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Research article

Traveling wave phenomena in a nonlocal dispersal predator-prey system with the Beddington-DeAngelis functional response and harvesting

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Abstract: This paper is devoted to studying the existence and nonexistence of traveling wave solution for a nonlocal dispersal delayed predator-prey system with the Beddington-DeAngelis functional response and harvesting. By constructing the suitable upper-lower solutions and applying Schauder's fixed point theorem, we show that there exists a positive constant c^* such that the system possesses a traveling wave solution for any given $c > c^*$. Moreover, the asymptotic behavior of traveling wave solution at infinity is obtained by the contracting rectangles method. The existence of traveling wave solution for $c = c^*$ is established by means of Corduneanu's theorem. The nonexistence of traveling wave solution in the case of $c < c^*$ is also discussed.

Keywords: predator-prey model; Beddington-DeAngelis functional response; traveling wave solution; nonlocal dispersal equation; upper-lower solutions; asymptotic behavior

1. Introduction

Nowadays predator-prey models have been widely applied in biological and ecological phenomena. The most general prey-predator population model is represented by

$$\begin{cases} \dot{x}(t) = xG(x) - yP(x,y), \\ \dot{y}(t) = yH(x,y), \end{cases}$$

where $x(t)$ and $y(t)$ denote the density of the prey and predator at time t, respectively. $G(x)$ is the per capita growth rate of the prey in the absence of predator, $P(x,y)$ represents the functional response of predators and $H(x,y)$ measures the growth rate of predators.

A prototype of $G(x)$ is the logistic growth pattern of $G(x) = r\left(1 - \frac{x}{N}\right)$, where $r > 0$ denotes the prey intrinsic growth rate and N means the carrying capacity in the absence of predator [1]. One of known growth rate of predators is the Leslie-Gower type: $H(x, y) = \alpha\left(1 - k\frac{y}{x}\right)$ [2, 3], where α is the intrinsic growth rates of predator and k is the conversion factor of prey into predators.

Lotka-Volterra response was used by Lotka [4] in studying a hypothetical chemical reaction and by Volterra [5] in modeling a predator-prey interaction. Lotka-Volterra response function is a straight line through the origin and is unbounded. The solutions of Lotka-Volterra model are not structural stable, thus a small perturbation can have a very marked effect [6]. The Holling-type II functional responses function is $P(x, y) = \frac{cx}{a+bx}$, where c is the maximum number of prey consumed per predator per unit time [7, 8]. When $a = 1$ and $b = 0$, the functional response is of Lotka-Volterra type. In 1975, Beddington [9] and DeAngelis et al. [10] developed a predator-prey model of the mutual interference effects, in which the relationship between predators' searching efficiency and both prey and predator is presented. The Beddington-DeAngelis (B-D) functional response is defined by

$$P(x, y) = \frac{sx}{1 + ax + by},$$

where $s, a, b > 0$, s is the consumption rate, a means the saturation constant for an alternative prey and b stands for the predator interference. The predator-prey models with the B-D functional response have been well-studied in the literature, for example, see [11–14] and references therein.

From the view of human needs, the exploitation of biological resources and harvest of population are commonly practised in the fields of fishery, wildlife, and forestry management. Many mathematical models have been proposed and developed to better describe the relationship between predator and prey populations by taking into account the harvesting, for instance, see [14–18]. In a very general way, harvesting for predator-prey models can be divided into three types. If the harvesting function $h(t)$ is a constant, it is called constant-rate or constant yield harvesting. It arises when a quota is specified (for example, through permits, as in deer hunting seasons in many areas, or by agreement as sometimes occurring in whaling) [19, 20]. If the function $h(t)$ is a linear function of population size, it is called proportional or constant-effort harvesting [16–18]. The harvesting function $h(t)$ can be of nonlinear form, for example, one of which is the so-called Michaelis-Menten type harvesting used in ecology and economics [21, 22].

Movements of some individuals usually cannot be restricted to a small area, and they are often free, so integral operators have been widely applied to model the long-distance dispersal problem [23]. That is, the diffusion process depends on the distance between two niches of population, such as the model:

$$\frac{\partial u}{\partial t}(x, t) = \int_{\mathbb{R}} J(x - y)(u(y, t) - u(x, t))\mathrm{d}y + f(u),$$

where $\int_{\mathbb{R}} J(x-y)(u(y, t)-u(x, t))\mathrm{d}y$ represents the nonlocal dispersal process [24,25]. Such model arises not only in biological phenomena, but also in many other fields, such as phase transition modelling [25–28].

There is, however, considerable evidence that time delay should not be neglected in biological and ecological phenomena. The growth rate of population of species and the response of one species to the interactions with other species are mediated by some time delay. Other causes of response delays include differences in resource consumption with respect to age structure, migration and diffusion of populations, gestation and maturation periods, delays in behavioral response to environmental changes, and dependence of a population on a food supply that requires time to recover from grazing [15, 25]. Hence, in order to make the modeling of interactions between predator and prey more realistic, time delay is often necessarily incorporated into predator-prey models [22, 28–31].

The purpose of this paper is to study the existence and nonexistence of traveling wave solution of a nonlocal dispersal delayed predator-prey model with the B-D functional response and harvesting:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1\left((J*u)(x,t) - u(x,t)\right) + ru(x,t)\left(1 - \dfrac{u(x,t)}{K}\right) - \dfrac{su(x,t)v(x,t-\tau)}{1 + au(x,t) + bv(x,t-\tau)} - qu(x,t), \\ \dfrac{\partial v}{\partial t} = d_2((J*v)(x,t) - v(x,t)) + v(x,t)\left(\alpha - \beta\dfrac{v(x,t)}{u(x,t-\tau)}\right), \end{cases} \tag{1.1}$$

where

$$(J*w)(x,t) = \int_{\mathbb{R}} J(y)w(x-y,t)\mathrm{d}y,$$

q represents the prey harvesting, τ denotes the time delay, and $a,\ b,\ r,\ d_1,\ d_2,\ s,\ K,\ \alpha$ and β are positive real constants. To reduce the number of parameters in system (1.1), we make the following transformations:

$$\bar{t} = rt,\ \ \bar{\tau} = r\tau,\ \ \bar{u} = \frac{u}{K},\ \ \bar{v} = \frac{sv}{r},\ \ \bar{d}_1 = \frac{d_1}{r},\ \ \bar{d}_2 = \frac{d_2}{r},$$
$$\bar{a} = aK,\ \ \bar{b} = \frac{rb}{s},\ \ \bar{\alpha} = \frac{\alpha}{r},\ \ \bar{\beta} = \frac{\beta}{sK},\ \ \bar{q} = \frac{q}{r}.$$

For the sake of convenience, we ignore the bars on u, v and other parameters, then system (1.1) can be re-expressed as

$$\begin{cases} u_t = d_1(J*u - u) + u(1-u) - \dfrac{uv(x,t-\tau)}{1 + au + bv(x,t-\tau)} - qu, \\ v_t = d_2(J*v - v) + v\left(\alpha - \beta\dfrac{v}{u(x,t-\tau)}\right). \end{cases} \tag{1.2}$$

Biologically, we require $0 < q < 1$. It is easy to see that system (1.2) has two spatially constant equilibria $(1-q,\ 0)$ and $(u^*,\ v^*)$, where $u^* = (1-q)\,\frac{\kappa-\beta-\alpha+\sqrt{(\kappa-\beta-\alpha)^2+4\beta\kappa}}{2\kappa}$, $v^* = \frac{\alpha u^*}{\beta}$ and $\kappa = (a\beta + b\alpha)(1-q)$.

In biology and ecology, traveling wave solutions are often used to describe the spatial-temporal process where the predator invades the territory of prey and they eventually coexist [25]. A solution of system (1.2) is called a traveling wave with the speed $c > 0$ if there exist positive function ϕ_1 and ϕ_2 defined on $\mathbb{R}$ such that

$$u(x,t) = \phi_1(z), \quad v(x,t) = \phi_2(z), \quad z = x + ct.$$

Here ϕ_1 and ϕ_2 represent the wave profiles and $(\phi_1,\ \phi_2)$ satisfies the resultant system:

$$\begin{cases} c\phi_1'(z) = d_1\left(J*\phi_1(z) - \phi_1(z)\right) + \phi_1(z)(1-\phi_1(z)) - \dfrac{\phi_1(z)\phi_2(z-c\tau)}{1 + a\phi_1(z) + b\phi_2(z-c\tau)} - q\phi_1(z), \\ c\phi_2'(z) = d_2\left(J*\phi_2(z) - \phi_2(z)\right) + \phi_2(z)\left(\alpha - \beta\dfrac{\phi_2(z)}{\phi_1(z-c\tau)}\right), \end{cases} \tag{1.3}$$

and

$$J*\phi(z) = \int_{\mathbb{R}} J(y)\phi(z-y)\mathrm{d}y.$$

Our primary interest lies in the traveling wave solution of system (1.3) connecting $(1-q, 0)$ and (u^*,v^*) with the asymptotic behavior:

$$\lim_{z\to-\infty}(\phi_1(z),\phi_2(z)) = (1-q,0), \quad \lim_{z\to+\infty}(\phi_1(z),\phi_2(z)) = (u^*,v^*). \tag{1.4}$$

The asymptotic behavior of traveling wave solution plays an important role in dispersion models of biological populations, because it describes the propagation processes of different species and enables us to understand how some species migrate from one area into another area until the density attains a certain value.

Recently, the existence of traveling wave solution for the nonlocal dispersal systems with the time delay has been extensively studied [28–33]. We can see that system (1.3) is non-monotone system and Schauder's fixed point theorem is a quiet powerful technique for constructing a suitable invariant set (see, for example [31, 33–36]). To explore the existence of traveling wave solution of nonlocal dispersal systems with $c > c^*$, we need to construct an invariant cone in a large bounded domain with the initial functions [33–35], where the nonlocal dispersal kernel function J is assumed to be compactly supported. For the existence of traveling wave solution at the critical point $c = c^*$, Corduneanu's theorem and the limiting method are useful techniques [33, 36].

Throughout this paper, for the nonlocal dispersal kernel function J of system (1.3), we make the following assumptions:

(G1) J is a smooth function in $\mathbb{R}$, Lebesgue measurable with $J \in C^1(\mathbb{R})$ and

$$J(x) = J(-x) \geq 0, \quad \int_{\mathbb{R}} J(x)\mathrm{d}x = 1.$$

(G2) $\int_{\mathbb{R}} J(x)e^{\lambda x}\mathrm{d}x < +\infty$, $\lambda \in \mathbb{R}$.

