_2(s))|^p ds] \\
&\le (t_2 - t_1)^{p-1} \int_{t_1}^{t_2} \frac{1}{2}(\mathbb{E}[|x_1(s)|^{2p}] + \mathbb{E}[|a_1 - b_1 x_1(s) + c_1 m_2(s)|^{2p}])ds \\
&\le (t_2 - t_1)^{p-1} \int_{t_1}^{t_2} \frac{1}{2}(\mathbb{E}[|x_1(s)|^{2p}] + 3^{2p-1}(a_1^{2p} + b_1^{2p}\mathbb{E}[|x_1(s)|^{2p}] + c_1^{2p}\mathbb{E}[|m_2(s)|^{2p}]))ds \\
&\le (t_2 - t_1)^{p-1} \int_{t_1}^{t_2} \frac{1}{2}[Q_{2p} + 3^{2p-1}(a_1^{2p} + b_1^{2p} Q_{2p} + c_1^{2p} Q_{2p})]ds \\
&= \tfrac{1}{2}(t_2 - t_1)^p [Q_{2p} + 3^{2p-1}(a_1^{2p} + b_1^{2p} Q_{2p} + c_1^{2p} Q_{2p})].
\end{aligned} \tag{4.8}$$

In addition, using the Moment inequality and Lemma 4.2 leads to

$$\begin{aligned}
&\mathbb{E}[|\int_{t_1}^{t_2} \alpha_{11} x_1(s)dB_1(s)|^p] + \mathbb{E}[|\int_{t_1}^{t_2} \alpha_{12} x_1(s)dB_2(s)|^p] \\
&\le (\alpha_{11}^p + \alpha_{12}^p)(\frac{p(p-1)}{2})^{\frac{p}{2}} (t_2 - t_1)^{\frac{p-2}{2}} \int_{t_1}^{t_2} \mathbb{E}[|x_1(s)|^p]ds \\
&= (\alpha_{11}^p + \alpha_{12}^p)(\frac{p(p-1)}{2}(t_2 - t_1))^{\frac{p}{2}} Q_p.
\end{aligned} \tag{4.9}$$

Substituting Eqs (4.8) and (4.9) into Eq (4.7), we have

$$\mathbb{E}[|x_1(t_2) - x_1(t_1)|^p] \le F_1(t_2 - t_1)^{\frac{p}{2}}, \tag{4.10}$$

where

$$F_1 = 3^{p-1}[\frac{1}{2}(t_2 - t_1)^{\frac{p}{2}}(Q_{2p} + 3^{2p-1}(a_1^{2p} + b_1^{2p} Q_{2p} + c_1^{2p} Q_{2p})) + (\alpha_{11}^p + \alpha_{12}^p)(\frac{p(p-1)}{2})^{\frac{p}{2}} Q_p].$$

Next, we continue to consider $m_1(t)$. For any $0 \leq t_1 \leq t_2$, integrating the third equation of system (2.5) from t_1 to t_2 yields that

$$m_1(t_2) - m_1(t_1) = \int_{t_1}^{t_2} \sigma_1(n_1(s) - m_1(s))ds.$$

Similar to Eq (4.7), we have from the Hölder inequality and Lemma 4.2 that

$$\begin{aligned}
&\mathbb{E}[|m_1(t_2) - m_1(t_1)|^p] = \mathbb{E}[|\int_{t_1}^{t_2} \sigma_1(n_1(s) - m_1(s))ds|^p] \\
&\leq \mathbb{E}[|(\int_{t_1}^{t_2} 1^{\frac{p}{p-1}} ds)^{\frac{p-1}{p}} (\int_{t_1}^{t_2} \sigma_1^p(n_1(s) - m_1(s))^p ds)^{\frac{1}{p}}|^p] \\
&\leq (t_2 - t_1)^{p-1} \int_{t_1}^{t_2} \mathbb{E}[|\sigma_1(n_1(s) - m_1(s))|^p] ds \\
&\leq (t_2 - t_1)^{p-1} \int_{t_1}^{t_2} 2^{p-1}(\sigma_1^p \mathbb{E}[|n_1(s)|^p] + \sigma_1^p \mathbb{E}[|m_1(s)|^p]) ds \\
&\leq F_2(t_2 - t_1)^{\frac{p}{2}},
\end{aligned} \tag{4.11}$$

where $F_2 = 2^p(t_2 - t_1)^{\frac{p}{2}} \sigma_1^p Q_p$.

Finally, we investigate $n_1(t)$. For any $0 \leq t_1 \leq t_2$, by integrating the fifth equation of system (2.5) one has

$$n_1(t_2) - n_1(t_1) = \int_{t_1}^{t_2} \sigma_1(\frac{x_1(s)}{1 + x_1(s)} - n_1(s))ds.$$

Similar to Eq (4.11), one obtains

$$\begin{aligned}
&\mathbb{E}[|n_1(t_2) - n_1(t_1)|^p] = \mathbb{E}[|\int_{t_1}^{t_2} \sigma_1(\frac{x_1(s)}{1 + x_1(s)} - n_1(s))ds|^p] \\
&\leq \mathbb{E}[|(\int_{t_1}^{t_2} 1^{\frac{p}{p-1}} ds)^{\frac{p-1}{p}} (\int_{t_1}^{t_2} \sigma_1^p(\frac{x_1(s)}{1 + x_1(s)} - n_1(s))^p ds)^{\frac{1}{p}}|^p] \\
&\leq (t_2 - t_1)^{p-1} \int_{t_1}^{t_2} \mathbb{E}[|\sigma_1(\frac{x_1(s)}{1 + x_1(s)} - n_1(s))|^p] ds \\
&\leq (t_2 - t_1)^{p-1} \int_{t_1}^{t_2} 2^{p-1}(\sigma_1^p \mathbb{E}[|\frac{x_1(s)}{1 + x_1(s)}|^p] + \sigma_1^p \mathbb{E}[|n_1(s)|^p]) ds \\
&\leq (t_2 - t_1)^{p-1} \int_{t_1}^{t_2} 2^{p-1}(\sigma_1^p \mathbb{E}[|x_1(s)|^p] + \sigma_1^p \mathbb{E}[|n_1(s)|^p]) ds \\
&\leq F_2(t_2 - t_1)^{\frac{p}{2}}.
\end{aligned} \tag{4.12}$$

