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Research article

Bernstein collocation method for neutral type functional differential equation

Ishtiaq Ali*

Department of Mathematics and Statistics, College of Science, King Faisal University, P. O. Box 400, Al-Ahsa 31982, Saudi Arabia.

* **Correspondence:** Email: iamirzada@kfu.edu.sa.

Abstract: Functional differential equations of neutral type are a class of differential equations in which the derivative of the unknown functions depends on the history of the function and its derivative as well. Due to this nature the explicit solutions of these equations are not easy to compute and sometime even not possible. Therefore, one must use some numerical technique to find an approximate solution to these equations. In this paper, we used a spectral collocation method which is based on Bernstein polynomials to find the approximate solution. The disadvantage of using Bernstein polynomials is that they are not orthogonal and therefore one cannot use the properties of orthogonal polynomials for the efficient evaluation of differential equations. In order to avoid this issue and to fully use the properties of orthogonal polynomials, a change of basis transformation from Bernstein to Legendre polynomials is used. An error analysis in infinity norm is provided, followed by several numerical examples to justify the efficiency and accuracy of the proposed scheme.

Keywords: Bernstein collocation method; functional differential equation; convergence analysis; numerical examples

1. Introduction

Delay differential equations (DDEs) are used numerously in many applications of engineering sciences and technology. They are used to describe the propagation of transport phenomena in dynamical systems, especially those dynamical systems which are nonlinear in nature. As in DDEs the unknown functions also depend on the history, therefore it is natural to use DDEs in the mathematical modeling of some biological processes (cell growth etc.) and economical system (evaluation of market, investment policy etc.). Functional differential equation also known as pantograph type delay differential equation is an important class of DDEs arises in many application, for example, immunology, physiology, electrodynamics, communication and neural network where signal transmission is carried by time interval (nonzero) between the initial and delivery time of a signal or message, where such systems are often described by functional spaces in mathematical framework. The delay term in these models is related to some hidden processes and therefore one must use a high order numerical technique to capture these hidden processes. A comprehensive list of applications of DDEs can be found in [12, 22]. Consider the DDEs of the form

$$\begin{cases} u'(t) &= \alpha(t)u(t) + \beta(x)u(rt) + \gamma(x)u'(rt), \quad t \in I := [0,\ T] \\ u(0) &= u_0. \end{cases} \tag{1.1}$$

where $\alpha(t)$,$\beta(t)$ and $\gamma(t)$ are smooth functions on $I := [0,\ T\,]$ and $r \in (0, 1)$ is a fixed constant known as proportional delay. Equation (1.1), which is a special type DDEs called the general pantograph type DDE with neutral term. Due to the transcendental nature of Eq (1.1) most of the author's used approximate methods to solve it numerically. In the start of twenty's, the researcher uses the application of collocation method and continuous Runge-Kutta (CRK) method [12, 15]. The CRK method does not achieve the required accuracy due the insufficient information on the right hand side of Eq (1.1), while using the collocation method with piecewise polynomials having degree $n \geq 1$ with meshes to be uniform does not achieve the classical superconvergence rate of $O(h^{2n})$-; for $n \geq 2$ the order (optimal) is only $n + 2$ [13, 14]. Thus, it was natural to switch to some methods to avoid these difficulties and get the exponential order of convergence with less computational efforts [10, 11]. To this end, the use of orthogonal polynomials and their properties are more useful to achieve the required accuracy. The idea of rational polynomial approximation was introduced in [24], while the Legendre collocation methods with detail convergence analysis results was used for the variety of DDEs and stochastic DDE including the pantograph type in [8, 9, 25–28]. Similarly the Tau method based on Chebyshev approximation and their operational matrix is used in [23]. The main aim of this work is to use Bernstein polynomial to find the approximate solution of Eq (1.1). The disadvantage of using Bernstein polynomial is that these polynomials are not orthogonal in nature. For this reason the change of basis function from Bernstein to Legendre polynomial will be used with the help of some matrix transformation [4–6]. The Bernstein approximation method is a powerful numerical technique used by a number of authors for the numerical approximation of different type of differential equations [17–20, 29–31].

