Research article

Qualitative analysis of a nonautonomous stochastic $S IS$ epidemic model with Lévy jumps

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Abstract: In this paper, we study a nonautonomous stochastic $S IS$ epidemic model with Lévy jumps. We first establish that this model has a unique global positive solution with the positive initial condition. Then, we investigate the condition for extinction of the disease. Moreover, by constructing suitable stochastic Lyapunov function, sufficient conditions for persistence and existence of Nontrivial T-periodic solution of system are obtained. Finally, numerical simulations are also presented to illustrate the main results.

1. Introduction

Infectious diseases are the public enemy of human population and seriously threatening the health and life. In many infectious disease research methods, epidemic models are widely used for analyzing the spread and control of infectious diseases. Here, the researchers construct an idea of compartmental model to study the dynamics of infectious diseases, initially proposed and investigated by Kermack and McKendrick. That is, $S IS$ epidemic model[1]. Now, Most of models are descended from the classical $S IS$ model. Some authors [2 − 4] analyzed the dynamic behavior of the $S IS$ epidemic model based on variable population, delay, vaccination, pulse, etc. Papers [5 − 6] have investigated the influence of age structure and stage structure on the dynamic behavior of the $S IS$ epidemic model, and studied the extinction and persistence of the disease. In addition, many results on the $S IS$ epidemic models with network have been reported [7 − 9].

However, In the natural world, epidemic systems are inevitably infected by some stochastic environmental noise. Hence, many authors introduce stochastic interference into deterministic models

to reveal the effect of environmental variability on the dynamic of infectious disease. In general the environmental fluctuations can be modeled by Gaussian white noises. The authors [10 − 12] have qualitatively analyzed stochastic SIS epidemic model with vaccination and vertical transmission. In [13], Zhang et al. investigated stochastic SIS epidemic model with vaccination under regime switching, and sufficient conditions for the existence of a unique ergodic stationary distribution has been established. The authors [14, 15] studied stochastic models of different diseases, and completed dynamic analysis and optimal control research. But Gaussian white noise can only describe a class of continuous stationary random disturbances. Otherwise, the population may suffer from sudden environmental shocks, namely, some jump type stochastic perturbation, such as, earthquakes, hurricanes, floods, tsunami, epidemic diseases (SARS, avian influenza), commercial harvesting, and so on. These phenomena are discontinuous and jumping at random time $t_i, i \in N$, and the waiting time of jumps is independent. Therefore, these phenomena cannot be described by Gaussian white noise alone. For this kind of sudden, The discontinuous random disturbance of Lévy jumps can more truly reflect the changing law of such sudden things. So, it's necessary to introduce Lévy jumps process into the dynamics of infectious disease. Bao et al.[16] considered stochastic Lotka-Volterra population systems with jumps for the first time. Zhang et al.[17] studied a stochastic $SIQS$ epidemic model with Lévy jumps. Recently, T. Caraballo et al.[18] considered a stochastic SIS model with Lévy noise perturbation. M. Naim et al.[19] discussed a stochastic SIS epidemic model with vertical transmission, specific functional response and Lévy jumps. About the knowledge of jumps, the readers can further refer to [20, 21].

For example, Zhou et al [22] proposed a stochastic SIS model with Lévy jumps. as follows:

$$
\begin{cases}
dS(t) = (\Lambda - \beta S(t) I(t) - \mu S(t) + \lambda I(t)) dt + \sigma_1 S(t) dB_1(t) + \int_Y \gamma_1(\delta) S(t-) \widetilde{N}(dt, d\delta), \\
dI(t) = (\beta S(t) I(t) - (\lambda + \mu + \alpha) I(t)) dt + \sigma_2 I(t) dB_2(t) + \int_Y \gamma_2(\delta) I(t-) \widetilde{N}(dt, d\delta),
\end{cases}
\tag{1.1}
$$

where $dB_i(t), i = 1, 2$ are independent Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$. And it is increasing and right continuous with $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets, $\sigma_i^2, i = 1, 2$ are the intensities of $B_i(t)$. Λ stands for recruitment rate of the population; μ is the natural death rate; β reprents the disease transmission rate; α is the mortality rate from disease; λ is the recovery rate. $S(t-)$ and $I(t-)$ are the left limit of $S(t)$ and $I(t)$. $\gamma_i(\delta) > -1, i = 1, 2$. Y is a measurable subset of $(0, +\infty)$, ν is a Lévy measure on Y with $\nu(\delta) < +\infty$, such that $\widetilde{N}(dt, d\delta) = N(dt, d\delta) - \nu(d\delta) dt$. Therein, N is an independent Poisson counting measure with characteristic measure ν.

In (1.1), they obtain a unique positive and global solution of the system. Moreover, the sufficient conditions for extinction and persistence of the disease are established.

Furthermore, many disease are not only disturbed by various environmental noises, but also affected by time and seasonal variation. such as, rubella, measles, chickenpox etc. So, it is more realistic to assume that all of the coefficients in the models are positive T-periodic continuous functions [23 − 25]. In [26], Qi et al. investigated the dynamics of a nonautonomous stochastic SIS epidemic model with nonlinear incidence rate and double epidemic hypothesis. In [27], Zhang et al. formulated a new stochastic nonautonomous $SIRI$ epidemic model, and the thresholds of the stochastic $SIRI$ epidemic model have been obtained. To the best of our knowledge, there are few papers to deal with nonautonomous epidemic model with Lévy jumps. Motivated by the above analysis, we assume the following nonautonomous stochastic SIS model:

$$\begin{cases} dS(t) = (\Lambda(t) - \beta(t)S(t)I(t) - \mu(t)S(t) + \lambda(t)I(t))\,dt + \sigma_1(t)S(t)\,dB_1(t) + \int_Y \gamma_1(\delta)S(t-)\widetilde{N}(dt,d\delta), \\ dI(t) = (\beta(t)S(t)I(t) - (\lambda(t) + \mu(t) + \alpha(t))I(t))\,dt + \sigma_2(t)I(t)\,dB_2(t) + \int_Y \gamma_2(\delta)I(t-)\widetilde{N}(dt,d\delta). \end{cases}$$
$$(1.2)$$

The rest of this paper is organized as follows: In Section 2, by constructing Lyapunov function, we show the existence and uniqueness of the positive solution of the system (1.2); The conditions for the extinction of the system (1.2) is given In Section 3; In Section 4, we discuss the conditions for the existence of at least one nontrivial positive T-periodic solution of the system (1.2); In Section 5, we obtain the sufficient conditions of the system (1.2) for the persistence of the epidemic disease; The conclusion and numerical experiments of our theoretical results are given in Section 6.

2. Existence and uniqueness of the positive solution

In this section, based on the theorem of Mao [28], the coefficients of system (1.2) are locally Lipschitz continuous [29]. In what follows, we shall prove that the solution of system (1.2) is positive and global.(i.e. the solution of system (1.2) will not explode in a finite time with probability one). For a bounded integrable function $f(t)$ on $[0, \infty)$, we define $f^l = \inf\limits_{t \in [0,\infty)} f(t)$, $f^u = \sup\limits_{t \in [0,\infty)} f(t)$ and $\langle f \rangle_t = \frac{1}{t} \int_0^t f(t)\,dt$.

Theorem 2.1. There is a unique and positive solution $(S(t), I(t)) \in R_+^2$ of system (1.2) with initial condition $(S(0), I(0)) \in R_+^2$ with probability one, assumptions A_1 and A_2 hold.

