



Research article

The asymptotic stability of numerical analysis for stochastic age-dependent cooperative Lotka-Volterra system

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Abstract: In this study, we explore a stochastic age-dependent cooperative Lotka-Volterra (LV) system with an environmental noise. By applying the theory of M-matrix, we prove the existence and uniqueness of the global solution for the system. Since the stochastic age-dependent cooperative LV system cannot be solved explicitly, we then construct an Euler-Maruyama (EM) numerical solution to approach the exact solution of the system. The convergence rate and the p th-moment boundedness of the scheme have also been obtained. Additionally, numerical experiments have been conducted to verify our theoretical results.

Keywords: age-dependent cooperative Lotka-Volterra system; brownian motion; euler-Maruyama numerical solution; convergence rate; p th-moment; error analysis; discrete time approximation

1. Introduction

The Lotka-Volterra (LV) system is often applied to depict the evolutionary process in population dynamic, physics, and economics [1,2,9,13]. Particularly, a cooperative LV system explains the interactions and benefits together among species in social animals and in human society, such as bees and flowers, algal-fungal associations of lichens, etc [4–7]. In literature, many studies focused on the deterministic cooperative LV system [8–13]. In [9], the authors considered the permanence in a two-species delay LV cooperative system. Also, Lu et al. [10] showed the certain and general delay effects on the performance of a LV cooperative system. Furthermore, in [11], the authors considered a non-autonomous cooperative LV system and established sufficient conditions to keep the system permanent. Xu et al. [12] studied the discrete LV cooperative system with periodic boundary conditions from the aspect of stability. Since environmental noise is nonnegligible in the ecosystem, some scholars introduced the stochastic noise into the cooperative LV model and revealed the effects of stochastic noise

on the model (see e.g. [2,3]). In particular, Zuo et al. [2] introduced the white noise to the intrinsic growth rates of the certain delay cooperative LV model, then they established the sufficient conditions for the persistence and obtained the existence and uniqueness of the stationary distribution. In [3], Liu studied a stochastic food-limited LV cooperative model and formulated the sufficient conditions of p -moment persistence and extinction. However, all these works mentioned above mainly focused on the stochastic ordinary differential equations (SODEs) rather than the stochastic partial differential equations (SPDEs).

Age-dependent theory of populations was first studied by Lotka in 1907 [14]. From then on, age-dependent has been introduced in many models to describe the dynamical behavior of species, such as the predator-prey LV model, population model and epidemic model [16,17,20–26,31,32,37,38]. Zhang and Liu [16] investigated the age-dependent in the predator-prey model and analysed the non-trivial periodic oscillation phenomenon. Solis and Ku-Carrillo [17] studied a predator-prey LV model with age-dependent and presented the theoretical results which preserve the properties of the solutions of the model. Chekroun and Kuniya [34] concerned with the global asymptotic behavior of an infection age-structured SIR epidemic model with diffusion in a general n -dimensional bounded spatial domain. Then they showed that the basic reproduction R_0 depended on the shape of the spatial domain in the 2-dimensional case. Lu and Wei [35] formulated a stochastic epidemic model with age of vaccination and adopted a generalized incidence rate to make the model more realistic. They got the sufficient conditions for extinction and two types of permanence. Yang and Wang [36] introduced an age-dependent predation system with prey harvesting and provided the explicit formulae which can determine the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions.

However, as far as we know, most of the studies in literature focused on the predator-prey LV and population models instead of cooperative LV system. Based on the importance of the cooperative LV system and to fill the gap of lacks of studying the SPDEs in these systems, in this study, we introduce the age-dependent into a stochastic cooperative LV system and study the properties of this system.

It is extremely difficult to get the explicit solutions of the stochastic age-dependent system, so in this study, we mainly focus on the numerical solutions of the system. The numerical method we use in this study is the Euler-Maruyama (EM) method. There are various advantages of the EM method, chief among them are the simple algebraic expression, low computational cost, and good convergence rate, etc [25–28]. However, the EM method usually requires the system to have Lipschitz continuous and linear growth condition on both the drift and diffusion coefficients. Since our system does not meet the Lipschitz continuous condition, we could not apply the EM algorithm directly. Instead, we use the theory of equilibrium point to avoid the restrictions on the coefficients and apply the EM method to the stochastic age-dependent cooperative LV system skilfully to obtain the approximate solution. This is one of the innovations of this paper.

We provide below a brief summary of, and comments on, our results.

- We introduce the age-dependent and stochastic white noise into a cooperative LV system and study the existence and uniqueness of the solutions. In particular, we have proven that a unique positive global solution exists for this stochastic age-dependent cooperative LV system;
- Using the positive equilibrium, we apply EM method to the stochastic age-dependent cooperative LV system in an appropriate region and study the numerical solution of the system;
- The p th-moment boundedness and strong convergence of the EM scheme for the stochastic age-dependent cooperative LV system under certain conditions are proved;

- Numerical experiments are presented, and they support our theoretical results.

The construction of this paper is designed as follows. In Section 2, we establish the stochastic age-dependent cooperative LV system and present some basic preliminaries. In Section 3, we introduce the EM approximation method to the stochastic age-dependent cooperative LV system. The p th-moment estimations of this scheme have been provided. Section 4 shows that the scheme is convergent and the strong convergence order is provided. In Section 5, some numerical examples are presented to verify our theories in Section 4. Section 6 gives the conclusions of this paper.

2. Preliminaries

2.1. Model formulation

The typical cooperative LV system can be described in the form:

$$\begin{cases} \frac{dx(t)}{dt} = x(t)(-\alpha_{11}x(t) + \alpha_{12}y(t) + r_1), \\ \frac{dy(t)}{dt} = y(t)(\alpha_{21}x(t) - \alpha_{22}y(t) + r_2), \end{cases} \quad (2.1)$$

where $x(t)$, $y(t)$ represent the density of the two cooperative species, r_1, r_2 are the intrinsic growth rates of the two species, α_{11}, α_{22} denote the intraspecific competition rates, α_{12}, α_{21} are the interspecific cooperation rates. All the rates are assumed to be positive constants [2].

Since birth rate is sensitive to age, according to [18,19], we introduce the age-dependent fertility rate into Eq (2.1) and obtain the age-dependent cooperative LV system as follows:

$$\begin{cases} \frac{\partial X(t, a)}{\partial t} + \frac{\partial X(t, a)}{\partial a} = X(t, a)(-\alpha_{11}(a)X(t, a) + \alpha_{12}(a)Y(t, a) + r_1(a)), & (t, a) \in Q, \\ \frac{\partial Y(t, a)}{\partial t} + \frac{\partial Y(t, a)}{\partial a} = Y(t, a)(\alpha_{21}(a)X(t, a) - \alpha_{22}(a)Y(t, a) + r_2(a)), & (t, a) \in Q, \\ X(t, 0) = \int_0^A \gamma(t, a)X(t, a)da, & t \in [0, T], \\ Y(t, 0) = \int_0^A \beta(t, a)Y(t, a)da, & t \in [0, T], \\ X(0, a) = X_0(a), Y(0, a) = Y_0(a), & a \in [0, A], \end{cases} \quad (2.2)$$

where $X(t, a)$ and $Y(t, a)$ denote the density of the two species at time t , age a . $Q = (0, T) \times (0, A)$.

Now, we consider the effect of random environment noise in Eq (2.2) and assume $r_i(a) \rightarrow r_i(a) + \sigma_i \dot{w}(t)$, ($i = 1, 2$), where $w(t)$ is a standard Brownian motion. So, we have the stochastic age-dependent cooperative LV system.

$$\left\{ \begin{array}{l} d_t X(t, a) = -\frac{\partial X(t, a)}{\partial a} dt + X(t, a)(-\alpha_{11}(a)X(t, a) + \alpha_{12}(a)Y(t, a) + r_1(a))dt + \sigma_1 X(t, a)dw(t), \quad (t, a) \in Q, \\ d_t Y(t, a) = -\frac{\partial Y(t, a)}{\partial a} dt + Y(t, a)(\alpha_{21}(a)X(t, a) - \alpha_{22}(a)Y(t, a) + r_2(a))dt + \sigma_2 Y(t, a)dw(t), \quad (t, a) \in Q, \\ X(t, 0) = \int_0^A \gamma(t, a)X(t, a)da, \quad t \in [0, T], \\ Y(t, 0) = \int_0^A \beta(t, a)Y(t, a)da, \quad t \in [0, T], \\ X(0, a) = X_0(a), \quad Y(0, a) = Y_0(a), \quad a \in [0, A], \end{array} \right. \quad (2.3)$$

where $d_t X$ and $d_t Y$ are the differential of $X(t, a)$ and $Y(t, a)$ relative to t , i.e., $d_t X = \frac{\partial X}{\partial t} dt$, and $d_t Y = \frac{\partial Y}{\partial t} dt$. Features of the parameters in Eq (2.3) are showed in Table 1.

Table 1. Symbols and their meaning in Eq (2.3)

Parameters	Meanings
$r_i(a)$	the intrinsic growth rates of the species at age a , $i = 1, 2$
$\alpha_{i,i}(a)$	the intraspecific competition rates at age a , $i = 1, 2$
$\alpha_{12}(a)$	the interspecific cooperation rate at age a
$\alpha_{21}(a)$	the interspecific cooperation rate at age a
$\gamma(t, a)$	the fertility rate of females of $X(t, a)$ at age a
$\beta(t, a)$	the fertility rate of females of $Y(t, a)$ at age a
σ_i	the size of uncertainty from the population growth, $i = 1, 2$

The integral form of Eq (2.3) is given by

$$\left\{ \begin{array}{l} X(t) = X_0 - \int_0^t \frac{\partial X(s)}{\partial a} ds + \int_0^t X(s)(-\alpha_{11}(a)X(s) + \alpha_{12}(a)Y(s) + r_1(a))ds + \int_0^t \sigma_1 X(s)dw(s), \\ Y(t) = Y_0 - \int_0^t \frac{\partial Y(s)}{\partial a} ds + \int_0^t Y(s)(\alpha_{21}(a)X(s) - \alpha_{22}(a)Y(s) + r_2(a))ds + \int_0^t \sigma_2 Y(s)dw(s), \\ X(t, 0) = \int_0^A \gamma(t, a)X(t, a)da, \quad t \in [0, T], \\ Y(t, 0) = \int_0^A \beta(t, a)Y(t, a)da, \quad t \in [0, T], \end{array} \right. \quad (2.4)$$

where $X(t) := X(t, a)$, $Y(t) := Y(t, a)$, $X_0 := X(0, a)$, and $Y_0 := Y(0, a)$.

Remark 2.1. Compared with Eq (2.1), the advantages of Eq (2.3) are summarized as follows:

- The intrinsic growth rates r_1, r_2 in Eq (2.1) are constant. However, the intrinsic growth rate should not be a certain number due to the random noise in the natural environment. Therefore, we describe the influence of the intrinsic growth rate of the environmental noise through a standard Brownian motion, that is, $r_i(a) \rightarrow r_i(a) + \sigma_i \dot{\omega}(t)$, ($i = 1, 2$) in Eq (2.3).

- The density of the species is not only affected by interspecific relationships, but also by the reproductive capacity of females. Eq (2.1) does not take the latter into account. So we introduce the female fertility into the Eq (2.3) and can be expressed as $X(t, 0) = \int_0^A \gamma(t, a)X(t, a)da$, $Y(t, 0) = \int_0^A \beta(t, a)Y(t, a)da$, which makes the Eq (2.3) more practical than model Eq (2.1).
- The density of the two species are the function of time t in model Eq (2.1). While in Eq (2.3), the density of the two species are the binary function of time t and age a . This is because the species density are not only affected by the time, but also affected by the change of birth rate and death rate which are caused by the age a .

