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Research article

New formulation for discrete dynamical type inequalities via $\hbar$-discrete fractional operator pertaining to nonsingular kernel

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Abstract: Discrete fractional calculus (DFC) use to analyse nonlocal behaviour of models has acquired great importance in recent years. The aim of this paper is to address the discrete fractional operator underlying discrete Atangana-Baleanu (AB)-fractional operator having $\hbar$-discrete generalized Mittag-Leffler kernels in the sense of Riemann type (ABR). In this strategy, we use the $\hbar$-discrete AB-fractional sums in order to obtain the Grüss type and certain other related variants having discrete generalized $\hbar$-Mittag-Leffler function in the kernel. Meanwhile, several other variants found by means of Young, weighted-arithmetic-geometric mean techniques with a discretization are formulated in the time domain $\hbar\mathbb{Z}$. At first, the proposed technique is compared to discrete AB-fractional sums that uses classical approach to derive the numerous inequalities, showing how the parameters used in the proposed discrete $\hbar$-fractional sums can be estimated. Moreover, the numerical meaning of the suggested study is assessed by two examples. The obtained results show that the proposed technique can be used efficiently to estimate the response of the neural networks and dynamic loads.

Keywords: discrete fractional calculus; Atangana-Baleanu fractional differences and sums; discrete Mittag-Leffler function; Grüss type inequality; Young inequality

1. Introduction

DFC has captivated a lot of consideration across various analysis and engineering disciplines, particularly in modelling [1], neural networks [2] and image encryption [3]. The developing approach portraying real-world problems have been exhibited to be helpful in numerical devices to analyze, comprehend and predict the nature within humankind live [4–10]. In 1974, Daiz et al. [11] introduced the idea of DFC and composed it with an infinite sum. Later on, in 1988, Gray et al. [12] extended this concept and implemented it on the finite sum. This concept is known as the nabla difference operator in the literature. Atici and Eloe [13] proposed the theory of fractional difference equations, although the practical implementation is presented in [14]. Yilmazer [15] proposed discrete fractional solution of a nonhomogeneous non-Fuchsian differential equations. Yilmazer and Ali [16] derived the discrete fractional solutions of the Hydrogen atom type equations. Many researchers' focus is directed towards modeling and analysis of various problems in bio-mathematical sciences. This field demonstrates several distinguished kernels depending on discrete power law, discrete exponential-law and discrete Mittag-Leffler law kernels which correspond to the Liouville-Caputo, Caputo-Fabrizio and the Atangana-Baleanu nabla(delta) difference operators generalized $\hbar\mathbb{Z}$ time scale [17–19].

Numerous utilities have been developed via DFC such as the solution of fractional difference equations and discrete boundary value problems are proposed in terms of new mathematical techniques [20–23]. Therefore, the conventional methodology of DFC have some intriguing and less-acknowledged opportunities for modelling. DFC is proposed to depict the customary practice of time scale analysis, with discussing its numerical approximations in $\check{\hbar}\mathbb{Z}$. Furthermore, we observe that $\check{\hbar}$-discrete fractional calculus is tremendously momentous in applied sciences and can also address the requirements of synchronous operation of various mechanisms, see [24–26].

Among the computational models formulated in fractional calculus, discrete AB-fractional operators, which is a universal operator of fractional calculus that has been traditionally employed to develop modern operators and their characterizations have been proposed in research article [27, 28]. Moreover, DFC has been theoretically presented more by introducing and analyzing discrete forms of these fractional operators [13]. Here, we intend to find the discrete fractional inequalities analogous to fractional operators having $\hbar$-discrete Mittag-Leffler kernels, encompassing and simplifying these operators in such a manner as to recuperate certain appropriate traits such as discrete inequalities for AB-fractional sums.

Mathematical inequalities [29–38] initially alluded to adjust, harmony, and coordination. Until modern times, refinements of inequalities were characterized as invariance to change [39–43]. Physics comprehends fractional inequalities as predictability, while Psychology accentuates that inequality is the trait of magnificence and art [44].

Numerous investigations have been directed on fractional inequalities in the natural science [45], engineering sciences, see [41, 46–48] and the references cited therein. Landscapes, structures, and mechanical equipment all demonstrate inequalities attributes. Therefore, we intend to find the discrete version of the Grüss type and some further connected modifications by the $\hbar$-discrete AB-fractional sums depending on $\hbar$-discrete generalized Mittag-Leffler kernel. This stands as an inspiration for the current paper. The intensively investigated Grüss inequality can be presented as follows:

Theorem 1.1. *(See [49]) Let $\mathcal{F}, \mathcal{G} : [c, d] \to \mathbb{R}$ be two positive functions such that $\alpha \le \mathcal{F}(x) \le \mathcal{A}$ and $\beta \le \mathcal{G}(x) \le \mathcal{B}$ for all $x \in [c, d]$ and $\alpha, \beta, \mathcal{A}, \mathcal{B} \in \mathbb{R}$. Then*

$$\Big|\frac{1}{d-c}\int_c^d \mathcal{F}(x)\mathcal{G}(x)dx - \frac{1}{(d-c)^2}\int_c^d \mathcal{F}(x)dx\int_c^d \mathcal{G}(x)dx\Big| \tag{1.1}$$

$$\leq \frac{1}{4}(\mathcal{A}-\alpha)(\mathcal{B}-\beta),$$

where the constant 1/4 *can not be improved.*

The Grüss inequality Eq (1.1) has been broadly and intensely investigated in engineering and applied analysis, and various developed consequences have been acquired so far. Nevertheless, the prevalent existence of Grüss inequality in scientific fields is not in direct proportion to the consideration it has acknowledged. In application viewpoint, practically all mechanical structures are found to have inequality Eq (1.1), and the vast majority of them have the qualities of discrete and continuous fractional operators [50–63].

Inspired by the excellent dynamical properties of $\hbar$-discrete AB-fractional sums differences formulation [64], the limitations of fractional calculus can be ameliorated via discrete and continuous state-of-the-art techniques for effective information chaotic map applications, that can be inferred as a generalization of nonlocal/nonsingular type kernels. These investigations promote further sum/difference operators and related inequalities. It is our aim in this investigation to explore the discrete version of the Grüss type and certain other associated variants with some traditional and forthright inequalities in the frame of $\hbar$-discrete AB-fractional sums. We also would like to mention that besides these variants, several other intriguing generalizations are derived. The comparison of Grüss type with other discrete fractional calculus frameworks is currently under investigation. Finally, two examples are presented that correlate with some well-known inequalities in the relative literature and with the proposed strategy.

2. Preliminaries on discrete fractional calculus

In this section, we evoke some basic ideas related to fractional operator, discrete generalized Mittag Leffler functions and the time scale calculus, see the detailed information in [13]. For the sake of simplicity, we use the notation, for $c, d \in \mathbb{R}$ and $\hbar > 0$, $\mathbb{N}_{c,\hbar} = \{c, c+\hbar, c+2\hbar, ...\}$ and $\mathbb{N}_{d,\hbar} = \{d, d+\hbar, d+2\hbar, ...\}$.

2.1. Basics on delta and nabla $\hbar$-factorials

Definition 2.1. ([65])The backward difference operator of a function $\mathcal{F}$ on $\hbar\mathbb{Z}$ is stated as

$$\widehat{\nabla}_{\hbar}\mathcal{F}(t) = \frac{\mathcal{F}(t)-\mathcal{F}(\rho_{\hbar}(t))}{\hbar}, \tag{2.1}$$

where $\rho_{\hbar}(t) = t-\hbar$ denotes the backward jump operator. Also, the forward difference operator of a function $\mathcal{F}$ on $\hbar\mathbb{Z}$ is stated as

$$\widehat{\Delta}_{\hbar}\mathcal{F}(t) = \frac{\mathcal{F}(\rho_{\hbar}(t))-\mathcal{F}(t)}{\hbar}, \tag{2.2}$$

where $\sigma_{\hbar}(t) = t+\hbar$ denotes the forward jump operator.