The rest of the paper is organized as: section 2 of the paper consist of preliminaries, followed by Bernstein collocation method in section 3. Section 4 describe the Bernstein-Legendre basis transformation. The error analysis is presented in section 5. Numerical examples are given in section 6, followed by conclusion in section 7.

2. Preliminaries

First we will introduce some basic of Bernstein polynomials and their properties. For any $\bar{t} \in [0, 1]$, Bernstein polynomials are define as [3].

$$\bar{B}^{\bar{n}}_{\bar{i}}(\bar{t}) = \binom{\bar{n}}{\bar{i}}\bar{t}(1 - \bar{t})^{\bar{n}-\bar{i}}, \bar{i} = 0, ..., \bar{n}, \quad \text{where}\binom{\bar{n}}{\bar{i}} = \frac{\bar{n}!}{\bar{i}!(\bar{n} - \bar{i})!}, \tag{2.1}$$

satisfying the following 3-term recurrence relation

$$\bar{B}^{\bar{n}}_{\bar{i}}(\bar{t}) = \bar{B}^{\bar{n}-1}_{\bar{i}}(\bar{t}) - \bar{t}\bar{B}^{\bar{n}-1}_{\bar{i}}(\bar{t}) + \bar{t}\bar{B}^{\bar{n}-1}_{\bar{i}-1}(\bar{t}). \tag{2.2}$$

First few terms of Bernstein polynomials are given by

$$\bar{B}_1^0 = 1 - \bar{t}, \quad \bar{B}_1^1 = \bar{t}, \quad \bar{B}_2^0 = 1 - \bar{t}^2, \quad \bar{B}_2^1 = 2\bar{t}(1 - \bar{t}), \quad \bar{B}_2^2 = \bar{t}^2,$$

$$\bar{B}_3^0 = (1 - \bar{t})^3, \quad \bar{B}_3^1 = 3\bar{t}(1 - \bar{t})^2, \quad \bar{B}_3^2 = 3\bar{t}^2(1 - \bar{t}), \quad \bar{B}_3^3 = \bar{t}^3.$$

Bernstein polynomials also satisfying the following properties.

i. $\sum_{\bar{i}=0}^{\bar{n}} \bar{B}_{\bar{i}}^{\bar{n}}(\bar{t}) \equiv 1$, (Unitary).

ii. $\bar{B}_{\bar{i}}^{\bar{n}}(\bar{t}) \geq 0, \quad \bar{t} \in [0, 1]$ (non negative).

iii. $\bar{B}_{\bar{i}}^{\bar{n}}(\bar{t}) = \bar{B}_{\bar{n}-\bar{i}}^{\bar{n}}(1 - \bar{t})$,(symmetric).

iv. $\bar{B}_{\bar{i}}^{\bar{n}}(\bar{t})$, has maximum value at $\bar{t} = \frac{\bar{i}}{\bar{n}}$ (uni-modality).

There product and integral is given

$$\bar{B}_{\bar{i}}^{\bar{n}}(\bar{t})\bar{B}_{\bar{j}}^{\bar{m}}(\bar{t}) = \frac{\binom{\bar{n}}{\bar{i}}\binom{\bar{m}}{\bar{j}}}{\binom{\bar{n}+\bar{m}}{\bar{i}+\bar{j}}}\binom{\bar{n}+\bar{m}}{\bar{i}+\bar{j}},$$

$$\int_0^1 \bar{B}_{\bar{i}}^{\bar{n}}(\bar{t})d\bar{t} = \frac{1}{\bar{n}+1}.$$

Bernstein polynomial form a complete basis over the interval $[a, b]$. Any unknown function $u(t)$ which is define on $[a, b]$ can be approximated with Bernstein polynomials having n degree basis function as

$$u(t) \equiv \sum_{\bar{i}=0}^{\bar{n}} \bar{C}_i \bar{B}_{\bar{i}}^{\bar{n}}(\bar{t}) = \bar{C}^T \bar{B}(\bar{t}), \tag{2.3}$$

where $\bar{C}$ and $\bar{B}(\bar{t})$ are $(\bar{n} + 1) \times 1$ given as

$$\bar{C} = [\bar{c}_0, \bar{c}_1, \bar{c}_2, ..., \bar{c}_{\bar{n}}]^T,$$

$$\bar{B}(\bar{t}) = [\bar{B}_{\bar{0}}^{\bar{n}}, \bar{B}_{\bar{1}}^{\bar{n}}, \bar{B}_{\bar{2}}^{\bar{n}}, ..., \bar{B}_{\bar{n}}^{\bar{n}}].$$