2.2. Basic concepts

Throughout the paper, let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space. Here, the filtration $\{\mathfrak{F}_t\}_{t \geq 0}$ satisfies the usual conditions (that is, it is increasing and right continuous with \mathfrak{F}_0 containing all \mathbb{P} -null sets), \mathbb{E} denotes the expectation corresponding to \mathbb{P} . For a set \mathbb{A} , its indicator function is denoted

by $\mathbf{1}_{\mathbb{A}} = \begin{cases} 1, & x \in \mathbb{A}, \\ 0, & x \notin \mathbb{A}. \end{cases}$ We also denote by \mathbb{R}_+^n the positive cone in \mathbb{R}^n , that is $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_i > 0 \text{ for all } 1 \leq i \leq n\}$. If $x \in \mathbb{R}^n$, its norms are denoted as $|x|_l = \begin{cases} |x_1| + |x_2| + \cdots + |x_n|, & l = 1, \\ (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}, & l = 2. \end{cases}$

Let $V \equiv \{\varphi | \varphi \in L^p([0, A]), \frac{\partial \varphi}{\partial a} \in L^p([0, A])\}$, where $\frac{\partial \varphi}{\partial a}$ is generalized partial derivatives with respect to age a . $H = L^2([0, A])$ satisfy that $V \hookrightarrow H \equiv H' \hookrightarrow V'$, where V' is the dual space of V , H' is the dual space of H . The norm in H is denoted as $|\cdot|$. The duality product between V, V' is written as $\langle \cdot, \cdot \rangle$, the scalar product in H is denoted by (\cdot, \cdot) . $\mathbb{C} = \mathbb{C}([0, T]; H)$ is the space of all continuous functions from $[0, T]$ into H . $I^p([0, T]; V)$ denotes the space of all V -valued processes $(P_t)_{t \in [0, T]}$, $L_V^p = L^p([0, T]; V)$. $W := (I^p([0, T]; V) \cap L^2(\Omega; \mathbb{C}([0, T]; H))) \times (I^p([0, T]; V) \cap L^2(\Omega; \mathbb{C}([0, T]; H)))$.

In order to analyze the stochastic age-dependent cooperative LV system Eq (2.3), we make the following basic assumptions:

(A1) $\lim_{\alpha \rightarrow A^-} \int_{\alpha}^A \gamma(t, a)X(t, a)da = \lambda_x > 0$; $\lim_{\alpha \rightarrow A^-} \int_{\alpha}^A \beta(t, a)Y(t, a)da = \lambda_y > 0$.

(A2) The fertility rate of females $\gamma(t, a), \beta(t, a) \in \mathbb{C}([0, T] \times [0, A]; H)$.

(A3) $\alpha_{ij}(a) \in \mathbb{C}([0, A]; \mathbb{R}_+)$, $i, j \in \{1, 2\}$ and satisfy $\sup_{a \in [0, A]} \alpha_{12}(a)\alpha_{21}(a) < \inf_{a \in [0, A]} \alpha_{11}(a)\alpha_{22}(a)$.

(A4) $r_i(a) \in \mathbb{C}([0, A]; \mathbb{R})$ and $r_i(a)$ is nondecreasing with $-\infty < r_i(0) < 0 < r_i(A) < \infty$, for $i = 1, 2$.

(A5) $\sup_{a \in [0, A]} \alpha_{12}(a)\alpha_{21}(a) < 1$ and $r_1(0)(1 - \alpha_{12}\alpha_{21}) + \alpha_{12}(r_2(A) + \alpha_{21}r_1(A)) < 0$; $r_2(0)(1 - \alpha_{12}\alpha_{21}) + \alpha_{21}(r_1(A) + \alpha_{12}r_2(A)) < 0$.

Remark 2.2. The biological background significance of assumptions (A1)-(A5) are described as follows:

(A1) When the age a approaches to point A for species X , the number of new individuals born at time t is λ_x , which is a positive constant. This means that the species X at age A has not lost its reproductive capacity.

(A2) $\gamma(t, a)$ and $\beta(t, a)$ are the fertility rate of females in species X and Y , respectively. From biological point of view, the fertility rate of females in X and Y are both continuous functions, i.e., there is no sudden external noise that may increase or decrease the fertility rate of the species within a short time.

(A3) The product of interspecific cooperation rate is less than the intraspecific competition rate. This assumption is mainly used to prove the existence of global positive solution of the system.

(A4) $r_i(0)$ represents the intrinsic growth rate at age 0. “ $r_i(0) < 0$ ” means that the birth rate is less than the death rate. “ $0 < r_i(A) < \infty$ ” means that the birth rate is greater than the death rate at age A . Combined with the actual biological background, “ $0 < r_i(A) < \infty$ ” also indicates that the species will not overreproduce at age A .

(A5) The value of α_{12} and α_{21} are estimated by the actual data based on the least-square method. From biological point of view, this means that the intrinsic growth rate of species X at age 0 is less than $-\frac{\alpha_{12}r_2(A)+\alpha_{21}r_1(A)}{1-\alpha_{12}\alpha_{21}}$.

Theorem 2.1. Under the assumptions (A1)–(A3), for any given initial value $(X_0(a_0), Y_0(a_0)) \in \mathbb{R}_+^2$, there exists a unique global solution $(X(t), Y(t))$ of Eq (2.3) on W . Moreover, the solution is nonnegative with probability 1.

Proof. It is easy to see that for any given initial value $(X_0(a_0), Y_0(a_0)) \in \mathbb{R}_+^2$, there is a unique local solution $(X(t, a), Y(t, a)) \in W$ when $t \in [0, \tau_e)$, where τ_e is the explosion time of Eq (2.3). Now we need to proof that $\tau_e = \infty$, a.s. Choosing a positive constant $k_0 > 1$ such that for any initial value $(X_0(a_0), Y_0(a_0))$, we have $(X_0(a_0), Y_0(a_0)) \in [\frac{1}{k_0}, k_0] \times [\frac{1}{k_0}, k_0]$. Then, for each $k \geq k_0$, $a \in [0, A]$, we define the stopping time τ_k as $\tau_k = \inf\{t \in [0, \tau_e) : (X, Y) \notin (\frac{1}{k}, k) \times (\frac{1}{k}, k)\}$.

We can deduce that τ_k is increasing when $k \rightarrow \infty$ and $\lim_{k \rightarrow \infty} \tau_k = \tau_\infty$ with $\tau_\infty \leq \tau_e$. Assuming that there are two constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon$, then, for the constant k_0 , there is an integer k_1 such that

$$\mathbb{P}\{\tau_k \leq T\} \geq \varepsilon, \text{ for all } k \geq k_1. \tag{2.5}$$

Now, for any positive constants c_x, c_y , we define a function $V : W \rightarrow \mathbb{R}^+$ as

$$V(X, Y) = c_x(X - \log X - 1) + c_y(Y - \log Y - 1).$$

we can check that $V(X, Y) \geq 0$. By the Itô formula [15],

$$\begin{aligned} LV(X, Y) &= c_x \left\langle 1 - \frac{1}{X}, -\frac{\partial X}{\partial a} + X(-\alpha_{11}X + \alpha_{12}Y + r_1) \right\rangle + \frac{1}{2}(c_x\sigma_1^2 + c_y\sigma_2^2) \\ &\quad + c_y \left\langle 1 - \frac{1}{Y}, -\frac{\partial Y}{\partial a} + Y(\alpha_{21}X - \alpha_{22}Y + r_2) \right\rangle \\ &\leq \Psi^T \text{diag}(c_x, c_y)\bar{R} - \bar{C}(A'\Psi + \bar{R}) + \frac{1}{2}\Psi^T(\text{diag}(c_x, c_y) \cdot A' + A'^T \cdot \text{diag}(c_x, c_y))\Psi \\ &\quad + c_x \int_0^A \frac{1}{X} d_a X + c_y \int_0^A \frac{1}{Y} d_a Y + \frac{1}{2}(c_x\sigma_1^2 + c_y\sigma_2^2) \\ &\leq c_x \log \lambda_x + c_y \log \lambda_y - [c_x \log | \int_0^A \gamma(t, a)X(t, a)da | + c_y \log | \int_0^A \beta(t, a)Y(t, a)da |] \\ &\quad + \frac{1}{2}(c_x\sigma_1^2 + c_y\sigma_2^2) + \Psi^T \text{diag}(c_x, c_y)\bar{R} - \bar{C}(A'\Psi + \bar{R}) \\ &\quad + \frac{1}{2}\Psi^T(\text{diag}(c_x, c_y) \cdot A' + A'^T \cdot \text{diag}(c_x, c_y))\Psi \end{aligned}$$

where $\Psi = \begin{pmatrix} X(t, a) \\ Y(t, a) \end{pmatrix}$, $\bar{R} = \begin{pmatrix} r_1(a) \\ r_2(a) \end{pmatrix}$, $\text{diag}(c_x, c_y) = \begin{pmatrix} c_x & 0 \\ 0 & c_y \end{pmatrix}$, $A' = \begin{pmatrix} -\alpha_{11}(a) & \alpha_{12}(a) \\ \alpha_{21}(a) & -\alpha_{22}(a) \end{pmatrix}$, $\bar{C}^T = \begin{pmatrix} c_x \\ c_y \end{pmatrix}$.

By assumption **(A3)**, since $-A'$ is a nonsingular M-matrix, we deduce

$$\begin{aligned} LV(X, Y) &\leq c_x \log \lambda_x + c_y \log \lambda_y - \left[c_x \log \left| \int_0^A \gamma(t, a) X(t, a) da \right| + c_y \log \left| \int_0^A \beta(t, a) Y(t, a) da \right| \right] \\ &\quad + \frac{1}{2}(c_x \sigma_1^2 + c_y \sigma_2^2) + \Psi^\top \text{diag}(c_x, c_y) \bar{R} - \bar{C}(A' \Psi + \bar{R}) \\ &\leq c_x \log \lambda_x + c_y \log \lambda_y - \left[c_x \log \left| \int_0^A \gamma(t, a) X(t, a) da \right| + c_y \log \left| \int_0^A \beta(t, a) Y(t, a) da \right| \right] \\ &\quad + \{\bar{R}^\top \text{diag}(c_x, c_y) - \bar{C}A'\} \Psi + \frac{1}{2}(c_x \sigma_1^2 + c_y \sigma_2^2) - r_1 c_x - r_2 c_y. \end{aligned}$$

According to assumptions **(A1)** and **(A2)**, we have

$$LV(x, y) \leq |\bar{R}^\top \text{diag}(c_x, c_y) - \bar{C}A'| \cdot |\Psi| + C \leq M_1(1 + |\Psi|). \quad (2.6)$$

Now, let $M_2 = \frac{c_x \vee c_y}{c_x \wedge c_y}$, $M_3 = c_x \wedge c_y$, then

$$M_2 V(X, Y) = \frac{c_x \vee c_y}{c_x \wedge c_y} [c_x(X - 1 - \log X) + c_y(Y - 1 - \log Y)] \geq V(X, Y).$$

Since $|\Psi| = (X^2 + Y^2)^{\frac{1}{2}} \leq X + Y$, we have

$$\begin{aligned} |\Psi| &\leq 2(X - 1 - \log X) + 2(Y - 1 - \log Y) + 4 \\ &\leq 4 + \frac{2}{M_3} V(X, Y). \end{aligned} \quad (2.7)$$

By Eqs (2.6) and (2.7), we derive that

$$LV(X, Y) \leq M_4(1 + V(X, Y)),$$

where $M_4 = M_1(5 \vee \frac{2}{M_3})$. Then, we have

$$\begin{aligned} \mathbb{E}V(X(t \wedge \tau_k), Y(t \wedge \tau_k)) &\leq V(X_0, Y_0) + \mathbb{E} \int_0^{t \wedge \tau_k} M_4(1 + V(X(s), Y(s))) ds \\ &\leq M_5 + M_4 \int_0^t \mathbb{E}V(X(s \wedge \tau_k), Y(s \wedge \tau_k)) ds, \end{aligned}$$

where $M_5 = V(X_0, Y_0) + M_4 T$.