We suppose the jump diffusion coefficient that for each $h > 0$ there exists $L_h > 0$. i.e.

(A_1) $\int_Y |F_i(x, \delta) - F_i(y, \delta)|^2 \nu(d\delta) \leqslant L_h |x - y|^2, i = 1, 2$, where $F_i(x, \delta) = \gamma_i(\delta)x(t-)$ with $|x| \vee |y| \leqslant h$.

(A_2) $\int_Y [\gamma_i(\delta) - \ln(1 + \gamma_i(\delta))]\nu(d\delta) \leqslant \kappa_i, i = 1, 2$, where κ_i is positive constant, which biologically interpret intensities of Lévy jumps are not too large.

Proof. According to the theorem of Mao [28] and (A_1), the system (1.2) exist a unique local solution $(S(0), I(0))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. To prove that this solution is global, i.e. $\tau_e = +\infty$. Let the solution defined on the $\left[\frac{1}{h_0}, h_0\right]$, where h_0 is a sufficiently large positive number.

Define the stop-time

$$\tau_h = \inf\left\{t \in [0, \tau_e) : \min\{S(t), I(t)\} \leqslant \frac{1}{h} \quad or \quad \max\{S(t), I(t)\} \geqslant h\right\}.$$

where all $h \geqslant h_0$.

We denote $\emptyset$ is the empty set. Let $\emptyset = \infty$. It is clear that τ_h is increasing as $h \to \infty$. Set $\tau_\infty = \lim\limits_{h \to \infty} \tau_h$. Then $\tau_\infty \leqslant \tau_h$. If $\tau_\infty = \infty$ is true, then $\tau_h = \infty$. Otherwise, there exist constants $T > 0$ and $\varepsilon \in (0, 1)$, such that $P(\tau_\infty \leqslant T) \geqslant \varepsilon$. Hence, there exists an integer $h_1 \geqslant h_0$ such that $P(\tau_h \leqslant T) \geqslant \varepsilon, h \geqslant h_1$.

Define the function $V(S(t), I(t))$ for $(S(t), I(t)) \in R_+^2$ as follows

$$V(S(t), I(t)) = S(t) - a(t) - \ln(S(t) + a(t)) + I(t) - 1 - \ln I(t),$$

where $a(t)$ is a positive function on $[0, 1]$.

Using the Itô's formula, we have

$$dV(S(t), I(t)) = LVdt + \sigma_1(t) S(t)\left(1 - \frac{1}{S(t) + a(t)}\right) dB_1(t) + \sigma_2(t)(I(t) - 1) dB_2(t)$$

$$+ \int_Y \left[\gamma_1(\delta) S(t-) + \ln \frac{S(t) + a(t)}{S(t) + a(t) + \gamma_1(\delta) S(t-)}\right] \widetilde{N}(dt, d\delta)$$

$$+ \int_Y [\gamma_2(\delta) I(t-) - \ln(1 + \gamma_2(\delta))] \widetilde{N}(dt, d\delta), \tag{2.1}$$

where $LV(S, I) : R_+^2 \to R_+$ by

$$LV = -a'(t)\left(1 + \frac{1}{S(t) + a(t)}\right) + \left(1 - \frac{1}{S(t) + a(t)}\right)(\Lambda(t) - \beta(t) S(t) I(t) - \mu(t) S(t) + \lambda(t) I(t))$$

$$+ \frac{\sigma_1^2(t) S^2(t)}{2(S(t) + a(t))^2} + \left(1 - \frac{1}{I(t)}\right)(\beta(t) S(t) I(t) - (\lambda(t) + \mu(t) + \alpha(t)) I(t)) + \frac{\sigma_2^2(t)}{2}$$

$$+ \int_Y \left[\gamma_1(\delta) S(t-) + \ln \frac{S(t) + a(t)}{S(t) + a(t) + \gamma_1(\delta) S(t-)} - \left(1 - \frac{1}{S(t) + a(t)}\right)\gamma_1(\delta) S(t-)\right] \nu(d\delta)$$

$$+ \int_Y \left[\gamma_2(\delta) I(t-) - \ln(1 + \gamma_2(\delta)) - \gamma_2(\delta) I(t-)\left(1 - \frac{1}{I(t)}\right)\right] \nu(d\delta)$$

$$\leqslant \Lambda(t) + \lambda(t) + \mu(t) + \alpha(t) + \beta(t) \frac{S(t)}{S(t) + a(t)} I(t) + \mu(t) \frac{S(t)}{S(t) + a(t)} + \frac{\sigma_1^2(t) S^2(t)}{2(S(t) + a(t))^2}$$

$$+ \frac{\sigma_2^2(t)}{2} + \int_Y \left[\frac{\gamma_1(\delta) S(t-)}{S(t) + a(t)} + \ln \frac{S(t) + a(t)}{S(t) + a(t) + \gamma_1(\delta) S(t-)}\right] \nu(d\delta)$$

$$+ \int_Y [\gamma_2(\delta) - \ln(1 + \gamma_2(\delta))] \nu(d\delta) - (\mu(t) + \alpha(t)) I(t).$$

Denote $a(t) = \frac{\beta(t)}{\mu(t) + \alpha(t)}$ such that $\frac{\beta(t)}{a(t)} - \mu(t) - \alpha(t) = 0$. By using of inequality $x - \ln(1 + x) \geqslant 0$ for $x > -1$ and (A_2), we conclude

$$LV \leqslant \Lambda(t) + \lambda(t) + 2\mu(t) + \alpha(t) + \frac{\sigma_1^2(t)}{2} + \frac{\sigma_2^2(t)}{2}$$

$$+ \int_Y \left[\frac{\gamma_1(\delta) S(t-)}{S(t) + a(t)} - \ln\left(1 + \frac{\gamma_1(\delta) S(t-)}{S(t) + a(t)}\right)\right] \nu(d\delta)$$

$$+ \int_Y [\gamma_2(\delta) - \ln(1 + \gamma_2(\delta))] \nu(d\delta), \tag{2.2}$$

$$LV \leqslant \Lambda(t) + \lambda(t) + 2\mu(t) + \alpha(t) + \frac{\sigma_1^2(t)}{2} + \frac{\sigma_2^2(t)}{2} + 2d$$

$$\leqslant \Lambda^u + \lambda^u + 2\mu^u + \alpha^u + \frac{\sigma_1^{2u}}{2} + \frac{\sigma_2^{2u}}{2} + 2d =: D, \tag{2.3}$$

where $d = \max\left\{\int_Y \left[\gamma_1(\delta) - \ln\left(1 + \frac{\gamma_1(\delta)}{1+a^u}\right)\right] \nu(d\delta), \int_Y [\gamma_2(\delta) - \ln(1 + \gamma_2(\delta))] \nu(d\delta)\right\}$ and D is a positive constant which is independent of S, I and t. we can get that

$$dV \leqslant Ddt + \sigma_1(t) S(t)\left(1 - \frac{1}{S(t) + a(t)}\right) dB_1(t) + \sigma_2(t)(I(t) - 1) dB_2(t)$$

$$+ \int_Y \left[\gamma_1 (\delta) S(t-) + \ln \frac{S(t) + a(t)}{S(t) + a(t) + \gamma_1 (\delta) S(t-)} \right] \widetilde{N}(dt, d\delta)$$

$$+ \int_Y \left[\gamma_2 (\delta) I(t-) - \ln (1 + \gamma_2 (\delta)) \right] \widetilde{N}(dt, d\delta). \tag{2.4}$$