Definition 2.2. ([65]) (i) For any $t, \alpha \in \mathbb{R}$ and $\hbar > 0$, the delta $\hbar$-factorial function is stated as

$$t_{\hbar}^{(\alpha)} = \hbar^{\alpha}\frac{\Gamma(\frac{t}{\hbar}+1)}{\Gamma(\frac{t}{\hbar}+1-\alpha)}, \tag{2.3}$$

where Γ denotes the Euler gamma function. For $\hbar = 1$, then $t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}$. Also, the division by a pole leads to zero.

(ii) For any $t, \alpha \in \mathbb{R}$ and $\hbar > 0$, the nabla $\hbar$-factorial function is stated as

$$t_{\hbar}^{(\alpha)} = \hbar^{\alpha}\frac{\Gamma(\frac{t}{\hbar} + \alpha)}{\Gamma(\frac{t}{\hbar})}. \tag{2.4}$$

For $\hbar = 1$, we observe that $t^{(\alpha)} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}$.

Lemma 2.3. *([64]) Let $t \in \mathbb{T} = \mathbb{N}_{c,\hbar}$, then for all $t \in \mathbb{T}^{\iota}$, we obtain*

$$\widehat{\nabla}_{x,\hbar}\left\{\frac{(x-t)_{\hbar}^{\overline{\iota+1}}}{(\iota+1)!}\right\} = \frac{(x-t)_{\hbar}^{\bar{\iota}}}{\iota!}. \tag{2.5}$$

Lemma 2.4. *([66]) For the time scale $\mathbb{T} = \mathbb{N}_{c,\hbar}$ then the nabla Taylor polynomial*

$$\widehat{\mathbb{B}}_{\iota}(x,t) = \frac{(x-t)_{\hbar}^{\bar{\iota}}}{\iota!}, \qquad \iota \in \mathbb{N}_0. \tag{2.6}$$

2.2. Nabla $\hbar$-discrete Mittag-Leffler function

Now we present the concept of nabla $\hbar$-discrete Mittag-Leffler function which is introduced by [6].

Definition 2.5. ([6]) Let $\alpha, \varrho, \Omega \in \mathbb{C}$ having $\Re(\alpha) > 0$ such that $\lambda \in \mathbb{R}$ with $|\lambda\hbar^{\alpha}| < 1$, then the nabla discrete Mittag-leffler function is defined

$${}_{\hbar}\check{E}_{\overline{\alpha,\varrho}}(\lambda,\Omega) = \sum_{\iota=0}^{\infty}\lambda^{\iota}\frac{\Omega_{\hbar}^{\overline{\iota\alpha+\varrho-1}}}{\Gamma(\alpha\iota+\varrho)}, \quad |\lambda\hbar^{\alpha}| < 1. \tag{2.7}$$

For $\varrho = 1$, we have

$${}_{\hbar}\check{E}_{\bar{\alpha}}(\lambda,y) \triangleq {}_{\hbar}\check{E}_{\overline{\alpha,1}}(\lambda,y) = \sum_{\iota=0}^{\infty}\lambda^{\iota}\frac{y_{\hbar}^{\overline{\iota\alpha}}}{\Gamma(\alpha\iota+1)}, \quad |\lambda\hbar^{\alpha}| < 1. \tag{2.8}$$

The following remark illustrates the strengthening properties why $\hbar\mathbb{Z}$ is important.

Remark 1. In view of $\hbar\mathbb{Z}$:

I. letting $\hbar = 1$, we attain the nabla discrete Mittag-Leffler function stated in [67, 68].

II. letting $0 < \hbar < 1$, the interval of convergence to which λ lies. Observe that, when $\hbar \mapsto 0$, then $\alpha \in (0,1)$. Moreover, when $\hbar \mapsto 1$ guarantee convergence for $\lambda = \frac{-\alpha}{1-\alpha}$, $\alpha \in (0, \frac{1}{2})$.

For further investigation of the discrete Mittag-Leffler function we refer the reader to [4].

2.3. Left and right delta fractional sums on $\hbar\mathbb{Z}$

Definition 2.6. ([26]) For some $\iota \in \mathbb{N}$, $\alpha > 0$ and let $d = c + \iota\hbar$. Assume that a function $\mathcal{F}$ be defined on $\mathbb{T} = \mathbb{N}_{c,\hbar} \cap \mathbb{N}_{d,\hbar}$. Then the delta $\hbar$-fractional sums in the left and right case are defined as follows

$$({}_{c}\widehat{\Delta}_{\hbar}^{-\alpha}\mathcal{F})(t) = \frac{1}{\Gamma(\alpha)}\sum_{\iota=c/\hbar}^{x/\hbar-\alpha}(x-\sigma(\iota\hbar))_{\hbar}^{(\alpha-1)}\mathcal{F}(\iota\hbar)\hbar, \qquad x \in \{x+\alpha\hbar : x \in \mathbb{T}\}$$

and

$$({}_{\hbar}\widehat{\Delta}_{d}^{-\alpha}\mathcal{F})(t) = \frac{1}{\Gamma(\alpha)} \sum_{\iota=x/\hbar+\alpha}^{d/\hbar-\alpha} (\iota\hbar - \sigma(x))_{\hbar}^{(\alpha-1)} \mathcal{F}(\iota\hbar)\hbar, \qquad x \in \{x - \alpha\hbar : x \in \mathbb{T}\},$$

respectively.

2.4. Left and right nabla fractional sums on $\hbar\mathbb{Z}$

Definition 2.7. ([6,66]) Assume that $\hbar > 0$ and the backward jump operator is $\rho(x) = x-\hbar$. A function $\mathcal{F} : \mathbb{N}_{c,\hbar} \mapsto \mathbb{R}$ is said to be nabla $\hbar$-fractional sum of order α, if

$$({}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}\mathcal{F})(t) = \frac{1}{\Gamma(\alpha)} \sum_{\iota=c/\hbar+1}^{x/\hbar-\alpha} (x - \rho(\iota\hbar))_{\hbar}^{(\alpha-1)} \mathcal{F}(\iota\hbar)\hbar, \quad x \in \mathbb{N}_{c+\hbar,\hbar}.$$

Also, the nabla right $\hbar$-fractional sum of order $\alpha > 0$(ending at d) for $\mathcal{F} : \mathbb{N}_{d,\hbar} \mapsto \mathbb{R}$ is described as follows

$$({}_{\hbar}\widehat{\nabla}_{d}^{-\alpha}\mathcal{F})(t) = \frac{1}{\Gamma(\alpha)} \sum_{\iota=x/\hbar}^{d/\hbar-1} (\iota\hbar - \rho(x))_{\hbar}^{(\alpha-1)} \mathcal{F}(\iota\hbar)\hbar.$$

2.5. Nabla $\hbar$-fractional differences with $\hbar$-discrete Mittag Leffler kernels

Now, we demonstrate the some new concepts which we will be utilized for proving coming results of this paper, see [4]. Also, we use the notation, $\lambda = -\frac{\alpha}{1-\alpha}$ and $\rho(x) = x - \hbar$.