Since we are interested in the Legendre form of Bernstein polynomials. The Legendre polynomials form orthonormal basis on $[-1, 1]$, while Bernstein polynomials are define over $[0, 1]$. In order to use the orthogonality properties of Legendre polynomials with very sophisticated geometric properties of Bernstein polynomials, the recurrence relation of Legendre polynomials $\bar{L}(\bar{t})$ on $\bar{t} \in [0, 1]$ is given by

$$\bar{L}_{\bar{n}}(\bar{t}) = \frac{2\bar{n}-1}{\bar{n}}(2\bar{t} - 1)\bar{L}_{\bar{n}-1}(\bar{t}) - \frac{\bar{n}-2}{\bar{n}}\bar{L}_{\bar{n}-2}(\bar{t}).$$

The first few Legendre polynomials on $[0, 1]$ are given by

$$L_0(\bar{t}) = 1, \quad \bar{L}_1(\bar{t}) = \sqrt{3}(2\bar{t} - 1), \quad \bar{L}_2(\bar{t}) = \sqrt{5}(6\bar{t}^2 - 6\bar{t} + 1),$$

$$\bar{L}_3(\bar{t}) = \sqrt{7}(20\bar{t}^3 - 30\bar{t}^2 + 12\bar{t} - 1).$$

The orthonormal properties of the shifted Legendre polynomial is given by

$$\int_0^{\tau_f} \bar{L}_{\bar{j}}(\bar{t})L_{\bar{k}}(\bar{t}) = \begin{cases} \frac{\tau_f}{2\bar{k}+1}, & \text{if} \quad \bar{j} = \bar{k}, \\ 0, & \text{if} \quad \bar{j} \neq \bar{k}. \end{cases}$$

3. Bernstein-Legendre basis transformation

As we know that for non-negative bases polynomials orthogonality is not possible. To avoid this and in order to fully use the properties of orthogonal polynomials with the geometric properties of Bernstein basis, we will use matrices transformation between Bernstein and Legendre polynomials [1, 2].

Consider $\bar{P}_{\bar{n}}(\bar{t})$, a polynomial of degree $\bar{n}$ can be expressed in the degree $\bar{n}$ Bernstein and Legendre basis on $\bar{t} \in [0, 1]$ in the following form:

$$\bar{P}_{\bar{n}}(\bar{t}) = \sum_{\bar{j}=0}^{\bar{n}} \bar{c}_{\bar{j}}\bar{B}_{\bar{j}}^{\bar{n}}(\bar{t}) = \sum_{\bar{k}=0}^{\bar{n}} \bar{l}_{\bar{k}}\bar{L}_{\bar{k}}(\bar{t}). \tag{3.1}$$

The linear transformation that maps the Bernstein coefficients $\bar{c}_{\bar{0}}, \bar{c}_{\bar{1}}, ..., \bar{c}_{\bar{n}}$ into the Legendre coefficient $\bar{l}_{\bar{0}}, \bar{l}_{\bar{1}}, ..., \bar{l}_{\bar{n}}$ is given by Eqs (5) and (7) respectively.

