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Research article

Bayesian and maximum likelihood estimations of the Dagum parameters under combined-unified hybrid censoring

Walid Emam and Khalaf S. Sultan*

Department of Statistics and Operations Research, College of Science, King Saud University, P.O.Box 2455, Riyadh 11451, Saudi Arabia

* **Correspondence:** Email: ksultan@ksu.edu.sa; Tel: +966-1-467-6263; Fax: +966-1-467-6274.

Abstract: In this paper, we introduce a new form of hybrid censoring sample, that is called COMBINED-UNIFIED (C-U) hybrid sample. In this unified approach, we merge the combined hybrid censoring sampling that considered by Huang and Yang [1] and unified hybrid censoring sampling that considered by Balakrishnan et al. [2]. We apply the C-U hybrid censoring sampling to develop estimation procedures of the unknown parameters of Dagum distribution. The maximum likelihood method is used to estimate the unknown parameters and the asymptotic confidence intervals as well as the bootstrap confidence intervals are obtained. Also, we develop the Bayesian estimation of the unknown parameters of Dagum distribution under the squared error and linear-exponential (LINEX) loss functions. Since the closed forms of the Bayesian estimators are not available, so we encounter some computational difficulties to evaluate the Bayes estimates of the parameters involved in the model such as Tierney and Kadanes procedure as well as Markov Chain Monte Carlo (MCMC) procedure to compute approximate Bayes estimates. In addition, we show the usefulness of the theoretical findings thought some simulation experiments. Finally, a real data set have been analyzed for illustrative purposes of our results

Keywords: maximum likelihood estimator; Bayes estimator; squared error loss functions; LINEX loss functions; Dagum distribution; combined hybrid censored data; unified hybrid censored data

1. Introduction

Type-I and Type-II censoring schemes are the two most popular censoring schemes which are used in practice. The mixture of Type-I and Type-II censoring schemes has been discussed in the literature for this purpose, is known as the hybrid censoring scheme which was first introduced by Epstein [3]. The hybrid censoring scheme becomes quite popular in the reliability and life-testing experiments, see for example, Fairbanks et al. [4], Draper and Guttman [5], Chen and Bhattacharya [6], Jeong et al. [7], Childs et al. [8] and Gupta and Kundu [9]. Balakrishnan and Kundu [10] have extensively reviewed and discussed Type-I and Type-II hybrid censoring schemes and associated inferential issues. They have presented details on developments of generalized hybrid censoring and unified hybrid censoring schemes that have been introduced in the literature and they have presented several examples to illustrate the described results. From now on, we refer to this hybrid censoring scheme as Type-I hybrid censoring scheme (Type-I HCS). It is evident that the complete sample situation as well as Type-I and Type-II right censoring schemes are all special cases of this Type-I HCS.

Recently, Tripathic and Lodhi [11] have discussed inferential procedures for Weibull competing risks model with partially observed failure causes under generalized progressive hybrid censoring. Jeon and Kang [12] have estimated the half-logistic distribution based on multiply Type-II hybrid censoring. Nassar and Dobbah [13] have analyzed the reliability characteristics of bathtub-shaped distribution under adaptive Type-I progressive hybrid censoring. Algarni, Almarashi and Abd-Elmougoud [14] have considered the joint Type-I generalized hybrid censoring for estimation two Weibull distributions.

A three parameter Dagum distribution was proposed by Dagum [15, 16], which plays are important role in size distribution of personal income. This distribution offers a more flexible for modeling lifetime data, such as in reliability. The Dagum distribution is not much popular, perhaps, because of its difficult mathematical procedures. In the 1970s, Camilo Dagum embarked on a quest for a statistical distribution closely fitting empirical income and wealth distributions. Not satisfied with the classical distributions, he looked for a model accommodating the heavy tails present in empirical income and wealth distributions as well as permitting an interior mode. He end up with Dagum Type I distribution, a three-parameter distribution, and two four parameter generalizations see Dagum [16–18]. Dagum distribution is also called the inverse Burr, especially in the actuarial literature, as it is the reciprocal transformation of the Burr XII. Nevertheless, unlike the Burr XII, which is widely known in various fields of science. Since Dagum proposed his model as income distribution, its properties have been appreciated in economics and financial fields and its features have been extensively discussed in the studies of income and wealth. Kleiber and Kotz [19] and Kleiber [20] provided an exhaustive review on the origin of the Dagum model and its applications. Contributions from Quintano and D'Agostino [21] adjusted Dagum model for income distribution to account for individual characteristics, while Domma et al. [22, 23] studied the Fisher information matrix in doubly censored data from the Dagum distribution and reliability studies of the Dagum distribution. An important characteristic of Dagum distribution is that its hazard function can be monotonically decreasing, an upside-down bathtub, or bathtub and then upside-down bathtub shaped, see Domma [24]. This behavior has led several authors to study the model in different fields. In fact, Dagum distribution has been studied from a reliability point of view and used to analyze survival data, see Domma, et al. [23].

Dagum distribution specified by the probability density function (pdf)

$$f(x; \lambda, \beta, \theta) = \lambda\beta\theta x^{-(\beta+1)} \left(1 + \lambda x^{-\beta}\right)^{-(\theta+1)}, \quad x > 0; \quad \lambda, \beta, \theta > 0, \tag{1.1}$$

and cumulative distribution function (cdf)

$$F(x; \lambda, \beta, \theta) = \left(1 + \lambda x^{-\beta}\right)^{-\theta}, \quad x > 0; \quad \lambda, \beta, \theta > 0, \tag{1.2}$$

where λ is the scale parameter and β, θ are the shape parameters.

Huang and Yang [1] have considered a combined hybrid censoring sampling (CHCS) scheme which define as follows: For fixed m, $r \in \{1, 2, \ldots, n\}$, $(T_1, T_2) \in (0, \infty)$ such that $m < r$, $T_1 < T_2$ and T^*

denote the terminating time of the experiment. If the kth failure occurs before time T_1, the experiment terminates at $\min\{X_{r:n}, T_1\}$, if the mth failure occurs between T_1 and T_2, the experiment is terminated at $X_{m:n}$ and finally if the mth failure occurs after time T_2, then the experiment terminates at T_2. For our later convenience, we abbreviate this scheme as combined CHCS$(m, r; T_1, T_2)$. In fact, this system contains the following six cases, and obviously, in each case some part of data are unobservable as:

$$T^* = \begin{cases} X_{m:n}, & 0 < T_1 < X_{m:n} < (T_2 < X_{r:n}), \\ X_{m:n}, & 0 < T_1 < X_{m:n} < (X_{r:n} < T_2), \\ T_2, & 0 < T_1 < T_2 < (X_{m:n} < X_{r:n}), \\ X_{r:n}, & 0 < X_{m:n} < X_{r:n} < (T_1 < T_2), \\ T_1, & 0 < X_{m:n} < T_1 < (X_{r:n} < T_2), \\ T_1, & 0 < X_{m:n} < T_1 < (T_2 < X_{r:n}), \end{cases} \tag{1.3}$$

where the data in parentheses are unobservable.

Balakrishnan, et al. [2] have proposed an unified hybrid censoring sampling (UHCS) scheme as follows: For fixed m, $r \in \{1, 2, \ldots, n\}$, $(T_1, T_2) \in (0, \infty)$, $m < r$, $T_1 < T_2$ and T^* denote the terminating time of the experiment. If the kth failure occurs before time T_1, the experiment terminates at $min\{max\{X_{r:n}, T_1\}, T_2\}$, if the mth failure occurs between T_1 and T_2, the experiment is terminated at $min\{X_{r:n}, T_2\}$ and finally if the mth failure occurs after time T_2, then the experiment terminates at $X_{m:n}$. Again, for our later convenience, we abbreviate this scheme as UHCS$(m, r; T_1, T_2)$. Similarly, each type of these hybrid censored samples contains six cases, and obviously, in each case some part of data are unobservable as following:

$$T^* = \begin{cases} T_2, & 0 < T_1 < X_{m:n} < T_2 < (X_{r:n}), \\ X_{r:n}, & 0 < T_1 < X_{m:n} < X_{r:n} < (T_2), \\ X_{m:n}, & 0 < T_1 < T_2 < X_{m:n} < (X_{r:n}), \\ T_1, & 0 < X_{m:n} < X_{r:n} < T_1 < (T_2), \\ X_{r:n}, & 0 < X_{m:n} < T_1 < X_{r:n} < (T_2), \\ T_2, & 0 < X_{m:n} < T_1 < T_2 < (X_{r:n}), \end{cases} \tag{1.4}$$

where the data in parentheses are unobservable.

In this paper, we merge CHCS$(m, r; T_1, T_2)$ and UHCS$(m, r; T_1, T_2)$, in a unified approach known as a combined-unified hybrid censored scheme (C-UHCS$(m, r; T_1, T_2)$). To the best of our knowledge, no attempt has been made on estimation of the parameters of the Dagum distribution using CHCS$(m, r; T_1, T_2)$ or UHCS$(m, r; T_1, T_2)$, so, we apply C-UHCS$(m, r; T_1, T_2)$ to Dagum distribution. We first obtain the maximum likelihood estimate of the parameters and use them to construct asymptotic and bootstrap confidence intervals (CIs). Next, we obtain the Bayes estimates of λ, β and θ. The layout of this paper as follows. In Section 2, we first describe the construction of likelihood function based C-UHCS$(m, r; T_1, T_2)$ and obtain the MLEs of λ, β and θ. The asymptotic and bootstrap confidence intervals based on the observed Fisher information matrix is also discussed here. Next, Section 3, we consider Bayesian estimation of the unknown parameters under squared error and LINEX loss functions. Simulation studies are carried out in Section 4 to assess the performance of the proposed methods. Section 5 contains a brief conclusion.