Repeating the same analysis method as above, we obtain that x_2, m_2 and n_2 own similar results as those of Eqs (4.10)–(4.12), respectively. Thus, it follows from Lemma 3.4 in [50] that almost every sample path of $X(t)$ is uniformly continuous. □

Lemma 4.4. *[51]. Let $h(t)$ be a nonnegative function defined on $[0, +\infty)$ such that $h(t)$ is integrable on $[0, +\infty)$ and is uniformly continuous on $[0, +\infty)$. Then $\lim_{t\to+\infty} h(t) = 0$.*

Lemma 4.5. *If $b_1b_2 - c_1c_2 > 0$, then the solution $X(t) = (x_1(t), x_2(t), m_1(t), m_2(t), n_1(t), n_2(t)) > 0$ to system (2.5) is globally attractive, that is, for any solution $\bar{X}(t) = (\bar{x}_1(t), \bar{x}_2(t), \bar{m}_1(t), \bar{m}_2(t), \bar{n}_1(t), \bar{n}_2(t))$ to system (2.5) with $\bar{X}(0) > 0$, there exist $\lim_{t\to+\infty} |x_i(t) - \bar{x}_i(t)| = 0$, $\lim_{t\to+\infty} |m_i(t) - \bar{m}_i(t)| = 0$, $\lim_{t\to+\infty} |n_i(t) - \bar{n}_i(t)| = 0$ a.s., $i = 1, 2$.*

Proof. It follows from the last two equations of system (2.5) that

$$d(n_i(t) - \bar{n}_i(t)) = \{\sigma_i(\frac{x_i(t)}{1 + x_i(t)} - \frac{\bar{x}_i(t)}{1 + \bar{x}_i(t)}) - \sigma_i(n_i(t) - \bar{n}_i(t))\}dt,\ i = 1, 2. \tag{4.13}$$

Integrating both sides of Eq (4.13) over the interval $[0, t]$ yields that

$$n_i(t) - \bar{n}_i(t) = (n_i(0) - \bar{n}_i(0))e^{-\sigma_i t} + \sigma_i e^{-\sigma_i t}\int_0^t e^{\sigma_i s}(\frac{x_i(s)}{1 + x_i(s)} - \frac{\bar{x}_i(s)}{1 + \bar{x}_i(s)})ds.$$

As a consequence, one has

$$|n_i(t) - \bar{n}_i(t)| \le |n_i(0) - \bar{n}_i(0)|e^{-\sigma_i t} + \sigma_i e^{-\sigma_i t}\int_0^t e^{\sigma_i s}|\frac{x_i(s)}{1 + x_i(s)} - \frac{\bar{x}_i(s)}{1 + \bar{x}_i(s)}|ds.$$

Note that

$$|\frac{x_i(t)}{1 + x_i(t)} - \frac{\bar{x}_i(t)}{1 + \bar{x}_i(t)}| = |\frac{x_i(t) - \bar{x}_i(t)}{(1 + x_i(t))(1 + \bar{x}_i(t))}| \le |x_i(t) - \bar{x}_i(t)|,$$

we have

$$|n_i(t) - \bar{n}_i(t)| \le |n_i(0) - \bar{n}_i(0)|e^{-\sigma_i t} + \sigma_i e^{-\sigma_i t}\int_0^t e^{\sigma_i s}|x_i(s) - \bar{x}_i(s)|ds, \tag{4.14}$$

from which we conclude that

$$\begin{aligned}
\int_0^t |n_i(s) - \bar{n}_i(s)|ds &\le -\frac{1}{\sigma_i}(e^{-\sigma_i t} - 1)|n_i(0) - \bar{n}_i(0)| + \sigma_i\int_0^t d\nu\int_0^\nu e^{\sigma_i(s-\nu)}|x_i(s) - \bar{x}_i(s)|ds \\
&= \frac{1}{\sigma_i}(1 - e^{-\sigma_i t})|n_i(0) - \bar{n}_i(0)| + \sigma_i\int_0^t e^{\sigma_i s}|x_i(s) - \bar{x}_i(s)|ds\int_s^t e^{-\sigma_i \nu}d\nu \\
&= \frac{1}{\sigma_i}(1 - e^{-\sigma_i t})|n_i(0) - \bar{n}_i(0)| + \int_0^t |x_i(s) - \bar{x}_i(s)|(1 - e^{\sigma_i(s-t)})ds \\
&\le \frac{1}{\sigma_i}|n_i(0) - \bar{n}_i(0)| + \int_0^t |x_i(s) - \bar{x}_i(s)|ds,\ i = 1, 2.
\end{aligned} \tag{4.15}$$

Similarly, one has

$$\int_0^t |m_i(s) - \bar{m}_i(s)|ds \le \frac{1}{\sigma_i}|m_i(0) - \bar{m}_i(0)| + \int_0^t |n_i(s) - \bar{n}_i(s)|ds,\ i = 1, 2. \tag{4.16}$$

Assign

$$\mu = b_1 b_2 - c_1 c_2,\quad M(t) = \frac{b_2 + c_2}{\mu}|\ln x_1(t) - \ln \bar{x}_1(t)| + \frac{b_1 + c_1}{\mu}|\ln x_2(t) - \ln \bar{x}_2(t)|. \tag{4.17}$$