$$\bar{c}_{\bar{j}} = \sum_{j=0}^{\bar{n}} \bar{M}_{\bar{n}}(\bar{j}, \bar{k})\bar{l}_{\bar{k}}, \quad \bar{j} = 0, 1, ..., \bar{n}, \tag{3.2}$$

$$\bar{l}_{\bar{k}} = \sum_{\bar{k}=0}^{\bar{n}} \bar{M}_{\bar{n}}^{-1}(\bar{j}, \bar{k})\bar{c}_{\bar{j}}, \quad \bar{k} = 0, 1, ..., \bar{n}, \tag{3.3}$$

where

$$\bar{M} = \frac{1}{\binom{\bar{n}}{\bar{k}}} \sum_{\bar{i}=max(\bar{0},\bar{j}+\bar{k}-\bar{n})}^{min(\bar{j},\bar{k})} (-1)^{\bar{k}+\bar{i}} \binom{\bar{j}}{\bar{i}}\binom{\bar{k}}{\bar{i}}\binom{\bar{n}-\bar{k}}{\bar{j}-\bar{i}},$$

and

$$\bar{M}^{-1} = \frac{2\bar{j}+1}{\bar{n}+\bar{j}+\bar{k}}\binom{\bar{n}}{\bar{k}} \sum_{\bar{i}=0}^{\bar{j}} (-1)^{\bar{j}+\bar{i}} \frac{\binom{\bar{j}}{\bar{i}}\binom{\bar{j}}{\bar{i}}}{\binom{\bar{n}-\bar{j}}{\bar{j}-\bar{i}}},$$

As we are interested in the Bernstein form of Legendre polynomial, therefore the the Legendre polynomial in Bernstein form are given by

$$\bar{L}_{\bar{n}}(\bar{t}) = \sum_{\bar{i}=0}^{\bar{n}} (-1)^{\bar{n}+\bar{i}} \binom{\bar{n}}{\bar{i}} \bar{B}_{\bar{i}}^{\bar{n}}(\bar{t}), \tag{3.4}$$

where the first few Legendre polynomial in Bernstein form are given by:

$$\bar{L}_0(\bar{t}) = B_0^0(\bar{t}), \quad \bar{L}_1(\bar{t}) = -B_0^1(\bar{t}) + B_1^1(\bar{t}), \quad \bar{L}_2(\bar{t}) = B_0^2(\bar{t}) - 2B_1^2(\bar{t}) + B_2^2(\bar{t}).$$

$$\bar{L}_3(\bar{t}) = -B_0^3(\bar{t}) + 3B_1^3(\bar{t}) - 3B_2^3(\bar{t}) + B_3^3(\bar{t}).$$

4. Bernstein collocation method

In order to fully use the properties of orthogonal polynomials, we will apply spectral method to the integrated form of Eq (1.1). For the reason integrating Eq (1.1) from $[0, t]$, we get:

$$u(t) = u_0 + \int_0^t \alpha(s)u(s)ds + \int_0^t \beta(s)u(rs)ds + \int_0^t \gamma(s)u'(rs)ds. \tag{4.1}$$

Let Eq (4.1) holds at t_j, where $t_j = t_0 + kh$ are the collocation points with $t_0 = a$, $h = b - a/\bar{n}, k = 0, 1, 2, ..., \bar{n} - 1$, we get

$$\begin{aligned} u(t_j) = u_0 + \int_0^{t_j} \alpha(s)u(s)ds + \int_0^{t_j} \beta(s)u(rs)ds \\ + \int_0^{t_j} \gamma(s)u'(rs)ds, \end{aligned} \tag{4.2}$$

or

$$\begin{aligned} u(t_j) = u_0 + \gamma(t_j)u(rt_j) - \gamma(0)u(0) + \int_0^{t_j} \alpha(s)u(s)ds \\ + \int_0^{t_j} \beta(s)u(rs)ds + \int_0^{t_j} \gamma(s)u'(rs)ds, \end{aligned} \tag{4.3}$$

Using the linear transformation

$$s = s't_j/\tau_f, \quad 0 \leq t_j \leq \tau_f,$$

we get

$$\begin{aligned} u(t_j) = u_0 + \gamma(t_j)u(rt_j) - \gamma(0)u(0) + t_j \int_0^1 \alpha(s')u(s')ds' \\ + rt_j \int_0^1 \beta(s')u(rs')ds' + t_j \int_0^1 \gamma(s')u'(rs')ds'. \end{aligned} \tag{4.4}$$