Definition 2.8. ([64]) For $\alpha \in [0, 1]$, $\hbar > 0$ with $|\lambda\hbar^{\alpha}| < 1$ and let $\mathcal{F}$ be a function defined on $\mathbb{N}_{c,\hbar} \cap {}_{d,\hbar}\mathbb{N}$ with $c < d$ such that $c \equiv d(mod\ \hbar)$, then the left nabla $\mathcal{ABC}$-fractional difference (in the sense of Atangana and Baleanu) is described as

$$({}_{c}^{\mathcal{ABC}}\widehat{\nabla}_{\hbar}^{\alpha}\mathcal{F})(x) = \mathbb{B}(\alpha, \hbar)\frac{1 - \alpha + \alpha\hbar}{1 - \alpha} \sum_{\iota=c/\hbar+1}^{x/\hbar} \hbar\widehat{\nabla}_{\hbar}\mathcal{F}(\iota\hbar)\, {}_{\hbar}\check{E}_{\bar{\alpha}}(\lambda, x - \rho(\iota\hbar)) \tag{2.9}$$

and in the left Riemann sense by

$$({}_{c}^{ABR}\widehat{\nabla}_{\hbar}^{\alpha}\mathcal{F})(x) = \mathbb{B}(\alpha, \hbar)\frac{1 - \alpha + \alpha\hbar}{1 - \alpha}\widehat{\nabla}_{\hbar} \sum_{\iota=c/\hbar+1}^{x/\hbar} \hbar\mathcal{F}(\iota\hbar)\, {}_{\hbar}\check{E}_{\bar{\alpha}}(\lambda, x - \rho(\iota\hbar)). \tag{2.10}$$

Definition 2.9. ([64]) For $0 < \alpha < 1$ and let the left $\hbar$-fractional sum concern to $({}_{c}^{ABR}\widehat{\nabla}_{\hbar}^{\alpha}\mathcal{F})(x)$ defined on $\mathbb{N}_{c,\hbar}$ is stated as follows

$$({}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}\mathcal{F})(x) \qquad = \frac{1 - \alpha}{\mathbb{B}(\alpha, \hbar)(1 - \alpha + \alpha\hbar)}\mathcal{F}(x)$$

$$+\frac{\alpha}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)\Gamma(\alpha)}\sum_{\iota=c/\hbar+1}^{x/\hbar}(x-\rho(\iota\hbar))_{\hbar}^{\overline{\alpha-1}}\mathcal{F}(\iota\hbar)\hbar. \tag{2.11}$$

The right $\hbar$-fractional sum is defined on ${}_{d,\hbar}\mathbb{N}$ by

$$\begin{aligned}({}_{\hbar}^{AB}\widehat{\nabla}_{d}^{-\alpha}\mathcal{F})(x) &= \frac{1-\alpha}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)}\mathcal{F}(x)\\ &+\frac{\alpha}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)\Gamma(\alpha)}\sum_{\iota=x/\hbar}^{d/\hbar-1}(\iota\hbar-\rho(x))_{\hbar}^{\overline{\alpha-1}}\mathcal{F}(\iota\hbar)\hbar.\end{aligned} \tag{2.12}$$

3. Discrete Grüss type inequalities

In this section, we present a different concept of Grüss type inequalities, which consolidates the ideas of $\hbar$-discrete AB-fractional sums.

Theorem 3.1. *Let $\alpha \in (0,1)$ and let $\mathcal{F}$ be a positive function on $\mathbb{N}_{c,\hbar}$. Suppose that there exist two positive functions ϕ_1, ϕ_2 on $\mathbb{N}_{c,\hbar}$ such that*

$$\phi_1(x) \leq \mathcal{F}(x) \leq \phi_2(x), \quad \forall x \in \mathbb{N}_{c,\hbar}. \tag{3.1}$$

Then, for $x \in \{c, c+\hbar, c+2\hbar, ...\}$, one has

$$\begin{aligned}&{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\phi_2(x)]\,{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}(x)] + {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{F}(x)]\,{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\phi_1(x)]\\ &\geq {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\phi_2(x)]\,{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\phi_1(x)] + {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}(x)]\,{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{F}(x)].\end{aligned} \tag{3.2}$$

Proof. From Eq (3.1), for $\theta, \lambda \in \mathbb{N}_{c,\hbar}$, we have

$$(\phi_2(\theta) - \mathcal{F}(\theta))(\mathcal{F}(\lambda) - \phi_1(\lambda)) \geq 0. \tag{3.3}$$

Therefore,

$$\phi_2(\theta)\mathcal{F}(\lambda) + \phi_1(\lambda)\mathcal{F}(\theta) \geq \phi_1(\lambda)\phi_2(\theta) + \mathcal{F}(\theta)\mathcal{F}(\lambda). \tag{3.4}$$

Taking product both sides of Eq (3.4) by $\frac{1-\alpha}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)}$, we get

$$\frac{(1-\alpha)\phi_2(\theta)\mathcal{F}(\lambda)}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)} + \frac{(1-\alpha)\phi_1(\lambda)\mathcal{F}(\theta)}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)} \geq \frac{(1-\alpha)\phi_1(\lambda)\phi_2(\theta)}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)} + \frac{(1-\alpha)\mathcal{F}(\theta)\mathcal{F}(\lambda)}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)}. \tag{3.5}$$

Replacing λ by t in Eq (3.5) and conducting product both sides by $\frac{\alpha(x-\rho(t))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}$, we have

$$\frac{\alpha(x-\rho(t))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\phi_2(\theta)\mathcal{F}(t) + \frac{\alpha(x-\rho(t))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\phi_1(t)\mathcal{F}(\theta)$$

$$\geq \frac{\alpha(x-\rho(t))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\phi_1(t)\phi_2(\theta)+\frac{\alpha(x-\rho(t))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\mathcal{F}(\theta)\mathcal{F}(t).$$

Summing both sides for $t \in \{c, c+\hbar, c+2\hbar, ...\}$, we get

$$\sum_{\iota=c/\hbar+1}^{x/\hbar}\frac{\alpha(x-\rho(\iota\hbar))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\phi_2(\theta)\mathcal{F}(\iota\hbar)\hbar+\sum_{\iota=c/\hbar+1}^{x/\hbar}\frac{\alpha(x-\rho(\iota\hbar))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\phi_1(\iota\hbar)\hbar\mathcal{F}(\theta)$$
$$\geq \frac{\alpha(x-\rho(t))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\phi_1(\iota\hbar)\hbar\phi_2(\theta)+\sum_{\iota=c/\hbar+1}^{x/\hbar}\frac{\alpha(x-\rho(\iota\hbar))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\mathcal{F}(\theta)\mathcal{F}(\iota\hbar)\hbar. \tag{3.6}$$

Adding Eqs (3.5) and (3.6), we have

$$\frac{(1-\alpha)\phi_2(\theta)\mathcal{F}(\lambda)}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)}+\sum_{\iota=c/\hbar+1}^{x/\hbar}\frac{\alpha(x-\rho(\iota\hbar))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\phi_2(\theta)\mathcal{F}(\iota\hbar)\hbar$$
$$+\frac{(1-\alpha)\phi_1(\lambda)\mathcal{F}(\theta)}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)}+\sum_{\iota=c/\hbar+1}^{x/\hbar}\frac{\alpha(x-\rho(\iota\hbar))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\phi_1(\iota\hbar)\hbar\mathcal{F}(\theta)$$
$$\geq \frac{(1-\alpha)\phi_1(\lambda)\phi_2(\theta)}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)}+\sum_{\iota=c/\hbar+1}^{x/\hbar}\frac{\alpha(x-\rho(\iota\hbar))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\phi_1(\iota\hbar)\hbar\phi_2(\theta)$$
$$+\frac{(1-\alpha)\mathcal{F}(\theta)\mathcal{F}(\lambda)}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)}+\sum_{\iota=c/\hbar+1}^{x/\hbar}\frac{\alpha(x-\rho(\iota\hbar))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\mathcal{F}(\theta)\mathcal{F}(\iota\hbar)\hbar,$$

arrives at

$$\phi_2(\theta)\,{}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}(x)]+\mathcal{F}(\theta)\,{}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\phi_1(x)]\geq\phi_2(\theta)\,{}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\phi_1(x)]+\mathcal{F}(\theta)\,{}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}(x)]. \tag{3.7}$$