For convenience, we assume the parameters of system (1.3) satisfying

$$0 < d_1 \leq d_2, \quad 0 < q < 1, \quad b > 1, \quad a > \frac{1}{q}, \quad 0 < b\alpha \leq \beta.$$

The rest of this paper is structured as follows. We construct an appropriate pair of upper-lower solutions of system (1.3) for $c > c^*$ in Section 2. We apply Schauder's fixed point theorem to investigate the existence of traveling wave solution for $c > c^*$ and develop the contracting rectangles method to study the asymptotic behavior of system (1.3) in Section 3. The existence of traveling wave solution for $c = c^*$ is discussed by means of Corduneanu's theorem and Lebesgue's dominated convergence theorem in Section 4. Section 5 is dedicated to the nonexistence of traveling wave for $0 < c < c^*$. A brief conclusion is given in Section 6.

2. Upper-lower solutions

Definition 2.1. Assume that $\mathbb{Z} := \{z_1, z_2, \cdots, z_m\} \in \mathbb{R}$ contains finite points of $\mathbb{R}$. We say that the functions $(\overline{\phi}_1, \overline{\phi}_2)$ and $(\underline{\phi}_1, \underline{\phi}_2)$ are a pair of upper-lower solutions of system (1.3), if for any $z \in \mathbb{R} \setminus \mathbb{Z}$,

$\overline{\phi}_i'(z)$ and $\underline{\phi}_i'(z)$ $(i = 1, 2)$ are bounded and continuous such that

$$\begin{cases} F(\overline{\phi}_1, \underline{\phi}_2)(z) = d_1(J * \overline{\phi}_1(z) - \overline{\phi}_1(z)) - c\overline{\phi}_1'(z) + \overline{\phi}_1(z)(1 - \overline{\phi}_1(z)) \\ \qquad - \dfrac{\overline{\phi}_1(z)\underline{\phi}_2(z - c\tau)}{1 + a\overline{\phi}_1(z) + b\underline{\phi}_2(z - c\tau)} - q\overline{\phi}_1(z) \le 0, \\ F(\underline{\phi}_1, \overline{\phi}_2)(z) = d_1(J * \underline{\phi}_1(z) - \underline{\phi}_1(z)) - c\underline{\phi}_1'(z) + \underline{\phi}_1(z)(1 - \underline{\phi}_1(z)) \\ \qquad - \dfrac{\underline{\phi}_1(z)\overline{\phi}_2(z - c\tau)}{1 + a\underline{\phi}_1(z) + b\overline{\phi}_2(z - c\tau)} - q\underline{\phi}_1(z) \ge 0, \\ F(\overline{\phi}_1, \overline{\phi}_2)(z) = d_2(J * \overline{\phi}_2(z) - \overline{\phi}_2(z)) - c\overline{\phi}_2'(z) + \overline{\phi}_2(z)(\alpha - \beta\dfrac{\overline{\phi}_2(z)}{\overline{\phi}_1(z - c\tau)}) \le 0, \\ F(\underline{\phi}_1, \underline{\phi}_2)(z) = d_2(J * \underline{\phi}_2(z) - \underline{\phi}_2(z)) - c\underline{\phi}_2'(z) + \underline{\phi}_2(z)(\alpha - \beta\dfrac{\underline{\phi}_2(z)}{\underline{\phi}_1(z - c\tau)}) \ge 0. \end{cases} \tag{2.1}$$

Define

$$f_\sigma(d, c, \lambda) = d\left(\int_R J(y)e^{-\lambda y}\mathrm{d}y - 1\right) - c\lambda + \sigma,$$

where $\sigma \ge 0$. By a direct calculation, for $c > 0$ and $\lambda > 0$ we have

(F1) $f_0(d_1, c, 0) = 0$ and $f_\alpha(d_2, c, 0) > 0$;

(F2) $\frac{\partial f_\sigma}{\partial c} = -\lambda < 0$, $\frac{\partial f_\sigma}{\partial \lambda}\big|_{\lambda=0} = -c < 0$ and $\frac{\partial f_\sigma}{\partial d} = \int_R J(y)e^{-\lambda y}dy - 1 > 0$;

(F3) $\frac{\partial^2 f_\sigma}{\partial \lambda^2} > 0$.

From (F1)–(F3), it follows that there exist $c^* > 0$ and $\lambda^* > 0$ such that [35]

$$f_\alpha(d_2, c^*, \lambda^*) = 0 \quad \text{and} \quad \frac{\partial f_\alpha(d_2, c, \lambda)}{\partial \lambda}\bigg|_{(c^*,\lambda^*)} = 0.$$

Lemma 2.1. *There exist $c > c^*$ and positive constants $0 < \lambda_2 < \lambda^* < \lambda_3 < \lambda_1$ such that*

$$f_0(d_1, c, \lambda)\begin{cases} = 0 & \lambda = 0,\ \lambda = \lambda_1 \\ > 0 & \lambda \in (\lambda_1, +\infty) \\ < 0 & \lambda \in (0, \lambda_1) \end{cases}, \qquad f_\alpha(d_2, c, \lambda)\begin{cases} = 0 & \lambda = \lambda_2,\ \lambda = \lambda_3 \\ > 0 & \lambda \in (0, \lambda_2) \cup (\lambda_3, +\infty) \\ < 0 & \lambda \in (\lambda_2, \lambda_3) \end{cases}.$$

Proof. We only need to show $\lambda_1 > \lambda_3$. It is easy to see $f_\alpha(d_2, c, \lambda_3) = f_0(d_1, c, \lambda_1) = 0$ and $f_0(d_1, c, \lambda_1) < f_\alpha(d_1, c, \lambda_1)$. Due to $d_2 \ge d_1$ and (F2), we have $f_\alpha(d_1, c, \lambda_1) \le f_\alpha(d_2, c, \lambda_1)$. It indicates $f_\alpha(d_2, c, \lambda_3) < f_\alpha(d_2, c, \lambda_1)$, i.e., $\lambda_1 > \lambda_3$. □

Now, we will construct an appropriate pair of upper-lower solutions for system (1.3). We fix $c > c^*$. For any given constant $m > 1$, it is easy to check that the function

$$g(z) = e^{\lambda_2 z} - me^{\theta z}$$

has a unique zero point at $z_0 = -\frac{\ln m}{\theta-\lambda_2}$ where $\theta \in (\lambda_2, \min\{2\lambda_2, \lambda_3\})$, and a unique maximum point at $z_M = -\frac{\ln m}{\lambda_2(\theta-\lambda_2)} < z_0$. Clearly, g is continuous on $\mathbb{R}$ and positive on $(-\infty, z_0)$. For any given $y \in \mathbb{R}$ we let

$$\Theta(z) = \int_{-\infty}^{z} J(y-x)g(x)\mathrm{d}x - \frac{g(z)}{2}$$

with $z \in [z_M, z_0]$. Since $\Theta(z)$ is nondecreasing for $z \in [z_M, z_0]$ and $\Theta(z_0) > 0$, we can find a sufficiently small $\delta \in \left(0, \frac{\alpha(b-1)}{\beta}\right)$ and $z_2 \in (z_M, z_0)$ such that

$$\delta = g(z_2), \quad \Theta(z_2) > 0.$$

Let p and m satisfy the following conditions:

$(A1)$ $p > -\frac{1}{f_0(d_1,c,\lambda_2)}$.

$(A2)$ $m > -\frac{\beta}{(b-1)f_\alpha(d_2,c,\theta)}$.

Then, we introduce $\overline{\phi}_1(z)$, $\overline{\phi}_2(z)$, $\underline{\phi}_1(z)$, $\underline{\phi}_2(z)$ as follows:

$$\overline{\phi}_1(z) = 1-q \qquad z \in \mathbb{R}, \tag{2.2}$$

$$\underline{\phi}_1(z) = \begin{cases} (1-q)\left(1-\frac{1}{b}\right) & z \geq z_1, \\ (1-q)\left(1-\frac{1}{b}\left(e^{\lambda_1 z} + pe^{\lambda_2 z}\right)\right) & z \leq z_1, \end{cases} \tag{2.3}$$

$$\overline{\phi}_2(z) = \begin{cases} \frac{1-q}{b} & z \geq 0, \\ \frac{1-q}{b}e^{\lambda_2 z} & z \leq 0, \end{cases} \tag{2.4}$$

$$\underline{\phi}_2(z) = \begin{cases} \frac{1-q}{b}\delta & z \geq z_2, \\ \frac{1-q}{b}(e^{\lambda_2 z} - me^{\theta z}) & z \leq z_2, \end{cases} \tag{2.5}$$

where $z_1 < 0$ is defined by $e^{\lambda_1 z_1} + pe^{\lambda_2 z_1} = 1$.

Lemma 2.2. *Assume $c > c^*$. Then $\left(\overline{\phi}_1, \overline{\phi}_2\right)$ and $\left(\underline{\phi}_1, \underline{\phi}_2\right)$ defined by* (2.2)–(2.5) *are a pair of upper-lower solutions of system* (1.3).