Using Eq (4) in Eq (11), we get

$$\begin{aligned} \bar{C}^T \bar{B}'(\bar{t}) = \bar{C}^T \bar{B}(0) + \gamma(t_j)\bar{C}^T \bar{B}(\bar{t})(rt_j) - \gamma(0)\bar{C}^T \bar{B}(0) + t_j \int_0^1 \alpha(s')\bar{C}^T \bar{B}(\bar{t})(s')ds' \\ + rt_j \int_0^1 \beta(s')\bar{C}^T \bar{B}(\bar{t})(rs')ds' + t_j \int_0^1 \gamma(s')\bar{C}^T \bar{B}'(\bar{t})(rs')ds'. \end{aligned} \tag{4.5}$$

Thus together with the initial condition we get a linear system of $2\bar{n} + 2$ equations. As we are more interested in the Legendre form of Bernstein polynomial, therefore using the $(N + 1)$-point Gauss-Legendre points relative to the Legendre weight gives

$$u(t_j) = u_0 + t_j \sum_{k=0}^{N} \alpha(s')u(s')\omega_k + rt_j \sum_{k=0}^{N} \beta(s')u(rs')\omega_k + \gamma(t_j)u(rt_j) - u(0)\gamma(0) - t_j \sum_{k=0}^{N} \gamma(s')u'(rs')\omega_k. \tag{4.6}$$

Let $U_i \approx u(t_j)$ and assume that $U \in \mathcal{P}_N$ is of the form

$$U(t) = \sum_{j=0}^{N} U_j \mathcal{F}_j(t), \tag{4.7}$$

where $\mathcal{F}_j(t)$ is Lagrange interpolation polynomials associated with Legendre-Gauss points $\{t_j\}_{j=0}^{N}$. The numerical approximation for solving (1.1) is then given by

$$U_j = u_0(1-\gamma(0))/r + t_j \sum_{k=0}^{N} \alpha(s')U(s')\omega_k + rt_j \sum_{k=0}^{N} \beta(s')U(rs')\omega_k - t_j \sum_{k=0}^{N} \gamma(s')U'(rs')\omega_k. \tag{4.8}$$

Let $U = [U_0, \cdots, U_N]^T$ and $\mathcal{F}_N = [u_0(1-\gamma(0)), \cdots, u_N(1-\gamma(0))]^T$, we can obtain a matrix form:

$$U + AU = \mathcal{F}_N, \tag{4.9}$$

To compute $\mathcal{F}(s)$ in efficient way, we express it in terms of the Bernstein form of the Legendre functions given in Eq (5).

5. Error analysis

Theorem. If $u_{\bar{j}}(\bar{t}), j = 1, 2, ...\bar{n}$ denotes the exact solution to the neutral functional differential equation of pantograph type (1.1), while $U_{\bar{j},\bar{m}}(\bar{t})$ denotes its approximate solution, then the error between the exact and approximate solution converge exponentially that is

$$\left\| u_{\bar{j}}(\bar{t}) - U_{\bar{j},\bar{m}}(\bar{t}) \right\| \longrightarrow 0, \quad \bar{m} \longrightarrow \infty.$$

Proof. Let $U_{\bar{j},\bar{m}}(\bar{t}) = \sum_{\bar{p}=0}^{\bar{m}} \bar{c}_{\bar{j}}^{\bar{p}} \bar{B}_{\bar{p}}^{\bar{m}}(\bar{t})$, where $B_{\bar{p}}^{\bar{m}}(\bar{t})$ is the m degree Bernstein polynomial, denote the approximate solution to equation and $u_{\bar{j}}(\bar{t}), j = 1, 2, ...\bar{n}$ represent the exact solution. Assume that

$$u_{\bar{j}}(\bar{t}) = \lim_{m \longrightarrow \infty} U_{\bar{j},\bar{m}}(\bar{t}),$$

holds. Let

$$e_{\bar{m}}(\bar{t}) = \sum_{\bar{i}=0}^{\bar{n}} e_{\bar{i},\bar{m}}(\bar{x}), \tag{5.1}$$

where $e_{\bar{m}}(\bar{x})$ denotes the difference between the exact and approximate solution. From Eq (11), we have

$$e_{\bar{m}}(\bar{t}) \leq \sum_{i=0}^{\bar{n}} e_{\bar{i},\bar{m}}(\bar{t}) \leq \sum_{i=0}^{\bar{n}} \left\| u_{\bar{j}}(\bar{t}) - U_{\bar{j},\bar{m}}(\bar{t}) \right\|. \tag{5.2}$$

Since all the coefficient in (1.1) are smooth function and therefore are all bounded, hence $\|e_{\bar{m}}(\bar{t})\| \longrightarrow 0, as \quad \bar{m} \longrightarrow \infty$.