Taking product both sides of Eq (3.7) by $\frac{1-\beta}{\mathbb{B}(\beta,\hbar)(1-\beta+\beta\hbar)}$, we have

$$\frac{(1-\beta)\phi_2(\theta)}{\mathbb{B}(\beta,\hbar)(1-\beta+\beta\hbar)}\,{}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}(x)]+\frac{(1-\beta)\mathcal{F}(\theta)}{\mathbb{B}(\beta,\hbar)(1-\beta+\beta\hbar)}\,{}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\phi_1(x)]$$
$$\geq\frac{(1-\beta)\phi_2(\theta)}{\mathbb{B}(\beta,\hbar)(1-\beta+\beta\hbar)}\,{}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\phi_1(x)]+\frac{(1-\beta)\mathcal{F}(\theta)}{\mathbb{B}(\beta,\hbar)(1-\beta+\beta\hbar)}\,{}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}(x)]. \tag{3.8}$$

Also, replacing θ by $\bar{t}$ in Eq (3.8) and conducting product both sides by $\frac{\beta(x-\rho(\bar{t}))_{\hbar}^{\overline{\beta-1}}}{\mathbb{B}(\beta,\hbar)\Gamma(\beta)}$, we have

$$\frac{\beta(x-\rho(\bar{t}))_{\hbar}^{\overline{\beta-1}}}{\mathbb{B}(\beta,\hbar)\Gamma(\beta)}\phi_2(\theta)\,{}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}(x)]+\frac{\beta(x-\rho(\bar{t}))_{\hbar}^{\overline{\beta-1}}}{\mathbb{B}(\beta,\hbar)\Gamma(\beta)}\mathcal{F}(\theta)\,{}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\phi_1(x)]$$
$$\geq\frac{\beta(x-\rho(\bar{t}))_{\hbar}^{\overline{\beta-1}}}{\mathbb{B}(\beta,\hbar)\Gamma(\beta)}\phi_2(\theta)\,{}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\phi_1(x)]+\frac{\beta(x-\rho(\bar{t}))_{\hbar}^{\overline{\beta-1}}}{\mathbb{B}(\beta,\hbar)\Gamma(\beta)}\mathcal{F}(\theta)\,{}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}(x)].$$

Summing both sides for $\bar{t} \in \{c, c+\hbar, c+2\hbar, ...\}$, we get

$$\sum_{j=c/\hbar+1}^{x/\hbar}\frac{\beta(x-\rho(j\hbar))_{\hbar}^{\overline{\beta-1}}}{\mathbb{B}(\beta,\hbar)\Gamma(\beta)}\phi_2(j\hbar)\,{}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}(x)]+\sum_{j=c/\hbar+1}^{x/\hbar}\frac{\beta(x-\rho(j\hbar))_{\hbar}^{\overline{\beta-1}}}{\mathbb{B}(\beta,\hbar)\Gamma(\beta)}\mathcal{F}(j\hbar)\hbar\,{}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\phi_1(x)]$$

$$\geq \sum_{j=c/\hbar+1}^{x/\hbar} \frac{\beta(x-\rho(j\hbar))_{\hbar}^{\overline{\beta-1}}}{\mathbb{B}(\beta,\hbar)\Gamma(\beta)}\phi_2(j\hbar)\hbar\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\phi_1(x)] + \sum_{j=c/\hbar+1}^{x/\hbar} \frac{\beta(x-\rho(j\hbar))_{\hbar}^{\overline{\beta-1}}}{\mathbb{B}(\beta,\hbar)\Gamma(\beta)}\mathcal{F}(j\hbar)\hbar\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}(x)]. \tag{3.9}$$

Adding Eqs (3.8) and (3.9), then in view of Definition 2.9, yields the inequality Eq (3.11). This completes the proof.

Some special cases which can be derived immediately from Theorem 3.1.

Choosing $\hbar = 1$, then we attain a new result for discrete AB-fractional sum.

Corollary 1. *Let $\alpha \in (0,1)$ and let $\mathcal{F}$ be a positive function on $\mathbb{N}_c$. Suppose that there exist two positive functions ϕ_1, ϕ_2 on $\mathbb{N}_c$ such that*

$$\phi_1(x) \leq \mathcal{F}(x) \leq \phi_2(x), \quad \forall x \in \mathbb{N}_c. \tag{3.10}$$

Then, for $x \in \{c, c+1, c+2, ...\}$, one has

$$\begin{aligned} &{}_{c}^{AB}\widehat{\nabla}^{-\beta}[\phi_2(x)]\, {}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\mathcal{F}(x)] + {}_{c}^{AB}\widehat{\nabla}^{-\beta}[\mathcal{F}(x)]\, {}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\phi_1(x)] \\ &\geq {}_{c}^{AB}\widehat{\nabla}^{-\beta}[\phi_2(x)]\, {}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\phi_1(x)] + {}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\mathcal{F}(x)]\, {}_{c}^{AB}\widehat{\nabla}^{-\beta}[\mathcal{F}(x)]. \end{aligned} \tag{3.11}$$

Theorem 3.2. *Let $\alpha, \beta \in (0,1)$ and let $\mathcal{F}$ and $\mathcal{G}$ be two positive functions on $\mathbb{N}_{c,\hbar}$. Suppose that Eq (3.1) satisfies and also one assumes that there exist two positive functions Ω_1, Ω_2 on $\mathbb{N}_{c,\hbar}$ such that*

$$\Omega_1(x) \leq \mathcal{G}(x) \leq \Omega_2(x), \quad \forall x \in \mathbb{N}_{c,\hbar}. \tag{3.12}$$

Then, for $x \in \{c, c+\hbar, c+2\hbar, ...\}$, one has

$$\begin{aligned} (M_1) \quad &{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\phi_2(x)]\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}(x)] + {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{F}(x)]\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\Omega_1(x)] \\ &\geq {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\phi_2(x)]\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\Omega_1(x)] + {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{F}(x)]\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}(x)], \\ (M_2) \quad &{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\phi_1(x)]\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{G}(x)] + {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\Omega_2(x)]\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}(x)] \\ &\geq {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\phi_1(x)]\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\Omega_2(x)] + {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}(x)]\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{G}(x)], \\ (M_3) \quad &{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\Omega_2(x)]\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\phi_2(x)] + {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{F}(x)]\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}(x)] \\ &\geq {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\phi_2(x)]\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}(x)] + {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{F}(x)]\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\Omega_2(x)], \\ (M_4) \quad &{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\phi_1(x)]\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\Omega_1(x)] + {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{F}(x)]\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}(x)] \\ &\geq {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\phi_1(x)]\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}(x)] + {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\Omega_1(x)]\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{F}(x)]. \end{aligned} \tag{3.13}$$

Proof. To prove Eq (M_1), from Eqs (3.1) and (3.12), we have for $\lambda, \theta \in \mathbb{N}_{c,\hbar}$ that

$$(\phi_2(\theta) - \mathcal{F}(\theta))(\mathcal{G}(\lambda) - \Omega_1(\lambda)) \geq 0. \tag{3.14}$$

Therefore,

$$\phi_2(\theta)\mathcal{G}(\lambda) + \Omega_1(\lambda)\mathcal{F}(\theta) \geq \Omega_1(\lambda)\phi_2(\theta) + \mathcal{G}(\lambda)\mathcal{F}(\theta). \tag{3.15}$$

Taking product both sides of Eq (3.17) by $\frac{1-\alpha}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)}$, we get

$$\begin{aligned}&\frac{(1-\alpha)\phi_2(\theta)\mathcal{G}(\lambda)}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)}+\frac{(1-\alpha)\Omega_1(\lambda)\mathcal{F}(\theta)}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)}\\&\geq\frac{(1-\alpha)\Omega_1(\lambda)\phi_2(\theta)}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)}+\frac{(1-\alpha)\mathcal{G}(\lambda)\mathcal{F}(\theta)}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)}.\end{aligned}\tag{3.16}$$