A direct calculation of the right differential $D^+M(t)$ of $M(t)$ leads to

$$\begin{aligned}
&D^+M(t) \\
&= \tfrac{b_2+c_2}{\mu}\mathrm{sgn}\{x_1(t) - \bar{x}_1(t)\}d(\ln x_1(t) - \ln \bar{x}_1(t)) + \tfrac{b_1+c_1}{\mu}\mathrm{sgn}\{x_2(t) - \bar{x}_2(t)\}d(\ln x_2(t) - \ln \bar{x}_2(t)) \\
&\le \tfrac{b_2+c_2}{\mu}(c_1|m_2(t) - \bar{m}_2(t)| - b_1|x_1(t) - \bar{x}_1(t)|)dt + \tfrac{b_1+c_1}{\mu}(c_2|m_1(t) - \bar{m}_1(t)| - b_2|x_2(t) - \bar{x}_2(t)|)dt,
\end{aligned}$$

from which and Eqs (4.15) and (4.16) one can obtain

$$\begin{aligned}
&M(t) - M(0)\\
&\leq \frac{b_2 + c_2}{\mu}[\frac{c_1}{\sigma_2}|m_2(0) - \bar{m}_2(0)| + c_1 \int_0^t |n_2(s) - \bar{n}_2(s)|ds - b_1 \int_0^t |x_1(s) - \bar{x}_1(s)|ds]\\
&\quad + \frac{b_1 + c_1}{\mu}[\frac{c_2}{\sigma_1}|m_1(0) - \bar{m}_1(0)| + c_2 \int_0^t |n_1(s) - \bar{n}_1(s)|ds - b_2 \int_0^t |x_2(s) - \bar{x}_2(s)|ds]\\
&\leq \frac{c_1(b_2 + c_2)}{\sigma_2\mu}|m_2(0) - \bar{m}_2(0)| + \frac{c_1(b_2 + c_2)}{\mu}[\frac{1}{\sigma_2}|n_2(0) - \bar{n}_2(0)| + \int_0^t |x_2(s) - \bar{x}_2(s)|ds]\\
&\quad + \frac{c_2(b_1 + c_1)}{\sigma_1\mu}|m_1(0) - \bar{m}_1(0)| + \frac{c_2(b_1 + c_1)}{\mu}[\frac{1}{\sigma_1}|n_1(0) - \bar{n}_1(0)| + \int_0^t |x_1(s) - \bar{x}_1(s)|ds]\\
&\quad - \frac{b_1(b_2 + c_2)}{\mu}\int_0^t |x_1(s) - \bar{x}_1(s)|ds - \frac{b_2(b_1 + c_1)}{\mu}\int_0^t |x_2(s) - \bar{x}_2(s)|ds\\
&= \frac{c_1(b_2 + c_2)}{\sigma_2\mu}|m_2(0) - \bar{m}_2(0)| + \frac{c_1(b_2 + c_2)}{\sigma_2\mu}|n_2(0) - \bar{n}_2(0)| - \int_0^t |x_1(s) - \bar{x}_1(s)|ds\\
&\quad + \frac{c_2(b_1 + c_1)}{\sigma_1\mu}|m_1(0) - \bar{m}_1(0)| + \frac{c_2(b_1 + c_1)}{\sigma_1\mu}|n_1(0) - \bar{n}_1(0)| - \int_0^t |x_2(s) - \bar{x}_2(s)|ds.
\end{aligned}$$

Rearranging the above inequality leads to

$$\begin{aligned}
M(t) + \sum_{i=1}^{2}\int_0^t |x_i(s) - \bar{x}_i(s)|ds \leq M(0) &+ \frac{c_1(b_2 + c_2)}{\sigma_2\mu}(|m_2(0) - \bar{m}_2(0)| + |n_2(0) - \bar{n}_2(0)|)\\
&+ \frac{c_2(b_1 + c_1)}{\sigma_1\mu}(|m_1(0) - \bar{m}_1(0)| + |n_1(0) - \bar{n}_1(0)|) < +\infty,
\end{aligned}$$

from which one gets $|x_i(t) - \bar{x}_i(t)| \in L^1[0, +\infty)$. Similarly, it follows from Eqs (4.15) and (4.16) that $|m_i(t) - \bar{m}_i(t)|, |n_i(t) - \bar{n}_i(t)| \in L^1[0, +\infty)$. Thus, we obtain from Lemmas 4.3 and 4.4 that

$$\lim_{t\to+\infty} |x_i(t) - \bar{x}_i(t)| = \lim_{t\to+\infty} |m_i(t) - \bar{m}_i(t)| = \lim_{t\to+\infty} |n_i(t) - \bar{n}_i(t)| = 0,\ i = 1, 2,$$

which confirms Lemma 4.5. □

Theorem 4.1. *If $a_1 - \xi_1 - c_1 > 0$, $a_2 - \xi_2 - c_2 > 0$ and $b_1b_2 - c_1c_2 > 0$, then system* (2.5) *admits a unique stationary distribution.*

Proof. To finish this proof, we will consider the following two steps.

Step 1: We first prove the existence of stationary Markov process.

It follows from Lemma 4.1 and Remark 4.1 that we only need to find a nonnegative C^2-function $\mathcal{W}(X(x_1, x_2, m_1, m_2, n_1, n_2))$ and a closed set $\mathcal{U} \subset \mathbb{R}^6_+$ such that $L\mathcal{W}(X) \leq -1$ on $X \in \mathbb{R}^6_+ \setminus \mathcal{U}$. Let $q > 1$ and

$$\begin{aligned}
\mathcal{W}(X) = \delta_1(-\ln x_1 + \frac{c_1}{\sigma_2}n_2) + \delta_2(-\ln x_2 + \frac{c_2}{\sigma_1}n_1)&\\
+ \sum_{i=1}^{2}\{\frac{1}{q}x_i^q + \frac{1}{2\sigma_i}n_i^2 - \frac{1}{\sigma_i}\ln n_i + \frac{1}{4\sigma_i}m_i^2 - \frac{1}{2\sigma_i}\ln m_i\},&
\end{aligned} \tag{4.18}$$

where $\delta_i = \frac{2}{\lambda_i}\max\{2, H_i\}, \lambda_i = a_i - \xi_i - c_i, i = 1, 2$, and the constants $H_i > 0$ will be given later. An application of Itô's formula and Lemma 3.1 gives

$$\begin{aligned} L\mathcal{W}(X) \le & -\delta_1\lambda_1 - b_1 x_1^{q+1} + [a_1 + \tfrac{1}{2}(q-1)(\alpha_{11}^2 + \alpha_{12}^2) + c_1]x_1^q + b_1\delta_1 x_1 - \tfrac{1}{n_1}\tfrac{x_1}{1+x_1} \\ & - \delta_2\lambda_2 - b_2 x_2^{q+1} + [a_2 + \tfrac{1}{2}(q-1)(\alpha_{21}^2 + \alpha_{22}^2) + c_2]x_2^q + b_2\delta_2 x_2 - \tfrac{1}{n_2}\tfrac{x_2}{1+x_2} \\ & - \tfrac{1}{4}m_1^2 - \tfrac{1}{4}m_2^2 - \tfrac{3}{4}n_1^2 - \tfrac{3}{4}n_2^2 - \tfrac{1}{2m_1}n_1 - \tfrac{1}{2m_2}n_2 + 5. \end{aligned} \tag{4.19}$$