6. Numerical examples

Example 6.1. *Consider the the following constructed example [16]*

$$\begin{cases} u'(t) &= \alpha u(t) + \beta u(rt) + cos(t) - \alpha sin(t) - \beta sin(rt), \quad t \in I := [0,\ T\ ] \\ u(0) &= 0. \end{cases} \tag{6.1}$$

The error between numerical solution and exact solution for $\alpha = -1, \beta = 0.5, r = 0.5$ and $T = 5$ for different N is shown in Table 1.

Table 1. Example 6.1: the point-wise error in L_∞ norm.

N	Error	N	Error	N	Error
6	1.174e−001	12	6.204e−006	17	9.730e−011
7	3.635e−002	13	4.092e−007	18	8.243e−012
9	1.752e−003	15	1.103e−008	19	6.651e−012
10	2.291e−004	16	1.075e−009	20	5.508e−013

Example 6.2. Choose $\alpha(t) = \sin(t)$, $\beta(t) = \cos(rt)$, $\gamma(t) = -\sin(rx)$ in (1.1). Figure 1 indicates the error behavior between approximate and exact solution for $r = 0.05$ and $T = 5$. The comparison was made with the Legendre spectral method presented in [8]. We found that both the method has a very good agreement with each other.

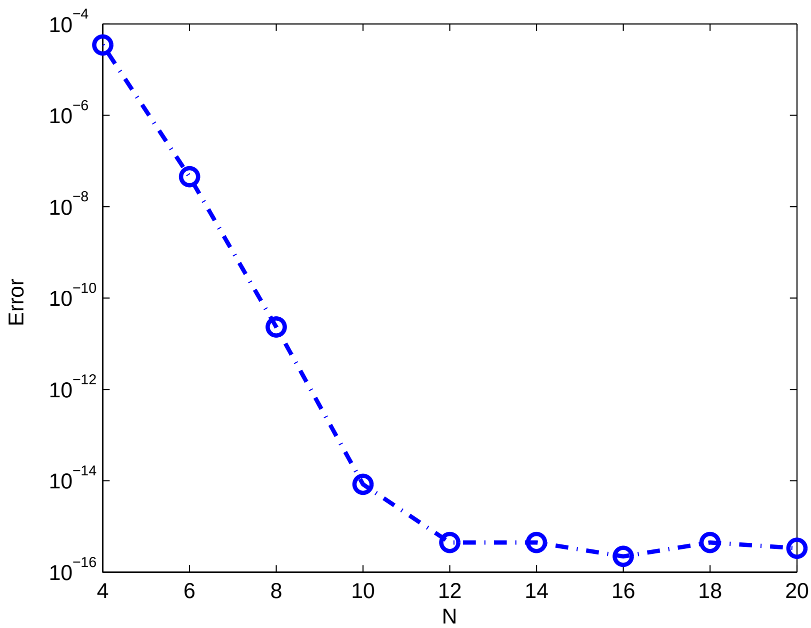

Figure 1. Example 6.2. The error behavior in L_∞ norm.

Example 6.3. *Consider the nonlinear equation of the form:*

$$u'(t) = \alpha u(t) + \beta u(rt)(1 - u(rt)).$$

The error behavior for $\alpha = 0.25, \beta = 1, r = 0.5$ *and* $T = 1$, *relative to N is shown in Figure 2.*