Integrating (2.4) from 0 to $\tau_h \wedge T$.

$$\int_0^{\tau_h \wedge T} dV \leqslant \int_0^{\tau_h \wedge T} Ddt + \int_0^{\tau_h \wedge T} \sigma_1 (t) S(t) \left(1 - \frac{1}{S(t) + a(t)} \right) dB_1 (t) + \int_0^{\tau_h \wedge T} \sigma_2 (t) (I(t) - 1) dB_2 (t)$$

$$+ \int_0^{\tau_h \wedge T} \int_Y \left[\gamma_1 (\delta) S(r-) - \ln \left(1 + \frac{\gamma_1 (\delta) S(r-)}{S(r) + a(r)} \right) \right] \widetilde{N}(dr, d\delta)$$

$$+ \int_0^{\tau_h \wedge T} \int_Y \left[\gamma_2 (\delta) I(r-) - \ln (1 + \gamma_2 (\delta)) \right] \widetilde{N}(dr, d\delta). \tag{2.5}$$

Taking the expectations of the above inequality leads to

$$E(V(S(\tau_h \wedge T), I(\tau_h \wedge T))) \leqslant V(S(0), I(0)) + DE(\tau_h \wedge T),$$

So

$$E(V(S(\tau_h \wedge T), I(\tau_h \wedge T))) \leqslant V(S(0), I(0)) + DT.$$

Let $\Omega_h = \{\tau_h \leqslant T\}$ for $h \geqslant h_1$. Then $P(\Omega_h) \geqslant \varepsilon$. In addition, for any $\omega \in \Omega_h$, there is at least one of $S(\tau_h, \omega)$ and $I(\tau_h, \omega)$ that equals either h or $\frac{1}{h}$. We have

$$
\begin{aligned}
V(S(0), I(0)) + DT &\geqslant E\{I_{\Omega_h}(\omega) V(S(\tau_h \wedge T), I(\tau_h \wedge T))\} \\
&= E\{I_{\Omega_h}(\omega) V(S(\tau_h, \omega), I(\tau_h, \omega))\} \\
&\geqslant \varepsilon \left[(2h - \ln h(a + h) - a - 1) \wedge \left(\frac{2}{h} + \ln \frac{h^2}{1 + ah} - a - 1 \right) \right],
\end{aligned}
$$

where $I_{\Omega_h}(\omega)$ is the indicator function of $\Omega_h(\omega)$.
Let $h \to \infty$, we have

$$\infty = V(S(0), I(0)) + DT < \infty.$$

Therefore, we obtain $\tau_\infty = \infty$. The proof is completed. $\qquad\square$

3. Extinction

In this section, we investigate the conditions for the extinction of disease $I(t)$.

Lemma 3.1. Let $(S(t), I(t))$ be a solution of system (1.2) with initial value $(S(0), I(0)) \in R_+^2$. Then $\limsup_{t \to \infty} (S(t) + I(t)) < \infty$, a.s.

$$\lim_{t \to \infty} \frac{S(t)}{t} = 0, \qquad \lim_{t \to \infty} \frac{I(t)}{t} = 0,$$

$$\limsup_{t \to \infty} \frac{\ln S(t)}{t} = 0, \qquad \limsup_{t \to \infty} \ln \frac{I(t)}{t} = 0.$$

and

$$\lim_{t\to\infty} \frac{\int_0^t S(r)\,dB_1(r)}{t} = 0, \qquad \lim_{t\to\infty} \frac{\int_0^t I(r)\,dB_2(r)}{t} = 0,$$

$$\lim_{t\to\infty} \frac{\int_0^t \int_Y \gamma_1(\delta) S(r-)\widetilde{N}(dr,d\delta)}{t} = 0, \qquad \lim_{t\to\infty} \frac{\int_0^t \int_Y \gamma_2(\delta) I(r-)\widetilde{N}(dr,d\delta)}{t} = 0.$$

Proof. From system (1.2), we obtain that

$$
\begin{aligned}
d(S(t)+I(t)) &= (\Lambda(t) - \mu(t)(S(t)+I(t)) - \alpha(t)I(t))\,dt + \sigma_1(t)S(t)\,dB_1(t) + \sigma_2(t)I(t)\,dB_2(t) \\
&\quad + \int_Y \gamma_1(\delta)S(t-)\widetilde{N}(dt,d\delta) + \int_Y \gamma_2(\delta)I(t-)\widetilde{N}(dt,d\delta).
\end{aligned}
$$

Solving above equation, we have

$$
\begin{aligned}
S(t)+I(t) &= e^{-\int_0^t \mu(m)dm} \int_0^t \left(\Lambda(r) - \alpha(r)I(r) + \sigma_1(r)S(r)\frac{dB_1(r)}{dr} + \sigma_2(r)I(r)\frac{dB_2(r)}{dr} \right) e^{\int_0^r \mu(m)dm}\,dr \\
&\quad + e^{-\int_0^t \mu(m)dm} \int_0^t \left(\frac{\int_Y \gamma_1(\delta)S(r-)\widetilde{N}(dr,d\delta)}{dr} + \frac{\int_Y \gamma_2(\delta)I(r-)\widetilde{N}(dr,d\delta)}{dr} \right) e^{\int_0^r \mu(m)dm}\,dr \\
&\quad + e^{-\int_0^t \mu(m)dm}(S(0)+I(0)),
\end{aligned}
$$

$$
\begin{aligned}
S(t)+I(t) &\leqslant e^{-\mu^u t}\cdot\frac{\Lambda^u}{\mu^u}\left(e^{\mu^u t}-1\right) - \alpha^l \int_0^t e^{-\mu^u(t-r)}I(r)\,dr + e^{-\mu^u t}(S(0)+I(0)) \\
&\quad + \sigma_1^u \int_0^t e^{-\mu^u(t-r)}S(r)\,dB_1(r) + \sigma_2^u \int_0^t e^{-\mu^u(t-r)}I(r)\,dB_2(r) \\
&\quad + \int_0^t \int_Y e^{-\mu^u(t-r)}\gamma_1(\delta)S(r-)\widetilde{N}(dr,d\delta) + \int_0^t \int_Y e^{-\mu^u(t-r)}\gamma_2(\delta)I(r-)\widetilde{N}(dr,d\delta) \\
&= \frac{\Lambda^u}{\mu^u} + \left(S(0)+I(0)-\frac{\Lambda^u}{\mu^u}\right)e^{-\mu^u t} - \alpha^l \int_0^t e^{-\mu^u(t-r)}I(r)\,dr + F(t) + G(t) \\
&\leqslant S(0)+I(0) + \frac{\Lambda^u}{\mu^u}\left(1-e^{-\mu^u t}\right) - (S(0)+I(0))\left(1-e^{-\mu^u t}\right) + F(t) + G(t),
\end{aligned}
$$

in which

$$F(t) = \sigma_1^u \int_0^t e^{-\mu^u(t-r)}S(r)\,dB_1(r) + \sigma_2^u \int_0^t e^{-\mu^u(t-r)}I(r)\,dB_2(r),$$

$$G(t) = \int_0^t \int_Y e^{-\mu^u(t-r)}\gamma_1(\delta)S(r-)\widetilde{N}(dr,d\delta) + \int_0^t \int_Y e^{-\mu^u(t-r)}\gamma_2(\delta)I(r-)\widetilde{N}(dr,d\delta).$$

Obviously, $F(t)$ and $G(t)$ are continuous local martingale with $F(0) = 0$ and $G(0) = 0$.