2. Likelihood function under C-UHCS

Let $X_1, X_2, ..., X_n$ denote a sequence of the lifetimes of reliability experiment units that placed on a life-test, we shall assume that these variables of this sample is iid from an absolutely continuous population with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. In this section, we construct the likelihood function under the censoring scheme C-UHCS$(m, r; T_1, T_2)$. Let D_j denote the maximum number of failures until T_j, $j = 1, 2$, obviously have $D_1 \leq D_2$. Then, the likelihood function of CHCS$(m, r; T_1, T_2)$, for a parameter space Ω, is given as

$$L^{(C)}(\Omega|\mathbf{x}) = \begin{cases} \dfrac{n!}{(n-m)!}[1-F(x_m)]^{n-m}\displaystyle\prod_{i=1}^{m} f(x_i); & D_1 = 0, \cdots, m-1, D_2 = m, \\ \dfrac{n!}{(n-D_2)!}[1-F(T_2)]^{n-D_2}\displaystyle\prod_{i=1}^{D_2} f(x_i); & D_1, D_2 = 0, \cdots, m-1, \\ \dfrac{n!}{(n-r)!}[1-F(x_r)]^{n-r}\displaystyle\prod_{i=1}^{r} f(x_i); & D_1 = D_2 = r, \\ \dfrac{n!}{(n-D_1)!}[1-F(T_1)]^{n-D_1}\displaystyle\prod_{i=1}^{D_1} f(x_i); & D_1 = D_2 = m, \cdots, r-1. \end{cases} \tag{2.1}$$

Similarly, the observed likelihood function based on UHCS$(m, r; T_1, T_2)$ is given as

$$L^{(U)}(\Omega|\mathbf{x}) = \begin{cases} \dfrac{n!}{(n-D)!}[1-F(T_1)]^{n-D}\displaystyle\prod_{i=1}^{m} f(x_i); & D_1 = D_2 = D = r, \cdots, n, \\ \dfrac{n!}{(n-r)!}[1-F(x_r)]^{n-r}\displaystyle\prod_{i=1}^{r} f(x_i); & D_1 = m, \cdots, r-1, D_2 = r, \\ \dfrac{n!}{(n-D_2)!}[1-F(T_2)]^{n-D_2}\displaystyle\prod_{i=1}^{D_2} f(x_i); & D_1, D_2 = m, \cdots, r-1, \\ \dfrac{n!}{(n-r)!}[1-F(x_r)]^{n-r}\displaystyle\prod_{i=1}^{r} f(x_i); & D_1 = 0, \cdots, m-1, D_2 = r, \\ \dfrac{n!}{(n-D_2)!}[1-F(T_2)]^{n-D_2}\displaystyle\prod_{i=1}^{D_2} f(x_i); D_1 = 0..m-1, D_2 = m...r-1, \\ \dfrac{n!}{(n-m)!}[1-F(x_m)]^{n-m}\displaystyle\prod_{i=1}^{m} f(x_i); D_1, D_2 = 0, \cdots, m-1. \end{cases} \tag{2.2}$$

Assume that, for any case, we terminate the experiment at T that may refer to time T_1, T_2, observation x_k or observation x_r, and let k denote the maximum number of failures until T which equal, respectively, D_1, D_2, k and r. The likelihood function of C-UHCS$(k, r; T_1, T_2)$, that represents all previous likelihood functions $L^{(C)}(\Omega|\mathbf{x})$ and $L^{(U)}(\Omega|\mathbf{x})$ under different values of k, T and $\mathbf{x}_k = (x_1, x_2, ..., x_k)$ can be written as

$$L(\Omega|\mathbf{x}_k) = \frac{n!}{(n-k)!}[1-F(T)]^{n-k}\prod_{i=1}^{k} f(x_i), \tag{2.3}$$

where k and T can be chosen as:

Cases	$L^{(C)}(\Omega\|\mathbf{x})$ k	T	$L^{(U)}(\Omega\|\mathbf{x})$ k	T
$1:\ 0 < T_1 < X_{k:n} < T_2 < X_{r:n}$	m	$X_{m:n}$	D_2	T_2
$2:\ 0 < T_1 < X_{k:n} < X_{r:n} < T_2$	m	$X_{m:n}$	r	$X_{r:n}$
$3:\ 0 < T_1 < T_2 < X_{k:n} < X_{r:n}$	D_2	T_2	m	$X_{m:n}$
$4:\ 0 < X_{k:n} < X_{r:n} < T_1 < T_2$	r	$X_{r:n}$	D_1	T_1
$5:\ 0 < X_{k:n} < T_1 < X_{r:n} < T_2$	D_1	T_1	r	$X_{r:n}$
$6:\ 0 < X_{k:n} < T_1 < T_2 < X_{r:n}$	D_1	T_1	D_2	T_2

3. Maximum likelihood estimation

Suppose that $\{x_1, x_2, ..., x_k\}$ be a sequence of observed data from Dagum distribution. Substituting (1.1) and (1.2) in (2.3), the observed likelihood function of λ, β and θ based on these C-UHCS$(k, r; T_1, T_2)$ becomes

$$L(\lambda, \beta, \theta|\mathbf{x}_k) = \frac{n!}{(n-k)!}\lambda^k\beta^k\theta^k\left[1-\left(1+\lambda T^{-\beta}\right)^{-\theta}\right]^{n-k}\prod_{i=1}^{k} x_i^{-(\beta+1)}\left(1+\lambda x_i^{-\beta}\right)^{-(\theta+1)}, \tag{3.1}$$

and the corresponding log-likelihood function (Ł) is

$$\begin{aligned} Ł &= \log L(\lambda, \beta, \theta|\mathbf{x}_k) \\ &= \log\frac{n!}{(n-k)!} + k\log\lambda + k\log\beta + k\log\theta + (n-k)\log\left[1-\left(1+\lambda T^{-\beta}\right)^{-\theta}\right] \\ &- \sum_{i=1}^{k}\left[(\beta+1)\log x_i + (\theta+1)\log\left(1+\lambda x_i^{-\beta}\right)\right]. \end{aligned} \tag{3.2}$$

Taking the first partial derivatives of log-likelihood (3.2) with respect to λ, β, θ and equating each to zero. Let $M_a = (1+\lambda T^{-\beta})^{-(\theta+a)}$, we obtain

$$\frac{\partial Ł}{\partial\lambda} = \frac{k}{\lambda} + \frac{(n-k)\theta T^{-\beta}M_1}{1-M_0} - \sum_{i=1}^{k}\frac{(\theta+1)x_i^{-\beta}}{1+\lambda x_i^{-\beta}} = 0, \tag{3.3}$$

$$\frac{\partial Ł}{\partial\beta} = \frac{k}{\beta} - \frac{(n-k)\theta\lambda T^{-\beta}M_1\log T}{1-M_0} - \sum_{i=1}^{k}\left[\log x_i - \frac{(\theta+1)\lambda x_i^{-\beta}\log x_i}{1+\lambda x_i^{-\beta}}\right] = 0, \tag{3.4}$$

$$\frac{\partial Ł}{\partial\theta} = \frac{k}{\theta} + \frac{(n-k)M_0\log\left(1+\lambda T^{-\beta}\right)}{1-M_0} - \sum_{i=1}^{k}\log(1+\lambda x_i^{-\beta}) = 0, \tag{3.5}$$

The solutions of the above nonlinear equations are the maximum likelihood estimators of the Dagum distribution parameters λ, β and θ. As the equations expressed in (3.3), (3.4) and (3.5) cannot be solved analytically, one must use a numerical procedure to solve them.

Then, we can use the asymptotic normality of the MLEs to compute the asymptotic confidence intervals of the parameters λ, β and θ. The observed variance-covariance matrix for the MLEs of the parameters $\hat{V} = [\sigma_{i,j}]$, $i, j = 1, 2, 3$,was taken as

$$\hat{V} = -\begin{bmatrix} \frac{\partial^2 Ł}{\partial\lambda^2} & \frac{\partial^2 Ł}{\partial\lambda\partial\beta} & \frac{\partial^2 Ł}{\partial\lambda\partial\theta} \\ \frac{\partial^2 Ł}{\partial\beta\partial\lambda} & \frac{\partial^2 Ł}{\partial\beta^2} & \frac{\partial^2 Ł}{\partial\beta\partial\theta} \\ \frac{\partial^2 Ł}{\partial\theta\partial\lambda} & \frac{\partial^2 Ł}{\partial\theta\partial\beta} & \frac{\partial^2 Ł}{\partial\theta^2} \end{bmatrix}^{-1}_{(\lambda=\hat{\lambda}_{ML},\beta=\hat{\beta}_{ML},\theta=\hat{\theta}_{ML})}, \tag{3.6}$$

where

$$\frac{\partial^2 Ł}{\partial\lambda^2} = -\frac{k}{\lambda^2} + \frac{(n-k)\left[\theta^2 M_1^2 - \theta(\theta+1)(1-M_0)M_2\right]}{T^{2\beta}(1-M_0)^2} + \sum_{i=1}^{k} \frac{(\theta+1)x_i^{-2\beta}}{\left(1+\lambda x_i^{-\beta}\right)^2}, \tag{3.7}$$

$$\begin{aligned} \frac{\partial^2 Ł}{\partial\beta\partial\lambda} &= \frac{(n-k)\theta\log T}{T^{\beta}(1-M_0)^2}\left\{\left[M_1 - (\theta+1)\lambda M_2 T^{-\beta}\right](1-M_0) + \theta\lambda M_1^2 T^{-\beta}\right\} \\ &- \sum_{i=1}^{k} \frac{(\theta+1)x_i^{-\beta}\log x_i}{\left(1+\lambda x_i^{-\beta}\right)^2}, \end{aligned} \tag{3.8}$$

$$\frac{\partial^2 Ł}{\partial\theta\partial\lambda} = \frac{(n-k)M_1\left(1 - M_0 - \theta\log(1+\lambda T^{-\beta})\right)}{T^{\beta}(1-M_0)^2} - \sum_{i=1}^{k} \frac{x_i^{-\beta}}{1+\lambda x_i^{-\beta}}, \tag{3.9}$$

$$\begin{aligned} \frac{\partial^2 Ł}{\partial\beta^2} &= -\frac{k}{\beta^2} - \sum_{i=1}^{k} \frac{(\theta+1)\lambda x_i^{-\beta}\log^2 x_i}{\left(1+\lambda x_i^{-\beta}\right)^2} \\ &+ \frac{(n-k)\log^2 T}{T^{\beta}(1-M_0)^2}\left\{\theta\lambda\left(M_1 - (\theta+1)M_2\lambda T^{-\beta}\right)(1-M_0) + T^{-\beta}(\theta\lambda M_1)^2\right\}, \end{aligned} \tag{3.10}$$

$$\frac{\partial^2 Ł}{\partial\theta\partial\beta} = \frac{(n-k)\lambda T^{-\beta} M_1 \log T\left(\theta\log(1+\lambda T^{-\beta}) + M_0 - 1\right)}{(1-M_0)^2} - \sum_{i=1}^{k} \frac{\lambda x_i^{-\beta}\log x_i}{1+\lambda x_i^{-\beta}}, \tag{3.11}$$

$$\frac{\partial^2 Ł}{\partial\theta^2} = \frac{-k}{\theta^2} - \frac{(n-k)M_0\log^2\left(1+\lambda T^{-\beta}\right)}{(1-M_0)^2}. \tag{3.12}$$

A $100(1-\alpha)\%$ two-sided approximate confidence intervals for the parameters λ, β and θ are then given by

$$\widehat{\lambda} \pm z_{\alpha/2}\sqrt{V(\widehat{\lambda})}, \tag{3.13}$$

$$\widehat{\beta} \pm z_{\alpha/2}\sqrt{V(\widehat{\beta})}, \tag{3.14}$$

and

$$\widehat{\theta} \pm z_{\alpha/2}\sqrt{V(\widehat{\theta})}, \tag{3.15}$$

respectively, where $V(\widehat{\lambda})$, $V(\widehat{\beta})$, and $V(\widehat{\theta})$ are the estimated variances of $\widehat{\lambda}_{ML}$, $\widehat{\beta}_{ML}$ and $\widehat{\theta}_{ML}$, which are given by the diagonal elements of $\widehat{V}$, and $z_{\alpha/2}$ is the upper $\left(\frac{\alpha}{2}\right)$ percentile of standard normal distribution.