Proof. Firstly, we show that

$$F\left(\overline{\phi}_1, \underline{\phi}_2\right)(z) \leq 0$$

holds for $z \in \mathbb{R}$. For any $z \in \mathbb{R}$, we have $\overline{\phi}_1(z) = 1 - qE$ and

$$\begin{aligned} F\left(\overline{\phi}_1, \underline{\phi}_2\right)(z) &= (1-q)q - \frac{(1-q)\underline{\phi}_2(z-c\tau)}{1+a(1-q)+b\underline{\phi}_2(z-c\tau)} - q(1-q) \\ &= -\frac{(1-q)\underline{\phi}_2(z-c\tau)}{1+a(1-q)+b\underline{\phi}_2(z-c\tau)} \\ &\leq 0. \end{aligned}$$

For $z \neq z_1$, we would like to show that

$$F\left(\underline{\phi}_1, \overline{\phi}_2\right)(z) \geq 0.$$

When $z > z_1$, we have $\underline{\phi}_1(z) = (1-q)\left(1-\frac{1}{b}\right), \quad \overline{\phi}_2(z) \le \frac{1-q}{b}$ and

$$F\left(\underline{\phi}_1, \overline{\phi}_2\right)(z) \ge (1-q)\frac{b-1}{b}\left[1-(1-q)\frac{b-1}{b} - \frac{1-q}{b+a(1-q)(b-1)+b(1-q)} - q\right] \ge 0.$$

In view of $f_0(d_1, c, \lambda_2) < f_0(d_2, c, \lambda_2) < f_\alpha(d_2, c, \lambda_2) = 0$ and (A1), for $z < z_1 < 0$ we have $\underline{\phi}_1 = (1-q)\left[1-\frac{1}{b}(e^{\lambda_1 z} + pe^{\lambda_2 z})\right]$, $\overline{\phi}_2 = \frac{1-q}{b}e^{\lambda_2 z}$ and

$$\begin{aligned}
F\left(\underline{\phi}_1, \overline{\phi}_2\right)(z) \ge{}& d_1\left(\int_{\mathbb{R}} J(y)(1-q)\left[1-\frac{1}{b}\left(e^{\lambda_1(z-y)} + pe^{\lambda_2(z-y)}\right)\right]\mathrm{d}y - (1-q)\left[1-\frac{1}{b}\left(e^{\lambda_1 z} + pe^{\lambda_2 z}\right)\right]\right) \\
&+ (1-q)\frac{c}{b}\left(\lambda_1 e^{\lambda_1 z} + p\lambda_2 e^{\lambda_2 z}\right) + (1-q)^2\left[1-\frac{1}{b}\left(e^{\lambda_1 z} + pe^{\lambda_2 z}\right)\right] \\
&- (1-q)^2\left[1-\frac{1}{b}\left(e^{\lambda_1 z} + pe^{\lambda_2 z}\right)\right]^2 - \frac{\underline{\phi}_1\overline{\phi}_2(z-c\tau)}{1+a\underline{\phi}_1 + b\overline{\phi}_2} \\
={}& -\frac{1-q}{b}e^{\lambda_1 z}\left[d_1\left(\int_R J(y)e^{-\lambda_1 y}\mathrm{d}y - 1\right) - c\lambda_1\right] - \frac{1-q}{b}pe^{\lambda_2 z}\left[d_1\left(\int_R J(y)e^{-\lambda_2 y}\mathrm{d}y - 1\right) - c\lambda_2\right] \\
&+ \frac{(1-q)^2}{b}\left(e^{\lambda_1 z} + pe^{\lambda_2 z}\right)\left[1-\frac{1}{b}\left(e^{\lambda_1 z} + pe^{\lambda_2 z}\right)\right] - \frac{\underline{\phi}_1\overline{\phi}_2(z-c\tau)}{1+a\underline{\phi}_1 + b\overline{\phi}_2(z-c\tau)} \\
>{}& -\frac{1-q}{b}pe^{\lambda_2 z}\left[d_1\left(\int_R J(y)e^{-\lambda_2 y}\mathrm{d}y - 1\right) - c\lambda_2\right] - \overline{\phi}_2(z-c\tau) \\
={}& -\frac{1-q}{b}pe^{\lambda_2 z}\left[d_1\left(\int_R J(y)e^{-\lambda_2 y}\mathrm{d}y - 1\right) - c\lambda_2\right] - \frac{1-q}{b}e^{\lambda_2(z-c\tau)} \\
={}& \frac{1-q}{b}e^{\lambda_2 z}\left[(-p)\left(d_1\left(\int_R J(y)e^{-\lambda_2 y}\mathrm{d}y - 1\right) - c\lambda_2\right) - e^{-\lambda_2 c\tau}\right] \\
>{}& \frac{1-q}{b}e^{\lambda_2 z}\left[(-p)f_0(d_1, c, \lambda_2) - 1\right] > 0.
\end{aligned} \tag{2.6}$$

Now, we show

$$F(\overline{\phi}_1, \overline{\phi}_2)(z) \le 0$$

for $z \ne 0$. In the case of $z > 0$, we have $\overline{\phi}_1 = 1-q$ and $\overline{\phi}_2 = \frac{1-q}{b}$. Then

$$\begin{aligned}
F(\overline{\phi}_1, \overline{\phi}_2)(z) &\le \frac{1-q}{b}\left[\alpha - \beta\frac{\frac{1-q}{b}}{1-q}\right] \\
&= \frac{1-q}{b}\left[\alpha - \frac{\beta}{b}\right] \le 0.
\end{aligned}$$

For $z<0$, we obtain $\overline{\phi}_2=\frac{1-q}{b}e^{\lambda_2 z}$ and

$$\begin{aligned}F(\overline{\phi}_1,\overline{\phi}_2)(z)\leq\ & d_2\left(\int_R J(y)\frac{1-q}{b}e^{\lambda_2(z-y)}\mathrm{d}y-\frac{1-q}{b}e^{\lambda_2 z}\right)-\frac{1-q}{b}c\lambda_2 e^{\lambda_2 z}\\ &+\frac{1-q}{b}e^{\lambda_2 z}\left[\alpha-\frac{\beta}{b}e^{\lambda_2 z}\right]\\ =\ &\frac{1-q}{b}e^{\lambda_2 z}\left[d_2\left(\int_R J(y)e^{-\lambda_2 y}\mathrm{d}y-1\right)-c\lambda_2+\alpha\right]-\frac{\beta(1-q)}{b^2}e^{2\lambda_2 z}\\ =\ &-\frac{\beta(1-q)}{b^2}e^{2\lambda_2 z}\leq 0.\end{aligned}$$

Finally, to show

$$F(\underline{\phi}_1,\underline{\phi}_2)(z)\geq 0$$

for $z\neq z_2$, we use the inequality $\underline{\phi}_1\geq(1-q)\left(1-\frac{1}{b}\right)$ and $\underline{\phi}_2=\frac{1-q}{b}\delta$ if $z>z_2$. Then

$$\begin{aligned}F(\underline{\phi}_1,\underline{\phi}_2)(z)\geq&\frac{d_2(1-q)}{b}\left[\int_{z-z_2}^{+\infty}J(y)\left(e^{\lambda_2(z-y)}-me^{\theta(z-y)}\right)\mathrm{d}y+\int_{-\infty}^{z-z_2}J(y)\delta\mathrm{d}y-\delta\right]\\ &+\frac{1-q}{b}\delta\left[\alpha-\frac{\beta}{(1-q)(1-\frac{1}{b})}\cdot\frac{1-q}{b}\delta\right]\\ \geq&\frac{d_2(1-q)}{b}\left(\int_{-\infty}^{z_2}J(z-y)\left(e^{\lambda_2 z}-me^{\theta z}\right)\mathrm{d}y-\frac{\delta}{2}\right)+\frac{1-q}{b}\delta\left[\alpha-\frac{\beta\delta}{b-1}\right]\\ =&\frac{d_2(1-q)}{b}\Theta(z_2)+\frac{1-q}{b}\delta\left[\alpha-\frac{\beta\delta}{b-1}\right]\\ \geq&\frac{1-q}{b}\delta\left[\alpha-\frac{\beta\delta}{b-1}\right]\geq 0,\end{aligned}$$

due to $0<\delta<\frac{\alpha(b-1)}{\beta}$.

On the other hand, if $z<z_2$, we have $\underline{\phi}_2=\frac{1-q}{b}\left(e^{\lambda_2 z}-me^{\theta z}\right)$ and thus

$$\begin{aligned}F\left(\underline{\phi}_1,\underline{\phi}_2\right)(z)\geq&\frac{d_2(1-q)}{b}\left[\int_R J(y)\left(e^{\lambda_2(z-y)}-me^{\theta(z-y)}\right)\mathrm{d}y-\left(e^{\lambda_2 z}-me^{\theta z}\right)\right]-\frac{c(1-q)}{b}\left(\lambda_2 e^{\lambda_2 z}-m\theta e^{\theta z}\right)\\ &+\frac{1-q}{b}\left(e^{\lambda_2 z}-me^{\theta z}\right)\left[\alpha-\frac{\beta}{(1-q)(1-\frac{1}{b})}\frac{1-q}{b}\left(e^{\lambda_2 z}-me^{\theta z}\right)\right]\\ =&\frac{1-q}{b}e^{\lambda_2 z}\left[d_2\left(\int_R J(y)e^{-\lambda_2 y}\mathrm{d}y-1\right)-c\lambda_2+\alpha\right]\\ &-m\frac{1-q}{b}e^{\theta z}\left[d_2\left(\int_R J(y)e^{-\theta z}\mathrm{d}y-1\right)-c\theta+\alpha\right]-\frac{\beta(1-q)}{b(b-1)}(e^{\lambda_2 z}-me^{\theta z})^2\\ >&-m\frac{1-q}{b}e^{\theta z}\left[d_2\left(\int_R J(y)e^{-\theta z}\mathrm{d}y-1\right)-c\theta+\alpha\right]-\frac{\beta(1-q)}{b(b-1)}e^{2\lambda_2 z}\\ =&\frac{1-q}{b}e^{\theta z}\left[(-m)\left(d_2\left(\int_R J(y)e^{-\theta z}\mathrm{d}y-1\right)-c\theta+\alpha\right)-\frac{\beta}{b-1}e^{(2\lambda_2-\theta)z}\right]\end{aligned}$$

$$
\begin{aligned}
&> \frac{1-q}{b} e^{\theta z} \left[(-m) \left(d_2 \left(\int_R J(y) e^{-\theta z} \mathrm{d}y - 1 \right) - c\theta + \alpha \right) - \frac{\beta}{b-1} \right] \\
&> \frac{1-q}{b} e^{\theta z} \left[(-m) f_\alpha(d_2, c, \theta) - \frac{\beta}{b-1} \right] > 0.
\end{aligned}
$$

The last inequality holds due to $\theta \in (\lambda_2, \min\{2\lambda_2, \lambda_3\})$ and condition (A2). □

3. Existence of traveling wave solution for $c > c^*$

In this section, we start with discussing the existence of traveling wave solution for system (1.3) with condition (1.4) by using the upper-lower solutions of system (1.3), which is defined in the preceding section, to construct an invariant set.

Let C be a set of bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}^2$ and

$$
\Gamma = \{(\phi_1, \phi_2) \in C : \underline{\phi}_i(z) \le \phi_i \le \overline{\phi}_i(z),\ z \in \mathbb{R},\ i = 1, 2\},
$$

where $\overline{\phi}_i(z)$ and $\underline{\phi}_i(z)$ (i=1,2) are defined by (2.2)–(2.5). Thus for any $(\phi_1, \phi_2) \in \Gamma$, we have $(1-q)\frac{b-1}{b} \le \phi_1(z) \le 1 - q$ and $0 \le \phi_2(z) \le \frac{1-q}{b}$.