Applying the Gronwall inequality, we obtain

$$\mathbb{E}V(X(\tau_k \wedge T), Y(\tau_k \wedge T)) \leq M_5 e^{M_4 T}.$$

Since $\mathbb{P}(\Omega_k) = \mathbb{P}\{\tau_k \leq T\} \geq \varepsilon$ for $k \geq k_1$, for every $\omega \in \Omega_k$, $(t, a) \in \mathcal{Q}$, $X(\tau_k, \omega)$ and $Y(\tau_k, \omega)$ equal either k or $\frac{1}{k}$, hence

$$V(X(\tau_k, \omega), Y(\tau_k, \omega)) \geq [c_x(k - 1 - \log k) + c_y(k - 1 - \log k)] \wedge [c_x(\frac{1}{k} - 1 + \log k) + c_y(\frac{1}{k} - 1 + \log k)]$$

and

$$\begin{aligned} M_5 e^{M_4 T} &\geq \mathbb{E}V(X(\tau_k \wedge T), Y(\tau_k \wedge T)) \\ &\geq \mathbb{E}[I_{\Omega_k(\omega)} V(X(\tau_k \wedge T), Y(\tau_k \wedge T))] \\ &\geq \varepsilon [c_x(k-1-\log k) + c_y(k-1-\log k)] \wedge [c_x(\frac{1}{k}-1+\log k) + c_y(\frac{1}{k}-1+\log k)]. \end{aligned}$$

Taking $k \rightarrow \infty$, we have the contradiction that

$$\infty > M_5 e^{M_4 T} \geq \infty.$$

The proof is completed.

3. The approximate EM methods

In this part, we apply the EM approximate method to the stochastic age-dependent cooperative LV Equation (2.3) and explore the moment boundedness of the scheme for Eq (2.3).

We begin with introducing the following notations in the Eq (2.3).

$$G_{1x}(X, Y) := \sigma_1 X(t, a), \quad G_{1y}(X, Y) := \sigma_2 Y(t, a),$$

$$H_{1x}(X, Y) := r_1(a)X(t, a), \quad H_{2x}(X, Y) := X(t, a)[- \alpha_{11}(a)X(t, a) + \alpha_{12}(a)Y(t, a)],$$

$$H_{1y}(X, Y) := r_2(a)Y(t, a), \quad H_{2y}(X, Y) := Y(t, a)[\alpha_{21}(a)X(t, a) - \alpha_{22}(a)Y(t, a)].$$

Equation (2.3) has four equilibria: $E_0(0, 0)$, $E_1(\frac{r_1(A)}{\alpha_{11}}, 0)$, $E_2(0, \frac{r_2(A)}{\alpha_{22}})$, and $E_*(x^*, y^*)$, where

$$x^* = \frac{\alpha_{22}(a)r_1(A) + \alpha_{12}(a)r_2(A)}{\alpha_{11}(a)\alpha_{22}(a) - \alpha_{12}(a)\alpha_{21}(a)}, \quad y^* = \frac{\alpha_{11}(a)r_2(A) + \alpha_{21}(a)r_1(A)}{\alpha_{11}(a)\alpha_{22}(a) - \alpha_{12}(a)\alpha_{21}(a)}.$$

Now, let us first present some lemmas.

Remark 3.1. We note $H_{ix}(\Psi_j) := H_{ix}(X_j, Y_j)$, $H_{iy}(\Psi_j) := H_{iy}(X_j, Y_j)$, $H_{ix}(\psi_j) := H_{ix}(x_j, y_j)$, $H_{iy}(\psi_j) := H_{iy}(x_j, y_j)$, ($i = 1, 2$), ($j = 1, 2$) as follows.

Lemma 3.1. Under assumptions (A4) and (A5), for any $\psi_1, \psi_2 \in W$, we have

$$|H_{1x}(\psi_1) - H_{1x}(\psi_2)| \vee |G_{1x}(\psi_1) - G_{1x}(\psi_2)| \leq L_1 |\psi_1 - \psi_2|, \quad (3.1)$$

$$|H_{1y}(\psi_1) - H_{1y}(\psi_2)| \vee |G_{1y}(\psi_1) - G_{1y}(\psi_2)| \leq L_2 |\psi_1 - \psi_2|, \quad (3.2)$$

$$|H_{2x}(\psi_1) - H_{2x}(\psi_2)| \leq \rho_1 |\psi_1 - \psi_2|_1 \leq 2^{\frac{1}{2}} \rho_1 |\psi_1 - \psi_2|_2, \quad (3.3)$$

and

$$|H_{2y}(\psi_1) - H_{2y}(\psi_2)| \leq \rho_2 |\psi_1 - \psi_2|_1 \leq 2^{\frac{1}{2}} \rho_2 |\psi_1 - \psi_2|_2, \quad (3.4)$$

where L_1, L_2 are constants and $\rho_1 = 2x^* - r_1(0) + \alpha_{12}(x^* + y^*)$, $\rho_2 = 2y^* - r_2(0) + \alpha_{21}(x^* + y^*)$, $\psi_i^\top(t, a) = (x_i(t, a), y_i(t, a))$, ($i = 1, 2$).

The proof of this lemma is similar to that in Yang et al. [30]. We skip it here.

Remark 3.2. From Lemma 3.1, we can easily derive that there exist constants K_1 and K_2 such that

$$|H_{1x}(\psi)| \vee |G_{1x}(\psi)| \leq K_1(1 + |\psi|), \quad (3.5)$$

$$|H_{1y}(\psi)| \vee |G_{1y}(\psi)| \leq K_2(1 + |\psi|), \quad (3.6)$$

and

$$x \cdot H_{2x}(\psi) \leq 2^{\frac{1}{2}}\rho_1(1 + |\psi|^2), \quad y \cdot H_{2y}(\psi) \leq 2^{\frac{1}{2}}\rho_2(1 + |\psi|^2), \quad (3.7)$$

where $\psi(t, a) = (x(t, a), y(t, a))^T$ is the numerical solution corresponding to the exact solution Ψ of Equation (2.3), $H_{1x}(\psi) = H_{1x}(x, y)$, $H_{1y}(\psi) = H_{1y}(x, y)$

Lemma 3.2. For any $p > 2$, we have the following inequalities.

$$x[H_{1x}(\psi) + H_{2x}(\psi)] + \frac{p-1}{2}|G_{1x}(\psi)|^2 \leq [2K_1 + K_1^2(p-1) + 2^{\frac{1}{2}}\rho_1](1 + |\psi|^2),$$

and

$$y[H_{1y}(\psi) + H_{2y}(\psi)] + \frac{p-1}{2}|G_{1y}(\psi)|^2 \leq [2K_2 + K_2^2(p-1) + 2^{\frac{1}{2}}\rho_2](1 + |\psi|^2).$$

Proof. From Eqs (3.1), (3.5) and (3.7), we have

$$\begin{aligned} x[H_{1x}(\psi) + H_{2x}(\psi)] + \frac{p-1}{2}|G_{1x}(\psi)|^2 &\leq K_1 \cdot |x| \cdot (1 + |\psi|) + \frac{p-1}{2}K_1^2(1 + |\psi|)^2 + 2^{\frac{1}{2}}\rho_1(1 + |\psi|^2) \\ &\leq K_1 \cdot |\psi|(1 + |\psi|) + (p-1) \cdot K_1^2(1 + |\psi|^2) + 2^{\frac{1}{2}}\rho_1(1 + |\psi|^2) \\ &\leq K_1|\psi| + K_1|\psi|^2 + K_1^2(p-1)(1 + |\psi|^2) + 2^{\frac{1}{2}}\rho_1(1 + |\psi|^2), \end{aligned}$$

by the definition of $|\psi|$ and the positive of $|\psi|^2 - |\psi| + 1$, we derive that

$$\begin{aligned} x[H_{1x}(\psi) + H_{2x}(\psi)] + \frac{p-1}{2}|G_{1x}(\psi)|^2 &\leq 2K_1(1 + |\psi|^2) + K_1^2(p-1)(1 + |\psi|^2) + 2^{\frac{1}{2}}\rho_1(1 + |\psi|^2) \\ &=[K_1^2(p-1) + 2K_1 + 2^{\frac{1}{2}}\rho_1](1 + |\psi|^2). \end{aligned}$$

Similarly, using Eqs (3.2), (3.6) and (3.7), we have

$$\begin{aligned} y[H_{1y}(\psi) + H_{2y}(\psi)] + \frac{p-1}{2}|G_{1y}(\psi)|^2 &\leq K_2 \cdot |y|(1 + |\psi|) + \frac{p-1}{2} \cdot K_2^2(1 + |\psi|)^2 + 2^{\frac{1}{2}}\rho_2(1 + |\psi|^2) \\ &\leq K_2|\psi| + K_2|\psi|^2 + K_2^2(p-1)(1 + |\psi|^2) + 2^{\frac{1}{2}}\rho_2(1 + |\psi|^2) \\ &\leq [K_2^2(p-1) + 2K_2 + 2^{\frac{1}{2}}\rho_2](1 + |\psi|^2). \end{aligned}$$

This completes the proof.

Now, we investigate the approximate solution of Eq (2.3) by using the EM method. Given the fixed time T and time step $\Delta \in (0, 1)$, let $t_k = k\Delta$ for $k = 0, 1, 2, \dots, [\frac{T}{\Delta}]$, where $[\frac{T}{\Delta}]$ denotes the integer part of $\frac{T}{\Delta}$. So, the discrete time EM approximate solution to Eq (2.3) is defined as

$$\begin{cases} x_{k+1} = x_k + \left\{ -\frac{\partial x_k}{\partial a} + r_1(a)x_k - \alpha_{11}(a)x_k x_k + \alpha_{12}(a)x_k y_k \right\} \Delta + \sigma_1 x_k \Delta w_k, \\ y_{k+1} = y_k + \left\{ -\frac{\partial y_k}{\partial a} + r_2(a)y_k + \alpha_{21}(a)x_k y_k - \alpha_{22}(a)y_k y_k \right\} \Delta + \sigma_2 y_k \Delta w_k, \end{cases}$$

where $\Delta w_k = w_{t_{k+1}} - w_{t_k}$, $X(0) = X_0(a)$, and $Y(0) = Y_0(a)$.

The corresponding continuous EM approximate solution of Eq (2.3) is formed as

$$\begin{cases} x(t) = x_0 - \int_0^t \frac{\partial \bar{x}(s)}{\partial a} ds + \int_0^t \{r_1(a)\bar{x}(s) - \alpha_{11}(a)\bar{x}(s)\bar{x}(s) + \alpha_{12}(a)\bar{x}(s)\bar{y}(s)\} ds + \int_0^t \sigma_1 \bar{x}(s) dw(s), \\ y(t) = y_0 - \int_0^t \frac{\partial \bar{y}(s)}{\partial a} ds + \int_0^t \{r_2(a)\bar{y}(s) + \alpha_{21}(a)\bar{x}(s)\bar{y}(s) - \alpha_{22}(a)\bar{y}(s)\bar{y}(s)\} ds + \int_0^t \sigma_2 \bar{y}(s) dw(s), \end{cases} \tag{3.8}$$

where

$$\bar{x}(t) = \sum_{k=0}^{\lfloor \frac{T}{\Delta} \rfloor} x_k 1_{[t_k, t_{k+1})}(t), \quad \bar{y}(t) = \sum_{k=0}^{\lfloor \frac{T}{\Delta} \rfloor} y_k 1_{[t_k, t_{k+1})}(t)$$

are called the step processes. We can easily see that $\bar{x}(t) = x_k$ and $\bar{y}(t) = y_k$ for $t \in [t_k, t_{k+1})$ when $k = 0, 1, 2, \dots, \lfloor \frac{T}{\Delta} \rfloor$. Without loss of generality, we denote $\psi(t) = (x(t), y(t))^T$ and $\bar{\psi}(t) = (\bar{x}(t), \bar{y}(t))^T$.