In order to construct the bootstrap confidence intervals Boot-p for the unknown parameters $\phi = (\lambda, \beta, \theta)$ based on C-UHCS scheme, we apply the following algorithms [For more details, one may refer to Kundu and Joarder [25] and Dube, Garg and Krishna [26]].

Boot-p interval's Algorithm:

step-1: Simulate $x_{1:n}, x_{2:n}, ..., x_{k:n}$ from Dagum distribution given in (1.1) and derive an estimate $\hat{\phi}$ of ϕ.

step-2: Simulate another sample $x^*_{1:n}, x^*_{2:n}, ..., x^*_{k:n}$ using $\hat{\phi}$, k and T. Then derive the updated bootstrap estimate $\hat{\phi}^*$ of ϕ.

step-3: Repeat the previous step, a prescribed B number of replications.

step-4: With $\hat{F}(x) = P(\hat{\phi}^* \leq x)$ denoting the distribution function of $\hat{\phi}^*$, the $100(1-\alpha)\%$ confidence interval of ϕ is given by

$$\left(\hat{\phi}_{Boot-p}(\frac{\alpha}{2}), \hat{\phi}_{Boot-p}(1-\frac{\alpha}{2})\right),$$

where $\hat{\phi}_{Boot-p}(x) = \hat{F}^{-1}(x)$ and x is prefixed.

4. Bayesian estimation

Bayesian inference is a convenient method to be used with C-UHCS$(k, r; T_1, T_2)$. Indeed, given that C-UHCS$(k, r; T_1, T_2)$ are so scarce, prior information is welcome. Risk functions are chosen depending on how one measures the distance between the estimate and the unknown parameter. In order to conduct the Bayesian analysis, usually quadratic loss function is considered. A very popular quadratic loss is the squared error (SE) loss function given by

$$L_S(g(\varphi), \hat{g}(\varphi)) = (g(\varphi) - \hat{g}(\varphi))^2 \tag{4.1}$$

where $\hat{g}(\varphi)$ is an estimate of the parametric function $g(\varphi)$. The Bayes estimate of $g(\varphi)$, say $\hat{g}_{ST}(\varphi)$, against SE loss function is the posterior mean given by

$$\hat{g}_S(\varphi) = E_{\varphi}\left[g(\varphi)|\mathbf{x}_k\right]. \tag{4.2}$$

Using SE loss function in the Bayesian approach leads to the equal penalization for underestimation and overestimation which is inappropriate in practical purposes. For instance, in estimating the reliability characteristics, overestimation is more serious than the underestimation. Therefore, different asymmetric loss functions are considered by researchers such as LINEX loss function. The LINEX loss function given by

$$L_L(g(\varphi), \hat{g}(\varphi)) = \exp\left[\rho\left(g(\varphi) - \hat{g}(\varphi)\right)\right] - \rho\left(g(\varphi) - \hat{g}(\varphi)\right) - 1, \quad \rho \neq 0 \tag{4.3}$$

is a popular asymmetric loss function that penalizes underestimation and overestimation for negative and positive values of ν, respectively. For ρ close to zero, the LINEX loss is approximately equal to the SE loss and therefore almost symmetric. The Bayes estimate of $g(\varphi)$ under LE loss function becomes

$$\hat{g}_L(\varphi) = -\frac{1}{\rho} \log\left(E_\varphi\left[\exp\left(-\rho g(\varphi)\right)|\mathbf{x}_k\right]\right), \tag{4.4}$$

Here, we derive different Bayes estimates by using the mentioned loss functions. Under the assumption that the parameters λ, β and θ are unknown and independent, we assume the joint prior density function, suggested by Al-Hussaini et al. [27] which gave good results, that is given by

$$\pi(\lambda,\beta,\theta) = \nu_1\nu_2\nu_3 \exp\left[-(\nu_1\lambda + \nu_2\beta + \nu_3\theta)\right], \quad \lambda,\beta,\theta > 0, \tag{4.5}$$

where ν_1, ν_2 and ν_3 are positive constants.

4.1. Tierney-Kadane's approximation

In order to use Tierney-Kadane's approximation technique, we set

$$\phi(\lambda,\beta,\theta) = \frac{1}{n}\log\left[L(\lambda,\beta,\theta|\mathbf{x}_k)\pi(\lambda,\beta,\theta)\right] \text{ and } \phi^{(g)}(\lambda,\beta,\theta) = \phi(\lambda,\beta,\theta) + \frac{1}{n}\log g(\lambda,\beta,\theta). \tag{4.6}$$

Now, assuming squared error loss functions, Bayes estimate of the function of parameters $g(\lambda,\beta,\theta)$ can be written in terms of (4.6) as

$$\begin{aligned} \hat{g}_{ST}(\lambda,\beta,\theta) &\propto \int_0^\infty\int_0^\infty\int_0^\infty g(\lambda,\beta,\theta)L(\lambda,\beta,\theta|\mathbf{x}_k)\pi(\lambda,\beta,\theta)\partial\lambda\partial\beta\partial\theta \\ &= \frac{\int_0^\infty\int_0^\infty\int_0^\infty \exp(n\phi^{(g)}(\lambda,\beta,\theta))\partial\lambda\partial\beta\partial\theta}{\int_0^\infty\int_0^\infty\int_0^\infty \exp(n\phi(\lambda,\beta,\theta))\partial\lambda\partial\beta\partial\theta}. \end{aligned} \tag{4.7}$$

By using Tierney and Kadane [28], the approximate form of (4.7) becomes

$$\hat{g}_{ST}(\lambda,\beta,\theta) = \left[\frac{detH^{(g)}}{detH}\right]^{\frac{1}{2}} \exp\left(n\left[\phi^{(g)}(\overline{\lambda}^{(g)},\overline{\beta}^{(g)},\overline{\theta}^{(g)}) - \phi(\overline{\lambda},\overline{\beta},\overline{\theta})\right]\right), \tag{4.8}$$

where $(\overline{\lambda}^{(g)},\overline{\beta}^{(g)},\overline{\theta}^{(g)})$ and $(\overline{\lambda},\overline{\beta},\overline{\theta})$ maximize $\phi^{(g)}(\lambda,\beta,\theta)$ and $\phi(\lambda,\beta,\theta)$, respectively, and $H^{(g)}$ and H are minus the inverse Hessians Matrix of $\phi^{(g)}(\lambda,\beta,\theta)$ and $\phi(\lambda,\beta,\theta)$ at $(\overline{\lambda}^{(g)},\overline{\beta}^{(g)},\overline{\theta}^{(g)})$ and $(\overline{\lambda},\overline{\beta},\overline{\theta})$, respectively. Here, from (3.2), (4.5) and (4.6), we have

$$\phi(\lambda,\beta,\theta) = \frac{1}{n}\left\{\log\nu_1\nu_2\nu_3 - \nu_1\lambda - \nu_2\beta - \nu_3\theta + \text{Ł}\right\}. \tag{4.9}$$

Now, $(\overline{\lambda},\overline{\beta},\overline{\theta})$ can be calculated from the simultaneous solution of the nonlinear equations

$$\frac{\partial}{\partial\lambda}\text{Ł}(\lambda,\beta,\theta) = \nu_1, \quad \frac{\partial}{\partial\beta}\text{Ł}(\lambda,\beta,\theta) = \nu_2 \quad \text{and} \quad \frac{\partial}{\partial\theta}\text{Ł}(\lambda,\beta,\theta) = \nu_3. \tag{4.10}$$

The second order derivatives of Ł, given in (3.7)-(3.12) can be used to determine the determinant of the negative of the inverse Hessian matrix of $\phi(\lambda,\beta,\theta)$ at $(\overline{\lambda},\overline{\beta},\overline{\theta})$ as

$$\det H = -\frac{1}{n^3}\det\begin{bmatrix} \frac{\partial^2 Ł}{\partial\lambda^2} & \frac{\partial^2 Ł}{\partial\lambda\partial\beta} & \frac{\partial^2 Ł}{\partial\lambda\partial\theta} \\ \frac{\partial^2 Ł}{\partial\beta\partial\lambda} & \frac{\partial^2 Ł}{\partial\beta^2} & \frac{\partial^2 Ł}{\partial\beta\partial\theta} \\ \frac{\partial^2 Ł}{\partial\theta\partial\lambda} & \frac{\partial^2 Ł}{\partial\theta\partial\beta} & \frac{\partial^2 Ł}{\partial\theta^2} \end{bmatrix}^{-1}_{(\lambda=\overline{\lambda},\beta=\overline{\beta},\theta=\overline{\theta})} \tag{4.11}$$