For $\Phi = (\phi_1, \phi_2) \in \Gamma$, we define

$$
\begin{cases}
H_1(\phi_1, \phi_2)(z) := d_1 J * \phi_1(z) + F_1(\phi_1(z), \phi_2(z - c\tau)), \\
H_2(\phi_1, \phi_2)(z) := d_2 J * \phi_2(z) + F_2(\phi_1(z - c\tau), \phi_2(z)),
\end{cases}
$$

where

$$
\begin{cases}
F_1(y_1, y_2) = (\gamma - d_1) y_1 + y_1 \left(1 - y_1 - \dfrac{y_2}{1 + a y_1 + b y_2} - q \right), \\
F_2(y_1, y_2) = (\gamma - d_2) y_2 + y_2 \left(\alpha - \beta \dfrac{y_2}{y_1} \right),
\end{cases}
$$

for some constant γ. For any fixed $\gamma > \max\left\{ d_1 + (1-q)\left(1 + \frac{1}{b}\right), d_2 + \frac{2\beta}{b-1} - \alpha \right\}$, it follows that F_1 is nondecreasing in y_1 and is decreasing in y_2 for $y_1 \in \left[(1-q)\frac{b-1}{b},\ 1-q\right]$ and $y_2 \in \left[0,\ \frac{1-q}{b}\right]$. Also, F_2 is nondecreasing with respect to y_1 and y_2 for $y_1 \in \left[(1-q)\frac{b-1}{b},\ 1-q\right]$ and $y_2 \in \left[0,\ \frac{1-q}{b}\right]$.

Define an operator $P = (P_1, P_2) : \Gamma \to C$ by

$$
\begin{cases}
P_1(\phi_1, \phi_2)(z) = \dfrac{1}{c} \displaystyle\int_{-\infty}^{z} e^{-\frac{\gamma(z-y)}{c}} H_1(\phi_1, \phi_2)(y) \mathrm{d}y, \\
P_2(\phi_1, \phi_2)(z) = \dfrac{1}{c} \displaystyle\int_{-\infty}^{z} e^{-\frac{\gamma(z-y)}{c}} H_2(\phi_1, \phi_2)(y) \mathrm{d}y.
\end{cases}
$$

Apparently, a fixed point of P is a solution of system (1.3). Let $\rho \in (0, \frac{\gamma}{c})$ and $\|\cdot\|$ denote the Euclidean norm in $\mathbb{R}^2$. We define

$$
B_\rho(\mathbb{R}, \mathbb{R}^2) = \{\Phi \in C : \sup_{z \in \mathbb{R}} \|\Phi(z)\| e^{-\rho|z|} < \infty\}
$$

and

$$
|\Phi|_\rho := \sup_{z \in \mathbb{R}} \|\Phi(z)\| e^{-\rho|z|}.
$$

It is easy to see that $(B_\rho(\mathbb{R}, \mathbb{R}^2)), |\cdot|_\rho)$ is a Banach space. Clearly, Γ is nonempty, bounded, convex and closed in $B_\rho(\mathbb{R}, \mathbb{R}^2)$.

Lemma 3.1. $P : \Gamma \to \Gamma$.

Proof. For any $\Phi(z) = (\phi_1, \phi_2)(z) \in \Gamma$, owing to the monotonicity of F_1 and F_2 we have

$$\begin{cases} H_1(\phi_1, \phi_2)(z) \geq d_1 J * \underline{\phi}_1(z) + \underline{F}_1(z) =: \underline{H}_1(z),\ z \in \mathbb{R}, \\ H_1(\phi_1, \phi_2)(z) \leq d_1 J * \overline{\phi}_1(z) + \overline{F}_1(z) =: \overline{H}_1(z),\ z \in \mathbb{R}, \end{cases}$$

and

$$\begin{cases} H_2(\phi_1, \phi_2)(z) \geq d_2 J * \underline{\phi}_2(z) + \underline{F}_2(z) =: \underline{H}_2(z),\ z \in \mathbb{R}, \\ H_2(\phi_1, \phi_2)(z) \leq d_2 J * \overline{\phi}_2(z) + \overline{F}_2(z) =: \overline{H}_2(z),\ z \in \mathbb{R}, \end{cases}$$

in which $\overline{F}_1$,$\underline{F}_1$, $\overline{F}_2$ and $\underline{F}_2$ are defined by

$$\begin{cases} \overline{F}_1(z) = (\gamma - d_1)\overline{\phi}_1(z) + \overline{\phi}_1(z)\left(1 - q - \overline{\phi}_1(z) - \dfrac{\underline{\phi}_2(z - c\tau)}{1 + a\overline{\phi}_1(z) + b\underline{\phi}_2(z - c\tau)}\right), \\ \underline{F}_1(z) = (\gamma - d_1)\underline{\phi}_1(z) + \underline{\phi}_1(z)\left(1 - q - \underline{\phi}_1(z) - \dfrac{\overline{\phi}_2(z - c\tau)}{1 + a\underline{\phi}_1(z) + b\overline{\phi}_2(z - c\tau)}\right), \end{cases}$$

and

$$\begin{cases} \overline{F}_2(z) = (\gamma - d_2)\overline{\phi}_2(z) + \overline{\phi}_2(z)\left(\alpha - \beta\dfrac{\overline{\phi}_2(z)}{\overline{\phi}_1(z - c\tau)}\right), \\ \underline{F}_2(z) = (\gamma - d_2)\underline{\phi}_2(z) + \underline{\phi}_2(z)\left(\alpha - \beta\dfrac{\underline{\phi}_2(z)}{\underline{\phi}_1(z - c\tau)}\right). \end{cases}$$

Let

$$\underline{P}_1(z) = \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{\gamma(z-y)}{c}}\underline{H}_1(y)\mathrm{d}y,\ \ \overline{P}_1(z) = \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{\gamma(z-y)}{c}}\overline{H}_1(y)\mathrm{d}y,\ z \in \mathbb{R},$$
$$\underline{P}_2(z) = \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{\gamma(z-y)}{c}}\underline{H}_2(y)\mathrm{d}y,\ \ \overline{P}_2(z) = \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{\gamma(z-y)}{c}}\overline{H}_2(y)\mathrm{d}y,\ z \in \mathbb{R}.$$

Obviously, $\underline{P}_i(z) \leq P_i(z) \leq \overline{P}_i(z)$ $(i = 1, 2)$. It suffices to prove that

$$\underline{\phi}_i(z) \leq \underline{P}_i(z), \quad \overline{P}_i(z) \leq \overline{\phi}_i(z),\ z \in \mathbb{R},\ i = 1, 2.$$

We denote $z_0 = -\infty$ and $z_{m+1} = \infty$. For any $z \in \mathbb{R} \setminus \mathbb{Z}$, there exists a $k \in \{0, 1, 2, ..., m\}$ such that $z \in (z_k, z_{k+1})$, and

$$\begin{aligned} \overline{P}_1(z) &= \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{\gamma(z-y)}{c}}\overline{H}_1(y)\mathrm{d}y \\ &= \left(\sum_{i=1}^{k}\frac{1}{c}\int_{z_{i-1}}^{z_i} + \frac{1}{c}\int_{z_k}^{z}\right)e^{-\frac{\gamma(z-y)}{c}}\overline{H}_1(y)\mathrm{d}y \\ &\leq \left(\sum_{i=1}^{k}\frac{1}{c}\int_{z_{i-1}}^{z_i} + \frac{1}{c}\int_{z_k}^{z}\right)e^{-\frac{\gamma(z-y)}{c}}\left[c\overline{\phi}_1'(y)\mathrm{d}y + \gamma\overline{\phi}_1(y)\right] \\ &= \overline{\phi}_1(z). \end{aligned}$$

Due to the continuity of both $\overline{P}_1(z)$ and $\overline{\phi}_1(z)$, we get

$$\overline{P}_1(z) \le \overline{\phi}_1(z),\ z \in \mathbb{R}.$$

Similarly, we have

$$\underline{\phi}_1(z) \le \underline{P}_1(z),\ z \in \mathbb{R},$$

and

$$\underline{\phi}_2(z) \le P_2(\phi)(z) \le \overline{\phi}_2(z),\ z \in \mathbb{R}.$$

Consequently, we obtain $P(\Gamma) \subset \Gamma$. □

Lemma 3.2. *P: $\Gamma \to \Gamma$ is continuous with respect to $|\cdot|_\rho$.*

Proof. For any $\Phi = (\phi_1, \phi_2)$ and $\Psi = (\psi_1, \psi_2) \in \Gamma$, we have

$$\begin{aligned}
&|H_1(\phi_1,\phi_2)(z) - H_1(\psi_1,\psi_2)(z)| \\
\le & d_1 \int_{\mathbb{R}} J(z-y)|\phi_1(y) - \psi_1(y)|\mathrm{d}y + (\gamma - d_1 + 1 - q)|\phi_1(z) - \psi_1(z)| \\
&+ |\phi_1(z) + \psi_1(z)|\,|\phi_1(z) - \psi_1(z)| + \left|\frac{\phi_1(z)\phi_2(z-c\tau)}{1 + a\phi_1(z) + b\phi_2(z-c\tau)} - \frac{\psi_1(z)\psi_2(z-c\tau)}{1 + a\psi_1(z) + b\psi_2(z-c\tau)}\right|
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{\phi_1(z)\phi_2(z-c\tau)}{1 + a\phi_1(z) + b\phi_2(z-c\tau)} - \frac{\psi_1(z)\psi_2(z-c\tau)}{1 + a\psi_1(z) + b\psi_2(z-c\tau)}\right| \\
<&\frac{\frac{1-q}{b}(2-q)}{\left(1 + a(1-q)\frac{b-1}{b}\right)^2}|\phi_1(z) - \psi_1(z)| + \frac{(1-q)(1+a(1-q))}{\left(1 + a(1-q)\frac{b-1}{b}\right)^2}|\phi_2(z-c\tau) - \psi_2(z-c\tau)| \\
<&\frac{1-q}{b}(2-q)|\phi_1(z) - \psi_1(z)| + (1-q)(1+a(1-q))|\phi_2(z-c\tau) - \psi_2(z-c\tau)| \\
<&\frac{1}{b}(1-q)(2-q)|\phi_1(z) - \psi_1(z)| + a(1-q)(2-q)|\phi_2(z-c\tau) - \psi_2(z-c\tau)|.
\end{aligned}$$