Moreover, replacing λ by t in Eq (3.17) and conducting product both sides by $\frac{\alpha(x-\rho(t))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}$, we have

$$\begin{aligned}&\frac{\alpha(x-\rho(t))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\phi_2(\theta)\mathcal{G}(\lambda)+\frac{\alpha(x-\rho(t))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\Omega_1(\lambda)\mathcal{F}(\theta)\\&\geq\frac{\alpha(x-\rho(t))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\Omega_1(\lambda)\phi_2(\theta)+\frac{\alpha(x-\rho(t))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\mathcal{G}(\lambda)\mathcal{F}(\theta).\end{aligned}\tag{3.17}$$

Summing both sides for $t\in\{c,c+\hbar,c+2\hbar,...\}$, we get

$$\begin{aligned}&\sum_{\iota=c/\hbar+1}^{x/\hbar}\frac{\alpha(x-\rho(\iota\hbar))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\phi_2(\theta)\mathcal{G}(\iota\hbar)\hbar+\sum_{\iota=c/\hbar+1}^{x/\hbar}\frac{\alpha(x-\rho(\iota\hbar))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\Omega_1(\iota\hbar)\hbar\mathcal{F}(\theta)\\&\geq\sum_{\iota=c/\hbar+1}^{x/\hbar}\frac{\alpha(x-\rho(\iota\hbar))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\Omega_1(\iota\hbar)\hbar\phi_2(\theta)+\sum_{\iota=c/\hbar+1}^{x/\hbar}\frac{\alpha(x-\rho(\iota\hbar))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\mathcal{G}(\iota\hbar)\hbar\mathcal{F}(\theta).\end{aligned}$$

Then, we have

$$\begin{aligned}&{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}(x)]\phi_2(\theta)+{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\Omega_1(x)]\mathcal{F}(\theta)\\&\geq{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\Omega_1(x)]\phi_2(\theta)+{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}(x)]\mathcal{F}(\theta).\end{aligned}\tag{3.18}$$

Taking product both sides of Eq (3.18) by $\frac{1-\beta}{\mathbb{B}(\beta,\hbar)(1-\beta+\beta\hbar)}$, we have

$$\begin{aligned}&\frac{1-\beta}{\mathbb{B}(\beta,\hbar)(1-\beta+\beta\hbar)}{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{G}(x)]\phi_2(\theta)+\frac{1-\beta}{\mathbb{B}(\beta,\hbar)(1-\beta+\beta\hbar)}{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\Omega_1(x)]\mathcal{F}(\theta)\\&\geq\frac{1-\beta}{\mathbb{B}(\beta,\hbar)(1-\beta+\beta\hbar)}{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\Omega_1(x)]\phi_2(\theta)+\frac{1-\beta}{\mathbb{B}(\beta,\hbar)(1-\beta+\beta\hbar)}{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{G}(x)]\mathcal{F}(\theta).\end{aligned}\tag{3.19}$$

Further, replacing θ by $\bar{t}$ in Eq (3.19) and conducting product both sides by $\frac{\beta(x-\rho(\bar{t}))_{\hbar}^{\overline{\beta-1}}}{\mathbb{B}(\beta,\hbar)\Gamma(\beta)}$, we have

$$\begin{aligned}&{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}(x)]\frac{\beta(x-\rho(\bar{t}))_{\hbar}^{\overline{\beta-1}}}{\mathbb{B}(\beta,\hbar)\Gamma(\beta)}\phi_2(\theta)+{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\Omega_1(x)]\frac{\beta(x-\rho(\bar{t}))_{\hbar}^{\overline{\beta-1}}}{\mathbb{B}(\beta,\hbar)\Gamma(\beta)}\mathcal{F}(\theta)\\&\geq{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\Omega_1(x)]\frac{\beta(x-\rho(\bar{t}))_{\hbar}^{\overline{\beta-1}}}{\mathbb{B}(\beta,\hbar)\Gamma(\beta)}\phi_2(\theta)+{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}(x)]\frac{\beta(x-\rho(\bar{t}))_{\hbar}^{\overline{\beta-1}}}{\mathbb{B}(\beta,\hbar)\Gamma(\beta)}\mathcal{F}(\theta).\end{aligned}\tag{3.20}$$

Summing both sides for $\bar{t}\in\{c,c+\hbar,c+2\hbar,...\}$, we get

$${}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}(x)]\sum_{j=c/\hbar+1}^{x/\hbar}\frac{\beta(x-\rho(j\hbar))_{\hbar}^{\overline{\beta-1}}}{\mathbb{B}(\beta,\hbar)\Gamma(\beta)}\phi_2(j\hbar)\hbar+{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\Omega_1(x)]\sum_{j=c/\hbar+1}^{x/\hbar}\frac{\beta(x-\rho(j\hbar))_{\hbar}^{\overline{\beta-1}}}{\mathbb{B}(\beta,\hbar)\Gamma(\beta)}\mathcal{F}(j\hbar)\hbar$$

$$\geq {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\Omega_1(x)] \sum_{j=c/\hbar+1}^{x/\hbar} \frac{\beta(x-\rho(j\hbar))_{\hbar}^{\overline{\beta-1}}}{\mathbb{B}(\beta,\hbar)\Gamma(\beta)} \phi_2(j\hbar)\hbar + {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}(x)] \sum_{j=c/\hbar+1}^{x/\hbar} \frac{\beta(x-\rho(j\hbar))_{\hbar}^{\overline{\beta-1}}}{\mathbb{B}(\beta,\hbar)\Gamma(\beta)} \mathcal{F}(j\hbar)\hbar. \tag{3.21}$$

Adding Eqs (3.19) and (3.21), we conclude the desired inequality Eq (M_1).

To prove Eqs (M_2)–(M_4), we utilize the following inequalities:

$$(M_2) \quad (\Omega_2(\theta) - \mathcal{G}(\theta))(\mathcal{F}(\lambda) - \phi_1(\lambda)) \geq 0,$$

$$(M_3) \quad (\phi_2(\theta) - \mathcal{F}(\theta))(\mathcal{G}(\lambda) - \Omega_2(\lambda)) \leq 0,$$

$$(M_4) \quad (\phi_1(\theta) - \mathcal{F}(\theta))(\mathcal{G}(\lambda) - \Omega_1(\lambda)) \leq 0.$$

Some special cases which can be derived immediately from Theorem 3.2.

Choosing $\hbar = 1$, then we attain a new result for discrete AB-fractional sums.

Corollary 2. *Let $\alpha, \beta \in (0,1)$ and let $\mathcal{F}$ and $\mathcal{G}$ be two positive functions on $\mathbb{N}_c$. Suppose that Eq* (3.1) *satisfies and also one assumes that there exist two positive functions Ω_1, Ω_2 on $\mathbb{N}_c$ such that*

$$\Omega_1(x) \leq \mathcal{G}(x) \leq \Omega_2(x), \quad \forall x \in \mathbb{N}_c.$$