Then, the Bayesian estimate of λ,β and θ based on square error loss function can be obtained by replacing $g(\lambda,\beta,\theta)$ by λ,β and θ, respectively and corresponding $\phi^{(g)}_{ST}(\lambda,\beta,\theta)$ take the forms:

$$\phi^{(g)}_{ST}(\lambda,\beta,\theta) = \begin{cases} \phi^{(\lambda)}_{ST}(\lambda,\beta,\theta) = \phi(\lambda,\beta,\theta) + \frac{1}{n}\log\lambda, & g(\lambda,\beta,\theta) = \lambda, \\ \phi^{(\beta)}_{ST}(\lambda,\beta,\theta) = \phi(\lambda,\beta,\theta) + \frac{1}{n}\log\beta, & g(\lambda,\beta,\theta) = \beta, \\ \phi^{(\theta)}_{ST}(\lambda,\beta,\theta) = \phi(\lambda,\beta,\theta) + \frac{1}{n}\log\theta, & g(\lambda,\beta,\theta) = \theta. \end{cases} \tag{4.12}$$

Hence, $(\overline{\lambda}^{(\lambda)}_{ST},\overline{\beta}^{(\lambda)}_{ST},\overline{\theta}^{(\lambda)}_{ST})$, $(\overline{\lambda}^{(\beta)}_{ST},\overline{\beta}^{(\beta)}_{ST},\overline{\theta}^{(\beta)}_{ST})$ and $(\overline{\lambda}^{(\theta)}_{ST},\overline{\beta}^{(\theta)}_{ST},\overline{\theta}^{(\theta)}_{ST})$ can be computed by maximizing $\phi^{(\lambda)}_{ST}(\lambda,\beta,\theta)$, $\phi^{(\beta)}_{ST}(\lambda,\beta,\theta)$ and $\phi^{(\theta)}_{ST}(\lambda,\beta,\theta)$, respectively, through the simultaneous solution of the each of the following systems:
System 1: $\frac{\partial}{\partial\lambda}Ł(\lambda,\beta,\theta) - \nu_1 + \frac{1}{n\lambda} = 0,\quad \frac{\partial}{\partial\beta}Ł(\lambda,\beta,\theta) - \nu_2 = 0$ and $\frac{\partial}{\partial\theta}Ł(\lambda,\beta,\theta) - \nu_3 = 0$,
System 2: $\frac{\partial}{\partial\lambda}Ł(\lambda,\beta,\theta) - \nu_1 = 0,\quad \frac{\partial}{\partial\beta}Ł(\lambda,\beta,\theta) - \nu_2 + \frac{1}{n\beta} = 0$ and $\frac{\partial}{\partial\theta}Ł(\lambda,\beta,\theta) - \nu_3 = 0$,
System 3: $\frac{\partial}{\partial\lambda}Ł(\lambda,\beta,\theta) - \nu_1 = 0,\quad \frac{\partial}{\partial\beta}Ł(\lambda,\beta,\theta) - \nu_2 = 0$ and $\frac{\partial}{\partial\theta}Ł(\lambda,\beta,\theta) - \nu_3 + \frac{1}{n\theta} = 0$.

Again, using the second order derivative of $\phi^{(\lambda)}_{ST}(\lambda,\beta,\theta)$, $\phi^{(\beta)}_{ST}(\lambda,\beta,\theta)$ and $\phi^{(\theta)}_{ST}(\lambda,\beta,\theta)$ at $(\overline{\lambda}^{(\lambda)}_{ST},\overline{\beta}^{(\lambda)}_{ST},\overline{\theta}^{(\lambda)}_{ST})$, $(\overline{\lambda}^{(\beta)}_{ST},\overline{\beta}^{(\beta)}_{ST},\overline{\theta}^{(\beta)}_{ST})$ can be used to calculate $(\overline{\lambda}^{(\theta)}_{ST},\overline{\beta}^{(\theta)}_{ST},\overline{\theta}^{(\theta)}_{ST})$, the elements of $H^{(\lambda)}_{ST}$, $H^{(\beta)}_{ST}$ and $H^{(\theta)}_{ST}$, respectively, as:

$$\det H^{(\lambda)}_{ST} = -\frac{1}{n^3}\det\begin{bmatrix} \frac{\partial^2 Ł}{\partial\lambda^2} - \frac{1}{\lambda^2} & \frac{\partial^2 Ł}{\partial\lambda\partial\beta} & \frac{\partial^2 Ł}{\partial\lambda\partial\theta} \\ \frac{\partial^2 Ł}{\partial\beta\partial\lambda} & \frac{\partial^2 Ł}{\partial\beta^2} & \frac{\partial^2 Ł}{\partial\beta\partial\theta} \\ \frac{\partial^2 Ł}{\partial\theta\partial\lambda} & \frac{\partial^2 Ł}{\partial\theta\partial\beta} & \frac{\partial^2 Ł}{\partial\theta^2} \end{bmatrix}^{-1}_{(\lambda=\overline{\lambda}^{(\lambda)}_{ST},\beta=\overline{\beta}^{(\lambda)}_{ST},\theta=\overline{\theta}^{(\lambda)}_{ST})}, \tag{4.13}$$

$$\det H_{ST}^{(\beta)} = -\frac{1}{n^3}\det\begin{bmatrix} \frac{\partial^2 Ł}{\partial\lambda^2} & \frac{\partial^2 Ł}{\partial\lambda\partial\beta} & \frac{\partial^2 Ł}{\partial\lambda\partial\theta} \\ \frac{\partial^2 Ł}{\partial\beta\partial\lambda} & \frac{\partial^2 Ł}{\partial\beta^2} - \frac{1}{\beta^2} & \frac{\partial^2 Ł}{\partial\beta\partial\theta} \\ \frac{\partial^2 Ł}{\partial\theta\partial\lambda} & \frac{\partial^2 Ł}{\partial\theta\partial\beta} & \frac{\partial^2 Ł}{\partial\theta^2} \end{bmatrix}^{-1}_{(\lambda=\overline{\lambda}_{ST}^{(\beta)},\beta=\overline{\beta}_{ST}^{(\beta)},\theta=\overline{\theta}_{ST}^{(\beta)})}, \tag{4.14}$$

and

$$\det H_{ST}^{(\theta)} = -\frac{1}{n^3}\det\begin{bmatrix} \frac{\partial^2 Ł}{\partial\lambda^2} & \frac{\partial^2 Ł}{\partial\lambda\partial\beta} & \frac{\partial^2 Ł}{\partial\lambda\partial\theta} \\ \frac{\partial^2 Ł}{\partial\beta\partial\lambda} & \frac{\partial^2 Ł}{\partial\beta^2} & \frac{\partial^2 Ł}{\partial\beta\partial\theta} \\ \frac{\partial^2 Ł}{\partial\theta\partial\lambda} & \frac{\partial^2 Ł}{\partial\theta\partial\beta} & \frac{\partial^2 Ł}{\partial\theta^2} - \frac{1}{\theta^2} \end{bmatrix}^{-1}_{(\lambda=\overline{\lambda}_{ST}^{(\theta)},\beta=\overline{\beta}_{ST}^{(\theta)},\theta=\overline{\theta}_{ST}^{(\theta)})}. \tag{4.15}$$

Therefore, the approximate Bayes estimate of λ based on square error loss function are:

$$\left.\begin{aligned} \hat{\lambda}_{ST} &= \left[\frac{detH_{ST}^{(\lambda)}}{detH}\right]^{\frac{1}{2}} \exp\left(n\left[\phi_{ST}^{(\lambda)}(\overline{\lambda}_{ST}^{(\lambda)},\overline{\beta}_{ST}^{(\lambda)},\overline{\theta}_{ST}^{(\lambda)}) - \phi(\overline{\lambda},\overline{\beta},\overline{\theta})\right]\right), \\ \hat{\beta}_{ST} &= \left[\frac{detH_{ST}^{(\beta)}}{detH}\right]^{\frac{1}{2}} \exp\left(n\left[\phi_{ST}^{(\beta)}(\overline{\lambda}_{ST}^{(\beta)},\overline{\beta}_{ST}^{(\beta)},\overline{\theta}_{ST}^{(\beta)}) - \phi(\overline{\lambda},\overline{\beta},\overline{\theta})\right]\right), \\ \hat{\theta}_{ST} &= \left[\frac{detH_{ST}^{(\theta)}}{detH}\right]^{\frac{1}{2}} \exp\left(n\left[\phi_{ST}^{(\theta)}(\overline{\lambda}_{ST}^{(\theta)},\overline{\beta}_{ST}^{(\theta)},\overline{\theta}_{ST}^{(\theta)}) - \phi(\overline{\lambda},\overline{\beta},\overline{\theta})\right]\right). \end{aligned}\right\} \tag{4.16}$$

Next, in order to obtain the Bayesian estimates of λ, β and θ based on LINEX loss function we replacing $g(\lambda,\beta,\theta)$ by $e^{-\rho\lambda}$, $e^{-\rho\beta}$ and $e^{-\rho\theta}$, respectively, and corresponding $\phi_{LT}^{(g)}(\lambda,\beta,\theta)$ take the forms:

$$\phi_{LT}^{(g)}(\lambda,\beta,\theta) = \begin{cases} \phi_{LT}^{(\lambda)}(\lambda,\beta,\theta) = \phi(\lambda,\beta,\theta) - \frac{\rho\lambda}{n}, & g(\lambda,\beta,\theta) = e^{-\rho\lambda}, \\ \phi_{LT}^{(\beta)}(\lambda,\beta,\theta) = \phi(\lambda,\beta,\theta) - \frac{\rho\beta}{n}, & g(\lambda,\beta,\theta) = e^{-\rho\beta}, \\ \phi_{LT}^{(\theta)}(\lambda,\beta,\theta) = \phi(\lambda,\beta,\theta) - \frac{\rho\theta}{n}, & g(\lambda,\beta,\theta) = e^{-\rho\theta}. \end{cases} \tag{4.17}$$