A straightforward calculation yields

$$\begin{aligned}
&|P_1(\phi_1,\phi_2)(z) - P_1(\psi_1,\psi_2)(z)|e^{-\rho|z|} \\
\le&\frac{d_1 e^{-\rho|z|}}{c}\int_{-\infty}^{z} e^{-\frac{\gamma(z-s)}{c}}\left(\int_{\mathbb{R}} J(s-y)|\phi_1(y) - \psi_1(y)|\mathrm{d}y\right)\mathrm{d}s \\
&+ \frac{(\gamma - d_1 + 1 - q)e^{-\rho|z|}}{c}\int_{-\infty}^{z} e^{-\frac{\gamma(z-y)}{c}}|\phi_1(y) - \psi_1(y)|\mathrm{d}y \\
&+ \frac{2(1-q)e^{-\rho|z|}}{c}\int_{-\infty}^{z} e^{-\frac{\gamma(z-y)}{c}}|\phi_1(y) - \psi_1(y)|\mathrm{d}y \\
&+ \frac{(1-q)(2-q)e^{-\rho|z|}}{cb}\int_{-\infty}^{z} e^{-\frac{\gamma(z-y)}{c}}|\phi_1(y) - \psi_1(y)|\mathrm{d}y \\
&+ \frac{a(1-q)(2-q)e^{-\rho|z|}}{c}\int_{-\infty}^{z} e^{-\frac{\gamma(z-y)}{c}}|\phi_2(y-c\tau) - \psi_2(y-c\tau)|\mathrm{d}y \\
=&d_1\frac{e^{-\rho|z|}}{c}\int_{-\infty}^{z} e^{-\frac{\gamma(z-s)}{c}}\left(\int_{\mathbb{R}} J(s-y)|\phi_1(y) - \psi_1(y)|\mathrm{d}y\right)\mathrm{d}s
\end{aligned}$$

$$+\left[\gamma - d_1 + (1-q)\left(3 + \frac{2-q}{b}\right)\right]\frac{e^{-\rho|z|}}{c}\int_{-\infty}^{z} e^{-\frac{\gamma(z-y)}{c}}|\phi_1(y) - \psi_1(y)|\mathrm{d}y$$
$$+ a(1-q)(2-q)\frac{e^{-\rho|z|}}{c}\int_{-\infty}^{z} e^{-\frac{\gamma(z-y)}{c}}|\phi_2(y-c\tau) - \psi_2(y-c\tau)|\mathrm{d}y.$$

We further have

$$\begin{aligned}
&\frac{e^{-\rho|z|}}{c}\int_{-\infty}^{z} e^{-\frac{\gamma(z-s)}{c}}\left(\int_{\mathbb{R}} J(s-y)|\phi_1(y) - \psi_1(y)|\mathrm{d}y\right)\mathrm{d}s\\
=&\frac{e^{-\rho|z|}}{c}\int_{-\infty}^{z} e^{-\frac{\gamma(z-s)}{c}}\left(\int_{\mathbb{R}} J(s-y)e^{\rho|y|}|\phi_1(y) - \psi_1(y)|e^{-\rho|y|}\mathrm{d}y\right)\mathrm{d}s\\
\leq&\frac{|\Phi-\Psi|_\rho}{c}\int_{-\infty}^{z} e^{-(\frac{\gamma}{c}-\rho)(z-s)}\left(\int_{\mathbb{R}} J(y)e^{\rho|y|}\mathrm{d}y\right)\mathrm{d}s\\
\leq&\frac{2\int_{\mathbb{R}} J(y)e^{\rho y}\mathrm{d}y}{\gamma - c\rho}|\Phi-\Psi|_\rho
\end{aligned}$$

and

$$\begin{aligned}
\frac{e^{-\rho|z|}}{c}\int_{-\infty}^{z} e^{-\frac{\gamma(z-y)}{c}}|\phi_1(y) - \psi_1(y)|\mathrm{d}y =&\frac{e^{-\rho|z|}}{c}\int_{-\infty}^{z} e^{-\frac{\gamma(z-y)}{c}}e^{\rho|y|}|\phi_1(y) - \psi_1(y)|e^{-\rho|y|}\mathrm{d}y\\
\leq&\frac{|\Phi-\Psi|_\rho}{c}\int_{-\infty}^{z} e^{-(\frac{\gamma}{c}-\rho)(z-y)}\mathrm{d}y\\
\leq&\frac{1}{\gamma - c\rho}|\Phi-\Psi|_\rho.
\end{aligned}$$

Processing in an analogous manner, we can derive

$$\frac{e^{-\rho|z|}}{c}\int_{-\infty}^{z} e^{-\frac{\gamma(z-y)}{c}}|\phi_2(y-c\tau) - \psi_2(y-c\tau)|\mathrm{d}y \leq \frac{e^{\rho c\tau}}{\gamma - c\rho}|\Phi-\Psi|_\alpha.$$

We now choose

$$L_1 = \frac{2d_1\int_{\mathbb{R}} J(y)e^{\rho y}\mathrm{d}y + \gamma - d_1 + (1-q)\left[3 + \left(\frac{1}{b} + ae^{\rho c\tau}\right)(2-q)\right]}{\gamma - c\rho}$$

such that

$$|P_1(\phi_1,\phi_2)(z) - P_1(\psi_1,\psi_2)(z)|e^{-\rho|z|} \leq L_1|\Phi-\Psi|_\alpha. \tag{3.1}$$

On the other hand, we have

$$\begin{aligned}
&|P_2(\phi_1,\phi_2)(z) - P_2(\psi_1,\psi_2)(z)|e^{-\rho|z|}\\
\leq&\frac{d_2e^{-\rho|z|}}{c}\int_{-\infty}^{z} e^{-\frac{\gamma(z-s)}{c}}\left(\int_{\mathbb{R}} J(y-s)|\phi_2(y) - \psi_2(y)|dy\right)\mathrm{d}s\\
&+\frac{\left(\gamma - d_2 + \alpha + \frac{2b\beta}{(b-1)^2}\right)e^{-\rho|z|}}{c}\int_{-\infty}^{z} e^{-\frac{\gamma(z-y)}{c}}|\phi_2(y) - \psi_2(y)|\mathrm{d}y
\end{aligned}$$

$$+\frac{\beta e^{-\alpha|z|}}{(b-1)^2c}\int_{-\infty}^{z}e^{-\frac{\gamma(z-y)}{c}}|\phi_1(y-c\tau)-\psi_1(y-c\tau)|\mathrm{d}y$$
$$\leq L_2|\Phi-\Psi|_\rho, \tag{3.2}$$

where

$$L_2=\frac{2d_2\int_{\mathbb{R}}J(y)e^{\rho y}\mathrm{d}y+\gamma-d_2+\alpha+\frac{\beta(2b+e^{\rho c\tau})}{(b-1)^2}}{\gamma-c\rho}.$$

In view of (3.1)–(3.2), there exists some constant $L^*>0$ such that

$$|P(\phi)-P(\Psi)|_\rho\leq L^*|\Phi-\Psi|_\rho.$$

Hence, P is a continuous operator from Γ to Γ. □

For any given $N\in\mathbb{R}$, let $\mathbb{R}_N^-:=(-\infty,N]$ and consider the domain of the functions of the space B_ρ on $\mathbb{R}_N^-$:

$$B_\rho\left(\mathbb{R}_N^-,\mathbb{R}^2\right)=\left\{\Phi\in C|_{\mathbb{R}_N^-}:\ \sup_{z\in\mathbb{R}_N^-}\|\Phi(z)\|e^{-|\rho|z}<\infty\right\}.$$

Then $\left(B_\rho\left(\mathbb{R}_N^-,\mathbb{R}^2\right),|\cdot|_\rho^N\right)$ is a Banach space equipped with the norm $|\cdot|_\rho^N$ defined by

$$|\Phi|_\alpha^N:=\sup_{z\in\mathbb{R}_N^-}\|\Phi(z)\|e^{-|\rho|z}.$$

Let us recall Corduneanu's Theorem [37, §2.12].

Lemma 3.3. *Let $F\subset B_\alpha\left(\mathbb{R}_N^-,\mathbb{R}^2\right)$ be a set satisfying the following conditions:*

(1) *F is bounded in $B_\alpha\left(\mathbb{R}_N^-,\mathbb{R}^2\right)$);*

(2) *the functions belonging to F are equicontinuous on any compact interval of $\mathbb{R}_N^-$;*

(3) *the functions in F are equiconvergent, i.e., for any given $\varepsilon>0$, there is a corresponding $Z(\varepsilon)<0$ such that $\|\Phi(z)-\Phi(-\infty)\|e^{-|\rho|z}<\varepsilon$ for $z\leq Z(\varepsilon)$ and $\Phi\in F$.*

Then F is compact in $B_\alpha\left(\mathbb{R}_N^-,\mathbb{R}^2\right)$.