Define

$$H(t) = H(0) + W(t) - Y(t) + F(t) + G(t),$$

where $H(0) = S(0)+I(0)$, $W(t) = \frac{\Lambda^u}{\mu^u}\left(1-e^{-\mu^u t}\right)$ and $Y(t) = (S(0)+I(0))\left(1-e^{-\mu^u t}\right)$.
Note that $S(t)+I(t) \leqslant H(t)$ for all $t \geqslant 0$. $W(t)$ and $Y(t)$ are continuous adapted increasing process on

$t \geqslant 0$ with $W(0) = Y(0) = 0$. By Theorem 3.9 in [28], we have $\lim_{t\to\infty} H(t) < \infty$ a.s., then for almost all $t \geqslant 0$

$$\limsup_{t\to\infty} (S(t) + I(t)) < \infty. \tag{3.1}$$

Denote

$$k_1(t) = \int_0^t S(r)\,dB_1(r), \quad k_2(t) = \int_0^t I(r)\,dB_2(r),$$

$$k_3(t) = \int_0^t \int_Y \gamma_1(\delta) S(r-) \widetilde{N}(dr, d\delta), \quad k_4(t) = \int_0^t \int_Y \gamma_2(\delta) I(r-) \widetilde{N}(dr, d\delta).$$

Then, their quadratic variations

$$\langle k_1, k_1 \rangle = \int_0^t S^2(r)\,dr \leqslant \left(\sup_{t\geqslant 0} S^2(t)\right) t,$$

$$\langle k_2, k_2 \rangle = \int_0^t I^2(r)\,dr \leqslant \left(\sup_{t\geqslant 0} I^2(t)\right) t,$$

$$\langle k_3, k_3 \rangle \leqslant \int_Y \gamma_1^2(\delta) \nu(d\delta) \cdot \left(\sup_{t\geqslant 0} S(t)\right) t < \rho \left(\sup_{t\geqslant 0} S(t)\right) t,$$

$$\langle k_4, k_4 \rangle \leqslant \int_Y \gamma_2^2(\delta) \nu(d\delta) \cdot \left(\sup_{t\geqslant 0} I(t)\right) t < \rho \left(\sup_{t\geqslant 0} I(t)\right) t.$$

By the Theorem 3.4 in [28] and (3.1), one has

$$\lim_{t\to\infty} \frac{\int_0^t S(r)\,dB_1(r)}{t} = 0, \qquad \lim_{t\to\infty} \frac{\int_0^t I(r)\,dB_2(r)}{t} = 0,$$

$$\lim_{t\to\infty} \frac{\int_0^t \int_Y \gamma_1(\delta) S(r-) \widetilde{N}(dr, d\delta)}{t} = 0, \qquad \lim_{t\to\infty} \frac{\int_0^t \int_Y \gamma_2(\delta) I(r-) \widetilde{N}(dr, d\delta)}{t} = 0.$$

which together with the positivity of the solution imply

$$\lim_{t\to\infty} \frac{S(t)}{t} = 0, \qquad \lim_{t\to\infty} \frac{I(t)}{t} = 0,$$

$$\limsup_{t\to\infty} \frac{\ln S(t)}{t} = 0, \qquad \limsup_{t\to\infty} \ln \frac{I(t)}{t} = 0.$$

The proof is therefore completed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Theorem 3.1. Let $(S(t), I(t))$ be a solution of system (1.2) with initial value $(S(0), I(0)) \in R_+^2$. If
(i)

$$\eta > \frac{\beta^u \Lambda^u}{\mu^l},$$

or
(ii)

$$\eta \leqslant \frac{\beta^u \Lambda^u}{\mu^l}, \quad R_0 < 1.$$

where

$$R_0 = \frac{\beta^u \Lambda^u}{\mu^l (\lambda^l + \mu^l + \alpha^l)} - \frac{\eta}{\lambda^l + \mu^l + \alpha^l},$$

$$\eta = \frac{\sigma_2^{2l}}{2} + \int_Y [\gamma_2(\delta) - \ln(1 + \gamma_2(\delta))] \nu(d\delta).$$

Then

$$\limsup_{t \to \infty} \ln \frac{I(t)}{t} \leqslant (\lambda^l + \mu^l + \alpha^l)(R_0 - 1) < 0,$$

a.s.

$$\lim_{t \to \infty} I(t) = 0.$$

Proof. By the system (1.2), one has

$$dS(t) + dI(t) = (\Lambda(t) - \mu(t)S(t) - \mu(t)I(t) - \alpha(t)I(t))dt + \sigma_1(t)S(t)dB_1(t) + \sigma_2(t)I(t)dB_2(t)$$
$$+ \int_Y \gamma_1(\delta)S(t-)\widetilde{N}(dt, d\delta) + \int_Y \gamma_2(\delta)I(t-)\widetilde{N}(dt, d\delta). \tag{3.2}$$

Intergrating (3.2), we obtain

$$\int_0^t dS(r)dr + \int_0^t dI(r)dr = \int_0^t (\Lambda(r) - \mu(r)S(r) - \mu(r)I(r) - \alpha(r)I(r))dr$$
$$+ \int_0^t \sigma_1(r)S(r)dB_1(r) + \int_0^t \sigma_2(r)I(r)dB_2(r)$$
$$+ \int_0^t \int_Y \gamma_1(\delta)S(r-)\widetilde{N}(dr, d\delta) + \int_0^t \int_Y \gamma_2(\delta)I(r-)\widetilde{N}(dr, d\delta). \tag{3.3}$$

Note that

$$\frac{S(t) - S(0)}{t} + \frac{I(t) - I(0)}{t} \leqslant \Lambda^u - \mu^l \langle S(t) \rangle - \mu^l \langle I(t) \rangle - \alpha^l \langle I(t) \rangle + \frac{\sigma_1^u}{t} \int_0^t S(r)dB_1(r)$$
$$+ \frac{\sigma_2^u}{t} \int_0^t I(r)dB_2(r) + \frac{1}{t} \int_0^t \int_Y \gamma_1(\delta)S(r-)\widetilde{N}(dr, d\delta) + \frac{1}{t} \int_0^t \int_Y \gamma_2(\delta)I(r-)\widetilde{N}(dr, d\delta),$$

where

$$\langle S(t) \rangle = \frac{1}{t} \int_0^t S(r)dr, \qquad \langle I(t) \rangle = \frac{1}{t} \int_0^t I(r)dr.$$

Clearly, we have

$$\langle S(t) \rangle \leqslant \frac{\Lambda^u}{\mu^l} - \frac{\mu^l + \alpha^l}{\mu^l} \langle I(t) \rangle + \Phi(t), \tag{3.4}$$

where

$$\Phi(t) = -\frac{1}{\mu^l} \left[\frac{S(t) - S(0)}{t} + \frac{I(t) - I(0)}{t} - \frac{\sigma_1^u}{t} \int_0^t S(r)dB_1(r) - \frac{\sigma_2^u}{t} \int_0^t I(r)dB_2(r) \right]$$

$$- \frac{1}{\mu^l} \left[\frac{1}{t} \int_0^t \int_Y \gamma_1(\delta) S(r-) \widetilde{N}(dr, d\delta) + \frac{1}{t} \int_0^t \int_Y \gamma_2(\delta) I(r-) \widetilde{N}(dr, d\delta) \right].$$