Hence, $(\overline{\lambda}_{LT}^{(\lambda)},\overline{\beta}_{LT}^{(\lambda)},\overline{\theta}_{LT}^{(\lambda)})$, $(\overline{\lambda}_{LT}^{(\beta)},\overline{\beta}_{LT}^{(\beta)},\overline{\theta}_{LT}^{(\beta)})$ and $(\overline{\lambda}_{LT}^{(\theta)},\overline{\beta}_{LT}^{(\theta)},\overline{\theta}_{LT}^{(\theta)})$ can be computed by maximizing $\phi_{LT}^{(\lambda)}(\lambda,\beta,\theta)$, $\phi_{LT}^{(\beta)}(\lambda,\beta,\theta)$ and $\phi_{LT}^{(\theta)}(\lambda,\beta,\theta)$, respectively, through solving simultaneously the following systems:
System 4: $\frac{\partial}{\partial\lambda}Ł(\lambda,\beta,\theta) - \nu_1 - \frac{\rho}{n} = 0$, $\frac{\partial}{\partial\beta}Ł(\lambda,\beta,\theta) - \nu_2 = 0$ and $\frac{\partial}{\partial\theta}Ł(\lambda,\beta,\theta) - \nu_3 = 0$.
System 5: $\frac{\partial}{\partial\lambda}Ł(\lambda,\beta,\theta) - \nu_1 = 0$, $\frac{\partial}{\partial\beta}Ł(\lambda,\beta,\theta) - \nu_2 - \frac{\rho}{n} = 0$ and $\frac{\partial}{\partial\theta}Ł(\lambda,\beta,\theta) - \nu_3 = 0$,
System 6: $\frac{\partial}{\partial\lambda}Ł(\lambda,\beta,\theta) - \nu_1 = 0$, $\frac{\partial}{\partial\beta}Ł(\lambda,\beta,\theta) - \nu_2 = 0$ and $\frac{\partial}{\partial\theta}Ł(\lambda,\beta,\theta) - \nu_3 - \frac{\rho}{n} = 0$.
Once again, we can derive that $H_{LT}^{(\lambda)} = H_{LT}^{(\beta)} = H_{LT}^{(\theta)} = H_{LT}$ by calculating the second order derivative of $\phi_{LT}^{(\lambda)}(\lambda,\beta,\theta)$, $\phi_{LT}^{(\beta)}(\lambda,\beta,\theta)$ and $\phi_{LT}^{(\theta)}(\lambda,\beta,\theta)$ at $(\overline{\lambda}_{LT}^{(\lambda)},\overline{\beta}_{LT}^{(\lambda)},\overline{\theta}_{LT}^{(\lambda)})$, $(\overline{\lambda}_{LT}^{(\beta)},\overline{\beta}_{LT}^{(\beta)},\overline{\theta}_{LT}^{(\beta)})$ and $(\overline{\lambda}_{LT}^{(\theta)},\overline{\beta}_{LT}^{(\theta)},\overline{\theta}_{LT}^{(\theta)})$ and by

using the same manner as in (4.12)-(4.15). Therefore, the approximate Bayes estimate of λ based on LINEX loss function are:

$$\left.\begin{aligned}
\hat{\lambda}_{LT} &= -\tfrac{1}{\rho}\log\left(n\left[\phi^{(\lambda)}_{LT}(\overline{\lambda}^{(\lambda)}_{LT},\overline{\beta}^{(\lambda)}_{LT},\overline{\theta}^{(\lambda)}_{LT}) - \phi(\overline{\lambda},\overline{\beta},\overline{\theta})\right]\right),\\
\hat{\beta}_{LT} &= -\tfrac{1}{\rho}\log\left(n\left[\phi^{(\beta)}_{LT}(\overline{\lambda}^{(\beta)}_{LT},\overline{\beta}^{(\beta)}_{LT},\overline{\theta}^{(\beta)}_{LT}) - \phi(\overline{\lambda},\overline{\beta},\overline{\theta})\right]\right),\\
\hat{\theta}_{LT} &= -\tfrac{1}{\rho}\log\left(n\left[\phi^{(\theta)}_{LT}(\overline{\lambda}^{(\theta)}_{LT},\overline{\beta}^{(\theta)}_{LT},\overline{\theta}^{(\theta)}_{LT}) - \phi(\overline{\lambda},\overline{\beta},\overline{\theta})\right]\right).
\end{aligned}\right\} \tag{4.18}$$

In order to calculate $100(1-\alpha)\%$ HPD credible intervals for the Bayesian estimates using both of SE and LE loss functions for any parameter, for example δ, we follow the steps below:

HPD credible interval:

1. Simulate censored sample of size n from Dagum distribution given in (1.1) and calculate the estimate of δ under a certain choice of k, r, T_1 and T_2, say δ^*.
2. Repeat the previous step M times to get $\delta^*_1, \delta^*_2, \ldots, \delta^*_M$, and the order values are: $\delta^*_{1:M}, \delta^*_{2:M}, \ldots, \delta^*_{M:M}$
3. $100(1-\alpha)\%$ HPD credible interval for δ is shortest length through the intervals $(\delta^*_{j:M}, \delta^*_{j+(1-\alpha)M:M})$, $j = 1, 2, \ldots, \alpha M$.

4.2. MCMC method

In the previous subsection, we have used Tierney-Kadane's approximation to derive the Bayes estimates of the parameters. However, it is not possible to obtain HPD credible intervals using this method. In this subsection, we adopt a Metropolis-Hastings within Gibbs sampling approach to generate random samples from the conditional densities of the parameters and use them to obtain the HPD credible intervals and point Bayes estimates. From (3.1) and (4.5), the posterior density of λ,β, and θ can be extracted as

$$\begin{aligned}
\pi^*(\lambda,\beta,\theta) &\propto \lambda^k\beta^k\theta^k \exp\left[-(\nu_1\lambda + \nu_2\beta + \nu_3\theta)\right]\\
&\times \left[1-\left(1+\lambda T^{-\beta}\right)^{-\theta}\right]^{n-k} \prod_{i=1}^{k} x_i^{-(\beta+1)}\left(1+\lambda x_i^{-\beta}\right)^{-(\theta+1)}.
\end{aligned} \tag{4.19}$$

In the following algorithm, we employ Metropolis-Hastings (M-H) technique with normal proposal distribution to generate samples from these distributions.

1. Start with initial values of the parameters $(\lambda^{(0)},\beta^{(0)},\theta^{(0)})$. Then, simulate censored sample of size k under a certain choice of m, r, T_1 and T_2 from Dagum distribution given in (1.1) and set $l = 1$.
2. Generate $\lambda^{(*)},\beta^{(*)},\theta^{(*)}$ using the proposal distributions $N(\lambda^{(l-1)}, 1), N(\beta^{(l-1)}, 1)$ and $N(\theta^{(l-1)}, 1)$, respectively.
3. Calculate the acceptance probability $r = min\left(1, \frac{\pi^*(\lambda^{(*)},\beta^{(*)},\theta^{(*)})}{\pi^*(\lambda^{(l-1)},\beta^{(l-1)},\theta^{(l-1)})}\right)$.
4. Generate U from uniform(0, 1).
5. Accept the proposal distribution and set $\left(\lambda^{(l)},\beta^{(l)},\theta^{(l)}\right) = \left(\lambda^{(*)},\beta^{(*)},\theta^{(*)}\right)$ if $U < r$. Otherwise, reject the proposal distribution and set $\left(\lambda^{(l)},\beta^{(l)},\theta^{(l)}\right) = \left(\lambda^{(l-1)},\beta^{(l-1)},\theta^{(l-1)}\right)$.

6. Set $l = 1 + 1$.
7. Repeat Steps 2-6, M times, and obtain $\lambda^{(l)}, \beta^{(l)}$ and $\theta^{(l)}$ for $l = 1, ..., M$.

By using the generated random samples from the above Gibbs sampling technique and for N is the nburn, the approximate Bayes estimate of the parameters under squared error and LINEX loss functions can be obtained as

$$\hat{\lambda}_{SM} = \frac{1}{M-N}\sum_{l=N+1}^{M}\lambda^{(l)}, \qquad \hat{\beta}_{SM} = \frac{1}{M-N}\sum_{l=N+1}^{M}\beta^{(l)}, \qquad \hat{\theta}_{SM} = \frac{1}{M-N}\sum_{l=N+1}^{M}\theta^{(l)},$$

$$\hat{\lambda}_{LM} = \frac{-1}{\rho}\log\left(\frac{\sum_{l=N+1}^{M} exp\left(-\rho\lambda^{(l)}\right)}{M-N}\right), \qquad \hat{\beta}_{LM} = \frac{-1}{\rho}\log\left(\frac{\sum_{l=N+1}^{M} exp\left(-\rho\beta^{(l)}\right)}{M-N}\right),$$

and

$$\hat{\theta}_{LM} = \frac{-1}{\rho}\log\left(\frac{\sum_{l=N+1}^{M} exp\left(-\rho\theta^{(l)}\right)}{M-N}\right).$$

MCMC HPD credible interval Algorithm:

1. Arrange the values of $\lambda^{(*)}, \beta^{(*)}$ and $\theta^{(*)}$ in increasing magnitude.
2. Find the positions of the lower bounds which is $(M - N) * \alpha/2$, then determine the lower bounds of λ, β and θ.
3. Find the positions of the upper bounds which is $(M - N) * (1 - \alpha/2)$, then determine the upper bounds of λ, β and θ.
4. Repeat the above steps M times. Find the average value of the lower and upper bounds MCMC HPD credible interval of λ, β and θ.
5. Get n number of MCMC HPD credible intervals. Find the average value of the lower and upper bounds credible interval of λ, β and θ.

5. Simulation experiments

In this section, we show the usefulness of the theatrical findings in this paper by conducting series of simulation experiments. The simulations show the bias and estimated risk of the maximum likelihood and Bayesian estimates, respectively. The Bayesian estimate are calculated based on mean squared and LINEX loss functions. In addition, 95% and 90% the confidence, Bootstrap and HPD credible intervals are calculated with the corresponding width. The simulation experiments can be explained though the following steps:

Evaluate the performances of Bayes predictors obtained from LINEX and squared error loss functions. In this case, to investigate the sensitivity of the predictors with respect to the choices of hyper parameters, the above mentioned priors are considered. We perform simulations to investigate the behavior of the different methods for $n = 30$ and for various r, k, T_1, T_2.

1. Fix different censoring cases as given in (7) as $X_{r=10:n}$, $X_{r=15:n}$, $X_{k=20:n}$, $X_{k=25:n}$, $T_1 = 18$, $T_1 = 24$, $T_2 = 30$ and $T_2 = 36$, next generate the censored samples from Dagum distribution using $\lambda = 5, \beta = 2$ and $\theta = 2$.