Choose $\epsilon > 0$ sufficiently small such that

$$0 < \epsilon < \min\{\frac{\lambda_i}{4b_i}, [\frac{b_i}{2(H_3+6)}]^{\frac{1}{q+1}}, [\frac{1}{4(H_3+6)}]^{\frac{1}{4}}, [\frac{3}{4(H_3+6)}]^{\frac{1}{4}}, \frac{1}{2(H_3+6)}, \frac{\sqrt{1+4/(H_3+6)}-1}{2}\},$$

$i = 1, 2$, where the constant $H_3 > 0$ is supplied later. A bounded closed set is defined by

$$\mathcal{U}^\epsilon = \{X \in \mathbb{R}_+^6 | \epsilon \le x_i \le \tfrac{1}{\epsilon},\ \epsilon^3 \le m_i \le \tfrac{1}{\epsilon^2},\ \epsilon^2 \le n_i \le \tfrac{1}{\epsilon^2}\},\ i = 1, 2.$$

Assign

$$\begin{aligned} &\mathcal{U}_1^\epsilon = \{X \in \mathbb{R}_+^6 | 0 < x_1 < \epsilon\},\ \mathcal{U}_2^\epsilon = \{X \in \mathbb{R}_+^6 | 0 < x_2 < \epsilon\}, \\ &\mathcal{U}_3^\epsilon = \{X \in \mathbb{R}_+^6 | x_1 > \tfrac{1}{\epsilon}\},\ \mathcal{U}_4^\epsilon = \{X \in \mathbb{R}_+^6 | x_2 > \tfrac{1}{\epsilon}\}, \\ &\mathcal{U}_5^\epsilon = \{X \in \mathbb{R}_+^6 | m_1 > \tfrac{1}{\epsilon^2}\},\ \mathcal{U}_6^\epsilon = \{X \in \mathbb{R}_+^6 | m_2 > \tfrac{1}{\epsilon^2}\}, \\ &\mathcal{U}_7^\epsilon = \{X \in \mathbb{R}_+^6 | n_1 > \tfrac{1}{\epsilon^2}\},\ \mathcal{U}_8^\epsilon = \{X \in \mathbb{R}_+^6 | n_2 > \tfrac{1}{\epsilon^2}\}, \\ &\mathcal{U}_9^\epsilon = \{X \in \mathbb{R}_+^6 | 0 < m_1 < \epsilon^3, n_1 > \epsilon^2, n_2 > \epsilon^2\},\ \mathcal{U}_{10}^\epsilon = \{X \in \mathbb{R}_+^6 | 0 < m_2 < \epsilon^3, n_1 > \epsilon^2, n_2 > \epsilon^2\}, \\ &\mathcal{U}_{11}^\epsilon = \{X \in \mathbb{R}_+^6 | 0 < n_1 < \epsilon^2, x_1 > \epsilon, x_2 > \epsilon\},\ \mathcal{U}_{12}^\epsilon = \{X \in \mathbb{R}_+^6 | 0 < n_2 < \epsilon^2, x_1 > \epsilon, x_2 > \epsilon\}. \end{aligned}$$

To prove that $L\mathcal{W}(X) \le -1$ for $X \in \mathbb{R}_+^6 \setminus \mathcal{U}^\epsilon$, we will consider the following six cases.
Case 1. When $X \in \mathcal{U}_1^\epsilon$, one has from Eq (4.19) that

$$L\mathcal{W}(X) \le -\frac{1}{2}b_1 x_1^{q+1} - \frac{\delta_1\lambda_1}{4} - \frac{\delta_1\lambda_1}{4} + b_1\delta_1\epsilon - \frac{\delta_1\lambda_1}{2} + H_1,$$

where

$$\begin{aligned} H_1 = \sup_{(x_1,x_2)\in\mathbb{R}_+^2} \{ & -\tfrac{1}{2}b_1 x_1^{q+1} + [a_1 + \tfrac{1}{2}(q-1)(\alpha_{11}^2 + \alpha_{12}^2) + c_1]x_1^q \\ & - b_2 x_2^{q+1} + [a_2 + \tfrac{1}{2}(q-1)(\alpha_{21}^2 + \alpha_{22}^2) + c_2]x_2^q + b_2\delta_2 x_2 + 5\}. \end{aligned}$$

We have from $\delta_1 = \frac{2}{\lambda_1}\max\{2, H_1\}$ that $\delta_1\lambda_1/4 \ge 1$. Then

$$L\mathcal{W}(X) \le -\frac{1}{2}b_1 x_1^{q+1} - \frac{\delta_1\lambda_1}{4} \le -\frac{\delta_1\lambda_1}{4} \le -1.$$

Similarly, for $X \in \mathcal{U}_2^\epsilon$ and $\delta_2 = \frac{2}{\lambda_2}\max\{2, H_2\}$, one has

$$L\mathcal{W}(X) \le -\frac{1}{2}b_2 x_2^{q+1} - \frac{\delta_2\lambda_2}{4} - \frac{\delta_2\lambda_2}{4} + \delta_2 b_2\epsilon - \frac{\delta_2\lambda_2}{2} + H_2 \le -\frac{\delta_2\lambda_2}{4} \le -1,$$

where

$$\begin{aligned} H_2 = \sup_{(x_1,x_2)\in\mathbb{R}_+^2} \{ & -\tfrac{1}{2}b_2 x_2^{q+1} + [a_2 + \tfrac{1}{2}(q-1)(\alpha_{21}^2 + \alpha_{22}^2) + c_2]x_2^q \\ & - b_1 x_1^{q+1} + [a_1 + \tfrac{1}{2}(q-1)(\alpha_{11}^2 + \alpha_{12}^2) + c_1]x_1^q + b_1\delta_1 x_1 + 5\}. \end{aligned}$$