From Lemma 3.1, it follows that

$$\lim_{t \to \infty} \Phi(t) = 0. \tag{3.5}$$

Applying Itô's formula to system (1.2) leads to

$$d \ln I(t) = \left[\beta(t) S(t) - \lambda(t) - \mu(t) - \alpha(t) - \frac{\sigma_2^2(t)}{2} + \int_Y (\ln(1 + \gamma_2(\delta)) - \gamma_2(\delta)) \nu(d\delta) \right] dt$$

$$+ \sigma_2(t) dB_2(t) + \int_Y \ln(1 + \gamma_2(\delta)) \widetilde{N}(dr, d\delta). \tag{3.6}$$

Intergrating (3.6), we obtain

$$\ln I(t) - \ln I(0) \leqslant \beta^u \int_0^t S(r) dr - \left(\lambda^l + \mu^l + \alpha^l \right) t - \frac{\sigma_2^{2l}}{2} t + t \int_Y (\ln(1 + \gamma_2(\delta)) - \gamma_2(\delta)) \nu(d\delta)$$

$$+ \sigma_2^u B_2(t) + \int_0^t \int_Y \ln(1 + \gamma_2(\delta)) \widetilde{N}(dr, d\delta). \tag{3.7}$$

Substituting (3.4) into (3.7), we get

$$\begin{aligned} \frac{\ln I(t)}{t} &\leqslant \beta^u \langle S(t) \rangle - \left(\lambda^l + \mu^l + \alpha^l + \frac{\sigma_2^{2l}}{2} \right) + \int_Y (\ln(1 + \gamma_2(\delta)) - \gamma_2(\delta)) \nu(d\delta) + \sigma_2^u \frac{B_2(t)}{t} \\ &\quad + \frac{\int_0^t \int_Y \ln(1 + \gamma_2(\delta)) \widetilde{N}(dr, d\delta)}{t} + \frac{\ln I(0)}{t} \\ &\leqslant \frac{\beta^u \Lambda^u}{\mu^l} + \beta^u \Phi(t) - \left(\lambda^l + \mu^l + \alpha^l + \frac{\sigma_2^{2l}}{2} \right) \\ &\quad + \int_Y (\ln(1 + \gamma_2(\delta)) - \gamma_2(\delta)) \nu(d\delta) + \sigma_2^u \frac{B_2(t)}{t} + \frac{\int_0^t \int_Y \ln(1 + \gamma_2(\delta)) \widetilde{N}(dr, d\delta)}{t} + \frac{\ln I(0)}{t}. \end{aligned}$$

According to the large number of theorem [28], we have

$$\lim_{t \to \infty} \frac{B_2(t)}{t} = 0. \tag{3.8}$$

Then

$$\begin{aligned} \limsup_{t \to \infty} \frac{\ln I(t)}{t} &\leqslant \frac{\beta^u \Lambda^u}{\mu^l} - \left(\lambda^l + \mu^l + \alpha^l + \frac{\sigma_2^{2l}}{2} \right) + \int_Y (\ln(1 + \gamma_2(\delta)) - \gamma_2(\delta)) \nu(d\delta) \\ &= \left(\lambda^l + \mu^l + \alpha^l \right) \left[\frac{\beta^u \Lambda^u + \mu^l \int_Y (\ln(1 + \gamma_2(\delta)) - \gamma_2(\delta)) \nu(d\delta) - \mu^l \frac{\sigma_2^{2l}}{2}}{\mu^l (\lambda^l + \mu^l + \alpha^l)} - 1 \right]. \end{aligned}$$

Define

$$\eta = \frac{\sigma_2^{2l}}{2} + \int_Y (\gamma_2(\delta) - \ln(1 + \gamma_2(\delta))) \nu(d\delta),$$

$$R_0 = \frac{\beta^u \Lambda^u}{\mu^l (\lambda^l + \mu^l + \alpha^l)} - \frac{\eta}{\lambda^l + \mu^l + \alpha^l}.$$

Clearly, if $R_0 < 1$, then

$$\limsup_{t \to \infty} \frac{\ln I(t)}{t} \leqslant (\lambda^l + \mu^l + \alpha^l)(R_0 - 1) < 0,$$

$$\lim_{t \to \infty} I(t) = 0. \tag{3.9}$$

Moreover, Deriving from (3.4) and (3.5) that

$$\lim_{t \to \infty} S(t) = \frac{\Lambda^u}{\mu^l}.$$

This completes the proof. $\qquad\square$

Remark 3.1. Theorem 3.1 shows that the disease will die out under the influence of random perturbation. Note that even if $\sigma_2(t) = 0$ only Lévy process is large, the disease also will die out. That is, Lévy process can suppress the spread of the disease.

4. Existence of Nontrivial T-periodic Solution

In this section, we discuss the existence of the nontrivial positive T-periodic solution.

Lemma 4.1[30]. Assume that

(A_1) System has a unique global solution;

(A_2) There is a function $V(t, x) \in C^2$ which is T-periodic in t, and satisfies the following conditions:

$$\inf_{|x| > R} V(t, x) \to \infty \quad as \quad R \to \infty,$$

and $LV \leqslant -1$ outside some compact set, where the operator L is given by

$$LV = V_t(t, x) + V_x(t, x) f(t, x) + \frac{1}{2} trace\left(g^T(t, x) V_{xx}(t, x) g(t, x)\right).$$

Then the system has a T-periodic solution.

Theorem 4.1. Assume that

$$R_1 = \frac{3\langle \beta(t) b(t) \rangle_T}{\left\langle \mu(t) + \frac{\sigma_1^2(t)}{2} + \int_Y (\gamma_1(\delta) - \ln(1 + \gamma_1(\delta))) \nu(d\delta) \right\rangle_T \left\langle \lambda(t) + \mu(t) + \alpha(t) + \frac{\sigma_2^2(t)}{2} + \int_Y (\gamma_2(\delta) - \ln(1 + \gamma_2(\delta))) \nu(d\delta) \right\rangle_T} > 1.$$

then there exists a nontrivial positive T-periodic solution of system (1.2).