Theorem 3.1. *Under assumptions (A1)–(A5) and $p > 2$, there exists a constant $C > 0$ such that*

$$\sup_{0 \leq t \leq T} \mathbb{E} |\psi(t)|^p \leq C,$$

where C only depends on T and p .

Proof. For any $p > 2, t \in (0, T)$, applying the Itô formula to the first equation of (3.8), we have

$$\begin{aligned} \mathbb{E}|x(t)|^p - |x_0|^p &\leq \mathbb{E} \int_0^t p|x(s)|^{p-2} \cdot \left\langle -\frac{\partial x(s)}{\partial a}, x(s) \right\rangle ds + \mathbb{E} \int_0^t \frac{p(p-1)}{2} |x(s)|^{p-2} |G_{1,x}(\bar{x}(s), \bar{y}(s))|^2 ds \\ &\quad + \mathbb{E} \int_0^t p|x(s)|^{p-2} \{ |x(s)| [H_{1,x}(\bar{x}(s), \bar{y}(s)) + H_{2,x}(\bar{x}(s), \bar{y}(s))] \} ds \\ &\leq \mathbb{E} \int_0^t p|x(s)|^{p-2} \{ |\bar{x}(s)| [H_{1,x}(\bar{x}(s), \bar{y}(s)) + H_{2,x}(\bar{x}(s), \bar{y}(s))] + \frac{p-1}{2} |G_{1,x}(\bar{x}(s), \bar{y}(s))|^2 \} ds \\ &\quad + \mathbb{E} \int_0^t p|x(s)|^{p-2} \{ |x(s) - \bar{x}(s)| [H_{1,x}(\bar{x}(s), \bar{y}(s)) + H_{2,x}(\bar{x}(s), \bar{y}(s))] \} ds \\ &\quad + \mathbb{E} \int_0^t p|x(s)|^{p-2} \left\langle -\frac{\partial x(s)}{\partial a}, x(s) \right\rangle ds. \end{aligned}$$

Since

$$\begin{aligned} \left\langle -\frac{\partial x(s)}{\partial a}, x(s) \right\rangle &= - \int_0^A x(s) d_a x(s) = \frac{1}{2} \left[\int_0^A \gamma(s, a) x(s, a) da \right]^2 \\ &\leq \frac{1}{2} \int_0^A \gamma^2(s, a) da \cdot \int_0^A x^2(s, a) da \\ &\leq \frac{1}{2} A \cdot \check{\gamma}^2(s, a) |x(s, a)|^2, \end{aligned}$$

where $\check{\gamma}$ means the maximum value of continuous function $\gamma(t, a)$. Then we have

$$\begin{aligned} & \mathbb{E}|x(t)|^p - |x_0|^p \\ & \leq \mathbb{E} \int_0^t p|x(s)|^{p-2} \{ |\bar{x}(s)| [H_{1x}(\bar{x}(s), \bar{y}(s)) + H_{2x}(\bar{x}(s), \bar{y}(s))] + \frac{p-1}{2} |G_{1x}(\bar{x}(s), \bar{y}(s))|^2 \} ds \\ & \quad + \mathbb{E} \int_0^t p|x(s)|^{p-2} \{ |x(s) - \bar{x}(s)| [H_{1x}(\bar{x}(s), \bar{y}(s)) + H_{2x}(\bar{x}(s), \bar{y}(s))] \} ds \\ & \quad + \frac{1}{2} A \check{\gamma}^2 p \mathbb{E} \int_0^t |x(s)|^p ds. \end{aligned} \quad (3.9)$$

Let

$$Q_1 = \mathbb{E} \int_0^t p|x(s)|^{p-2} \{ |\bar{x}(s)| [H_{1x}(\bar{x}(s), \bar{y}(s)) + H_{2x}(\bar{x}(s), \bar{y}(s))] + \frac{p-1}{2} |G_{1x}(\bar{x}(s), \bar{y}(s))|^2 \} ds,$$

$$Q_2 = \mathbb{E} \int_0^t p|x(s)|^{p-2} |x(s) - \bar{x}(s)| |H_{1x}(\bar{x}(s), \bar{y}(s))| ds,$$

$$Q_3 = \mathbb{E} \int_0^t p|x(s)|^{p-2} |x(s) - \bar{x}(s)| |H_{2x}(\bar{x}(s), \bar{y}(s))| ds.$$

By Lemma 3.2 and Young inequality $a^\beta + b^{1-\beta} \leq \beta a + (1-\beta)b$, for $a, b \geq 0$ and $\beta \in (0, 1)$, we have

$$\begin{aligned} Q_1 & \leq (p-2)[K_1^2(p-1) + 2K_1 + 2^{\frac{1}{2}}\rho_1] \mathbb{E} \int_0^t |x(s)|^p ds \\ & \quad + 2[K_1^2(p-1) + 2K_1 + 2^{\frac{1}{2}}\rho_1] \mathbb{E} \int_0^t (1 + |\bar{\psi}(s)|^2)^{\frac{p}{2}} ds \\ & \leq (p-2)[K_1^2(p-1) + 2K_1 + 2^{\frac{1}{2}}\rho_1] \mathbb{E} \int_0^t |x(s)|^p ds \\ & \quad + 2^p [K_1^2(p-1) + 2K_1 + 2^{\frac{1}{2}}\rho_1] \mathbb{E} \int_0^t (1 + |\bar{\psi}(s)|^p) ds \\ & \leq 2^p [K_1^2(p-1) + 2K_1 + 2^{\frac{1}{2}}\rho_1] T \{ 1 + \int_0^t \mathbb{E} |\psi(s)|^p ds + \int_0^t \mathbb{E} |\bar{\psi}(s)|^p ds \}. \end{aligned} \quad (3.10)$$

Similarly, we can show that

$$Q_2 \leq C \left[1 + \int_0^t (\mathbb{E} |\psi(s)|^p + \mathbb{E} |\bar{\psi}(s)|^p) ds \right]. \quad (3.11)$$

Moreover, by the Young inequality and Eq (3.3), we have

$$\begin{aligned} Q_3 & \leq 2^{\frac{1}{2}}\rho_1 \mathbb{E} \int_0^t \{ (p-2)|x(s)|^p + 2|x(s) - \bar{x}(s)|^{\frac{p}{2}} |\psi(s)|^{\frac{p}{2}} \} ds \\ & \leq 2^{\frac{1}{2}}\rho_1 (p-2) \mathbb{E} \int_0^t |x(s)|^p ds + 2^{\frac{3}{2}}\rho_1 \mathbb{E} \int_0^t |x(s) - \bar{x}(s)|^{\frac{p}{2}} |\psi(s)|^{\frac{p}{2}} ds. \end{aligned} \quad (3.12)$$

On the other hand, there is a unique nonnegative constant k which satisfies that $t_k \leq s \leq t_{k+1}$ for any $s \in [0, T]$. By Itô integral and the Hölder, Burkholder-Davis-Gundy (BDG) inequalities, we have

$$\begin{aligned}
 & \mathbb{E}|x(s) - \bar{x}(s)|^{\frac{p}{2}} \\
 &= \mathbb{E} \left| \int_{t_k}^s (H_{1x}(\bar{x}(u), \bar{y}(u)) + H_{2x}(\bar{x}(u), \bar{y}(u)))du + \int_{t_k}^s G_{1x}(\bar{x}(u), \bar{y}(u))dw(u) + \int_{t_k}^s \frac{\partial \bar{x}(u)}{\partial a} du \right|^{\frac{p}{2}} \\
 &\leq 3^{\frac{p}{2}-1} \mathbb{E} \left| \int_{t_k}^s (H_{1x}(\bar{x}(u), \bar{y}(u)) + H_{2x}(\bar{x}(u), \bar{y}(u)))du \right|^{\frac{p}{2}} + 3^{\frac{p}{2}-1} \mathbb{E} \left| \int_{t_k}^s \frac{\partial \bar{x}(u)}{\partial a} du \right|^{\frac{p}{2}} \\
 &\quad + 3^{\frac{p}{2}-1} \mathbb{E} \left| \int_{t_k}^s G_{1x}(\bar{x}(u), \bar{y}(u))dw(u) \right|^{\frac{p}{2}} \\
 &\leq 3^{\frac{p}{2}-1} \Delta^{\frac{p}{2}-1} \mathbb{E} \left| \int_{t_k}^s 2^{\frac{p}{2}-1} (H_{1x}^{\frac{p}{2}}(\bar{x}(u), \bar{y}(u)) + H_{2x}^{\frac{p}{2}}(\bar{x}(u), \bar{y}(u)))du \right| + 3^{\frac{p}{2}-1} \mathbb{E} \left| \int_{t_k}^s \frac{\partial \bar{x}(u)}{\partial a} du \right|^{\frac{p}{2}} \\
 &\quad + C_p 3^{\frac{p}{2}-1} K_1^{\frac{p}{2}} \mathbb{E} \left| \int_{t_k}^s (1 + |\bar{\psi}(u)|)^2 du \right|^{\frac{p}{4}} \tag{3.13} \\
 &\leq 3^{p-2} \Delta^{\frac{p-2}{2}} \mathbb{E} \int_{t_k}^s (H_{1x}^{\frac{p}{2}}(\bar{x}(u), \bar{y}(u)) + H_{2x}^{\frac{p}{2}}(\bar{x}(u), \bar{y}(u)))du + C \Delta^{\frac{p}{2}} \\
 &\quad + C_p 3^{\frac{p}{2}-1} K_1^{\frac{p}{2}} \mathbb{E} \left| \int_{t_k}^s 2(1 + |\bar{\psi}(u)|^2)du \right|^{\frac{p}{4}} \\
 &\leq 3^{p-2} \Delta^{\frac{p-2}{2}} K_1^{\frac{p}{2}} \mathbb{E} \int_{t_k}^s (1 + |\bar{\psi}(u)|)^{\frac{p}{2}} du + C \Delta^{\frac{p-2}{2}} \mathbb{E} \int_{t_k}^s \rho_1^{\frac{p}{2}} |\bar{\psi}(u)|^{\frac{p}{2}} du + C \Delta^{\frac{p}{2}} \\
 &\quad + C_p 3^{p-1} K_1^{\frac{p}{2}} (\Delta + \mathbb{E} \int_{t_k}^s |\bar{\psi}(u)|^2 du)^{\frac{p}{4}} \\
 &\leq C \Delta^{\frac{p}{4}} + C \Delta^{\frac{p}{4}} \mathbb{E} |\bar{\psi}(s)|^{\frac{p}{2}},
 \end{aligned}$$

where C is a positive constant and C_p is defined by

$$C_p = \begin{cases} \left(\frac{64}{p}\right)^{\frac{p}{4}}, & 0 < p < 4, \\ 4, & p = 4, \\ \left[\frac{\left(\frac{p}{2}\right)^{\frac{p}{2}+1}}{2\left(\frac{p}{2}-1\right)^{\frac{p}{2}-1}}\right]^{\frac{p}{4}}, & p > 4. \end{cases}$$