Then, for $x \in \{c, c+1, c+2, ...\}$, one has

$$\begin{aligned}
(M_5) \quad & {}_{c}^{AB}\widehat{\nabla}^{-\beta}[\phi_2(x)]\, {}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\mathcal{G}(x)] + {}_{c}^{AB}\widehat{\nabla}^{-\beta}[\mathcal{F}(x)]\, {}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\Omega_1(x)] \\
& \geq {}_{c}^{AB}\widehat{\nabla}^{-\beta}[\phi_2(x)]\, {}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\Omega_1(x)] + {}_{c}^{AB}\widehat{\nabla}^{-\beta}[\mathcal{F}(x)]\, {}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\mathcal{G}(x)], \\
(M_6) \quad & {}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\phi_1(x)]\, {}_{c}^{AB}\widehat{\nabla}^{-\beta}[\mathcal{G}(x)] + {}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\Omega_2(x)]\, {}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\mathcal{F}(x)] \\
& \geq {}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\phi_1(x)]\, {}_{c}^{AB}\widehat{\nabla}^{-\beta}[\Omega_2(x)] + {}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\mathcal{F}(x)]\, {}_{c}^{AB}\widehat{\nabla}^{-\beta}[\mathcal{G}(x)], \\
(M_7) \quad & {}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\Omega_2(x)]\, {}_{c}^{AB}\widehat{\nabla}^{-\beta}[\phi_2(x)] + {}_{c}^{AB}\widehat{\nabla}^{-\beta}[\mathcal{F}(x)]\, {}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\mathcal{G}(x)] \\
& \geq {}_{c}^{AB}\widehat{\nabla}^{-\beta}[\phi_2(x)]\, {}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\mathcal{G}(x)] + {}_{c}^{AB}\widehat{\nabla}^{-\beta}[\mathcal{F}(x)]\, {}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\Omega_2(x)], \\
(M_8) \quad & {}_{c}^{AB}\widehat{\nabla}^{-\beta}[\phi_1(x)]\, {}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\Omega_1(x)] + {}_{c}^{AB}\widehat{\nabla}^{-\beta}[\mathcal{F}(x)]\, {}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\mathcal{G}(x)] \\
& \geq {}_{c}^{AB}\widehat{\nabla}^{-\beta}[\phi_1(x)]\, {}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\mathcal{G}(x)] + {}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\Omega_1(x)]\, {}_{c}^{AB}\widehat{\nabla}^{-\beta}[\mathcal{F}(x)].
\end{aligned}$$

Theorem 3.3. *Let $\alpha, \beta \in (0,1)$ and let $\mathcal{F}$ and $\mathcal{G}$ be two positive functions on $\mathbb{N}_{c,\hbar}$ with $p, q > 0$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then, for $x \in \{c, c+\hbar, c+2\hbar, ...\}$, one has*

$$\begin{aligned}
(M_9) \quad & \frac{1}{p}\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{F}^p(x)]\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}^p(x)] + \frac{1}{q}\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{G}^q(x)]\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}^q(x)] \\
& \geq {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{F}\mathcal{G}(x)]\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}\mathcal{F}(x)], \\
(M_{10}) \quad & \frac{1}{p}\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}^q(x)]\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{F}^p(x)] + \frac{1}{q}\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}^p(x)]\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{G}^q(x)] \\
& \geq {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}^{q-1}\mathcal{F}^{p-1}(x)]\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{F}\mathcal{G}(x)], \\
(M_{11}) \quad & \frac{1}{p}\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}^2(x)]\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{F}^p(x)] + \frac{1}{q}\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}^2(x)]\, {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{G}^q(x)]
\end{aligned}$$

$$\geq {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}^{\frac{2}{q}}\mathcal{G}^{\frac{2}{p}}(x)]\,{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{F}\mathcal{G}(x)],$$

$$(M_{12})\qquad \frac{1}{p}\,{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}^{q}(x)]\,{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{F}^{2}(x)]+\frac{1}{q}\,{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}^{p}(x)]\,{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{G}^{2}(x)]$$
$$\geq {}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}^{p-1}\mathcal{G}^{q-1}(x)]\,{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{F}^{\frac{2}{p}}\mathcal{G}^{\frac{2}{q}}(x)]. \tag{3.22}$$

Proof. According to the well-known Young's inequality:

$$\frac{1}{p}a^{p}+\frac{1}{q}b^{q}\geq ab,\quad \forall a,b\geq 0,\ \ p,q>0,\ \ \frac{1}{p}+\frac{1}{q}=1, \tag{3.23}$$

setting $a=\mathcal{F}(\theta)\mathcal{G}(\lambda)$ and $b=\mathcal{F}(\lambda)\mathcal{G}(\theta)$, $\theta,\lambda>0$, we have

$$\frac{1}{p}(\mathcal{F}(\theta)\mathcal{G}(\lambda))^{p}+\frac{1}{q}(\mathcal{F}(\lambda)\mathcal{G}(\theta))^{q}\geq(\mathcal{F}(\theta)\mathcal{G}(\lambda))(\mathcal{F}(\lambda)\mathcal{G}(\theta)). \tag{3.24}$$

Taking product both sides of Eq (3.24) by $\frac{1-\alpha}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)}$, we have

$$\frac{1}{p}\frac{(1-\alpha)\mathcal{F}^{p}(\theta)\mathcal{G}^{p}(\lambda)}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)}+\frac{1}{q}\frac{(1-\alpha)\mathcal{F}^{q}(\lambda)\mathcal{G}^{q}(\theta)}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)}\geq\frac{(1-\alpha)\mathcal{F}(\theta)\mathcal{G}(\lambda))(\mathcal{F}(\lambda)\mathcal{G}(\theta)}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)}. \tag{3.25}$$

Moreover, replacing λ by t in Eq (3.25) and conducting product both sides by $\frac{\alpha(x-\rho(t))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}$, we have

$$\frac{\mathcal{F}^{p}(\theta)}{p}\frac{\alpha(x-\rho(t))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\mathcal{G}^{p}(t)+\frac{\mathcal{G}^{q}(\theta)}{q}\frac{\alpha(x-\rho(t))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}v^{q}(t)\geq\mathcal{F}(\theta)\mathcal{G}(\theta)\frac{\alpha(x-\rho(t))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\mathcal{F}(t)\mathcal{G}(t). \tag{3.26}$$

Summing both sides for $t\in\{c,c+\hbar,c+2\hbar,...\}$, we get

$$\frac{\mathcal{F}^{p}(\theta)}{p}\sum_{\iota=c/\hbar+1}^{x/\hbar}\frac{\alpha(x-\rho(\iota\hbar))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\mathcal{G}^{p}(\iota\hbar)\hbar+\frac{\mathcal{G}^{q}(\theta)}{q}\sum_{\iota=c/\hbar+1}^{x/\hbar}\frac{\alpha(x-\rho(\iota\hbar))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\mathcal{F}^{q}(\iota\hbar)\hbar$$
$$\geq\mathcal{F}(\theta)\mathcal{G}(\theta)\sum_{\iota=c/\hbar+1}^{x/\hbar}\frac{\alpha(x-\rho(\iota\hbar))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\mathcal{F}(\iota\hbar)\hbar\mathcal{G}(\iota\hbar)\hbar. \tag{3.27}$$

Adding Eqs (3.24) and (3.27), we get

$$\frac{1}{p}\frac{(1-\alpha)\mathcal{F}^{p}(\theta)\mathcal{G}^{p}(\lambda)}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)}+\frac{\mathcal{F}^{p}(\theta)}{p}\sum_{\iota=c/\hbar+1}^{x/\hbar}\frac{\alpha(x-\rho(\iota\hbar))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\mathcal{G}^{p}(\iota\hbar)\hbar$$
$$+\frac{1}{q}\frac{(1-\alpha)\mathcal{F}^{q}(\lambda)\mathcal{G}^{q}(\theta)}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)}+\frac{\mathcal{G}^{q}(\theta)}{q}\sum_{\iota=c/\hbar+1}^{x/\hbar}\frac{\alpha(x-\rho(\iota\hbar))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\mathcal{F}^{q}(\iota\hbar)\hbar$$
$$\geq\frac{(1-\alpha)\mathcal{F}(\theta)\mathcal{G}(\lambda)\mathcal{F}(\lambda)\mathcal{G}(\theta)}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)}+\mathcal{F}(\theta)\mathcal{G}(\theta)\sum_{\iota=c/\hbar+1}^{x/\hbar}\frac{\alpha(x-\rho(\iota\hbar))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\mathcal{F}(\iota\hbar)\hbar\mathcal{G}(\iota\hbar)\hbar. \tag{3.28}$$

In view of Definition 2.9, yields

$$\frac{\mathcal{F}^{p}(\theta)}{p}\,{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}^{p}(x)]+\frac{\mathcal{G}^{q}(\theta)}{q}\,{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}^{p}(x)]\geq\mathcal{F}(\theta)\mathcal{G}(\theta)\,{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}(x)\mathcal{G}(x)]. \tag{3.29}$$