Proof. Define

$$V = M(-C \ln S(t) - \ln I(t) - I(t) + \omega(t)) + (S(t) + I(t))^\theta - \ln S(t),$$

where $V_1 = -C \ln S(t) - \ln I(t) - I(t)$ and C is a positive constant. By the Itô's formula, we have

$$LV_1 = -C\frac{\Lambda(t)}{S(t)} + C\beta(t) I(t) + C\mu(t) - C\lambda(t)\frac{I(t)}{S(t)} + C\frac{\sigma_1^2(t)}{2} - \beta(t) S(t) + \lambda(t) + \mu(t) + \alpha(t) + \frac{\sigma_2^2(t)}{2}$$

$$
\begin{aligned}
&+ \quad C \int_Y (\gamma_1 (\delta) - \ln (1 + \gamma_1 (\delta))) \, v (d\delta) + \int_Y (\gamma_2 (\delta) - \ln (1 + \gamma_2 (\delta))) \, v (d\delta) \\
&- \quad \beta (t) S (t) I (t) + (\lambda (t) + \mu (t) + \alpha (t)) I (t) \\
&\leqslant \quad -2 \sqrt{C \Lambda (t) \beta (t)} - 2 \sqrt{C \lambda (t) \beta (t)} I (t) + C \left(\mu (t) + \frac{\sigma_1^2 (t)}{2} + \int_Y (\gamma_1 (\delta) - \ln (1 + \gamma_1 (\delta))) \, v (d\delta) \right) \\
&+ \quad (C\beta (t) + \lambda (t) + \mu (t) + \alpha (t)) I (t) + \lambda (t) + \mu (t) + \alpha (t) + \frac{\sigma_2^2 (t)}{2} + \int_Y (\gamma_2 (\delta) - \ln (1 + \gamma_2 (\delta))) \, v (d\delta) \\
&\leqslant \quad -4 \sqrt{C \beta (t) b (t)} + C \left(\mu (t) + \frac{\sigma_1^2 (t)}{2} + \int_Y (\gamma_1 (\delta) - \ln (1 + \gamma_1 (\delta))) \, v (d\delta) \right) \\
&+ \quad (C\beta (t) + \lambda (t) + \mu (t) + \alpha (t)) I (t) + \lambda (t) + \mu (t) + \alpha (t) + \frac{\sigma_2^2 (t)}{2} + \int_Y (\gamma_2 (\delta) - \ln (1 + \gamma_2 (\delta))) \, v (d\delta),
\end{aligned}
$$

where $b (t) = \Lambda (t) \bigwedge \lambda (t)$, we get

$$
LV_1 =: R_1 (t) + (C\beta (t) + \lambda (t) + \mu (t) + \alpha (t)) I (t).
$$

Define the T-periodic function $\omega (t)$ satisfying

$$
\omega' (t) = \langle R_1 (t) \rangle_T - R_1 (t).
$$

So

$$
L (V_1 + \omega (t)) \leqslant \langle R_1 (t) \rangle_T + (C\beta (t) + \lambda (t) + \mu (t) + \alpha (t)) I (t),
$$

Choose

$$
C = \frac{\left\langle \sqrt{\beta (t) b (t)} \right\rangle_T}{\left\langle \mu (t) + \frac{\sigma_1^2 (t)}{2} + \int_Y (\gamma_1 (\delta) - \ln (1 + \gamma_1 (\delta))) \, v (d\delta) \right\rangle_T}. \tag{4.1}
$$

Then

$$
\begin{aligned}
L (V_1 + \omega (t)) &\leqslant -\left[\left\langle \lambda (t) + \mu (t) + \alpha (t) + \frac{\sigma_2^2 (t)}{2} + \int_Y (\gamma_2 (\delta) - \ln (1 + \gamma_2 (\delta))) \, v (d\delta) \right\rangle_T \right] (R_1 - 1) \\
&+ \quad (C\beta (t) + \lambda (t) + \mu (t) + \alpha (t)) I (t) \\
&\leqslant \quad -\langle Q (t) \rangle_T (R_1 - 1) + (C\beta (t) + \lambda (t) + \mu (t) + \alpha (t)) I (t),
\end{aligned}
$$

in which $R_1 = \dfrac{3 \langle \beta(t) b(t) \rangle_T}{\left\langle \mu(t) + \frac{\sigma_1^2(t)}{2} + \int_Y (\gamma_1(\delta) - \ln(1+\gamma_1(\delta))) v(d\delta) \right\rangle_T \left\langle \lambda(t) + \mu(t) + \alpha(t) + \frac{\sigma_2^2(t)}{2} + \int_Y (\gamma_2(\delta) - \ln(1+\gamma_2(\delta))) v(d\delta) \right\rangle_T}$,

$$
Q (t) = \lambda (t) + \mu (t) + \alpha (t) + \frac{\sigma_2^2 (t)}{2} + \int_Y (\gamma_2 (\delta) - \ln (1 + \gamma_2 (\delta))) \, v (d\delta).
$$

Define $Z (t) = S (t) + I (t)$ and $V_2 = Z^\theta (t), \theta > 1$, by Itô's formula, we obtain

$$
\begin{aligned}
LV_2 &\leqslant \theta Z^{\theta-1} (t) (\Lambda (t) - \mu (t) Z (t) - \alpha (t) I (t)) + \frac{\theta (\theta - 1)}{2} Z^{\theta-2} (t) \left(\sigma_1^2 (t) S^2 (t) + \sigma_2^2 (t) I^2 (t) \right) \\
&+ \quad \int_Y Z^\theta (t) \left[\left(1 + \gamma_1 (\delta) \bigvee \gamma_2 (\delta) \right)^\theta - 1 - \gamma_1 (\delta) \bigwedge \gamma_2 (\delta) \right] v (d\delta)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \ \theta \Lambda (t) Z^{\theta-1} (t) \\
&- \ \theta Z^{\theta} (t) \left[\mu (t) - \frac{\theta-1}{2} \sigma^2 (t) - \frac{1}{\theta} \int_Y \left(\left(1 + \gamma_1 (\delta) \bigvee \gamma_2 (\delta) \right)^{\theta} - 1 - \gamma_1 (\delta) \bigwedge \gamma_2 (\delta) \right) \nu (d\delta) \right] \\
&\leqslant \ \theta Z^{\theta-2} (t) \left(\Lambda (t) Z (t) - \eta (t) Z^2 (t) \right) \\
&\leqslant \ B - \frac{\theta \eta (t)}{2} \left(S^{\theta} (t) + I^{\theta} (t) \right),
\end{aligned}
$$

where

$$
\sigma (t) = \sigma_1 (t) \bigvee \sigma_2 (t),
$$

$$
\eta (t) = \mu (t) - \frac{\theta-1}{2} \sigma^2 (t) - \frac{1}{\theta} \int_Y \left(\left(1 + \gamma_1 (\delta) \bigvee \gamma_2 (\delta) \right)^{\theta} - 1 - \gamma_1 (\delta) \bigwedge \gamma_2 (\delta) \right) \nu (d\delta),
$$

and

$$
B = \sup_{(S,I) \in R_+^2} \left\{ \theta \Lambda (t) Z^{\theta-1} (t) - \frac{\theta \eta (t)}{2} Z^{\theta} (t) \right\}.
$$

Then,

$$
\begin{aligned}
LV \ \leqslant \ & M \left(- \langle Q(t) \rangle_T (R_1 - 1) \right) + (C\beta (t) + \lambda (t) + \mu (t) + \alpha (t)) I (t) \\
+ \ & B - \frac{\theta \eta (t)}{2} \left(S^{\theta} (t) + I^{\theta} (t) \right) - \frac{\Lambda (t)}{S (t)} + \beta (t) I (t) + \mu (t) + \frac{\sigma_1^2 (t)}{2} + \int_Y (\gamma_1 (\delta) - \ln (1 + \gamma_1 (\delta))) \nu (d\delta).
\end{aligned}
$$