Substituting Eq (3.13) into Eq (3.12), we have

$$\begin{aligned}
 Q_3 &\leq 2^{\frac{1}{2}} \rho_1 \mathbb{E} \int_0^t (p-2) |x(s)|^p ds + 2^{\frac{3}{2}} \rho_1 \mathbb{E} \int_0^t |x(s) - \bar{x}(s)|^{\frac{p}{2}} |\psi(s)|^{\frac{p}{2}} ds \\
 &\leq 2^{\frac{1}{2}} \rho_1 (p-2) \mathbb{E} \int_0^t |x(s)|^p ds + 2^{\frac{3}{2}} C \rho_1 \Delta^{\frac{p}{4}} \int_0^t (1 + \mathbb{E} |\bar{\psi}(s)|^{\frac{p}{2}}) |\psi(s)|^{\frac{p}{2}} ds \tag{3.14} \\
 &\leq 2^{\frac{1}{2}} \rho_1 (p-2) \int_0^t \mathbb{E} |x(s)|^p ds + 2^{\frac{5}{2}} C \rho_1 \Delta^{\frac{p}{4}} \int_0^t \mathbb{E} |\bar{\psi}(s)|^p ds.
 \end{aligned}$$

Now, substituting Eqs (3.10), (3.11) and (3.14) into Eq (3.9), we can derive that

$$\begin{aligned}
 \mathbb{E}|x(t)|^p &\leq |x_0|^p + \frac{1}{2} A^2 \check{\gamma}^2 p \mathbb{E} \int_0^t |x(s)|^p ds + C [1 + \mathbb{E} \int_0^t |\psi(s)|^p ds + \mathbb{E} \int_0^t |\bar{\psi}(s)|^p ds] \\
 &\leq C (1 + \int_0^t \sup_{0 \leq v \leq s} \mathbb{E} |\psi(v)|^p ds). \tag{3.15}
 \end{aligned}$$

Since the right-hand side of Eq (3.15) is non-decreasing for $t \in [0, T]$, a conclusion can be draw that

$$\sup_{0 \leq \nu \leq t} \mathbb{E}|x(\nu)|^p \leq C(1 + \int_0^t \sup_{0 \leq \nu \leq s} \mathbb{E}|\psi(\nu)|^p ds). \quad (3.16)$$

Similarly, we can get that

$$\begin{aligned} \mathbb{E}|y(t)|^p - |y_0|^p &\leq \mathbb{E} \int_0^t p|y(s)|^{p-2} \cdot \langle -\frac{\partial y(s)}{\partial a}, y(s) \rangle ds + \mathbb{E} \int_0^t \frac{p(p-1)}{2} |y(s)|^{p-2} |G_{1y}(\bar{x}(s), \bar{y}(s))|^2 ds \\ &+ \mathbb{E} \int_0^t p|y(s)|^{p-2} [|y(s)|(H_{1y}(\bar{x}(s), \bar{y}(s)) + H_{2y}(\bar{x}(s), \bar{y}(s)))] ds. \end{aligned}$$

Since

$$\begin{aligned} \langle -\frac{\partial y(s)}{\partial a}, y(s) \rangle &= - \int_0^A y(s) d_a y(s) = \frac{1}{2} [\int_0^A \beta(s, a) y(s, a) da]^2 \\ &\leq \frac{1}{2} \int_0^A \beta^2(s, a) da \cdot \int_0^A y^2(s, a) da \\ &\leq \frac{1}{2} A \cdot \check{\beta}^2(s, a) |y(s, a)|^2, \end{aligned}$$

where $\check{\beta}$ is the maximum value of continuous function $\beta(t, a)$, and

$$\begin{aligned} &\mathbb{E}|y(t)|^p - |y_0|^p \\ &\leq \mathbb{E} \int_0^t p|y(s)|^{p-2} \{ |\bar{y}(s)| [H_{1y}(\bar{x}(s), \bar{y}(s)) + H_{2y}(\bar{x}(s), \bar{y}(s))] + \frac{p-1}{2} |G_{1y}(\bar{x}(s), \bar{y}(s))|^2 \} ds \\ &+ \mathbb{E} \int_0^t p|y(s)|^{p-2} \{ |y(s) - \bar{y}(s)| [H_{1y}(\bar{x}(s), \bar{y}(s)) + H_{2y}(\bar{x}(s), \bar{y}(s))] \} ds \\ &+ \frac{1}{2} A \check{\beta}^2 p \mathbb{E} \int_0^t |y(s)|^p ds. \end{aligned} \quad (3.17)$$

Let

$$\begin{aligned} G_1 &= \mathbb{E} \int_0^t p|y(s)|^{p-2} \{ |\bar{y}(s)| [H_{1y}(\bar{x}(s), \bar{y}(s)) + H_{2y}(\bar{x}(s), \bar{y}(s))] + \frac{p-1}{2} \sigma_2^2 |\bar{y}(s)|^2 \} ds, \\ G_2 &= \mathbb{E} \int_0^t p|y(s)|^{p-2} |y(s) - \bar{y}(s)| |H_{1y}(\bar{x}(s), \bar{y}(s))| ds, \\ G_3 &= \mathbb{E} \int_0^t p|y(s)|^{p-2} |y(s) - \bar{y}(s)| |H_{2y}(\bar{x}(s), \bar{y}(s))| ds. \end{aligned}$$

By Lemma 3.2, we have

$$G_1 \leq p \mathbb{E} \int_0^t |y(s)|^{p-2} [K_2^2(p-1) + 2K_2 + 2^{\frac{1}{2}} \rho_2] (1 + |\bar{\psi}(s)|^2) ds. \quad (3.18)$$

By Eq (3.6), we have

$$\begin{aligned} G_2 &\leq p K_2 \mathbb{E} \int_0^t |y(s)|^{p-2} |y(s) - \bar{y}(s)| (1 + |\psi(s)|) ds \\ &\leq C [1 + \int_0^t (\mathbb{E}|\psi(s)|^p + \mathbb{E}|\bar{\psi}(s)|^p) ds]. \end{aligned} \quad (3.19)$$

Moreover, using the Young inequality, we can get

$$\begin{aligned} G_3 &\leq 2^{\frac{1}{2}} p \rho_2 \mathbb{E} \int_0^t |y(s)|^{p-2} |y(s) - \bar{y}(s)| |\psi(s)| ds \\ &\leq 2^{\frac{1}{2}} \rho_2 \mathbb{E} \int_0^t (p-2) |y(s)|^p + 2^{\frac{3}{2}} \rho_2 \mathbb{E} \int_0^t |y(s) - \bar{y}(s)|^{\frac{p}{2}} |\psi(s)|^{\frac{p}{2}} ds. \end{aligned} \quad (3.20)$$

Similarly, by the Hölder, BDG inequalities and the Itô integral, we derive that

$$\mathbb{E} |y(s) - \bar{y}(s)|^{\frac{p}{2}} \leq C \Delta^{\frac{p}{4}} + C \Delta^{\frac{p}{4}} \mathbb{E} |\bar{\psi}(u)|^{\frac{p}{2}}. \quad (3.21)$$

Substituting Eq (3.21) into Eq (3.20), we get

$$G_3 \leq 2^{\frac{1}{2}} \rho_2 (p-2) \mathbb{E} \int_0^t |y(s)|^p ds + 2^{\frac{3}{2}} \rho_2 C \Delta^{\frac{p}{4}} \int_0^t (1 + \mathbb{E} |\bar{\psi}(s)|^{\frac{p}{2}}) |\psi(s)|^{\frac{p}{2}} ds. \quad (3.22)$$

According to Eqs (3.18), (3.19), (3.22), and Eq (3.17), we have

$$\mathbb{E} |y(t)|^p \leq C \left(1 + \int_0^t \sup_{0 \leq \nu \leq s} \mathbb{E} |\psi(\nu)|^p ds \right). \quad (3.23)$$

For any $t \in [0, T]$, since the right-hand side of Eq (3.23) is non-decreasing function, we obtain

$$\sup_{0 \leq \nu \leq t} \mathbb{E} |y(\nu)|^p \leq C \left(1 + \int_0^t \sup_{0 \leq \nu \leq s} \mathbb{E} |\psi(\nu)|^p ds \right). \quad (3.24)$$

Using Eqs (3.16), (3.24) and the norm of $\psi(t)$, we have

$$\sup_{0 \leq \nu \leq t} \mathbb{E} |\psi(\nu)|^p \leq C \left(1 + \int_0^t \sup_{0 \leq \nu \leq s} \mathbb{E} |\psi(\nu)|^p ds \right).$$

Applying the continuous Gronwall inequality, it yields

$$\sup_{0 \leq t \leq T} \mathbb{E} |\psi(t)|^p \leq C.$$

Now, let us present a theorem which shows that $\psi(t)$ and $\bar{\psi}(t)$ are coincide with the discrete solution.

Theorem 3.2. *Under assumptions (A1)–(A5) and $p > 2$, there exists a constant C such that*

$$\mathbb{E} |\psi(t) - \bar{\psi}(t)|^p < C \Delta^{\frac{p}{2}}.$$

Proof. For any $t \in [0, T]$, there exists a positive integer k such that $t \in [t_k, t_{k+1}]$, and

$$\begin{aligned} &\mathbb{E} |x(t) - \bar{x}(t)|^p \\ &= \mathbb{E} \left| - \int_{t_k}^t \frac{\partial x(s)}{\partial a} ds + \int_{t_k}^t [H_{1,x}(\bar{x}(s), \bar{y}(s)) + H_{2,x}(\bar{x}(s), \bar{y}(s))] ds + \int_{t_k}^t G_{1,x}(\bar{x}(s), \bar{y}(s)) dw(s) \right|^p \\ &\leq 4^{p-1} (\mathbb{E} \left| \int_{t_k}^t \frac{\partial x(s)}{\partial a} ds \right|^p + \mathbb{E} \left| \int_{t_k}^t H_{1,x}(\bar{x}(s), \bar{y}(s)) ds \right|^p + \mathbb{E} \left| \int_{t_k}^t H_{2,x}(\bar{x}(s), \bar{y}(s)) ds \right|^p \\ &\quad + \mathbb{E} \left| \int_{t_k}^t G_{1,x}(\bar{x}(s), \bar{y}(s)) dw(s) \right|^p). \end{aligned}$$