2. Use each of the censoring cases in step (1) for calculating the MLEs by solving the system of nonlinear equations (3.3), (3.4) and (3.5).
3. Again, we use each of the censoring cases in step (1) for calculating the Bayesian estimates by using Tierney-Kadane's approximation in Section (4) for both cases of mean squared and LINEX loss functions. Let the hyper-parameters be the inverse of initial values which are $\nu_1 = 0.2$ and $\nu_2 = \nu_3 = 0.5$, respectively. The parameter ρ in LINEX is chosen as -0.5, 1.0 and 1.5.
4. The steeps (1)-(3) are repeated 1000 times, then the bias and estimated risk (ER) in each cases are calculated in Table 1. The ER of parameter φ under squared error and LINEX loss functions by:

$$\begin{aligned} ER_S &= \frac{1}{R}\sum_{i=1}^{R}(\hat{\varphi}_i - \varphi)^2, \\ ER_L &= \frac{1}{R}\sum_{i=1}^{R}(Exp(\rho(\hat{\varphi}_i - \varphi)) - (\hat{\varphi}_i - \varphi) - 1), \end{aligned}$$

where $\hat{\varphi}$ is the estimate of φ and R is the number of replication.
5. The 90% and 95% approximate confidence, bootstrap and HPD credible intervals with their width for the parameters λ, β and θ are calculated in Tables (2), (3) and (4), respectively.

From Tables 1, 2, 3 and 4, we see that:

1. The estimate of λ is overestimated except just few cases. Also, the Bayesian estimate of λ is best in terms of the bias in the case of LINEX loss function at $\rho = 1.0$. Also, the ER supports the Bayesian estimate in the case of LINEX loss function.
2. Again, the estimate of β is overestimated except just few cases. Also, the Bayesian estimate of λ behaves better in terms of the bias in the case of LINEX loss function. Similar argument is also can be stated for ER.
3. One again, the estimate of θ is underestimated for most of the cases. Also, the ER shows that the Bayesian estimate of θ is the best in terms of the bias in the case of LINEX loss function.
4. HPD credible interval estimation for λ behaves better in terms of the the interval width for the LINEX loss function when $\rho = -0.5$.
5. HPD credible interval estimation for β behaves better in terms of the the interval width for the SE loss function.
6. HPD credible interval estimation for θ behaves better in terms of the the interval width for the LINEX loss function when $\rho = 1.0$.

6. Data analysis

Here we use one data set that will be used for the purpose of making comparisons between the estimators presented in this paper. The data set is taken from Nichols and Padgett [29] consisting of 100 observations on breaking stress of carbon fibers (in Gba). Dey, et al. [30] have fitted the Dagum distribution to this data set. The data are:
3.7, 2.74, 2.73, 3.11, 3.27, 2.87, 4.42, 2.41, 3.19, 3.28, 3.09, 1.87, 3.75, 2.43, 2.95, 2.96, 2.3, 2.67, 3.39, 2.81, 4.2, 3.31, 3.31, 2.85, 3.15, 2.35, 2.55, 2.81, 2.77, 2.17, 1.41, 3.68, 2.97, 2.76, 4.91, 3.68, 3.19, 1.57, 0.81, 1.59, 2, 1.22, 2.17, 1.17, 5.08, 3.51, 2.17, 1.69, 1.84, 0.39, 3.68, 1.61, 2.79, 4.7, 1.57,

Table 1. Bias, MSE and estimated risk (ER) of the parameters estimation when $n = 30, \lambda = 5, \beta = 2$ and $\theta = 2$.

T	λ_{ML}	λ_{ST}	λ_{LT}			β_{ML}	β_{ST}	β_{LT}			θ_{ML}	θ_{ST}	θ_{LT}		
			$\rho = -0.5$	$\rho = 1.0$	$\rho = 1.5$			$\rho = -0.5$	$\rho = 1.0$	$\rho = 1.5$			$\rho = -0.5$	$\rho = 1.0$	$\rho = 1.5$
$X_{r=10:n}$	0.2363	0.2324	0.2149	-0.0605	0.0144	-0.0284	0.0214	0.0264	0.0157	0.0085	0.1374	-0.4014	-0.2774	-0.3524	-0.2393
	8.0744	0.1993	0.1928	0.1553	0.1618	0.6231	0.2048	0.1938	0.0941	0.0976	0.4086	0.6174	0.2512	0.3189	0.2156
$X_{r=15:n}$	0.0067	0.3116	0.2985	0.0925	0.1149	0.0702	0.0753	0.0834	0.078	0.0866	-0.1417	-0.1755	-0.1803	-0.2811	0.1547
	4.0237	0.1795	0.1768	0.1428	0.1529	0.3233	0.1876	0.1851	0.0802	0.0879	0.2084	0.3061	0.1431	0.2539	0.1394
$X_{k=20:n}$	0.4301	0.3551	0.3714	0.2116	0.2363	0.1619	0.1027	0.1113	0.1141	0.1085	-0.106	-0.1258	-0.1694	-0.2164	0.1096
	2.359	0.1517	0.1435	0.1339	0.1421	0.179	0.1419	0.1471	0.0726	0.0676	0.1209	0.1637	0.1128	0.1953	0.0987
$X_{k=25:n}$	0.1065	0.4041	0.3928	0.2801	0.3183	0.0599	0.131	0.134	0.1255	0.133	-0.0684	-0.0768	-0.1111	-0.1882	-0.0928
	1.5964	0.1407	0.1327	0.1216	0.1158	0.1059	0.1081	0.0555	0.0628	0.0646	0.0811	0.0993	0.1002	0.1698	0.0835
$T_1 = 18$	0.4675	0.3669	0.3542	0.2379	0.2477	0.1206	0.1256	0.1258	0.1235	0.1209	-0.1262	0.1234	-0.1131	-0.1814	0.1035
	1.9223	0.1461	0.1381	0.1237	0.1224	0.1505	0.1168	0.1011	0.0711	0.0668	0.0952	0.112	0.1021	0.1737	0.0932
$T_1 = 24$	-0.1587	0.2724	0.2641	0.06	0.1081	-0.0359	0.0564	0.0765	0.0665	0.0544	0.1528	-0.1616	-0.157	-0.3176	0.1755
	5.1027	0.1839	0.1871	0.1535	0.1565	0.3924	0.1959	0.1888	0.0898	0.0949	0.2527	0.3225	0.1621	0.287	0.1581
$T_2 = 30$	0.3561	0.4094	0.4098	0.3132	0.2822	0.0627	0.1302	0.1395	0.1346	0.1357	-0.0716	-0.0833	-0.1149	-0.1548	-0.0851
	1.4869	0.1339	0.1309	0.2813	0.1135	0.0952	0.0877	0.0455	0.0521	0.0532	0.0618	0.0777	0.0936	0.1396	0.0767
$T_2 = 36$	0.2507	0.3533	0.3359	0.1864	0.2058	0.1621	0.0937	0.1057	0.092	0.0951	-0.1454	0.1328	-0.1532	-0.2664	0.1301
	3.1372	0.1598	0.1516	0.1372	0.1446	0.2343	0.1507	0.1781	0.0727	0.0756	0.1525	0.2219	0.1385	0.2406	0.1172

The first row represents the Bias while the second row represents the MSE and ER.

Table 2. Bias and the estimated risk (ER) of the parameters estimation using MCMC when $n = 30, \lambda = 5, \beta = 2$ and $\theta = 2$.

T	λ_{SE}	λ_{LE}			β_{SE}	β_{LE}			θ_{SE}	θ_{LE}		
		$\rho = -0.5$	$\rho = 1.0$	$\rho = 1.5$		$\rho = -0.5$	$\rho = 1.0$	$\rho = 1.5$		$\rho = -0.5$	$\rho = 1.0$	$\rho = 1.5$
$X_{r=10:n}$	-2.3890	-2.1640	-2.8200	-2.7040	-1.1830	-1.1730	-1.2130	-1.2030	-0.1960	-0.1470	-0.3210	-0.2830
	0.0276	0.0373	0.0188	0.0181	0.0637	0.0653	0.0564	0.0593	0.0081	0.0089	0.0062	0.0064
$X_{r=15:n}$	-2.7440	-2.6220	-3.0110	-2.9340	-1.1810	-1.1740	-1.2030	-1.1960	-0.4460	-0.4160	-0.5270	-0.5020
	0.0222	0.0270	0.0151	0.0184	0.0382	0.0386	0.0336	0.0346	0.0063	0.0065	0.0047	0.0047
$X_{k=20:n}$	-2.9700	-2.8930	-3.1550	-3.1000	-1.0840	-1.0770	-1.1050	-1.0980	-0.6160	-0.5960	-0.6740	-0.6550
	0.0118	0.0091	0.0068	0.0065	0.0143	0.0147	0.0118	0.0129	0.0027	0.0018	0.0014	0.0025
$X_{k=25:n}$	-3.1700	-3.1150	-3.3110	-3.2680	-1.0820	-1.0770	-1.0990	-1.0940	-0.7550	-0.7380	-0.8000	-0.7850
	0.0083	0.0038	0.0021	0.0038	0.0073	0.0059	0.0066	0.0055	0.0003	0.0013	0.0011	0.0017
$T_1 = 18$	-2.6400	-2.4940	-2.9540	-2.8650	-1.2480	-1.2410	-1.2700	-1.2630	-0.3480	-0.3110	-0.4470	-0.4160
	0.0187	0.0200	0.0118	0.0113	0.0351	0.0354	0.0299	0.0311	0.0059	0.0060	0.0040	0.0048
$T_1 = 24$	-2.8350	-2.7320	-3.0700	-3.0010	-1.1650	-1.1590	-1.1860	-1.1790	-0.5250	-0.4990	-0.5960	-0.5740
	0.0092	0.0112	0.0065	0.0086	0.0254	0.0240	0.0213	0.0229	0.0034	0.0036	0.0031	0.0025
$T_2 = 30$	-3.1250	-3.0620	-3.2830	-3.2350	-1.1130	-1.1070	-1.1300	-1.1250	-0.7110	-0.6930	-0.7610	-0.7450
	0.0100	0.0083	0.0043	0.0068	0.0201	0.0208	0.0195	0.0184	0.0017	0.0018	0.0012	0.0014
$T_2 = 36$	-3.2540	-3.2040	-3.3820	-3.3430	-1.1130	-1.1080	-1.1280	-1.1230	-0.8060	-0.7920	-0.8470	-0.8340
	0.0096	0.0075	0.0087	0.0072	0.0061	0.0057	0.0064	0.0065	0.0020	0.0008	0.0013	0.0012

The first row represents the Bias while the second row represents ER.