Lemma 3.4. *$P(\Gamma)$ is compact in B_ρ.*

Proof. For any $\Phi=(\phi_1,\phi_2)\in\Gamma$ and $n\in\mathbb{N}$, we define

$$P^n(\Phi)(z)=\begin{cases}P(\Phi)(n) & z>n,\\ P(\Phi)(z) & z\in(-\infty,n].\end{cases} \tag{3.3}$$

Clearly, $P^n(\Gamma)$ is compact if $P(\Gamma)(z)|_{\mathbb{R}_n^-}$ is compact. We will show that the functions belonging to $P(\Gamma)(z)|_{\mathbb{R}_n^-}$ satisfy all three conditions (1)–(3) in Lemma 3.3. Since $P(\Gamma)\subset\Gamma$, it is easy to see that $P(\Gamma)(z)|_{\mathbb{R}_n^-}$ is bounded. Indeed, for any $z_1,z_2\in(-\infty,n]$ we deduce

$$
\begin{aligned}
&\left|P_1(\phi_1,\phi_2)(z_1)\,e^{-\rho|z_1|}-P_1(\phi_1,\phi_2)(z_2)\,e^{-\rho|z_2|}\right|\\
&=\frac{1}{c}\left|e^{-\rho|z_1|}\int_{-\infty}^{z_1}e^{-\frac{\gamma(z_1-y)}{c}}H_1(\phi_1,\phi_2)(y)\,\mathrm{d}y-e^{-\alpha|z_2|}\int_{-\infty}^{z_2}e^{-\frac{\gamma(z_2-y)}{c}}H_1(\phi_1,\phi_2)(y)\,\mathrm{d}y\right|\\
&=\frac{1}{c}\left|e^{-\left(\rho|z_1|+\frac{\gamma}{c}z_1\right)}\int_{-\infty}^{z_1}e^{\frac{\gamma}{c}y}H_1(\phi_1,\phi_2)(y)\,\mathrm{d}y-e^{-\left(\rho|z_2|+\frac{\gamma}{c}z_2\right)}\int_{-\infty}^{z_2}e^{\frac{\gamma}{c}y}H_1(\phi_1,\phi_2)(y)\,\mathrm{d}y\right|\\
&\le\frac{1}{c}e^{-\frac{\gamma}{c}z_1}\left|\int_{z_1}^{z_2}e^{\frac{\gamma}{c}y}H_1(\phi_1,\phi_2)(y)\,\mathrm{d}y\right|\\
&\quad+\frac{1}{c}\left(e^{-\rho|z_2|}\left|e^{-\frac{\gamma}{c}z_1}-e^{-\frac{\gamma}{c}z_2}\right|+e^{-\frac{\gamma}{c}z_1}\left|e^{-\rho|z_1|}-e^{-\rho|z_2|}\right|\right)\cdot\left|\int_{-\infty}^{z_2}e^{\frac{\gamma}{c}y}H_1(\phi_1,\phi_2)(y)\,\mathrm{d}y\right|\\
&\le(1-q)e^{\frac{\gamma}{c}|z_2-z_1|}\left[\left(1+\frac{\gamma}{c}\right)|z_2-z_1|+1\right].
\end{aligned}
$$

Similarly, we have

$$
\left|P_2(\phi_1,\phi_2)(z_1)\,e^{-\rho|z_1|}-P_2(\phi_1,\phi_2)(z_2)\,e^{-\rho|z_2|}\right|\le\frac{1-q}{b\gamma}\left(\gamma+\alpha-\frac{\beta}{b}\right)e^{\frac{\gamma}{c}|z_2-z_1|}\left[\left(1+\frac{\gamma}{c}\right)|z_2-z_1|+1\right].
$$

This implies that $P(\Gamma)(z)|_{\mathbb{R}_n^-}$ is equicontinuous on any compact interval of $\mathbb{R}_n^-$.

For any $\Phi(z)=(\phi_1(z),\phi_2(z))\in\Gamma$, we find

$$
(1-q)\left(1-\frac{1}{b}\left(e^{\lambda_1 z}+pe^{\lambda_2 z}\right)\right)\le\phi_1(z)\le 1-q,\quad \frac{1-q}{b}\left(e^{\lambda_2 z}-me^{\theta z}\right)\le\phi_1(z)\le\frac{1-q}{b}e^{\lambda_2 z}
$$

for $z<\min\{z_1,\ z_2\}$. That is,

$$
\lim_{z\to-\infty}\phi_1(z)=1-q,\qquad \lim_{z\to-\infty}\phi_2(z)=0.
$$

Then

$$
|\phi_1(z)-(1-q)|e^{-\rho|z|}<\frac{1-q}{b}\left(e^{(\lambda_1+\rho)z}+pe^{(\lambda_2+\rho)z}\right),\quad |\phi_2(z)-0|e^{-\rho|z|}<\frac{1-q}{b}e^{(\lambda_2+\rho)z}
$$

for $z<\min\{z_1,\ z_2\}$. That is, condition (3) is satisfied. According to Lemma 3.3, $P^n(\Gamma)(z)$ is compact in the sense of the norm $|\cdot|_\rho$. Note that

$$
|P^n(\Phi)(z)-P(\Phi)(z)|e^{-\rho|z|}\le 2(1-q)\sqrt{1+\left(\frac{\gamma+\alpha-\frac{\beta}{b}}{b\gamma}\right)^2}\,e^{-\rho n}\to 0,\ \ as\ n\to\infty.
$$

Hence, $P^n(\Phi)(z)$ converge to $P(\Phi)(z)$ with respect to the norm $|\cdot|_\rho$, and $P(\Gamma)$ is compact. □

From Lemmas 3.1–3.4 and Schauder's fixed point theorem, we can see that P has a fixed point $\Phi\in\Gamma$ such that $P(\Phi)=\Phi$, which is a solution of system (1.3). Hence, we obtain the following theorem immediately.

Theorem 3.5. *Assume that conditions (G1)–(G2) hold. Then for any fixed $c>c^*$, system* (1.3) *has a positive solution* $(\phi_1(z),\phi_2(z))\in\Gamma$. *That is,* $\underline{\phi}_i(z)\le\phi_i(z)\le\overline{\phi}_i(z)$ $(i=1,2)$*, where* $\overline{\phi}_i$ *and* $\underline{\phi}_i$ $(i=1,2)$ *are defined by* (2.2)–(2.5).

We now discuss the asymptotic behavior of traveling wave solution described in Theorem 3.5. For $z \to -\infty$, it is easy to see that

$$\lim_{z\to-\infty} \phi_1(z) = 1 - q, \quad \lim_{z\to-\infty} \phi_2(z) = 0.$$

By applying the contracting rectangles method, we analyze the asymptotic behavior of traveling wave solution as $z \to \infty$. We define

$$\begin{cases} E_1(\xi,\eta) := \xi\left(1 - qE - \xi - \dfrac{\eta}{1 + a\xi + b\eta}\right), \\ E_2(\xi,\eta) := \eta(\alpha - \beta\dfrac{\eta}{\xi}), \end{cases} \tag{3.4}$$

and

$$\begin{aligned} & u_1(\theta) := u^*\theta, & & u_2(\theta) := \left(1 + \frac{\beta a}{\alpha b}(1-\epsilon)(1-\theta)\right)u^*, \\ & v_1(\theta) := \begin{cases} v^*\frac{\theta}{2(1-\epsilon)} & \theta < 2\epsilon \\ v^*\frac{\theta-\epsilon}{1-\epsilon} & \theta \geq 2\epsilon \end{cases}, & & v_2(\theta) := v^* + \frac{a}{b}u^*(1-\theta), \end{aligned} \tag{3.5}$$

for $\theta \in [0,1]$, where (u^*, v^*) is the equilibrium point of system (1.3) and $0 < \epsilon < \min\left\{\frac{1}{4}, \frac{bv^*}{1+au^*}, 1 - \frac{1}{a}\right\}$.

Theorem 3.6. *The following three statements are true.*

(C1) *$u_1(\theta)$ and $v_1(\theta)$ are continuous and strictly increasing while $u_2(\theta)$ and $v_2(\theta)$ are continuous and strictly decreasing for $\theta \in [0,1]$.*

(C2) *For $\theta \in [0,1]$, we have*

$$\begin{cases} u_1(0) \leq u_1(\theta) \leq u_1(1) = u^* = u_2(1) \leq u_2(\theta) \leq u_2(0), \\ v_1(0) \leq v_1(\theta) \leq v_1(1) = v^* = v_2(1) \leq v_2(\theta) \leq v_2(0). \end{cases}$$

(C3) *If $\xi_1 = u_1(\theta_0)$, $\eta_1 = v_1(\theta_0)$ for any $\theta_0 \in (0,1)$ and*

$$u_1(\theta_0)) \leq \xi \leq u_2(\theta_0), \quad v_1(\theta_0) \leq \eta \leq v_2(\theta_0),$$

then $E_1(\xi_1,\eta) > 0$ and $E_2(\xi,\eta_1) > 0$.

If $\xi_2 = u_2(\theta_0)$, $\eta_2 = v_2(\theta_0)$ for any $\theta_0 \in (0,1)$ and

$$u_1(\theta_0)) \leq \xi \leq u_2(\theta_0), \quad v_1(\theta_0) \leq \eta \leq v_2(\theta_0),$$

then $E_1(\xi_2,\eta) < 0$ and $E_2(\xi,\eta_2) < 0$.

Proof. It is easy to see that (C1)–(C2) are true.

To prove (C3), we claim that for any $\theta_0 \in (0,1)$, there holds $E_1(\xi_1,\eta) > 0$ with $\xi_1 = u_1(\theta_0) = u^*\theta_0$ and $v_1(\theta_0) \leq \eta \leq v_2(\theta_0)$. As $E_1(\xi,\eta)$ is decreasing in η, we only need to show that $E_1(u^*\theta_0, v_2(\theta_0)) > 0$.

Let

$$\tilde{v}(\theta) = -\frac{a}{b}u^*\theta + \frac{a-b}{b^2} + \frac{\frac{a}{b}(1-q) - \frac{a-b}{b^2}}{1 - b(1-q) + bu^*\theta}.$$

Then $E_1\left(u^*\theta, \tilde{v}(\theta)\right) \equiv 0$ for $\theta \in [0,1]$. In view of $a > \frac{1}{q}$, it follows that

$$\begin{aligned} v_2(\theta_0) &= v^* + \frac{a}{b}u^*(1-\theta_0) \\ &= -\frac{a}{b}u^*\theta_0 + \frac{a-b}{b^2} + \frac{\frac{a}{b}(1-q) - \frac{a-b}{b^2}}{1-b(1-q)+bu^*} \\ &< -\frac{a}{b}u^*\theta_0 + \frac{a-b}{b^2} + \frac{\frac{a}{b}(1-q) - \frac{a-b}{b^2}}{1-b(1-q)+bu^*\theta_0} \\ &= \tilde{v}(\theta_0). \end{aligned}$$

Thus, $E_1(u^*\theta_0, v_2(\theta_0)) > E_1(u^*\theta_0, \tilde{v}(\theta_0)) = 0$.