According to the Hölder, BDG, and elementary inequalities, we have

$$\begin{aligned}
\mathbb{E}|x(t) - \bar{x}(t)|^p &\leq 4^{p-1} \left[\Delta^{p-1} \mathbb{E} \int_{t_k}^t \left| \frac{\partial x(s)}{\partial a} \right|^p ds + K_1^p \Delta^{p-1} \mathbb{E} \int_{t_k}^t (1 + |\psi(s)|)^p ds \right. \\
&\quad \left. + 2^{\frac{p}{2}} \rho_1^p \Delta^{p-1} \mathbb{E} \int_{t_k}^t |\psi(s)|^p ds + C_p \mathbb{E} \left| \int_{t_k}^t G_{1,x}^2(s) ds \right|^{\frac{p}{2}} \right] \\
&\leq 4^{p-1} \left[C \Delta^p + 2^{p-1} K_1^p \Delta^{p-1} (\Delta + \mathbb{E} \int_{t_k}^t |\psi(s)|^p ds) \right. \\
&\quad \left. + 2^{\frac{p}{2}} \rho_1^p \Delta^{p-1} \mathbb{E} \int_{t_k}^t |\psi(s)|^p ds + C_p \mathbb{E} \left| \int_{t_k}^t G_{1,x}^2(s) ds \right|^{\frac{p}{2}} \right] \\
&\leq 4^{p-1} \left[(C + 2^{p-1} K_1^p + C_p 2^{p-1} K_1^p) \Delta^{\frac{p}{2}} \right. \\
&\quad \left. + (2^{p-1} K_1^p + 2^{\frac{p}{2}} \rho_1^p + C_p 2^{p-1} K_1^p) \Delta^{\frac{p}{2}} \mathbb{E} |\psi(t)|^p \right] \\
&\leq C \Delta^{\frac{p}{2}} (1 + \mathbb{E} |\psi(t)|^p),
\end{aligned} \tag{3.25}$$

where C_p is defined in the proof of Theorem 3.1. Similarly, we have

$$\begin{aligned}
&\mathbb{E}|y(t) - \bar{y}(t)|^p \\
&= \mathbb{E} \left| - \int_{t_k}^t \frac{\partial y(s)}{\partial a} ds + \int_{t_k}^t [H_{1y}(\bar{x}(s), \bar{y}(s)) + H_{2y}(\bar{x}(s), \bar{y}(s))] ds + \int_{t_k}^t G_{1y}(\bar{x}(s), \bar{y}(s)) dw(s) \right|^p \\
&\leq 4^{p-1} \left(\mathbb{E} \left| \int_{t_k}^t \frac{\partial y(s)}{\partial a} ds \right|^p + \mathbb{E} \left| \int_{t_k}^t H_{1y}(\bar{x}(s), \bar{y}(s)) ds \right|^p + \mathbb{E} \left| \int_{t_k}^t H_{2y}(\bar{x}(s), \bar{y}(s)) ds \right|^p \right. \\
&\quad \left. + \mathbb{E} \left| \int_{t_k}^t G_{1y}(\bar{x}(s), \bar{y}(s)) dw(s) \right|^p \right) \\
&\leq 4^{p-1} \left[\Delta^{p-1} \mathbb{E} \int_{t_k}^t \left| \frac{\partial y(s)}{\partial a} \right|^p ds + K_2^p \Delta^{p-1} \mathbb{E} \int_{t_k}^t (1 + |\psi(s)|)^p ds \right. \\
&\quad \left. + 2^{\frac{p}{2}} \rho_2^p \Delta^{p-1} \mathbb{E} \int_{t_k}^t |\psi(s)|^p ds + C_p \mathbb{E} \left| \int_{t_k}^t G_{1y}^2(s) ds \right|^{\frac{p}{2}} \right] \\
&\leq 4^{p-1} \left[C \Delta^p + 2^{p-1} K_2^p \Delta^{p-1} (\Delta + \mathbb{E} \int_{t_k}^t |\psi(s)|^p ds) \right. \\
&\quad \left. + 2^{\frac{p}{2}} \rho_2^p \Delta^{p-1} \mathbb{E} \int_{t_k}^t |\psi(s)|^p ds + C_p 2^{p-1} K_2^p \Delta^{\frac{p}{2}} (1 + \mathbb{E} |\psi(t)|^p) \right] \\
&\leq C \Delta^{\frac{p}{2}} (1 + \mathbb{E} |\psi(t)|^p).
\end{aligned} \tag{3.26}$$

Using Eqs (3.25) and (3.26), we deduce that

$$\mathbb{E}|\psi(t) - \bar{\psi}(t)|^p \leq C \Delta^{\frac{p}{2}} (1 + \mathbb{E} |\psi(t)|^p).$$

By Theorem 3.1, we have

$$\mathbb{E}|\psi(t) - \bar{\psi}(t)|^p \leq C \Delta^{\frac{p}{2}}.$$

Theorem 3.2 shows that the numerical solution converges to the step process. Now, we can discuss the convergence order rate between the exact and numerical solution of Eq (2.3).

4. Convergence rate over the time interval $[0, T]$

Theorem 4.1. *Under assumptions (A1)–(A5) and $p > 2$, we have*

$$\mathbb{E} \sup_{0 \leq t \leq T} |\Psi(t) - \psi(t)|^p \leq C\Delta^{\frac{p}{2}}.$$

where C is a constant and $\Psi(t) := (X(t), Y(t))$ is the true solution of Eq (2.4).

Proof. For any $t \in [0, T]$, an application of Itô formula yields

$$\begin{aligned} d|X(t) - x(t)|^p &= p|X(t) - x(t)|^{p-2} \langle X(t) - x(t), H_{1x}(\Psi(t)) - H_{1x}(\bar{\psi}(t)) \rangle dt \\ &\quad + p|X(t) - x(t)|^{p-2} \langle X(t) - x(t), H_{2x}(\Psi(t)) - H_{2x}(\bar{\psi}(t)) \rangle dt \\ &\quad + p|X(t) - x(t)|^{p-2} \langle X(t) - x(t), [G_{1x}(\Psi(t)) - G_{1x}(\bar{\psi}(t))] \rangle dw(t) \\ &\quad + \frac{p(p-1)}{2} (|X(t) - x(t)|^{p-2}, |G_{1x}(\Psi(t)) - G_{1x}(\bar{\psi}(t))|^2) dt \\ &\quad - p|X(t) - x(t)|^{p-2} \langle X(t) - x(t), \frac{\partial(X(t) - x(t))}{\partial a} \rangle dt \\ &\leq p(|X(t) - x(t)|^{p-1}, [G_{1x}(\Psi(t)) - G_{1x}(\bar{\psi}(t))] \rangle dw(t) \\ &\quad + \frac{p(p-1)}{2} L_1^2 |X(t) - x(t)|^{p-2} |\Psi(t) - \bar{\psi}(t)|^2 dt \\ &\quad - p|X(t) - x(t)|^{p-1}, \frac{\partial(X(t) - x(t))}{\partial a} \rangle dt \\ &\quad + p\rho_1 2^{\frac{1}{2}} |X(t) - x(t)|^{p-1} |\Psi(t) - \bar{\psi}(t)| dt \\ &\quad + pL_1 |X(t) - x(t)|^{p-1} |\Psi(t) - \bar{\psi}(t)| dt, \end{aligned}$$

hence we have

$$\begin{aligned} \mathbb{E} \sup_{s \in [0, t]} |X(s) - x(s)|^p &\leq (pL_1 + 2^{\frac{1}{2}} p\rho_1) \int_0^t \mathbb{E} \sup_{r \in [0, s]} |X(r) - x(r)|^{p-1} |\Psi(r) - \bar{\psi}(r)| ds \\ &\quad + p\mathbb{E} \int_0^s \sup_{s \in [0, t]} (|X(s) - x(s)|^{p-1}, [G_{1x}(\Psi(s)) - G_{1x}(\bar{\psi}(s))] \rangle dw(s)) \\ &\quad + \frac{p(p-1)}{2} L_1^2 \int_0^t \mathbb{E} \sup_{r \in [0, s]} |X(r) - x(r)|^{p-2} |\Psi(r) - \bar{\psi}(r)|^2 ds \\ &\quad + \frac{p}{2} A^2 \gamma^2(t, a) \int_0^t \mathbb{E} \sup_{r \in [0, s]} |X(r) - x(r)|^p ds \\ &\leq (2pL_1 + 2^{\frac{3}{2}} p\rho_1 + \frac{L_1^2}{2} p(p-1)) \int_0^t \mathbb{E} \sup_{r \in [0, s]} |X(r) - x(r)|^{p-2} |\Psi(r) - \bar{\psi}(r)|^2 ds \\ &\quad + p \sup_{s \in [0, t]} \int_0^s \mathbb{E} |X(s) - x(s)|^{p-2} \langle X(s) - x(s), [G_{1x}(\Psi(s)) - G_{1x}(\bar{\psi}(s))] \rangle dw(s) \\ &\quad + \left(\frac{p}{2} A^2 \gamma^2(t, a) + 2pL_1 + 2^{\frac{3}{2}} p\rho_1 \right) \int_0^t \mathbb{E} \sup_{r \in [0, s]} |X(r) - x(r)|^p ds. \end{aligned} \tag{4.1}$$

Using the Young inequality, we derive that

$$\int_0^t \mathbb{E} \sup_{r \in [0, s]} |X(r) - x(r)|^{p-2} |\Psi(r) - \bar{\psi}(r)|^2 ds \leq \int_0^t \mathbb{E} \sup_{r \in [0, s]} \left\{ \left(1 - \frac{2}{p}\right) |X(r) - x(r)|^p + \frac{2}{p} |\Psi(r) - \bar{\psi}(r)|^p \right\} ds. \quad (4.2)$$

By the BDG inequality, Young inequality, and elementary inequality, we have

$$\begin{aligned} & \mathbb{E} \sup_{s \in [0, t]} \int_0^s ((X(s) - x(s))^{p-1}, [G_{1x}(\Psi(s)) - G_{1x}(\bar{\psi}(s))] dw(s)) \\ & \leq C_1 \mathbb{E} \left\{ \int_0^t |((X(s) - x(s))^{p-1}, G_{1x}(\Psi(s)) - G_{1x}(\bar{\psi}(s)))|^2 ds \right\}^{\frac{1}{2}} \\ & \leq C_1 \mathbb{E} \sup_{s \in [0, t]} |X(s) - x(s)|^{p-1} \left\{ \int_0^t |G_{1x}(\Psi(s)) - G_{1x}(\bar{\psi}(s))|^2 ds \right\}^{\frac{1}{2}} \\ & \leq C_1 \mathbb{E} \sup_{s \in [0, t]} \frac{p-1}{p^2} |X(s) - x(s)|^p + \frac{C_1 L_1^p}{p^{2-p}} \mathbb{E} \sup_{s \in [0, t]} \left\{ \int_0^t |\Psi(s) - \bar{\psi}(s)|^2 ds \right\}^{\frac{p}{2}} \\ & \leq \frac{C_1(p-1)}{p^2} \mathbb{E} \sup_{s \in [0, t]} |X(s) - x(s)|^p + \frac{C_1 L_1^p T^{\frac{p-2}{2}}}{p^{2-p}} \mathbb{E} \sup_{s \in [0, t]} \int_0^t |\Psi(s) - \bar{\psi}(s)|^p ds. \end{aligned} \quad (4.3)$$

Now, substituting Eqs (4.2) and (4.3) into Eq (4.1), we can derive

$$\begin{aligned} & \mathbb{E} \sup_{s \in [0, t]} |X(s) - x(s)|^p \\ & \leq \left\{ \frac{p}{2} A^2 \check{\gamma}^2(t, a) + 2pL_1 + 2^{\frac{3}{2}} p \rho_1 + (2pL_1 + 2^{\frac{3}{2}} p \rho_1 + \frac{L_1^2}{2} p(p-1)) \left(1 - \frac{2}{p}\right) \right\} \int_0^t \mathbb{E} \sup_{r \in [0, s]} |X(r) - x(r)|^p ds \\ & \quad + \left\{ \frac{2^p}{p} (2pL_1 + 2^{\frac{3}{2}} p \rho_1 + \frac{L_1^2}{2} p(p-1)) + C_1 2^{p-1} L_1^p p^{p-1} T^{\frac{p-2}{2}} \right\} \mathbb{E} \sup_{s \in [0, t]} \int_0^t |\Psi(s) - \psi(s)|^p ds \\ & \quad + \left\{ \frac{2^p}{p} (2pL_1 + 2^{\frac{3}{2}} p \rho_1 + \frac{L_1^2}{2} p(p-1)) + C_1 2^{p-1} L_1^p p^{p-1} T^{\frac{p-2}{2}} \right\} \mathbb{E} \int_0^t |\psi(s) - \bar{\psi}(s)|^p ds \\ & \quad + \frac{p-1}{p} \mathbb{E} \sup_{s \in [0, t]} |X(s) - x(s)|^p. \end{aligned} \quad (4.4)$$