Table 3. 95% and 90% Interval estimation of the parameter λ when $n = 30, \lambda = 5, \beta = 2$ and $\theta = 2$.

T		ML	$Bootstrap$	HPD_S	HPD_{LT}		
					$\rho = -0.5$	$\rho = 1.0$	$\rho = 1.5$
$X_{r=10:n}$	95%	3.8809 7.1027 3.2218	4.301 5.85 1.549	4.2645 5.691 1.4266	4.703 5.6513 0.9483	4.322 5.6302 1.3082	4.2076 5.6923 1.4846
	90%	3.8901 7.0931 3.203	4.3333 5.8146 1.4813	4.2644 5.6723 1.4079	4.703 5.5978 0.8949	4.322 5.5623 1.2402	4.2076 5.6161 1.4085
$X_{r=15:n}$	95%	4.2337 5.8233 1.5896	4.6868 5.7555 1.0687	4.6531 5.6376 0.9845	4.9182 5.6107 0.6925	4.6543 5.5599 0.9057	4.6097 5.6464 1.0368
	90%	4.2337 5.8099 1.5762	4.7149 5.7202 1.0052	4.6431 5.6065 0.9634	4.9254 5.5946 0.6692	4.6543 5.5128 0.8585	4.6097 5.6421 1.0324
$X_{k=20:n}$	95%	4.6878 6.1134 1.4256	4.8764 5.6987 0.8222	4.8931 5.6201 0.727	4.9164 5.5526 0.6362	4.8637 5.5532 0.6895	4.8085 5.5939 0.7854
	90%	4.6896 6.0885 1.3989	4.898 5.6722 0.7743	4.8931 5.6135 0.7204	4.9204 5.5567 0.6363	4.8637 5.5449 0.6812	4.8085 5.5508 0.7423
$X_{k=25:n}$	95%	4.9406 5.8616 0.921	4.9989 5.655 0.6561	4.9981 5.5968 0.5987	4.9585 5.5467 0.5882	4.9862 5.5477 0.5614	4.9615 5.5974 0.6359
	90%	4.9501 5.8366 0.8865	4.9989 5.6381 0.6392	4.9984 5.5068 0.5084	4.9985 5.5381 0.5396	4.9862 5.5197 0.5335	4.9615 5.5821 0.6206
$T_1 = 18$	95%	4.5417 5.8111 1.2694	4.9665 5.676 0.7095	4.9704 5.5971 0.6267	4.9182 5.5517 0.6335	4.9683 5.5676 0.5993	4.909 5.6031 0.6942
	90%	4.5451 5.8043 1.2592	4.9865 5.6605 0.674	4.9904 5.5962 0.6058	4.9645 5.5421 0.5776	4.9683 5.5277 0.5594	4.909 5.5548 0.6458
$T_1 = 24$	95%	4.4725 5.9568 1.4843	4.3615 5.7797 1.4181	4.3455 5.6537 1.3082	4.885 5.6217 0.7367	4.5357 5.5698 1.0341	4.4527 5.6489 1.1962
	90%	4.4765 5.9381 1.4616	4.5019 5.743 1.1411	4.5445 5.6123 1.0678	4.885 5.6061 0.721	4.5357 5.516 0.9803	4.4527 5.5817 1.129
$T_2 = 30$	95%	4.8469 5.8925 1.0456	4.9994 5.651 0.6516	4.9983 5.5882 0.5899	4.9772 5.5389 0.5617	4.9832 5.5436 0.5604	4.9779 5.6068 0.6289
	90%	4.8547 5.8712 1.0165	4.9991 5.6347 0.6356	4.9993 5.4456 0.4463	4.9992 5.5373 0.5381	4.9972 5.5161 0.5189	4.9712 5.5067 0.5355
$T_2 = 36$	95%	4.3165 6.296 1.9795	4.7768 5.7342 0.9574	4.6733 5.6214 0.9481	4.9254 5.5959 0.6705	4.7816 5.575 0.7934	4.7175 5.6404 0.923
	90%	4.3255 6.2768 1.9513	4.8078 5.7113 0.9035	4.6893 5.5926 0.9033	4.9645 5.6077 0.6432	4.7816 5.5594 0.7778	4.7175 5.5734 0.856

HPD_S: HPD credible interval based on least squared error
HPD_{LT}: HPD credible interval based on LINEX loss function

1.08, 2.03, 1.89, 2.88, 2.82, 2.5, 3.6, 1.47, 3.11, 3.22, 1.69, 3.15, 4.9, 2.97, 3.39, 2.93, 3.22, 3.33, 2.55, 2.56, 3.56, 2.59, 2.38, 2.83, 1.92, 1.36, 0.98, 1.84, 1.59, 5.56, 1.73, 1.12, 1.71, 2.48, 1.18, 1.25, 4.38, 2.48, 0.85, 2.03, 1.8, 1.61, 2.12, 2.05, 3.65.

The point and estimation techniques in Section (3), (4) and (5) can be applied to this data set throughout the steps below:

1. Sorting the data set in ascending order.
2. Applying the censoring scheme C-UHCS($m, r; T_1, T_2$) using one arbitrary case of Type-II censor-

Table 4. 95% and 90% Interval estimation of the parameter β when $n = 30, \lambda = 5, \beta = 2$ and $\theta = 2$.

T		ML	$Bootstrap$	HPD_S	HPD_{LT}		
					$\rho = -0.5$	$\rho = 1.0$	$\rho = 1.5$
$X_{r=10:n}$	95%	1.7489 2.656 0.9071	1.9078 2.2267 0.3189	1.9125 2.1361 0.2236	1.4056 2.02 0.6144	1.1589 2.5683 1.4094	1.4042 2.0164 0.6122
	90%	1.7495 2.6351 0.8856	1.9145 2.2176 0.3031	1.9101 2.1315 0.2214	1.419 2.0052 0.5862	1.1669 2.5609 1.394	1.4208 2.0078 0.587
$X_{r=15:n}$	95%	1.8257 2.514 0.6883	1.9782 2.2249 0.2467	1.9681 2.1799 0.2118	1.5921 2.0159 0.4239	1.4358 2.3563 0.9205	1.5886 2.0187 0.4302
	90%	1.8301 2.5079 0.6778	1.9788 2.2192 0.2404	1.9491 2.1611 0.212	1.6049 2.0028 0.3979	1.4378 2.3499 0.9122	1.5977 2.0115 0.4138
$X_{k=20:n}$	95%	1.8536 2.4237 0.5701	1.9789 2.2207 0.2418	1.9671 2.1645 0.1974	1.6915 2.0137 0.3222	1.7138 2.2432 0.5294	1.6882 2.0141 0.3259
	90%	1.8584 2.4104 0.552	1.9912 2.2169 0.2257	1.9895 2.1515 0.162	1.7022 2.0032 0.301	1.7165 2.1922 0.4757	1.6973 2.0022 0.3049
$X_{k=25:n}$	95%	1.805 2.2553 0.4503	1.9888 2.2191 0.2303	1.9878 2.1742 0.1864	1.7504 2.0143 0.2639	1.6093 2.0064 0.3971	1.7519 2.013 0.2611
	90%	1.8102 2.2424 0.4322	1.9989 2.2144 0.2155	1.9978 2.1793 0.1815	1.7576 2.0076 0.25	1.6094 2.0009 0.3914	1.7599 2.0049 0.245
$T_1 = 18$	95%	1.8275 2.2544 0.4269	1.9882 2.2192 0.231	1.9718 2.1698 0.198	1.7269 2.0157 0.2888	1.6462 2.1718 0.5256	1.7317 2.0148 0.283
	90%	1.8337 2.2401 0.4064	1.9982 2.2156 0.2174	1.9899 2.175 0.1851	1.7356 2.01 0.2744	1.7526 2.156 0.4034	1.7401 2.007 0.2669
$T_1 = 24$	95%	1.8581 2.2525 0.3944	1.9785 2.2282 0.2497	1.9698 2.1857 0.2159	1.5349 2.016 0.4812	1.556 2.4197 0.8637	1.5334 2.0189 0.4855
	90%	1.8769 2.2362 0.3593	1.9868 2.2208 0.234	1.9581 2.1738 0.2157	1.5473 2.0055 0.4581	1.5628 2.4043 0.8415	1.5461 2.0107 0.4646
$T_2 = 30$	95%	1.8678 2.2417 0.3739	1.9978 2.217 0.2192	1.9998 2.1812 0.1814	1.7666 2.0137 0.2471	1.7155 2.0919 0.3764	1.7681 2.0111 0.243
	90%	1.899 2.2401 0.3411	1.9995 2.2141 0.2146	1.9971 2.1768 0.1797	1.7726 2.0079 0.2353	1.7156 2.0682 0.3526	1.7737 2.005 0.2313
$T_2 = 36$	95%	1.9684 2.264 0.2956	1.9784 2.2246 0.2462	1.9668 2.1652 0.1984	1.6365 2.0148 0.3784	1.3802 2.3056 0.9254	1.6354 2.0157 0.3803
	90%	1.9762 2.2549 0.2786	1.9802 2.2186 0.2384	1.988 2.1536 0.1656	1.6502 2.0056 0.3553	1.3855 2.2989 0.9134	1.6443 2.0019 0.3576

HPD_S: HPD credible interval based on least squared error
HPD_{LT}: HPD credible interval based on LINEX loss function

ing at $X_{25:100} = 1.74$ and arbitrary case of Type-I censoring at $T = 2.4$.

3. Applying the point estimations of the parameters λ,β and θ using MLE, Tierney-Kadane and MCMC methods (MCMC are based on 15000 repetitions and 5000 burns).
4. Calculating the 95% and 90% HPD credible intervals using MCMC based on squared error loss function.
5. The results of the point and interval estimation of the unknown parameters are displayed in Tables 7 and 8.

Table 5. 95% and 90% Interval estimation of the parameter θ when $n = 30, \lambda = 5, \beta = 2$ and $\theta = 2$.