Now, define a compact subset

$$
A = \left\{ (S,I) \in R_+^2 : \varepsilon \leq S \leq \frac{1}{\varepsilon}, \quad \varepsilon \leq I \leq \frac{1}{\varepsilon} \right\},
$$

where ε is a sufficiently small positive number. Set A^C such that $A \cup A^C = R_+^2$ and $A \cap A^C = \phi$. Then

$$
A^C = A_1^C \cup A_2^C \cup A_3^C \cup A_4^C,
$$

with

$$
A_1^C = \left\{ (S,I) \in R_+^2 : 0 < S < \varepsilon \right\},
$$

$$
A_2^C = \left\{ (S,I) \in R_+^2 : 0 < I < \varepsilon \right\},
$$

$$
A_3^C = \left\{ (S,I) \in R_+^2 : S > \frac{1}{\varepsilon} \right\},
$$

$$
A_4^C = \left\{ (S,I) \in R_+^2 : I > \frac{1}{\varepsilon} \right\}.
$$

In the set A^C, we choose appropriate positive constant M and a sufficiently small positive constant ε satisfying the following inequalities

$$
- MP + D + \mu^u + \frac{\sigma_1^{2u}}{2} + \kappa_1 + B - \frac{\Lambda^l}{\varepsilon} \leq -1, \tag{4.2}
$$

$$
- MP + [M (C\beta^u + \lambda^u + \mu^u + \alpha^u) + \beta^u] \varepsilon + \mu^u + \frac{\sigma_1^{2u}}{2} + \kappa_1 + B \leq -1, \tag{4.3}
$$

$$-MP + D + \mu^u + \frac{\sigma_1^{2u}}{2} + \kappa_1 + B - \frac{\theta\eta^l}{2\varepsilon^\theta} \leq -1, \tag{4.4}$$

$$-MP + D + \mu^u + \frac{\sigma_1^{2u}}{2} + \kappa_1 + B - \frac{\theta\eta^l}{4\varepsilon^\theta} \leq -1, \tag{4.5}$$

where

$$P = \langle Q(t) \rangle_T (R_1 - 1),$$

$$D = \sup_{I(t) \in [0,\infty)} \left\{ [M((C+1)\beta^u + \lambda^u + \mu^u + \alpha^u)] I(t) - \frac{\theta\eta^l}{4} I^\theta(t) \right\} < \infty.$$

Next, we prove $LV \leq -1$ for any $(S, I) \in A^C$.

Case 1. If $(S, I) \in A_1^C$, (4.1) and (4.2) imply that

$$\begin{aligned}
LV &\leq -MP + D + \mu^u + \frac{\sigma_1^{2u}}{2} + \kappa_1 + B - \frac{\Lambda^l}{S(t)} \\
&\leq -MP + D + \mu^u + \frac{\sigma_1^{2u}}{2} + \kappa_1 + B - \frac{\Lambda^l}{\varepsilon} \leq -1.
\end{aligned}$$

Case 2. If $(S, I) \in A_2^C$, (4.1) and (4.3) imply that

$$\begin{aligned}
LV &\leq -MP + \mu^u + \frac{\sigma_1^{2u}}{2} + \kappa_1 + B + [M(C\beta^u + \lambda^u + \mu^u + \alpha^u) + \beta^u] I(t) \\
&\leq -MP + \mu^u + \frac{\sigma_1^{2u}}{2} + \kappa_1 + B + [M(C\beta^u + \lambda^u + \mu^u + \alpha^u) + \beta^u] \varepsilon \leq -1.
\end{aligned}$$

Case 3. If $(S, I) \in A_3^C$, we have from (4.1) and (4.4)

$$\begin{aligned}
LV &\leq -MP + D + \mu^u + \frac{\sigma_1^{2u}}{2} + \kappa_1 + B - \frac{\theta\eta^l}{2} S^\theta(t) \\
&\leq -MP + D + \mu^u + \frac{\sigma_1^{2u}}{2} + \kappa_1 + B - \frac{\theta\eta^l}{2\varepsilon^\theta} \leq -1.
\end{aligned}$$

Case 4. If $(S, I) \in A_4^C$, from (4.1) and (4.5) it follows that

$$\begin{aligned}
LV &\leq -MP + D + \mu^u + \frac{\sigma_1^{2u}}{2} + \kappa_1 + B - \frac{\theta\eta^l}{4} I^\theta(t) \\
&\leq -MP + D + \mu^u + \frac{\sigma_1^{2u}}{2} + \kappa_1 + B - \frac{\theta\eta^l}{4\varepsilon^\theta} \leq -1.
\end{aligned}$$

From the above discussion, we can conclude $LV \leq -1$ on A^C. By Lemma 4.1, we show that there exists a T-periodic solution system (1.2). This completes the proof of Theorem 4.1. □

5. Permanence

In this section, we verify the conditions for the permanence of disease $I(t)$.

Theorem 5.1. If $R_1 > 1$ and $\liminf_{t \to \infty} \langle I \rangle_t \geq \langle Q(t) \rangle_T (R_1 - 1)$, then the disease of system (1.2) is permanent in mean.

Proof. Define

$$V_4 = -C \ln S(t) - \ln I(t) - I(t) + \omega(t),$$

$$dV_4 = LV_4 dt - C\sigma_1(t) dB_1(t) - \sigma_2(t) dB_2(t) - \sigma_2(t) I(t) dB_2(t)$$
$$- C \int_Y \ln(1 + \gamma_1(\delta)) \widetilde{N}(dr, d\delta) - \int_Y \ln(1 + \gamma_2(\delta)) \widetilde{N}(dr, d\delta), \quad (5.1)$$

$$LV_4 \leqslant -\langle Q(t) \rangle_T (R_1 - 1) + (C\beta^u + \lambda^u + \mu^u + \alpha^u) I(t).$$

Then integrating both side of (5.1), we have

$$\frac{V_4(S(t), I(t)) - V_4(S(0), I(0))}{t} \leqslant -\langle Q(t) \rangle_T (R_1 - 1) + (C\beta^u + \lambda^u + \mu^u + \alpha^u) \langle I \rangle_t$$
$$- C\sigma_1(t) \frac{B_1(t)}{t} - \sigma_2(t) \frac{B_2(t)}{t} - \frac{\int_0^t \sigma_2(r) I(r) dB_2(r)}{t}$$
$$- C \frac{\int_0^t \int_Y \ln(1 + \gamma_1(\delta)) \widetilde{N}(dr, d\delta)}{t} - \frac{\int_0^t \int_Y \ln(1 + \gamma_2(\delta)) \widetilde{N}(dr, d\delta)}{t}$$
$$=: -\langle Q(t) \rangle_T (R_1 - 1) + (C\beta^u + \lambda^u + \mu^u + \alpha^u) \langle I \rangle_t - \xi(t),$$

where

$$\xi(t) = C\sigma_1(t) \frac{B_1(t)}{t} + \sigma_2(t) \frac{B_2(t)}{t} + \frac{\int_0^t \sigma_2(r) I(r) dB_2(r)}{t}$$
$$+ C \frac{\int_0^t \int_Y \ln(1 + \gamma_1(\delta)) \widetilde{N}(dr, d\delta)}{t} + \frac{\int_0^t \int_Y \ln(1 + \gamma_2(\delta)) \widetilde{N}(dr, d\delta)}{t}.$$

By Lemma 3.1 and the large number of theorem [28], one can get that

$$\lim_{t \to \infty} \xi(t) = 0.$$

Then

$$\liminf_{t \to \infty} (C\beta^u + \lambda^u + \mu^u + \alpha^u) \langle I \rangle_t \geqslant \langle Q(t) \rangle_T (R_1 - 1) + \liminf_{t \to \infty} \frac{V_4(S(t), I(t)) - V_4(S(0), I(0))}{t}$$
$$\geqslant \langle Q(t) \rangle_T (R_1 - 1).$$

Remark 5.1. Theorem 5.1 shows that if the random perturbation is not large and $R_1 > 1$, the disease is persistent. □

6. Numerical simulations and conclusion

Now, we will introduce some numerical simulation to support our main theoretical results.