To discuss the following *Cases 2-6*, we reconsider Eq (4.19) and obtain

$$\begin{aligned} \mathcal{LW}(X) \leq & -\frac{b_1}{2}x_1^{q+1} - \frac{1}{n_1}\frac{x_1}{1+x_1} - \frac{1}{4}m_1^2 - \frac{3}{4}n_1^2 - \frac{n_1}{2m_1} \\ & - \frac{b_2}{2}x_2^{q+1} - \frac{1}{n_2}\frac{x_2}{1+x_2} - \frac{1}{4}m_2^2 - \frac{3}{4}n_2^2 - \frac{n_2}{2m_2} + H_3 + 5, \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} H_3 = \sup_{(x_1,x_2)\in\mathbb{R}_+^2} & \{-\frac{1}{2}b_1x_1^{q+1} + [a_1 + \frac{1}{2}(q-1)(\alpha_{11}^2 + \alpha_{12}^2) + c_1]x_1^q + b_1\delta_1x_1 \\ & - \frac{1}{2}b_2x_2^{q+1} + [a_2 + \frac{1}{2}(q-1)(\alpha_{21}^2 + \alpha_{22}^2) + c_2]x_2^q + b_2\delta_2x_2\}. \end{aligned}$$

Case 2. When $X \in \mathcal{U}_3^\epsilon$, it follows from Eq (4.20) that $\mathcal{LW}(X) \leq H_3 + 5 - \frac{1}{2}b_1\epsilon^{-(q+1)} \leq -1$. Similarly, if $X \in \mathcal{U}_4^\epsilon$, then $\mathcal{LW}(X) \leq H_3 + 5 - \frac{1}{2}b_2\epsilon^{-(q+1)} \leq -1$.
Case 3. When $X \in \mathcal{U}_5^\epsilon$ and $X \in \mathcal{U}_6^\epsilon$, one has $\mathcal{LW}(X) \leq H_3 + 5 - \frac{m_i^2}{4} < H_3 + 5 - \frac{1}{4\epsilon^4} \leq -1$.
Case 4. When $X \in \mathcal{U}_7^\epsilon$ and $X \in \mathcal{U}_8^\epsilon$, then $\mathcal{LW}(X) \leq H_3 + 5 - \frac{3n_i^2}{4} < H_3 + 5 - \frac{3}{4\epsilon^4} \leq -1$.
Case 5. When $X \in \mathcal{U}_9^\epsilon$ and $X \in \mathcal{U}_{10}^\epsilon$, we get $\mathcal{LW}(X) \leq H_3 + 5 - \frac{1}{2m_i}n_i < H_3 + 5 - \frac{1}{2\epsilon^3}\epsilon^2 \leq -1$.
Case 6. When $X \in \mathcal{U}_{11}^\epsilon$ and $X \in \mathcal{U}_{12}^\epsilon$, then $\mathcal{LW}(X) \leq H_3 + 5 - \frac{1}{n_i}\frac{x_i}{1+x_i} < H_3 + 5 - \frac{1}{\epsilon^2}\frac{\epsilon}{1+\epsilon} \leq -1$.

From the above discussions we know the closed set $\mathcal{U}^\epsilon$ satisfying $\sup_{X\in\mathbb{R}_+^6\setminus\mathcal{U}^\epsilon} \mathcal{LW}(X) \leq -1$.

Step 2: When $b_1b_2 - c_1c_2 > 0$, we know from Lemma 4.5 that the solution $X(t)$ is globally attractive.

Combining **Step 1** and **Step 2**, we complete the proof of Theorem 4.1. □

5. Numerical simulations

In this section, we will employ several specific examples to simulate the solutions to system (2.5), and verify the analytical results of the previous section.

For system (2.5), we first fix the parameter values as follows: $a_1 = 0.295$, $a_2 = 0.3$, $b_1 = 0.75$, $b_2 = 0.65$, $c_1 = 0.05$, $c_2 = 0.05$, $\sigma_1 = 0.1$, $\sigma_2 = 0.2$ and initial value $x_1(0) = 0.1$ and $x_2(0) = 0.12$. We will reveal how two coupling noise sources influence the long-time behaviors by choosing different noise intensities α_{11}^2, α_{12}^2, α_{21}^2 and α_{22}^2.

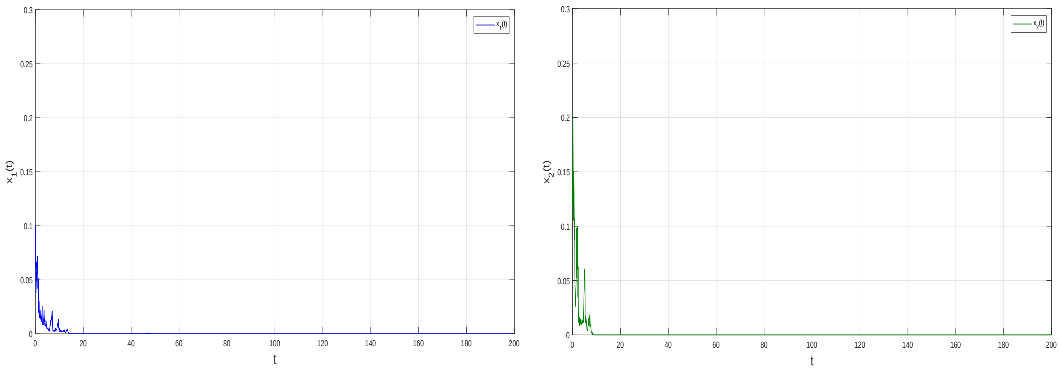

Figure 1. Species x_1 and x_2 are extinct exponentially in system (2.5).

Choose $\alpha_{11}^2 = 0.65^2$, $\alpha_{12}^2 = 0.75^2$, $\alpha_{21}^2 = 0.64^2$, $\alpha_{22}^2 = 0.74^2$. A calculation gives that $a_1 + c_1 = 0.3 < \xi_1 = 0.4925$ and $a_2 + c_2 = 0.35 < \xi_2 = 0.4786$, which surely satisfies Theorem 3.1. Thus, both species will be EE (see Figure 1).