Similarly, we can show

$$\begin{aligned}
 \mathbb{E} \sup_{s \in [0,t]} |Y(s) - y(s)|^p &\leq (pL_2 + 2^{\frac{1}{2}}p\rho_2) \int_0^t \mathbb{E} \sup_{r \in [0,s]} |Y(r) - y(r)|^{p-1} |\Psi(r) - \bar{\psi}(r)| ds \\
 &\quad + p \mathbb{E} \sup_{s \in [0,t]} \int_0^s (|Y(s) - y(s)|^{p-1}, [G_{1y}(\Psi(s)) - G_{1y}(\bar{\psi}(s))]) dw(s) \\
 &\quad + \frac{p(p-1)}{2} L_2^2 \int_0^t \mathbb{E} \sup_{r \in [0,s]} |Y(r) - y(r)|^{p-2} |\Psi(r) - \bar{\psi}(r)|^2 ds \\
 &\quad + \frac{p}{2} A^2 \check{\beta}^2(t, a) \int_0^t \mathbb{E} \sup_{r \in [0,s]} |Y(r) - y(r)|^p ds \tag{4.5} \\
 &\leq (2pL_2 + 2^{\frac{3}{2}}p\rho_2 + \frac{p(p-1)}{2} L_2^2) \int_0^t \mathbb{E} \sup_{r \in [0,s]} |Y(r) - y(r)|^{p-2} |\Psi(r) - \bar{\psi}(r)|^2 ds \\
 &\quad + p \mathbb{E} \sup_{s \in [0,t]} \int_0^s (|Y(s) - y(s)|^{p-1}, [G_{1y}(\Psi(s)) - G_{1y}(\bar{\psi}(s))]) dw(s) \\
 &\quad + (\frac{p}{2} A^2 \check{\beta}^2(t, a) + 2pL_2 + 2^{\frac{3}{2}}p\rho_2) \int_0^t \mathbb{E} \sup_{r \in [0,s]} |Y(r) - y(r)|^p ds.
 \end{aligned}$$

Therefore, due to Young inequality,

$$\int_0^t \mathbb{E} \sup_{r \in [0,s]} |Y(r) - y(r)|^{p-2} |\Psi(r) - \bar{\psi}(r)|^2 ds \leq \int_0^t \mathbb{E} \sup_{r \in [0,s]} [(1 - \frac{2}{p})|Y(r) - y(r)|^p + \frac{2}{p}|\Psi(r) - \bar{\psi}(r)|^p] ds. \tag{4.6}$$

Moreover, applying the BDG inequality, Young inequality, and Hölder’s inequality, it yields

$$\begin{aligned}
 &\mathbb{E} \sup_{s \in [0,s]} \int_0^s (|Y(s) - y(s)|^{p-1}, [G_{1y}(\Psi(s)) - G_{1y}(\bar{\psi}(s))]) dw(s) \\
 &\leq \frac{p-1}{p^2} \mathbb{E} \sup_{s \in [0,t]} |Y(s) - y(s)|^p + C_2 \mathbb{E} \sup_{s \in [0,t]} [\int_0^s |G_{1y}(\Psi(s)) - G_{1y}(\bar{\psi}(s))|^2 ds]^{\frac{p}{2}} \tag{4.7} \\
 &\leq \frac{p-1}{p^2} \mathbb{E} \sup_{s \in [0,t]} |Y(s) - y(s)|^p + C_2 L_2^p T^{\frac{p-2}{2}} \mathbb{E} \sup_{s \in [0,t]} \int_0^s |\Psi(s) - \bar{\psi}(s)|^p ds.
 \end{aligned}$$

Now, substituting Eqs (4.6) and (4.7) into Eq (4.5), we obtain

$$\begin{aligned}
 &\frac{1}{p} \mathbb{E} \sup_{s \in [0,t]} |Y(s) - y(s)|^p \\
 &\leq [\frac{p}{2} A^2 \check{\beta}^2 + 2pL_2 + 2^{\frac{3}{2}}p\rho_2 + (2pL_2 + 2^{\frac{3}{2}}p\rho_2 + \frac{p(p-1)}{2} L_2^2)(1 - \frac{2}{p})] \int_0^t \mathbb{E} \sup_{r \in [0,s]} |Y(r) - y(r)|^p ds \\
 &\quad + [\frac{2^p}{p} (2pL_2 + 2^{\frac{3}{2}}p\rho_2 + \frac{p(p-1)}{2} L_2^2) + 2^{p-1} C_2 p T^{\frac{p-2}{2}} L_2^p] \mathbb{E} \sup_{s \in [0,t]} \int_0^s |\Psi(s) - \psi(s)|^p ds \tag{4.8} \\
 &\quad + [\frac{2^p}{p} (2pL_2 + 2^{\frac{3}{2}}p\rho_2 + \frac{p(p-1)}{2} L_2^2) + 2^{p-1} C_2 p T^{\frac{p-2}{2}} L_2^p] \mathbb{E} \int_0^t |\psi(s) - \bar{\psi}(s)|^p ds.
 \end{aligned}$$

By applying Eqs (4.4) and (4.8), Theorem 3.2, and the Gronwall inequality, we have

$$\mathbb{E} \sup_{s \in [0,T]} |\Psi(s) - \psi(s)|^p \leq C \Delta^{\frac{p}{2}}.$$

Therefore, the proof is complete.

Corollary 4.2. Assume the assumptions (A1)–(A5) and $p > 2$ hold, the numerical solution of Eq (2.3) will converge to the exact solution in the sense

$$\lim_{\Delta t \rightarrow 0} \mathbb{E}(\sup_{0 \leq t \leq T} |\Psi(t) - \psi(t)|^p) = 0.$$

Remark 4.1. Theorem 4.1 and Corollary 4.2 show that the exact solution $\Psi(t)$ and the numerical solution $\psi(t)$ are close to each other, which means that the numerical scheme constructed in this paper is effective.

5. Numerical experiments

In this part, in order to simulate the numerical approximation of the EM scheme, we consider the following stochastic age-dependent cooperative LV system.

$$\begin{cases} d_t X = -\frac{\partial X}{\partial a} dt + X[-\frac{1}{(1-a)^2} X + \cos^2 a Y - 2] dt - \frac{1}{2} dw(t), & (t, a) \in Q, \\ d_t Y = -\frac{\partial Y}{\partial a} dt + Y[\sin^2 a X - e^{\frac{1}{a}} Y + 3a - 2] dt - \frac{1}{2} dw(t), & (t, a) \in Q, \\ X(t, 0) = \int_0^1 \frac{1}{1-a} X(t, a) da, & t \in [0, 1], \\ Y(t, 0) = \int_0^1 \frac{1}{1-a} Y(t, a) da, & t \in [0, 1], \\ X(0, a) = e^{\frac{-2}{1-a}}, Y(0, a) = e^{\frac{-2}{1-a}}, & a \in [0, 1], \end{cases} \quad (5.1)$$

where $\alpha_{11}(a) = \frac{1}{(1-a)^2}$, $\alpha_{12}(a) = \cos^2 a$, $\alpha_{21}(a) = \sin^2 a$, $\alpha_{22}(a) = e^{\frac{1}{a}}$, $\gamma(t, a) = \beta(t, a) = \frac{1}{1-a}$, $Q = (0, 1) \times (0, 1)$. All computations are run with $T = 1, \Delta = 0.005$, and $\Delta a = 0.05$ for Eq (5.1). In Equation (5.1), we simulate the EM numerical solution of the two species respectively (see Figure 1). Figure 1 shows that both the two species are bounded, which is consistent with Theorem 3.1. From the range of pictures (a) and (b) in Figure 1, we conclude that the fluctuation of species $Y(t, a)$ is larger than that of $X(t, a)$, which indicates that the number of species $X(t, a)$ is relatively stable.

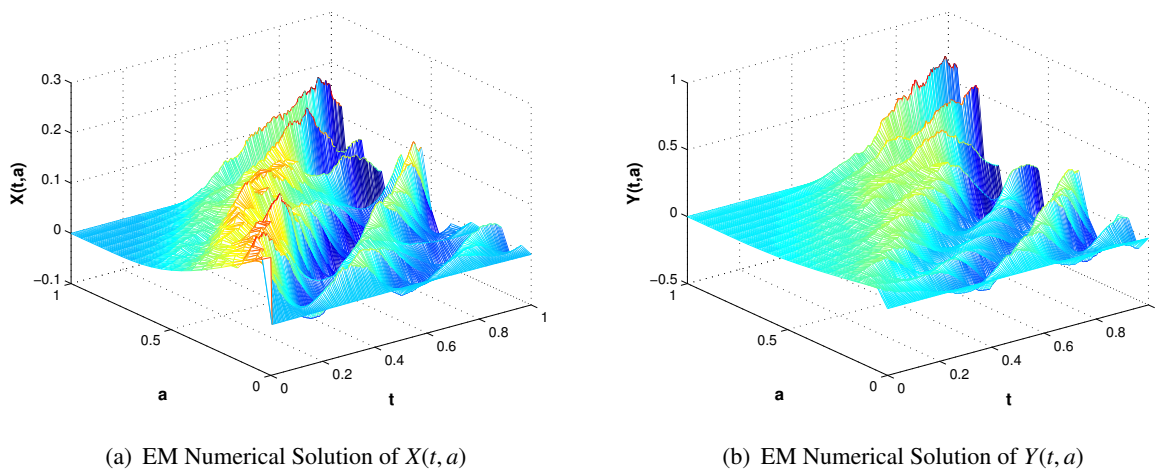


Figure 1. The EM numerical solutions of $X(t, a)$ and $Y(t, a)$ of Eq (5.1).

Now, we apply the exact solution of Eq (2.3) by the method [29] to demonstrate the effectiveness of the EM scheme. In Figure 2, we present the exact solutions of $X(t, a)$ and $Y(t, a)$ with perturbation respectively.

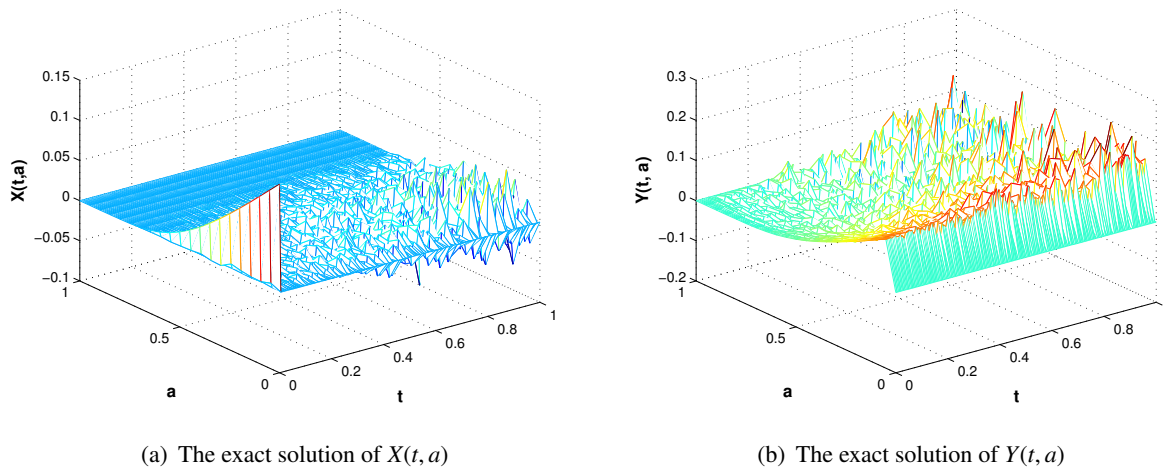


Figure 2. The exact solutions of $X(t, a)$ and $Y(t, a)$ with perturbation of Eq (5.1).