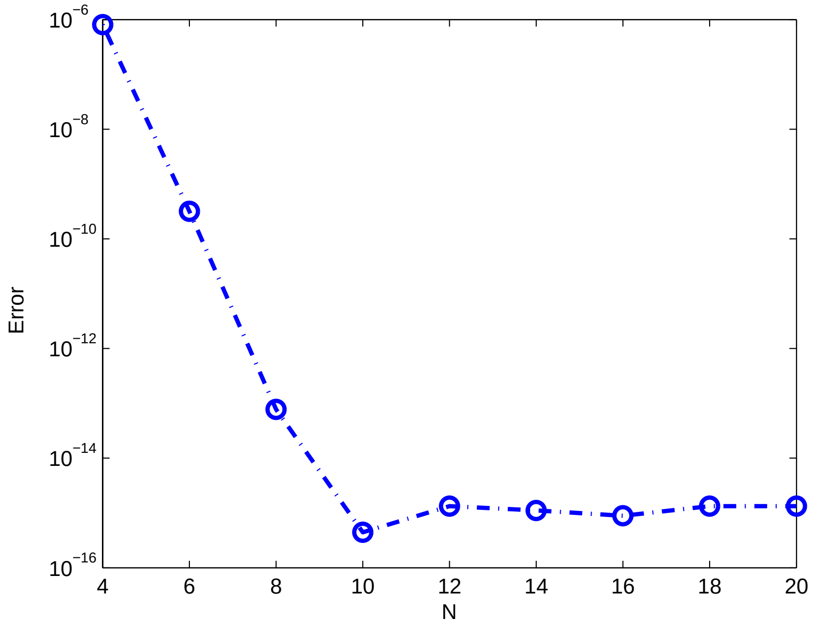

Figure 2. Example 6.3. The error behavior in L_∞ norm.

Example 6.4. *Consider the following initial value problem [7].*

$$\begin{cases} u'(t) &= -u(t) + \alpha y(rt) + u'(rt) + cos(t) - cos(rt) + sin(t), \quad t \in I := [0, \ T\ ] \\ u(0) &= 0 \end{cases}$$

The maximum point wise error for $\beta = 0, r = 0.5$ *and* $T = 2$ *for different N is given in Table 2.*

Table 2. Example 6.4: the point-wise error in L_∞ norm.

N	Error	N	Error	N	Error
8	6.800e−004	16	2.817e−011	24	2.665e−014
10	1.916e−005	18	2.317e−012	26	3.442e−014
12	2.198e−007	20	3.162e−013	28	3.442e−014
14	3.522e−009	22	1.998e−013	30	3.096e−014

Example 6.5. *Consider the the following example*

$$\begin{cases} u'(t) &= \alpha u(t) + \beta u(rt), \quad t \in I := [0, \ T\ ] \\ u(0) &= 1. \end{cases} \tag{6.2}$$

The error between numerical solution and exact solution for $\alpha = 0.5, \beta = 0.5, r = 0.5$ and $T = 3$, for different values of N is shown in Table 3. We compare the result with the Bernstein series solution method and found a very good agreement with it [29].

Table 3. Example 6.5: the point-wise error in L_∞ norm.

N	Error	N	Error	N	Error
6	1.023e−002	12	5.345e−007	17	8.971e−013
7	2.130e−003	13	3.021e−009	18	7.209e−013
9	1.567e−004	15	2.376e−010	19	3.312e−015
10	3.121e−006	16	3.218e−011	20	4.273e−016

7. Conclusions

A new method based on the Bernstein polynomials is introduced for the approximate solution of neutral functional differential equation of pantograph type with proportional delay. For better efficiency of the proposed scheme, a transformation from Bernstein to Legendre polynomial is used, which allow us to take the advantage of orthogonality of Legendre polynomials which is not possible in case of Bernstein polynomial directly. An error analysis is provided and a number of numerical experiments were performed to confirm the theoretical justification. The numerical as well as theoretical result shows that the method has a spectral accuracy. It is observed from our numerical experiments that while increasing the number of collocation points that is N one lose the spectral accuracy because of the fact that using Lagrange interpolating polynomials which is bounded by Lebesgue constant grows exponentially while increasing N. This is also because of the oscillating nature of orthogonal polynomials. In our proposed scheme one does not need to increase the number of collocation points as we achieve a spectral accuracy after a few collocations points.

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Conflict of interest

The author declares no competing interest regarding the publication of this paper.

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