Choose $\alpha_{11}^2 = 0.65^2$, $\alpha_{12}^2 = 0.75^2$, $\alpha_{21}^2 = 0.03^2$, $\alpha_{22}^2 = 0.02^2$. By calculating we have $a_1 + c_1 = 0.3 < \xi_1 = 0.4925$ and $a_2 = 0.3 > \xi_2 = 0.00065$, it then follows from Theorem 3.2 that species x_1 is EE while x_2 is PM (see Figure 2).

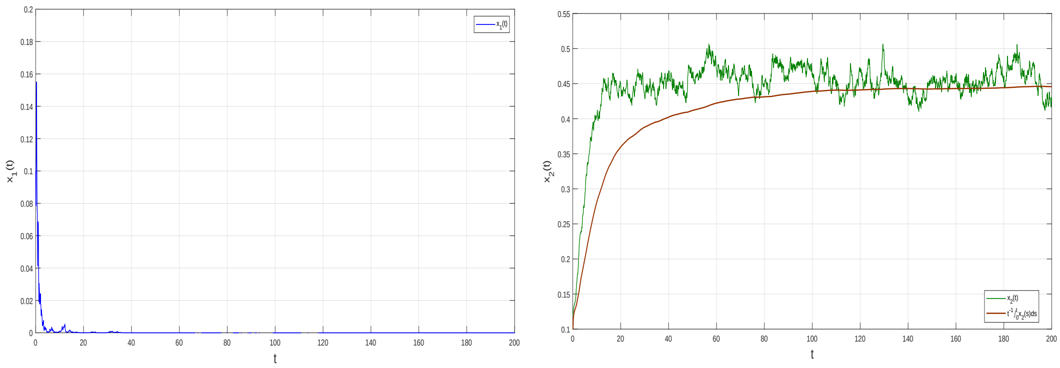

Figure 2. Species x_1 is extinct exponentially while x_2 is persistence in the mean in system (2.5).

Choose $\alpha_{11}^2 = 0.03^2$, $\alpha_{12}^2 = 0.02^2$, $\alpha_{21}^2 = 0.65^2$, $\alpha_{22}^2 = 0.75^2$. Since $a_1 = 0.295 > \xi_1 = 0.00065$ and $a_2 + c_2 = 0.35 < \xi_2 = 0.4925$, Theorem 3.3 indicates that species x_1 is PM while x_2 is EE (see Figure 3).

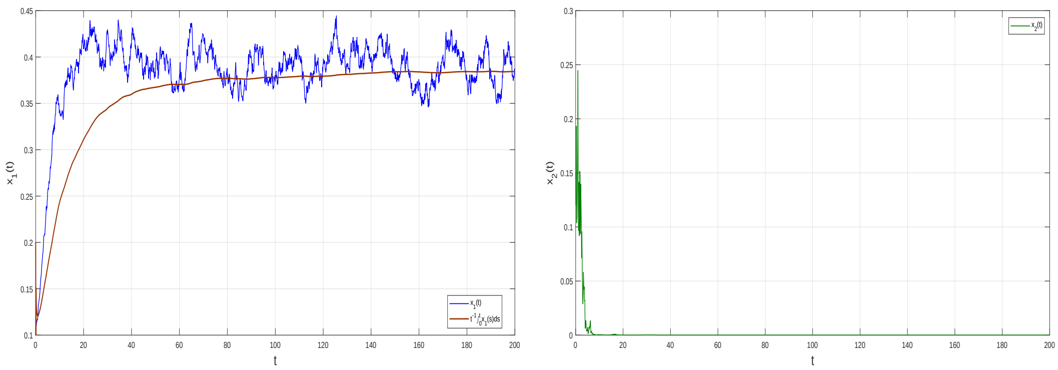

Figure 3. Species x_1 is persistence in the mean while x_2 is extinct exponentially in system (2.5).

Choose $\alpha_{11}^2 = 0.03^2$, $\alpha_{12}^2 = 0.02^2$, $\alpha_{21}^2 = 0.02^2$, $\alpha_{22}^2 = 0.01^2$. Together with Theorem 3.4, we have from $a_1 = 0.295 > \xi_1 = 0.00065$ and $a_2 = 0.3 > \xi_2 = 0.00025$ that both species are PTA (see Figure 4).

Furthermore, taking the same noise intensities as in Figure 4, a calculation shows that $a_1 - \xi_1 - c_1 = 0.24435 > 0$, $a_2 - \xi_2 - c_2 = 0.24975 > 0$ and $b_1 b_2 - c_1 c_2 = 0.485 > 0$. So we know, by Theorem 4.1, that system (2.5) owns a unique stationary distribution (see Figure 5).

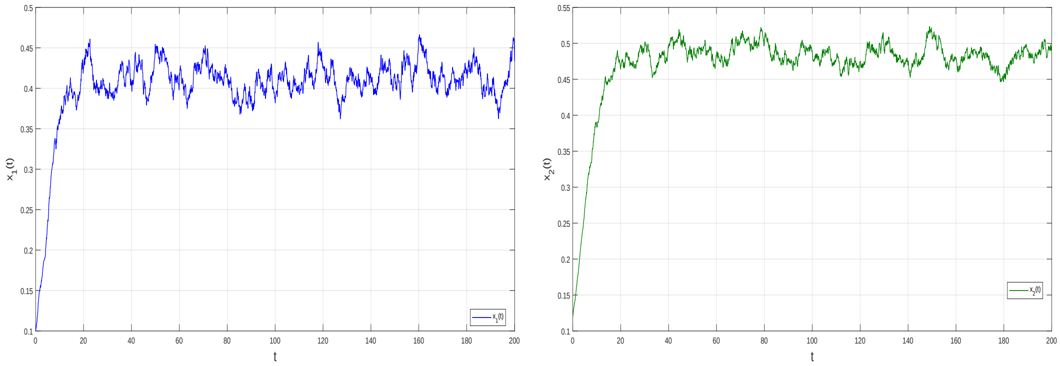

Figure 4. Species x_1 and x_2 are permanent in time average in system (2.5).