Again, taking product both sides of Eq (3.29) by $\frac{1-\beta}{\mathbb{B}(\beta,\hbar)(1-\beta+\beta\hbar)}$, we have

$$\frac{\mathcal{F}^p(\theta)}{p}\frac{(1-\beta)\,{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}^p(x)]}{\mathbb{B}(\beta,\hbar)(1-\beta+\beta\hbar)}+\frac{\mathcal{G}^q(\theta)}{q}\frac{(1-\beta)\,{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}^p(x)]}{\mathbb{B}(\beta,\hbar)(1-\beta+\beta\hbar)}\geq\frac{(1-\beta)\,{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}(x)\mathcal{G}(x)]}{\mathbb{B}(\beta,\hbar)(1-\beta+\beta\hbar)}\mathcal{F}(\theta)\mathcal{G}(\theta). \tag{3.30}$$

Further, replacing θ by $\bar{t}$ in Eq (3.29) and conducting product both sides by $\frac{\beta(x-\rho(\bar{t}))_{\hbar}^{\overline{\beta-1}}}{\mathbb{B}(\beta,\hbar)\Gamma(\beta)}$, we have

$$\begin{aligned}&\frac{1}{p}\,{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}^p(x)]\frac{\beta(x-\rho(\bar{t}))_{\hbar}^{\overline{\beta-1}}}{\mathbb{B}(\beta,\hbar)\Gamma(\beta)}\mathcal{F}^p(\bar{t})+\frac{1}{q}\,{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}^p(x)]\frac{\beta(x-\rho(\bar{t}))_{\hbar}^{\overline{\beta-1}}}{\mathbb{B}(\beta,\hbar)\Gamma(\beta)}\mathcal{G}^q(\bar{t})\\&\geq{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}(x)\mathcal{G}(x)]\frac{\beta(x-\rho(\bar{t}))_{\hbar}^{\overline{\beta-1}}}{\mathbb{B}(\beta,\hbar)\Gamma(\beta)}\mathcal{F}(\bar{t})\mathcal{G}(\bar{t}).\end{aligned} \tag{3.31}$$

After summing the above inequality Eq (3.31) both sides for $\bar{t}\in\{c,c+\hbar,c+2\hbar,...\}$, yields the desired assertion Eq (M_9).

The remaining variants can be derived by adopting the same technique and accompanying the selection of parameters in Young inequality.

$$\begin{aligned}&(M_{10})\quad a=\frac{\mathcal{F}(\theta)}{\mathcal{F}(\lambda)},\qquad b=\frac{\mathcal{G}(\theta)}{\mathcal{G}(\lambda)},\ \ \mathcal{F}(\lambda),\mathcal{G}(\lambda)\neq 0,\\&(M_{11})\quad a=\mathcal{F}(\theta)\mathcal{G}^{\frac{2}{p}}(\lambda),\quad b=\mathcal{F}^{\frac{2}{q}}(\lambda)\mathcal{G}(\theta),\\&(M_{12})\quad a=\mathcal{F}^{\frac{2}{p}}(\theta)\mathcal{F}(\lambda),\quad b=\mathcal{G}^{\frac{2}{q}}(\theta)\mathcal{G}(\lambda),\ \ \mathcal{F}(\lambda),\mathcal{G}(\lambda)\neq 0.\end{aligned}$$

Repeating the foregoing argument, we obtain Eqs (M_{10})–(M_{12}).

(I) Letting $\hbar=1$, then we attain a result for discrete AB-fractional sums.

Corollary 3. *Let $\alpha,\beta\in(0,1)$ and let $\mathcal{F}$ and $\mathcal{G}$ be two positive functions on $\mathbb{N}_c$ with $p,q>0$ satisfying $\frac{1}{p}+\frac{1}{q}=1$. Then, for $x\in\{c,c+1,c+2,...\}$, one has*

$$\begin{aligned}(M_{13})\qquad&\frac{1}{p}\,{}_{c}^{AB}\widehat{\nabla}^{-\beta}[\mathcal{F}^p(x)]\,{}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\mathcal{G}^p(x)]+\frac{1}{q}\,{}_{c}^{AB}\widehat{\nabla}^{-\beta}[\mathcal{G}^q(x)]\,{}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\mathcal{F}^q(x)]\\&\geq{}_{c}^{AB}\widehat{\nabla}^{-\beta}[\mathcal{F}\mathcal{G}(x)]\,{}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\mathcal{G}\mathcal{F}(x)],\\(M_{14})\qquad&\frac{1}{p}\,{}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\mathcal{G}^q(x)]\,{}_{c}^{AB}\widehat{\nabla}^{-\beta}[\mathcal{F}^p(x)]+\frac{1}{q}\,{}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\mathcal{F}^p(x)]\,{}_{c}^{AB}\widehat{\nabla}^{-\beta}[\mathcal{G}^q(x)]\\&\geq{}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\mathcal{G}^{q-1}\mathcal{F}^{p-1}(x)]\,{}_{c}^{AB}\widehat{\nabla}^{-\beta}[\mathcal{F}\mathcal{G}(x)],\\(M_{15})\qquad&\frac{1}{p}\,{}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\mathcal{G}^2(x)]\,{}_{c}^{AB}\widehat{\nabla}^{-\beta}[\mathcal{F}^p(x)]+\frac{1}{q}\,{}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\mathcal{F}^2(x)]\,{}_{c}^{AB}\widehat{\nabla}^{-\beta}[\mathcal{G}^q(x)]\\&\geq{}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\mathcal{F}^{\frac{2}{q}}\mathcal{G}^{\frac{2}{p}}(x)]\,{}_{c}^{AB}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{F}\mathcal{G}(x)],\\(M_{16})\qquad&\frac{1}{p}\,{}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\mathcal{G}^q(x)]\,{}_{c}^{AB}\widehat{\nabla}^{-\beta}[\mathcal{F}^2(x)]+\frac{1}{q}\,{}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\mathcal{F}^p(x)]\,{}_{c}^{AB}\widehat{\nabla}^{-\beta}[\mathcal{G}^2(x)]\\&\geq{}_{c}^{AB}\widehat{\nabla}^{-\alpha}[\mathcal{F}^{p-1}\mathcal{G}^{q-1}(x)]\,{}_{c}^{AB}\widehat{\nabla}^{-\beta}[\mathcal{F}^{\frac{2}{p}}\mathcal{G}^{\frac{2}{q}}(x)].\end{aligned} \tag{3.32}$$

Example 3.4. Let $\alpha,\beta \in (0,1)$ and let $\mathcal{F}$ and $\mathcal{G}$ be two positive functions on $\mathbb{N}_{c,\hbar}$ with $p,q > 0$ satisfying $p+q=1$. Then, for $x \in \{c, c+\hbar, c+2\hbar, ...\}$, one has

$$\begin{aligned}
(M_{17}) \quad & p\, {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{F}(x)]\, {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}(x)] + q\, {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}(x)]\, {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{G}(\eta) \\
& \geq {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{F}^{p}\mathcal{G}^{q}(x)]\, {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}^{q}\mathcal{G}^{p}(x)], \\
(M_{18}) \quad & p\, {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{F}^{p-1}(x)]\, {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\alpha}[(\mathcal{F}(x)]\mathcal{G}^{q}(x)]) + q\, {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}^{q-1}(x)]\, {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{F}^{q}\mathcal{G}(x)] \\
& \geq {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{G}^{q}(x)]\, {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}^{p}(x)], \\
(M_{19}) \quad & p\, {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{F}(x)]\, {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}^{\frac{2}{p}}(x)] + q\, {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{G}(x)]\, {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}^{\frac{2}{q}}(x)] \\
& \geq {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{F}^{p}\mathcal{G}(x)]\, {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}^{q}\mathcal{F}^{2}(x)], \\
(M_{20}) \quad & p\, {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{F}^{\frac{2}{p}}\mathcal{G}^{q}(x)]\, {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}^{p-1}(x)] + q\, {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{G}^{q-1}(x)]\, {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}^{\frac{2}{q}}\mathcal{G}^{p}(x)] \\
& \geq {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\beta}[\mathcal{F}^{2}(x)]\, {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}^{2}(x)].
\end{aligned} \tag{3.33}$$