To show $E_2(\xi, \eta_1) > 0$ for any $\theta_0 \in (0,1)$, $\eta_1 = v_1(\theta_0)$ and $u_1(\theta_0)) \le \xi \le u_2(\theta_0)$, we know that $E_2(\xi,\eta)$ is nondecreasing in ξ. So it is equivalent to prove $E_2(u_1(\theta_0), v_1(\theta_0)) > 0$. When $2\epsilon \le \theta_0 < 1$, in view of $v_1(\theta_0) = v^*\frac{\theta_0-\epsilon}{1-\epsilon}$ we have

$$E_2(u_1(\theta_0), v_1(\theta_0)) = v_1(\theta_0)\left[\alpha - \beta\frac{v^*\frac{\theta_0-\epsilon}{1-\epsilon}}{u^*\theta_0}\right] = \alpha v_1(\theta_0)\frac{\epsilon(1-\theta_0)}{\theta_0(1-\epsilon)} > 0.$$

For $0 < \theta_0 < 2\epsilon$, using $v_1(\theta_0) = v^*\frac{\theta_0}{2(1-\epsilon)}$ we have

$$E_2(u_1(\theta_0), v_1(\theta_0)) = v_1(\theta_0)\left[\alpha - \beta\frac{v^*\frac{\theta_0}{2(1-\epsilon)}}{u^*\theta_0}\right] = \alpha v_1(\theta_0)\frac{1-2\epsilon}{2(1-\epsilon)} > 0.$$

For any $\theta_0 \in (0,1)$, to show $E_1(\xi_2, \eta) < 0$, where $\xi_2 = u_2(\theta_0)$ and $v_1(\theta_0) \le \eta \le v_2(\theta_0)$, it suffices to prove that $E_1(u_2(\theta_0), v_1(\theta_0)) < 0$. Let $\varphi(\theta) := E_1(u_2(\theta), v_1(\theta))/u_2(\theta)$. Then $\varphi(1) = 0$. We proceed by considering two cases.

Case 1: for $2\epsilon \le \theta < 1$, from (3.5) we have $v_1(\theta) = v^*\frac{\theta-\epsilon}{1-\epsilon}$, $u_2(\theta) = \left(1 + \frac{\beta a}{\alpha b}(1-\epsilon)(1-\theta)\right)u^*$ for $\theta \in [2\epsilon, 1)$, and then

$$\begin{aligned} \frac{d\varphi}{d\theta} &= \frac{d}{d\theta}\left(1 - q - u_2(\theta) - \frac{v_1(\theta)}{1+au_2(\theta)+bv_1(\theta)}\right) \\ &= \frac{\rho_1(\theta)}{(1+au_2(\theta)+bv_1(\theta))^2}, \end{aligned}$$

where

$$\begin{aligned} \rho_1(\theta) &= -\frac{du_2}{d\theta}(1+au_2(\theta)+bv_1(\theta))^2 - \frac{dv_1}{d\theta}(1+au_2(\theta)) + av_1(\theta)\frac{du_2}{d\theta} \\ &= \frac{\beta a}{\alpha b}(1-\epsilon)\,u^*(1+au_2(\theta)+bv_1(\theta))^2 - \frac{v^*}{1-\epsilon}(1+au^*) - \frac{\beta a^2}{\alpha b}u^*v^*(1-\epsilon). \end{aligned}$$

Since $\alpha b \le \beta$ and $0 < \epsilon < 1 - \frac{1}{a}$, we get

$$\frac{d}{d\theta}(1+au_2(\theta)+bv_1(\theta)) = \frac{bv^*}{1-\epsilon} - \frac{\beta a^2}{\alpha b}u^*(1-\epsilon) = \frac{bv^*}{1-\epsilon}\left[1 - \left(\frac{\beta a}{\alpha b}\right)^2(1-\epsilon)^2\right] < 0.$$

It is easy to see that $\inf_{\theta\in[2\epsilon,1)} \rho_1(\theta) = \rho_1(1)$. In view of $0 < \epsilon < \min\left\{\frac{1}{4}, \frac{bv^*}{1+au^*}, 1-\frac{1}{a}\right\}$, we have

$$\begin{aligned}\rho_1(1) &= \frac{\beta a}{\alpha b}(1-\epsilon)u^*\left[(1+au^*+bv^*)^2 - av^*\right] - \frac{v^*}{1-\epsilon}(1+au^*)\\ &> u^*\left[(1+au^*+bv^*)^2 - av^* - \frac{1+au^*}{(1-\epsilon)}\right]\\ &> u^*\left[1+2au^*+2bv^* - av^* - (1+2\epsilon)(1+au^*)\right]\\ &> 2u^*\left[bv^* - \epsilon(1+au^*)\right]\\ &> 0.\end{aligned}$$

This implies that for $\theta \in [2\epsilon, 1)$, $\rho_1(\theta) > 0$ holds and $\varphi(\theta)$ is nondecreasing. That is, $\varphi(\theta) < 0$. Moreover, $E_1(u_2(\theta), v_1(\theta)) = \varphi(\theta)u_2(\theta) < 0$ for $\theta \in [2\epsilon, 1)$.

Case 2: for $0 < \theta < 2\epsilon$, from (3.5) we have $v_1(\theta) = v^*\frac{\theta}{2(1-\epsilon)}$ and $u_2(\theta) = \left(1+\frac{\beta a}{\alpha b}(1-\epsilon)(1-\theta)\right)u^*$ for $\theta \in (0, 2\epsilon)$. Then we get

$$\frac{d\varphi}{d\theta} = \frac{\rho_2(\theta)}{(1+au_2(\theta)+bv_1(\theta))^2},$$

where

$$\rho_2(\theta) = \frac{\beta a}{\alpha b}(1-\epsilon)\,u^*(1+au_2(\theta)+bv_1(\theta))^2 - \frac{v^*}{2(1-\epsilon)}(1+au^*) - \frac{\beta a^2}{2\alpha b}u^*v^*,$$

and $\inf_{\theta\in(0,2\epsilon)} \rho_2(\theta) = \rho_2(2\epsilon)$. In view of $0 < \epsilon < \min\left\{\frac{1}{4}, \frac{bv^*}{1+au^*}, 1-\frac{1}{a}\right\}$, there holds

$$\begin{aligned}\rho_2(2\epsilon) &> u^*\left[1+au^*\left(1+\frac{\beta a}{\alpha b}(1-\epsilon)(1-2\epsilon)\right)+\frac{\epsilon bv^*}{2(1-\epsilon)}\right]^2 - \frac{v^*}{2(1-\epsilon)}(1+au^*) - \frac{\beta a^2}{2\alpha b}u^*v^*\\ &> u^*\left[1+2au^*+2u^*\frac{\beta a^2}{\alpha b}(1-3\epsilon) - \frac{1+2\epsilon}{2}(1+au^*) - \frac{\beta a^2}{2\alpha b}u^*\right]\\ &= u^*\left[\left(1-\frac{1+2\epsilon}{2}\right)+\left(2-\frac{1+2\epsilon}{2}\right)au^* + \left(2(1-3\epsilon)-\frac{1}{2}\right)\frac{\beta a^2}{\alpha b}u^*\right]\\ &> 0.\end{aligned}$$

Since $\varphi(2\epsilon) < 0$, for $\theta \in (0, 2\epsilon)$ we have $\rho_2(\theta) > 0$ and $\varphi(\theta) < 0$. This leads to $E_1(u_2(\theta), v_1(\theta)) = \varphi(\theta)u_2(\theta) < 0$ for $\theta \in (0, 2\epsilon)$. Hence, $E_1(u_2(\theta), v_1(\theta)) < 0$ for $\theta \in (0, 1)$.

To prove $E_2(\xi, \eta_2) < 0$ for $\theta_0 \in (0, 1)$, $\eta_2 = v_2(\theta_0)$ and $u_1(\theta_0)) \le \xi \le u_2(\theta_0)$, from (3.5) we deduce

$$\begin{aligned}E_2(u_2(\theta_0), v_2(\theta_0)) &= v_2(\theta_0)\left[\alpha - \beta\frac{v^*+\frac{a}{b}u^*(1-\theta_0)}{u^*+\frac{\beta a}{\alpha b}(1-\epsilon)(1-\theta_0)u^*}\right]\\ &< v_2(\theta_0)\left[\alpha - \beta\frac{v^*+\frac{a}{b}u^*(1-\theta_0)}{u^*+\frac{\beta a}{\alpha b}u^*(1-\theta_0)}\right]\\ &= v_2(\theta_0)\left[\alpha - \beta\frac{v^*+\frac{a}{b}u^*(1-\theta_0)}{\frac{\beta}{\alpha}(v^*+\frac{a}{b}u^*(1-\theta_0))}\right]\\ &= 0.\end{aligned}$$

Hence, $E_2(u_2(\theta), v_2(\theta)) < 0$ for any $\theta \in (0, 1)$. □

Theorem 3.7. *Assume that conditions (G1)–(G2) hold and* $\Phi = (\phi_1, \phi_2) \in \Gamma$ *is a solution of system* (1.3). *Then we have*

$$\lim_{z\to\infty}(\phi_1(z), \phi_2(z)) = (u^*, v^*). \tag{3.6}$$

Proof. From (3.5), we observe

$$u_1(0) = 0, \qquad u_2(0) = u^* + \frac{\beta a}{\alpha b}(1-\epsilon)u^* > 2u^* > 1-q,$$
$$v_1(0) = 0, \qquad v_2(0) = v^* + \frac{a}{b}u^* > \frac{v^* + u^*}{b} > \frac{1-q}{b}.$$

In view of $(\phi_1, \phi_2) \in \Gamma$ for $z >> 0$, it follows that

$$(1-q)\left(1-\frac{1}{b}\right) \le \phi_1(z) \le 1-q, \quad \frac{1-q}{b}\delta \le \phi_2(z) \le \frac{1-q}{b}.$$

So we have

$$\begin{aligned} u_1(\theta_0)) &\le \liminf_{z\to\infty}\phi_1(z) \le \limsup_{z\to\infty}\phi_1(z) \le u_2(\theta_0), \\ v_1(\theta_0) &\le \liminf_{\xi\to\infty}\phi_2(z) \le \limsup_{z\to\infty}\phi_2(z) \le v_2(\theta_0), \end{aligned} \tag{3.7}$$

for some $\theta_0 \in (0, 1)$.