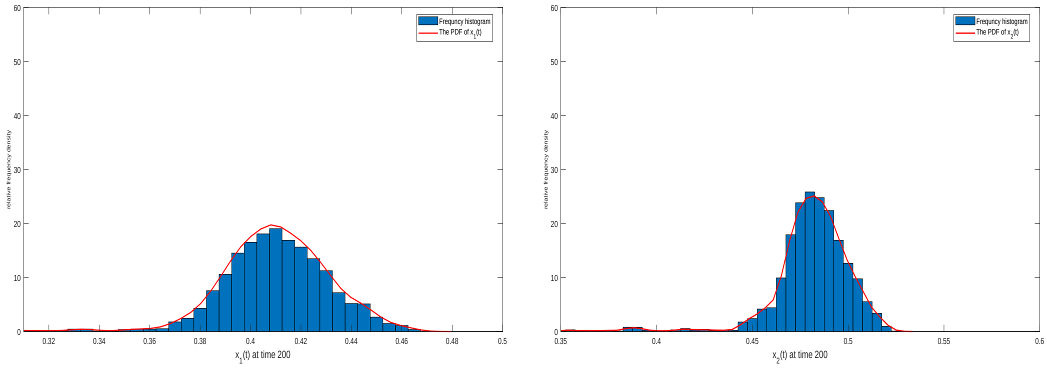

Figure 5. *Blue bar* frequency histogram of system (2.5) at time 200; *Red line* the probability density function (PDF) of its corresponding stationary distribution simulated by 2500 sample trajectories.

6. Discussion

This paper is concerned with a stochastic facultative mutualism model with saturation effect and distributed delays, in which strong kernel functions are incorporated (see model (2.3)). By analyzing a corresponding equivalent system (2.5), a set of easily verifiable sufficient conditions for the survival results and stationary distribution of system (2.5) is established. Note that $\xi_1 = 0.5(\alpha_{11}^2 + \alpha_{12}^2)$ and $\xi_2 = 0.5(\alpha_{21}^2 + \alpha_{22}^2)$ and from the above theoretical results, we have the following conclusions:

- Theorems 3.1–3.4 imply that large coupling noise intensities are harmful for the survival of both species while suitably small coupling noise intensities are advantage for them (see Figures 1–4).
- It follows from Theorems 3.2 and 3.3 that if the intrinsic growth rate of one species is small, the coupling noise intensities are large amplitude and the cooperation from the other species is not enough, then one species will be extinct exponentially (see Figures 2–3). However, if the other species only owns large intrinsic growth rate and relatively small coupling noise intensities, then it will be persistent in the mean (see Figures 2–3).
- Compared with the conditions of Theorems 3.1, 3.4 and 4.1, that is, $a_i + c_i < \xi_i$, $\xi_i < a_i$ and $\xi_i < a_i - c_i$ $(i = 1, 2)$, we find that the intrinsic growth rates a_i are larger and larger while the coupling noise intensity ξ_i is smaller and smaller, then both species go from exponent extinction

(see Figure 1) to permanence in time average (see Figure 4) to the existence of a unique stationary distribution (see Figure 5). In addition, $b_1b_2 - c_1c_2 > 0$ reveals that the effect of interspecific mutualism is less than that of intra-specific competition.

Acknowledgments

The authors thank the editor and referees for their careful reading and valuable comments. The work is supported by the College Innovation Team Project of Hubei Provincial Department of Education (No.T201812).

Conflict of interest

The authors declare there is no conflict of interest.

References

1. E. P. Odum, *Fundamental of Ecology*, 3rd edition, Saunders, Philadelphia, 1971.
2. D. H. Janzen, Coevolution of mutualism between ants and acacias in Central America, *Evolution*, **20** (1966), 249–275.
3. K. G. Poter, Enhancement of algal growth and productivity by grazing zooplankton, *Science*, **192** (1976), 1332–1334.
4. B. S. Goh, Stability in models of mutualism, *Amer. Natur.*, **113** (1979), 261–275.
5. X. Z. He, K. Gopalsamy, Persistence, attractivity, and delay in facultative mutualism, *J. Math. Anal. Appl.*, **215** (1997), 154–173.
6. H. L. Smith, On the asymptotic behavior of a class of deterministic models of cooperating species, *SIAM J. Appl. Math.*, **46** (1986), 368–375.
7. D. Mukherjee, Permanence and global attractivity for facultative mutualism system with delay, *Math. Methods Appl. Sci.*, **26** (2003), 1–9.
8. A. A. S. Zaghrout, Stability and persistence of facultative mutualism with populations interacting in a food chain, *Appl. Math. Comput.*, **45** (1991), 1–15.
9. R. Kumar, H. I. Freedman, A mathematical model of facultative mutualism with population interacting in a food chain, *Math. Biosci.*, **97** (1989), 235–261.
10. M. X. Liao, C. J. Xu, X. H. Tang, Stability and Hopf bifurcation for a competition and cooperation model of two enterprises with delay, *Commun. Nonlinear Sci. Numer. Simul.*, **19** (2014), 3845–3856.
11. D. Maxin, P. Georgescu, L. Sega, L. Berec, Global stability of the coexistence equilibrium for a general class of models of facultative mutualism, *J. Biol. Dyn.*, **11** (2017), 339–364.
12. P. Georgescu, H. Zhang, D. Martin, The global stability of coexisting equilibria for three models of mutualism, *Math. Biosci. Eng.*, **13** (2016), 101–118.
13. G. C. Lu, Z. Y. Lu, X. Z. Lian, Delay effect on the permanence for Lotka-Volterra cooperative systems, *Nonlinear Anal. Real World Appl.*, **11** (2010), 2810–2816.
14. Y. Nakata, Y. Muroya, Permanence for nonautonomous Lotka-Volterra cooperative systems with delays, *Nonlinear Anal. Real World Appl.*, **11** (2010), 528–534.