In Fig1, we choose the parameter in system (1.2) as follows
$\Lambda(t) = 0.5 + 0.1 \sin \pi t, \beta(t) = 0.0011 + 0.001 \sin \pi t, \mu(t) = 0.005 + 0.005 \sin \pi t,$

$\lambda(t) = 0.9 + 0.1 \sin \pi t, \alpha(t) = 0.03 + 0.03 \sin \pi t, \sigma_1(t) = 0.08 + 0.08 \sin \pi t,$
$\sigma_2(t) = 0.03 + 0.03 \sin \pi t, \gamma_1 = 0.03, \gamma_2 = 0.02.$

Then $R_1 = \dfrac{3\langle \beta(t)b(t)\rangle_T}{\left\langle \mu(t)+\frac{\sigma_1^2(t)}{2}+\int_Y(\gamma_1(\delta)-\ln(1+\gamma_1(\delta)))\nu(d\delta)\right\rangle_T \left\langle \lambda(t)+\mu(t)+\alpha(t)+\frac{\sigma_2^2(t)}{2}+\int_Y(\gamma_2(\delta)-\ln(1+\gamma_2(\delta)))\nu(d\delta)\right\rangle_T}.$

Note that $R_1 > 1$ and $\liminf\limits_{t\to\infty}\langle I\rangle_t \geqslant \langle Q(t)\rangle_T (R_1 - 1)$, That is, $I(t)$ will lead the disease to permanent. From Fig 1, the disease will go persistent when the noises are sufficiently small.

In Fig 2, we take the parameter values in system (1.2) as
$\Lambda(t) = 0.8 + 0.1 \sin \pi t, \beta(t) = 0.001 + 0.001 \sin \pi t, \mu(t) = 0.2 + 0.1 \sin \pi t,$
$\lambda(t) = 0.9 + 0.1 \sin \pi t, \alpha(t) = 0.1 + 0.1 \sin \pi t, \sigma_1(t) = 0.1 + 0.1 \sin \pi t,$
$\sigma_2(t) = 0.3 + 0.1 \sin \pi t, \gamma_1 = 0.3, \gamma_2 = 0.1.$

Note that $\eta > \frac{\beta^u \Lambda^u}{\mu^l}$, so, condition (i) of Theorem 3.1 holds. From Fig 2, the disease will go to extinction when the noise is large.
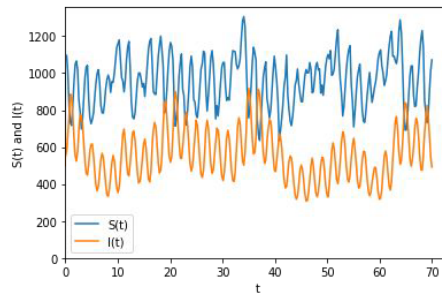

Figure 1. The trajectories of nonautonomous stochastic model (1.2) with Lévy jumps $(R_1 > 1)$.
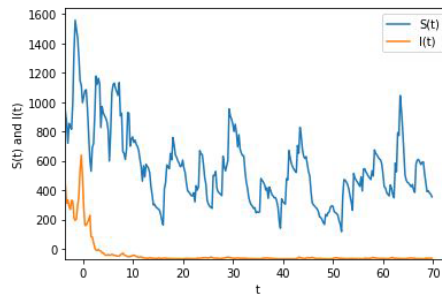

Figure 2. The trajectories of nonautonomous stochastic model (1.2) with Lévy jumps $\left(\eta > \frac{\beta^u \Lambda^u}{\mu^l}\right)$.

In addition, The difference between conditions of Fig 3 is that the values of $\lambda(t) = 0.8 + 0.1 \sin \pi t,$ $\sigma_2(t) = 0.1 + 0.1 \sin \pi t, \gamma_1 = 0.4$. Note that $\eta \leqslant \frac{\beta^u \Lambda^u}{\mu^l}$ and $R_0 < 1$. Therefore, conditions (ii) of Theorem 3.1 holds. Then the disease will also go extinct when the noise is large. Thus, it is necessary to consider the effect of the noise in model process.

From the above conclusion, we know that the effect of the noise cannot be ignored in model process. In addition, considering the sudden environmental shocks and the influence of seasonal changes, nonautonomous stochastic model with jumps is better than stochastic model be described by Gaussian
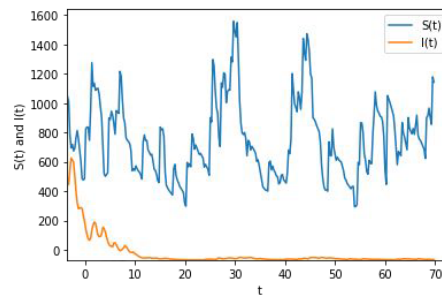
Figure 3. The trajectories of nonautonomous stochastic model (1.2) with Lévy jumps $\left(\eta \leqslant \frac{\beta^u \Lambda^u}{\mu^l}, R_0 < 1\right)$.

white noise alone.

In this paper, Firstly, By constructing suitable stochastic Lyapunov function, we show that the system (1.2) has a unique positive solution.

(1) When $\eta > \frac{\beta^u \Lambda^u}{\mu^l}$, where $\eta = \frac{\sigma_2^{2l}}{2} + \int_Y \left[\gamma_2(\delta) - \ln(1 + \gamma_2(\delta))\right] \nu(d\delta)$, we know η increase as σ_2^{2l} and $\gamma_2(\delta)$ increase, the disease will go extinct as long as the noises are large in mean. In addition, If $\eta \leqslant \frac{\beta^u \Lambda^u}{\mu^l}$ and $R_0 < 1$, where $R_0 = \frac{\beta^u \Lambda^u}{\mu^l(\lambda^l + \mu^l + \alpha^l)} - \frac{\eta}{\lambda^l + \mu^l + \alpha^l}$. The disease can also die out, when the noises are large in mean.

(2) If $R_1 > 1$ and $\liminf\limits_{t \to \infty} \langle I \rangle_t \geqslant \langle Q(t) \rangle_T (R_1 - 1)$,

where $R_1 = \dfrac{3\langle \beta(t)b(t) \rangle_T}{\left\langle \mu(t) + \frac{\sigma_1^2(t)}{2} + \int_Y (\gamma_1(\delta) - \ln(1 + \gamma_1(\delta)))\nu(d\delta) \right\rangle_T \left\langle \lambda(t) + \mu(t) + \alpha(t) + \frac{\sigma_2^2(t)}{2} + \int_Y (\gamma_2(\delta) - \ln(1 + \gamma_2(\delta)))\nu(d\delta) \right\rangle_T}$, the disease can be permanent in mean, when the noises are small enough.

(3) If $R_1 > 1$, there is at least one positive T-periodic solution of system (1.2). This means the occurrence of disease may be periodic in the ecosystem.

On the basis of this article, some issues deserve further investigation. For instance, We can consider the effects of impulsive or delay perturbations in system (1.2). Furthermore, we can also explore some complex nonautonoumous stochastic epidemic models , such as SIR and $SEIR$ model with non-Gaussian white noise. We leave these investigations for the future work.

Conflict of interest

The authors have no conflict of interest.

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