Denote

$$\theta^* := \sup\{\theta \in [\theta_0, 1) \,|\, (3.7) \text{ hold}\}.$$

Then, $\theta^* = 1$. Otherwise, we have $\theta^* < 1$ in (3.7). Namely, at least one of the following equalities is true:

$$u_1(\theta^*) = \liminf_{z\to\infty}\phi_1(z), \;\; u_2(\theta^*) = \limsup_{z\to\infty}\phi_1(z),$$
$$v_1(\theta^*) = \liminf_{z\to\infty}\phi_2(z), \;\; v_2(\theta^*) = \limsup_{z\to\infty}\phi_2(z).$$

Without loss of generality, we assume that

$$u_1(\theta^*) = \liminf_{z\to\infty}\phi_1(z).$$

It follows from Lebesgue's dominated convergence theorem that

$$\begin{aligned} \liminf_{z\to\infty}\phi_1(z) &= \liminf_{z\to\infty}\frac{1}{\gamma}\left[\gamma\phi_1(z) + E_1(\phi_1(z), \phi_2(z-c\tau))\right] \\ &\ge \liminf_{z\to\infty}\phi_1(z) + \frac{1}{\gamma}E_1(\liminf_{z\to\infty}\phi_1(z), \limsup_{z\to\infty}\phi_2(z)). \end{aligned}$$

That is,

$$E_1(\liminf_{z\to\infty}\phi_1(z), \; \limsup_{z\to\infty}\phi_2(z)) \le 0.$$

This implies that $E_1(u_1(\theta^*), \; \eta) \le 0$ with $v_1(\theta^*) \le \eta \le v_2(\theta^*)$, which yields a contradiction to (C3) of Theorem 3.6. The other three cases can be proceeded in an analogous manner. □

4. Existence of traveling wave solution for $c = c^*$

Let $z \in \mathbb{R}_N^-$ with $N \in \mathbb{R}$. We define

$$C_l\left(\mathbb{R}_N^-, \mathbb{R}^2\right) = \left\{(\phi_1, \phi_2) \in C|_{\mathbb{R}_N^-} : \lim_{z\to-\infty} \phi_1(z) = \phi_1(-\infty), \quad \lim_{z\to-\infty} \phi_2(z) = \phi_2(-\infty)\right\}.$$

It is not difficult to see that $C_l\left(\mathbb{R}_N^-, \mathbb{R}^2\right)$ is isomorphic to $C\left(\left[\frac{N}{N-1}, 1\right], \mathbb{R}^2\right)$. Indeed, if $x(s) \in C\left(\left[\frac{N}{N-1}, 1\right], \mathbb{R}^2\right)$, then $y(t) = x(s)$ for $t = \frac{s}{s-1}$, $s \in \left[\frac{N}{N-1}, 1\right)$, and $y(t) \in C_l\left(\mathbb{R}_N^-, \mathbb{R}^2\right)$. That is, $C_l\left(\mathbb{R}_N^-, \mathbb{R}^2\right)$ is a Banach space equipped with the superemum norm.

Theorem 4.1. *When $c = c^*$, system* (1.3) *has a positive traveling wave solution satisfying* (1.4).

Proof. Let $\{c_n\}$ be a decreasing sequence with $c_n < c^*+1$ and $\lim_{n\to\infty} c_n = c^*$. Then for each c_n, system (1.3) has a positive traveling wave solution $(\phi_{1n}(z), \phi_{2n}(z))$ satisfying (1.4) and

$$(1-q)\frac{b-1}{b} \le \phi_{1n}(z) \le 1-q, \;\; 0 \le \phi_{2n}(z) \le \frac{1-q}{b}.$$

Since a traveling wave solution is invariant in the sense of phase shift, we can assume that

$$\phi_{1n}(0) = (1-q)\iota_1, \;\; \phi_{1n}(z) > (1-q)\iota_1 \text{ for } z < 0 \text{ and } \phi_{2n}(0) = \iota_2, \;\; \phi_{2n}(z) < \iota_2 \text{ for } z < 0,$$

with $\frac{b-1}{b} < \iota_1 < 1$ and $0 < \iota_2 < \frac{1-q}{b}$. From (1.4), we know that the above expressions are admissible.

For $n \in \mathbb{N}$, it is evident that $(\phi_{1n}(z), \phi_{2n}(z))$ are equipcontinuous, bounded and equipconvergent in $C_l\left(\mathbb{R}_N^-, \mathbb{R}^2\right)$. According to Lemma 3.3, $\{(\phi_{1n}(z), \phi_{2n}(z))\}$ has a subsequence, still denoted by $\{(\phi_{1n}(z), \phi_{2n}(z))\}$, such that

$$\phi_{1n}(z) \to \phi_1(z), \;\; \phi_{2n}(z) \to \phi_2(z), \text{ as } n \to \infty$$

and

$$\lim_{z\to-\infty} \phi_1(z) = 1-q, \quad \lim_{z\to-\infty} \phi_2(z) = 0.$$

Here, $(\phi_1(z), \phi_2(z)) \in C_l\left(\mathbb{R}_N^-, \mathbb{R}^2\right)$ is continuous and the above limits converge uniformly on $\mathbb{R}_N^-$. It follows from Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty} J * \phi_{in}(z) = \phi_i(z), \quad i = 1, 2$$

on $z \in \mathbb{R}_N^-$. Thus, $(\phi_1(z), \phi_2(z))$ is a solution to system (1.3) which satisfies

$$\phi_1(0) = (1-q)\iota_1, \;\; \phi_1(z) > (1-q)\iota_1 \text{ for } z < 0 \text{ and } \phi_2(0) = \iota_2, \;\; \phi_2(z) < \iota_2 \text{ for } z < 0,$$

and

$$(1-q)\frac{b-1}{b} \le \phi_1(z) \le 1-q, \quad 0 \le \phi_2(z) \le \frac{1-q}{b}.$$

From $\phi_2(0) = \iota_2 > 0$, $\liminf_{z\to-\infty} \phi_2(z) > 0$ holds. By virtue of Theorem 3.7, we obtain

$$\lim_{z\to+\infty} \phi_1(z) = u^*, \;\; \lim_{z\to+\infty} \phi_2(z) = v^*.$$

□

5. Nonexistence of traveling wave solutions

Consider the Cauchy problem:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d\left(J * u(x,t) - u(x,t)\right) + u(x,t)\left(1 - ru(x,t)\right), \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{5.1}$$

where J satisfies condition (G1), $r > 0$ is constant and the initial value $u_0(x)$ is uniformly continuous and bounded for $x \in \mathbb{R}$.

Lemma 5.1. *[32] Assume that* $0 \le u_0(x) \le \frac{1}{r}$. *Then system* (5.1) *admits a solution for* $x \in \mathbb{R}$ *and* $t > 0$. *If* $\omega(x,0)$ *is uniformly continuous and bounded, and* $\omega(x,0)$ *satisfies*

$$\begin{cases} \dfrac{\partial \omega(x,t)}{\partial t} \ge (\le)\, d\left(J * \omega(x,t) - \omega(x,t)\right) + \omega(x,t)\left(1 - r\omega(x,t)\right), \\ \omega(x,0) \ge (\le)\, u_0(x), \quad x \in \mathbb{R}, \end{cases}$$

then we have

$$\omega(x,t) \ge (\le)\, u(x,t), \quad x \in \mathbb{R},\ t > 0.$$

Lemma 5.2. *[32] Assume that* $u_0(x) > 0$. *Then for any* $0 < c < c^*$ *there holds*

$$\liminf_{t\to\infty} \inf_{|x|<ct} u(x,t,u_0(x)) = \limsup_{t\to\infty} \sup_{|x|<ct} u(x,t,u_0(x)) = \frac{1}{r}.$$

Theorem 5.3. *For any speed* $0 < c < c^*$, *there is no nontrivial positive solution* $(\phi_1(z), \phi_2(z))$ *of system* (1.3) *satisfying condition* (1.4).

Proof. Suppose on the contrary that there exists some $0 < c_1 < c^*$, such that system (1.3) has a positive solution $(\phi_1(z), \phi_2(z))$ satisfying condition (1.4). Then $\phi_1(z)$ is bounded on $\mathbb{R}$ and we can find a positive constant K such that $\psi(x,t) = \phi_2(x + ct)$ satisfies

$$\begin{cases} \dfrac{\partial \psi(x,t)}{\partial t} \ge d_2\left(J * \psi(x,t) - \psi(x,t)\right) + \alpha\psi(x,t)\left(1 - K\psi(x,t)\right), \\ \psi(x,0) = \phi_2(x) > 0. \end{cases}$$

Let $x(t) = -\frac{c_1+c^*}{2}t$. From Lemmas 5.1 and 5.2 it follows that

$$\liminf_{t\to\infty} \inf_{2|x|=(c_1+c^*)t} \psi(x,t) \ge \frac{1}{K}.$$

Meanwhile, in view of $x(t) + c_1 t = \frac{c_1-c^*}{2}t$, we see $z = x(t) + c_1 t \to -\infty$ as $t \to +\infty$, and

$$\limsup_{t\to\infty} \psi(x(t),t) = \lim_{z\to-\infty} \phi_2(z) = 0.$$

This yields a contradiction. □

6. Conclusions

In this paper, we have studied the existence and nonexistence of traveling wave solution of a nonlocal delayed predator-prey model with the B-D functional response and harvesting. As we see, model (1.3) is nonmonotone or not quasimonotone. We employed Schauder's fixed point theorem and the upper-lower solutions method to discuss the existence of traveling wave solution for the speed $c > c^*$. Then, we investigated the asymptotic behavior of traveling wave solution by construction of the upper-lower solutions at $-\infty$ and by developing the contacting rectangles technique at $+\infty$. For the special case of $c = c^*$, one usually can not establish the existence of traveling wave solution directly by constructing a pair of upper-lower solutions. One of available methods is the limiting argument together with the Arzela-Ascoli Theorem [33, 36, 39]. In this study we have presented not only the existence of traveling wave solution but also the asymptotic behavior of traveling wave solution at $-\infty$ by Corduneanu's theorem. The nonexistence of traveling wave solution of system (1.3) with condition (1.4) was investigated by applying the comparison principle of nonlocal dispersal equations.

It is remarkable that for the parameters of system (1.3), we only need $b > 1$ and $0 < b\alpha \leq \beta$ to prove Theorem 3.5. These conditions were used to construct a pair of suitable upper-lower solutions of system (1.3). For $a > 1$ and $0 < a\alpha \leq \beta$, we could also construct the appropriate upper-lower solutions of system (1.3) in a similar way. To obtain the asymptotic behavior of traveling wave solution as $z \to \infty$, we additionally needed $a > \frac{1}{q}$.

When $q = 0$ in model (1.3), it means that there does not have any prey harvesting. By assuming $b > 1$, $0 < b\alpha \leq \beta$ and $a > \frac{b\alpha}{\beta}$, we can derive the same results as Theorems 3.5 and 3.7 in an analogous manner.

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Conflict of interest

The authors declare there is no conflicts of interest.

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