15. C. Y. Ji, D. Q. Jiang, Persistence and non-persistence of a mutualism system with stochastic perturbation, *Discrete Contin. Dyn. Syst.*, **32** (2012), 867–889.
16. M. Li, H. J. Gao, C. F. Sun, Y. Z. Gong, Analysis of a mutualism model with stochastic perturbations, *Int. J. Biomath.*, **8** (2015), 1550072.
17. Q. Liu, Q. M. Chen, Y. Y. Hu, Analysis of a stochastic mutualism model, *Commun. Nonlinear Sci. Numer. Simul.*, **29** (2015), 188–197.
18. G. Y. Lv, B. B. Zhang, Permanence and extinction of stochastic regime-switching mutualism model, *Int. J. Biomath.*, **13** (2020), 2050028.
19. W. X. Ning, Z. J. Liu, L. W. Wang, R. H. Tan, A stochastic mutualism model with saturation effect and impulsive toxicant input in a polluted environment, *J. Appl. Math. Comput.*, **65** (2021), 177–197.
20. Y. H. Yang, S. D. Huang, Permanence and almost periodic solution of two-species delayed Lotka-Volterra cooperative systems with impulsive perturbations, *Internat. J. Control*, **89** (2016), 2492–2506.
21. D. Ye, M. Fan, Periodicity in mutualism systems with impulse, *Taiwanese J. Math.*, **10** (2006), 723–737.
22. Z. J. Liu, R. H. Tan, P. Y. Chen, L. S. Chen, On the stable periodic solutions of a delayed two-species of facultative mutualism model, *Appl. Math. Comput.*, **196** (2008), 105–117.
23. Z. J. Liu, J. H. Wu, R. H. Tan, Y. P. Chen, Modeling and analysis of a periodic delayed two-species model of facultative mutualism, *Appl. Math. Comput.*, **217** (2010), 893–903.
24. C. J. Xu, P. L. Li, Almost periodic solutions for a competition and cooperative model of two enterprises with time-varying delays and feedback controls, *J. Appl. Math. Comput.*, **53** (2017), 397–411.
25. M. Liu, C. Z. Bai, Optimal harvesting of a stochastic mutualism model with regime-switching, *Appl. Math. Comput.*, **373** (2020), 125040.
26. M. Liu, C. Z. Bai, Optimal harvesting of a stochastic mutualism model with Lévy jumps, *Appl. Math. Comput.*, **276** (2016), 301–309.
27. G. D. Liu, H. K. Qi, Z. B. Chang, X. Z. Meng, Asymptotic stability of a stochastic May mutualism system, *Comput. Math. Appl.*, **79** (2020), 735–745.
28. X. H. Zhang, D. Q. Jiang, A. Alsaedi, T. Hayat, Periodic solutions and stationary distribution of mutualism models in random environments, *Phys. A*, **460** (2016), 270–282.
29. K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, Kluwer Academic, Dordrecht, 1992.
30. V. Volterra, Remarques sur la note de M.Régnier et Mlle. Lambin (Étude d'un cas d'antagonisme microbien), *C. R. Acad. Sci.*, **199** (1934), 1684–1686.
31. J. M. Cushing, Integrodifferential equations and delay models in population dynamics, in *Lecture Notes in Biomathematics*, Springer-Verlag, Berlin, 1977.
32. N. Macdonald, Time lags in biological models, in *Lecture Notes in Biomathematics*, Springer-Verlag, Berlin, 1978.
33. C. Lu, Q. Ma, Analysis of a stochastic Lotka-Volterra competitive model with infinite delay and impulsive perturbations, *Taiwanese J. Math.*, **21** (2017), 1413–1436.

34. J. Hu, Z. J. Liu, L. W. Wang, R. H. Tan, Extinction and stationary distribution of a competition system with distributed delays and higher order coupled noises, *Math. Biosci. Eng.*, **17** (2020), 3240–3251.
35. N. Wang, M. A. Han, Relaxation oscillations in predator-prey model with distributed delay, *Comput. Appl. Math.*, **37** (2018), 475–484.
36. W. J. Zuo, D. Q. Jiang, X. G. Sun, T. Hayat, A. Alsaedi, Long-time behaviors of a stochastic cooperative Lotka-Volterra system with distributed delay, *Phys. A*, **506** (2018), 542–559.
37. X. G. Sun, W. J. Zuo, D. Q. Jiang, T. Hayat, Unique stationary distribution and ergodicity of a stochastic logistic model with distributed delay, *Phys. A*, **512** (2018), 864–881.
38. Q. Liu, D. Q. Jiang, Stationary distribution and extinction of a stochastic predator-prey model with distributed delay, *Appl. Math. Lett.*, **78** (2018), 79–87.
39. Q. Liu, D. Q. Jiang, T. Hayat, A. Alsaedi, Dynamical behavior of stochastic predator-prey models with distributed delay and general functional response, *Stoch. Anal. Appl.*, **38** (2020), 403–426.
40. W. X. Ning, Z. J. Liu, L. W. Wang, R. H. Tan, Analysis of a stochastic competitive model with saturation effect and distributed delay, *Methodol. Comput. Appl. Probab.*, 2020.
41. S. G. Ruan, Delay differential equations in single species dynamics, in *Delay Differential Equations and Applications* (eds. Arino, O., et al.), Springer, New York, 2006.
42. R. M. Nisbet, W. S. C. Gurney, *Modelling Fluctuating Populations*, Wiley, New York, 1982.
43. R. M. May, *Stability and Complexity in Model Ecosystem*, Princeton University Press, Princeton, NJ, 2001.
44. J. Hu, Z. J. Liu, Incorporating coupling noises into a nonlinear competitive system with saturation effect, *Int. J. Biomath.*, **13** (2020), 2050012.
45. Q. Li, Z. J. Liu, S. L. Yuan, Cross-diffusion induced Turing instability for a competition model with saturation effect, *Appl. Math. Comput.*, **347** (2019), 64–77.
46. A. Bahar, X. R. Mao, Stochastic delay Lotka-Volterra model, *J. Math. Anal. App.*, **292** (2004), 364–380.
47. X. R. Mao, *Stochastic Differential Equations and Applications*, Horwood, Chichester, 2007.
48. M. Liu, K. Wang, Survival analysis of stochastic competitive models in a polluted environment and stochastic competitive exclusion principle, *Bull. Math. Biol.*, **73** (2011), 1969–2012.
49. R. Z. Khas'minskii, *Stochastic Stability of Differential Equations*, Sijthoff Noordhoff, Alphen aan den Rijn, Netherlands, 1980.
50. D. Y. Xu, Y. M. Huang, Z. G. Yang, Existence theorems for periodic Markov process and stochastic functional differential equations, *Discrete Contin. Dyn. Syst.*, **24** (2009), 1005–1023.
51. I. Barhalat, Systems d'equations differential d'oscillations nonlinearies, *Rev. Roum. Math. Pures Appl.*, **4** (1959), 267–270.

AIMS Press