Proof. The example can be proved with the aid of the weighted AM–GM inequality with the same technique as we did in Theorem 3.3 and utilizing the following assumptions:

$$\begin{aligned}
(M_{17}) \quad & a = \mathcal{F}(\theta)\mathcal{G}(\lambda), && b = \mathcal{F}(\lambda)\mathcal{G}(\theta). \\
(M_{18}) \quad & a = \frac{\mathcal{F}(\lambda)}{\mathcal{F}(\theta)}, && b = \frac{\mathcal{G}(\theta)}{\mathcal{G}(\lambda)}, \ \mathcal{F}(\theta), \mathcal{G}(\lambda) \neq 0. \\
(M_{19}) \quad & a = \mathcal{F}(\theta)\mathcal{G}^{\frac{2}{p}}(\lambda), && b = \mathcal{F}^{\frac{2}{q}}(\lambda)\mathcal{G}(\theta), \\
(M_{20}) \quad & a = \frac{\mathcal{F}^{\frac{2}{p}}(\theta)}{\mathcal{G}(\lambda)}, && b = \frac{\mathcal{F}^{\frac{2}{q}}(\lambda)}{\mathcal{G}(\theta)}, \ \mathcal{G}(\theta), \mathcal{G}(\theta) \neq 0.
\end{aligned}$$

Example 3.5. Let $\alpha \in (0,1)$ and let $\mathcal{F}$ and $\mathcal{G}$ be two positive functions on $\mathbb{N}_{c,\hbar}$ with $p,q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Let

$$\gamma = \min_{\theta\in\mathbb{N}_{c,\hbar}} \frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)} \qquad \textit{and} \qquad \Upsilon = \max_{\theta\in\mathbb{N}_{c,\hbar}} \frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)}. \tag{3.34}$$

Then, for $x \in \{c, c+\hbar, c+2\hbar, ...\}$, one has

$$\begin{aligned}
(i) \quad & 0 \leq {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}^{2}(x)]\, {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}^{2}(x)] \leq \frac{\gamma+\Upsilon}{4\gamma\Upsilon}\big({}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}\mathcal{G}(x)]\big)^{2}, \\
(ii) \quad & 0 \leq \sqrt{{}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}^{2}(x)]\, {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}^{2}(x)]} - \big({}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}\mathcal{G}(x)]\big) \leq \frac{\sqrt{\Upsilon}-\sqrt{\gamma}}{2\sqrt{\Upsilon\gamma}}\big({}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}\mathcal{G}(x)]\big), \\
(iii) \quad & 0 \leq {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}^{2}(x)]\, {}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}^{2}(x)] - \big({}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}\mathcal{G}(x)]\big)^{2} \leq \frac{\Upsilon-\gamma}{4\gamma\Upsilon}\big({}^{AB}_{\ c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}\mathcal{G}(x)]\big)^{2}.
\end{aligned}$$

Proof. From Eq (3.34) and the inequality

$$\Big(\frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)} - \gamma\Big)\Big(\Upsilon - \frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)}\Big)\mathcal{G}^{2}(\theta) \geq 0, \ \theta \in \mathbb{N}_{c,\hbar} \tag{3.35}$$

then we can write as,

$$\mathcal{F}^2(\theta) + \gamma\Upsilon\mathcal{G}^2(\theta) \leq (\gamma + \Upsilon)\mathcal{F}(\theta)\mathcal{G}(\theta). \tag{3.36}$$

Taking product both sides of Eq (3.36) by $\frac{1-\alpha}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)}$, we have

$$\frac{(1-\alpha)\mathcal{F}^2(\theta)}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)} + \frac{(1-\alpha)\mathcal{G}^2(\theta)}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)}\gamma x \leq \frac{1-\alpha}{\mathbb{B}(\alpha,\hbar)(1-\alpha+\alpha\hbar)}(\gamma + x)\mathcal{F}(\theta)\mathcal{G}(\theta). \tag{3.37}$$

Replacing θ by t in Eq (3.36) and conducting product both sides by $\frac{\alpha(x-\rho(t))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}$, we have

$$\frac{\alpha(x-\rho(t))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\mathcal{F}^2(t) + \gamma\Upsilon\frac{\alpha(x-\rho(t))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\mathcal{G}^2(\theta) \leq (\gamma + \Upsilon)\frac{\alpha(x-\rho(t))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\mathcal{F}(t)\mathcal{G}(t). \tag{3.38}$$

Summing both sides for $t \in \{c, c+\hbar, c+2\hbar, ...\}$, we get

$$\begin{aligned}&\sum_{\iota=c/\hbar+1}^{x/\hbar}\frac{\alpha(x-\rho(\iota\hbar))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\mathcal{F}^2(\iota\hbar)\hbar + \gamma\Upsilon\sum_{\iota=c/\hbar+1}^{x/\hbar}\frac{\alpha(x-\rho(t))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\mathcal{G}^2(\iota\hbar)\hbar\\ &\leq (\gamma+\Upsilon)\sum_{\iota=c/\hbar+1}^{x/\hbar}\frac{\alpha(x-\rho(\iota\hbar))_{\hbar}^{\overline{\alpha-1}}}{\mathbb{B}(\alpha,\hbar)\Gamma(\alpha)}\mathcal{F}(\iota\hbar)\hbar\mathcal{G}(\iota\hbar)\hbar.\end{aligned} \tag{3.39}$$

Adding Eqs (3.37) and (3.39), yields

$${}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}^2(x)] + \gamma\Upsilon\,{}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}^2(x)] \leq (\gamma+\Upsilon)\,{}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}\mathcal{G}(x)], \tag{3.40}$$

on the other hand, it follows from $\gamma\Upsilon > 0$ and

$$\left(\sqrt{{}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}^2(x)]} - \sqrt{\gamma\Upsilon\,{}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}^2(x)]}\right)^2 \geq 0, \tag{3.41}$$

that

$$2\sqrt{{}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}^2(x)]}\sqrt{\gamma\Upsilon\,{}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}^2(x)]} \leq \sqrt{{}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}^2(x)]} + \sqrt{\gamma\Upsilon\,{}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}^2(x)]} \tag{3.42}$$

then from Eqs (3.40) and (3.42), we obtain,

$$4\gamma\Upsilon\,{}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}^2(x)]\,{}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{G}^2(x)] \leq (\gamma+\Upsilon)^2({}^{AB}_{c}\widehat{\nabla}_{\hbar}^{-\alpha}[\mathcal{F}\mathcal{G}(x)]). \tag{3.43}$$

Which implies (i). By some change of (i), analogously, we get (ii) and (iii).

4. Conclusions

Unlike some known and established inequalities in the literature, the Grüss type inequalities have been presented via the $\hbar$-discrete AB-fractional sums with different values of parameters on the domain $\hbar\mathbb{Z}$ that can be implemented to solve the qualitative properties of difference equations. Our consequences can be applied to overcome the obstacle of obtaining estimation on the explicit bounds of unknown functions and also to extend and unify continuous inequalities by using the simple technique. Several novel consequences have been derived by the use of discrete $\hbar$-fractional sums. The noted consequences can also be extended to the weighted function case. Certainly, the case $\hbar \mapsto 1$ recaptures the outcomes of the discrete AB-fractional sums. For indicating the strength of the offered fallouts, we employ them to investigate numerous initial value problems of fractional difference equations.

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Conflict of interest

The authors declare no conflict of interest.

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