T		ML	$Bootstrap$	HPD_S	HPD_{LT}		
					$\rho = -0.5$	$\rho = 1.0$	$\rho = 1.5$
$X_{r=10:n}$	95%	0.7189 3.2086 2.4898	0.4627 2.8471 2.3844	1.4028 2.6683 1.2655	1.2825 2.2195 0.937	1.5545 2.3768 0.8223	1.625 2.7123 1.0873
	90%	0.8205 3.1491 2.3286	0.5627 2.9076 2.3449	1.4032 2.6546 1.2514	1.2825 2.1632 0.8806	1.5636 2.3573 0.7937	1.625 2.7123 1.0873
$X_{r=15:n}$	95%	1.1368 2.8548 1.7181	1.0812 2.6651 1.5839	1.7381 2.4344 0.6963	1.5355 2.1759 0.6404	1.5868 2.215 0.6282	1.7435 2.5074 0.7638
	90%	1.1726 2.8136 1.641	1.1012 2.6801 1.5789	1.7442 2.4119 0.6677	1.5555 2.1887 0.6332	1.5964 2.1953 0.5989	1.7435 2.47 0.7265
$X_{k=20:n}$	95%	1.3395 2.647 1.3076	1.2596 2.4642 1.2046	1.5794 2.2142 0.6348	1.6333 2.1204 0.4871	1.5763 2.0877 0.5114	1.8405 2.4217 0.5812
	90%	1.3623 2.6044 1.2421	1.3175 2.5177 1.2002	1.8941 2.516 0.6219	1.6333 2.104 0.4707	1.5833 2.0654 0.4821	1.8405 2.3672 0.5268
$X_{k=25:n}$	95%	1.4763 2.5274 1.0511	1.4401 2.3948 0.9546	1.7151 2.3253 0.6102	1.6987 2.0984 0.3997	1.766 2.0669 0.3009	1.87 2.3354 0.4654
	90%	1.5002 2.5044 1.0043	1.4401 2.3452 0.9051	1.723 2.2991 0.5761	1.6987 2.0823 0.3835	1.7685 2.0411 0.2726	1.87 2.3101 0.4401
$T_1 = 18$	95%	1.4217 2.5676 1.1459	1.3419 2.3888 1.0468	1.6474 2.2818 0.6344	1.6718 2.0971 0.4253	1.6078 2.0092 0.4014	1.8339 2.3407 0.5068
	90%	1.447 2.5303 1.0833	1.3419 2.3042 0.9623	1.589 2.1938 0.6048	1.6818 2.104 0.4222	1.615 1.9903 0.3753	1.8339 2.3349 0.501
$T_1 = 24$	95%	1.0018 2.9762 1.9744	0.9066 2.7038 1.7972	1.6136 2.6236 1.01	1.4692 2.2074 0.7382	1.4449 2.0777 0.6328	1.7081 2.5718 0.8638
	90%	1.082 2.8934 1.8114	1.1066 2.7118 1.6052	1.6187 2.6062 0.9875	1.4892 2.2104 0.7213	1.4482 2.0705 0.6223	1.7081 2.5337 0.8256
$T_2 = 30$	95%	1.4913 2.4924 1.001	1.4721 2.3811 0.909	1.7284 2.2596 0.5312	1.7319 2.1083 0.3765	1.7869 2.0398 0.2529	1.8609 2.2978 0.437
	90%	1.5199 2.4638 0.9438	1.4721 2.3679 0.8959	1.7331 2.2516 0.5185	1.7419 2.1107 0.3688	1.7968 2.0363 0.2395	1.8709 2.316 0.4251
$T_2 = 36$	95%	1.2225 2.7506 1.5281	1.1188 2.5164 1.3977	1.8853 2.5395 0.6542	1.5903 2.1688 0.5784	1.5935 2.1895 0.596	1.7732 2.4407 0.6675
	90%	1.2715 2.7046 1.4331	1.1188 2.4309 1.3121	1.648 2.2724 0.6244	1.5913 2.1692 0.5799	1.5942 2.1794 0.5852	1.7732 2.4396 0.6664

HPD_S: HPD credible interval based on least squared error
HPD_{LT}: HPD credible interval based on LINEX loss function

The underline selections in Tables (7) represent the best point estimation with minimum variances. Also,underline selections in Tables (8) represent the selected best interval estimates with minimum interval width.

7. Conclusion

In this paper, parameters point and interval estimation of C-U hybrid censored model of the Dagum model under classical and Bayesian perspectives are discussed. The MLEs and asymptotic CIs for the interested parameters are computed. Since the Bayesian estimates of the involved parameters could

Table 6. CI using McMc the parameters when $n = 30, \lambda = 5, \beta = 2$ and $\theta = 2$.

T	λ		β		θ	
	95%	90%	95%	90%	95%	90%
$X_{r=10:n}$	2.2612 5.7079 3.4467	2.397 5.2419 2.8449	1.4673 2.2299 0.7627	1.5146 2.1574 0.6427	1.089 2.7555 1.6665	1.1792 2.568 1.3888
$X_{r=15:n}$	3.2011 5.774 2.573	3.3156 5.4827 2.1672	1.5123 2.1693 0.657	1.5558 2.1016 0.5459	0.9799 2.3119 1.332	1.0516 2.1579 1.1063
$X_{k=20:n}$	3.1594 5.2554 2.096	3.2552 5.0265 1.7713	1.6151 2.2627 0.6476	1.6567 2.1992 0.5425	1.8853 2.991 1.1057	1.9577 2.8866 0.9289
$X_{r=25:n}$	4.0678 5.8652 1.7974	4.1599 5.6728 1.5129	1.6377 2.2228 0.5851	1.6794 2.1732 0.4938	1.8064 2.7845 0.9781	1.8634 2.6919 0.8285
$T_1 = 18$	2.2023 5.0454 2.8431	3.3361 5.7186 2.3825	1.4538 2.0967 0.6429	1.4946 2.0382 0.5436	1.0156 2.5102 1.4946	1.0914 2.3422 1.2508
$T_1 = 24$	1.1682 5.5887 2.4205	1.2789 5.2896 2.0107	1.5336 2.1702 0.6367	1.5748 2.1128 0.538	0.92 2.1664 1.2464	0.9965 2.036 1.0395
$T_1 = 30$	4.0586 5.978 1.9194	4.1574 5.7554 1.598	1.606 2.2067 0.6007	1.646 2.1437 0.4976	1.8271 2.8706 1.0434	1.8869 2.7562 0.8693
$T_1 = 36$	4.0154 5.7364 1.7209	4.1155 5.5376 1.4221	1.6228 2.1806 0.5577	1.6627 2.1322 0.4695	1.7817 2.7215 0.9398	1.8362 2.6184 0.7823

Table 7. Pint estimation and the estimated variances of the unknown parameters using the real data set.

Censoring		MLE	Tierney-Kadane				MCMC			
			SE	LE(-.5)	LE(1)	LE(1.5)	SE	LE(-.5)	LE(1)	LE(1.5)
$x_{25:100}$	λ	6.6126	1.7776	1.6093	1.6511	1.6632	1.4560	1.5198	1.5566	1.5972
		6.6948	0.0482	0.0434	0.0431	0.0432	0.0508	0.0432	0.0434	0.0433
	β	2.5055	1.3135	1.3900	1.3308	1.3922	1.3444	1.3639	1.3737	1.3837
		2.5544	0.0460	0.0455	0.0411	0.0488	0.0469	0.0465	0.0468	0.0409
	θ	2.4578	2.7547	2.8720	2.8539	2.7734	2.4850	2.5687	2.6144	2.6644
		1.6470	0.0268	0.0218	0.0271	0.0263	0.0272	0.0247	0.0242	0.0246
$T = 2.4$	λ	2.9549	1.3291	1.3711	1.4598	1.4453	1.4568	1.4973	1.5183	1.5396
		2.8270	0.0364	0.0367	0.0337	0.0361	0.0367	0.0327	0.0327	0.0326
	β	1.7964	1.3560	1.3494	1.3707	1.3223	1.3630	1.3763	1.3830	1.3898
		1.7344	0.0340	0.0360	0.0345	0.0370	0.0328	0.0456	0.0337	0.0316
	θ	1.9669	2.7632	2.7933	2.8013	2.7365	2.3428	2.4169	2.4573	2.4995
		1.4303	0.0142	0.0170	0.0169	0.0146	0.0197	0.0181	0.0188	0.0123

The first row represents the point estimation and the second row is the estimated variance.

LE(a)=Linex loss function when $\rho = a$

not be obtained analytically, so Tierney and Kadane's approach have employed to obtain approximate Bayes estimates. It is found that, the performances of the Bayesian estimates based on LINEX loss function are superior than those of the corresponding ML estimators. Similar improvements are observed for the Bayesian estimates evaluated for different loss functions. However, depending on the

Table 8. 90% and 95% C.I and the interval widths of the unknown parameters using the real data set.

Censoring		MLE		Tierney-Kadane		MCMC	
		95%	90%	95%	90%	95%	90%
$x_{25:100}$	λ	0.052 13.174	1.089 12.136	0.981 2.000	1.102 2.004	0.968 2.020	0.987 2.018
		13.122	11.046	1.019	0.902	1.052	1.031
	β	0.002 5.009	0.398 4.613	1.006 1.685	1.009 1.632	1.018 1.672	1.021 1.666
		5.007	4.215	0.639	0.622	0.654	0.645
	θ	0.844 4.072	1.099 3.817	1.693 3.200	1.884 3.173	1.833 3.183	1.900 3.129
		3.228	2.718	1.507	1.290	1.350	1.230
$T = 2.4$	λ	0.184 5.725	0.623 5.287	0.942 2.069	1.028 1.967	0.927 2.070	1.026 1.983
		5.541	4.665	1.127	0.938	1.144	0.957
	β	0.097 3.496	0.366 3.227	1.051 1.699	1.091 1.675	1.053 1.695	1.092 1.621
		3.399	2.862	0.648	0.584	0.642	0.529
	θ	0.565 3.369	0.787 3.147	1.713 3.148	1.793 3.057	1.742 3.131	1.790 3.057
		2.803	2.360	1.435	1.264	1.388	1.267

The first row represents the interval estimation while the second row is the interval width.

different values of asymmetry parameter ρ, the ER of LINEX loss function may be smaller than those of MLEs. The points estimates in the real data set show that both of Tierney-Kadanean approximation and MCMC are comparable in terms of the estimated variances as well as in the interval estimation in terms on the interval width.

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Conflict of interest

The authors have no conflict of interest.

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