Next, in order to test and verify the convergence of the EM method in the stochastic age-dependent model, we conduct the error analysis of Eq (5.1). Figure 3 shows the errors between the exact solution and numerical solution of species $X(t, a)$. It is easy to observe that the maximum value of the squared error is below 0.2, which means that the numerical solution tends to the exact solution.

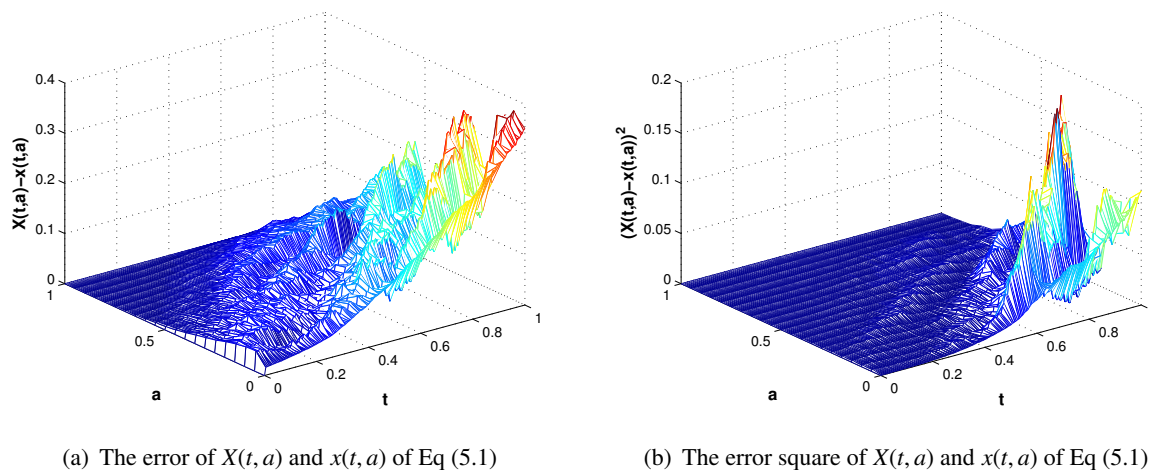
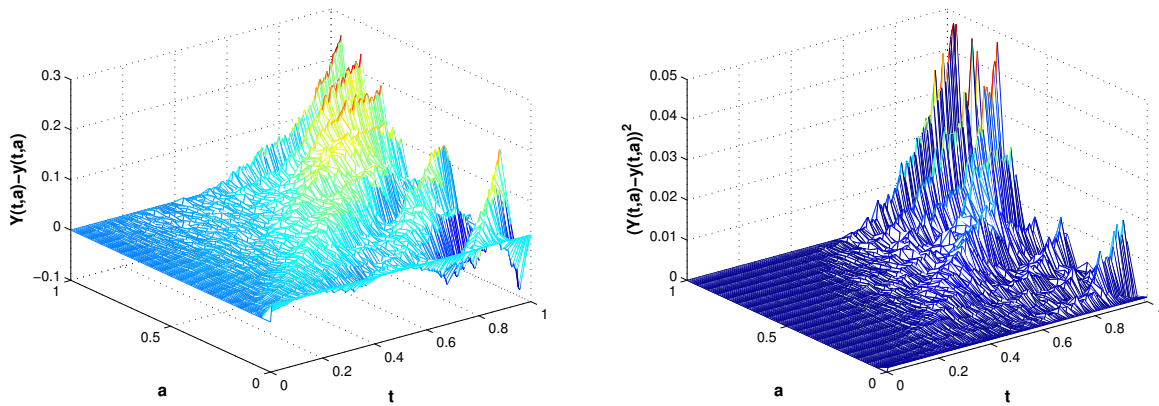


Figure 3. The error simulation of species $X(t, a)$ of Eq (5.1).

Similarly, we get the error estimate of species $Y(t, a)$. From Figure 4, it shows that the square error of $Y(t, a)$ is less than 0.05. This confirms our theoretical results in Theorem 4.1.



(a) The error of $Y(t, a)$ and $y(t, a)$ of Eq (5.1) (b) The error square of $Y(t, a)$ and $y(t, a)$ of Eq (5.1)

Figure 4. The error simulation of species $Y(t, a)$ of Eq (5.1).

To study the convergence of EM algorithm in Eq (5.1) more precisely, we calculate the ι -th order error between $x(t, a)$ and $X(t, a)$. From Table 2, we observe that under the same value of order ι , the error $(X(t, a) - x(t, a))^\iota$ becomes smaller as step size Δ gets smaller. In addition, we find that increasing the order ι can reduce the error of $X(t, a)$ and $x(t, a)$ under the same step size. Similarly, Table 3 illustrates the same phenomena.

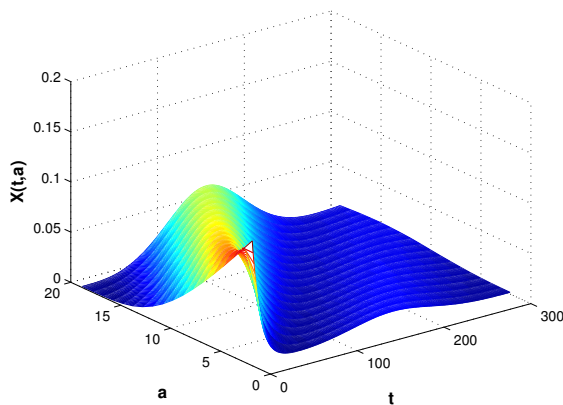
Table 2. Error simulation between $X(t, a)$ and $x(t, a)$.

$(X(t, a) - x(t, a))^\iota$	$\Delta = 0.01$		$\Delta = 0.005$		$\Delta = 0.002$		$\Delta = 0.001$	
	Error	Time/s	Error	Time/s	Error	Time/s	Error	Time/s
$\iota = 4$	1.2×10^{-2}	26.881276	1.5×10^{-2}	63.584526	5×10^{-3}	373.050756	2×10^{-3}	984.649714
$\iota = 6$	3×10^{-3}	22.164523	1.8×10^{-3}	76.028217	4×10^{-4}	1040.348042	2×10^{-4}	1323.191920
$\iota = 8$	1.6×10^{-4}	23.557376	1.2×10^{-4}	40.686763	1×10^{-4}	588.323881	2×10^{-5}	1570.733100
$\iota = 10$	1.2×10^{-4}	22.441634	0.5×10^{-4}	63.584526	1×10^{-5}	681.486228	1×10^{-6}	1608.228890

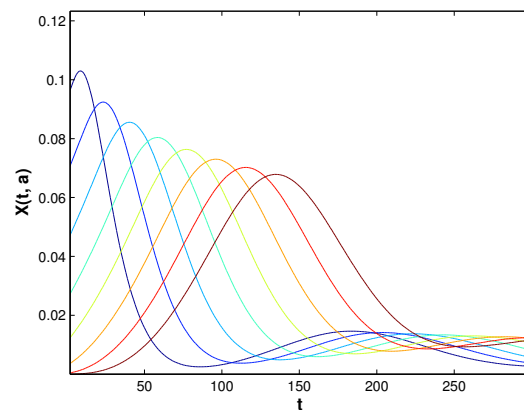
Table 3. Error simulation between $Y(t, a)$ and $y(t, a)$.

$(Y(t, a) - y(t, a))^\iota$	$\Delta = 0.01$		$\Delta = 0.005$		$\Delta = 0.002$		$\Delta = 0.001$	
	Error	Time/s	Error	Time/s	Error	Time/s	Error	Time/s
$\iota = 4$	1×10^{-2}	41.756866	5×10^{-3}	81.872743	4×10^{-3}	646.558116	2.5×10^{-3}	2773.836731
$\iota = 6$	6×10^{-4}	42.402488	1.5×10^{-4}	76.028217	6×10^{-5}	587.713828	2×10^{-5}	2851.378369
$\iota = 8$	2×10^{-5}	42.937773	8×10^{-6}	80.371846	4×10^{-6}	680.575583	2×10^{-6}	1570.733100
$\iota = 10$	1.2×10^{-6}	42.679908	1×10^{-6}	82.256243	3×10^{-7}	992.707197	1×10^{-7}	1608.228890

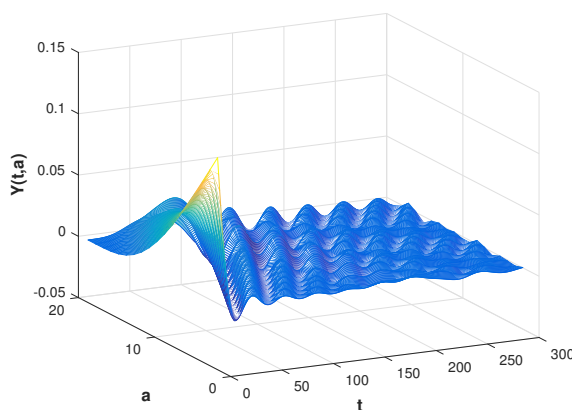
Inspired by [33], in order to examine the stability of the numerical scheme for Eq (5.1), we present 3-dimensional and 2-dimensional trajectory numerical solution respectively. The paths of $x(t, a)$ and $y(t, a)$ for Eq (5.1) are given in Figure 5(a) and Figure 5(c). The two figures show that the limit of $x(t, a)$ and $y(t, a)$ are 0. In order to study the relationship of t and $x(t, a)$, we fix the age $a \in [0, 1]$, then obtain the 2-dimensional trajectory numerical solution of $x(t, a)$. Similarly, we could obtain the trajectory of $y(t, a)$. Figure 5 shows that the numerical scheme is asymptotically stable, (a.s.).



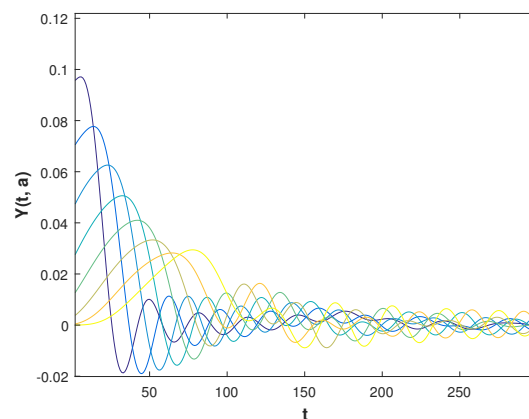
(a) The 3-D trajectory numerical solution of $x(t, a)$ for Eq (5.1).



(b) The 2-D trajectory numerical solution of $x(t, a)$ for Eq (5.1).



(c) The 3-D trajectory numerical solution of $y(t, a)$ for Eq (5.1).



(d) The 2-D trajectory numerical solution of $y(t, a)$ for Eq (5.1).

Figure 5. The stability of numerical solution for Eq (5.1).

6. Conclusions and future research

In this paper, the stochastic age-dependent cooperative LV system has been established. Since it is extremely difficult to obtain the explicit solution for this system, the main contribution of this article is to investigate a numerical scheme to approximate the exact solution of the stochastic age-dependent cooperative LV system. By using the theory of M-matrices, we obtain the sufficient conditions which ensure the stochastic age-dependent cooperative LV Eq (2.3) has a global unique positive solution (see Theorem (2.1)). Since the coefficients of Eq (2.3) do not satisfy the Lipschitz and linear growth conditions, we can not apply the EM scheme directly. However, we apply the EM scheme to Eq (2.3) in a suitable region successfully through studying the equilibrium points. After that, we show that the approximation algorithm constructed in this paper is p -th moments boundedness (see Theorem 3.1) and it converges to the exact solution in the strong sense (see Theorem 4.1). Lastly, we present some numerical experiments which support our theoretical results. In the future, we will construct new

algorithms which have a higher convergence order.

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Conflict of interest

The authors declare that they have no competing interests.

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