Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/MBE

MBE, 18(1): 518–529.
DOI: 10.3934/mbe.2021028
Received: 05 September 2020
Accepted: 22 November 2020
Published: 11 December 2020

Research article

Optimal strategies for a fishery model applied to utility functions

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Abstract: This work examines aquaculture-related activities in the commercial exploitation of fish reproduction. Fisheries' problem of maximizing utility is modeled for the state of Puebla, Mexico, to determine optimal fish production. The problem of maximizing utility subject to the fish production function is solved using an approach based on Euler's equation. The theoretical results are then applied, using data on aquaculture production and tilapia sales prices in the state of Puebla, Mexico. A logarithmic regression is used to approximate the utility function. The optimal fishing production and utility functions are thus explicitly obtained. Furthermore, this work shows how to obtain greater profits from the amount of fish that can be extracted without reducing the fish population.

Keywords: discounted Markov decision processes; dynamic programming; fishery model; Euler equation; fishery research; utility functions

1. Introduction

Aquaculture has had great success in the fishing subsector in recent years, using controlled reproduction through breeding and cultivation techniques that are amenable to commercial exploitation. Aquaculture has great potential in Mexico, since it has the necessary features for generating jobs and commercially exploiting different species of fish, given rising demand that is expected to increase even further over the following decades [1]. In order of importance, the main aquaculture species harvested in Mexico are mojarra, shrimp, oyster, carp, and trout.

In particular, Puebla has been one of the most successful states in aquaculture development. This progress has led to the consolidation of aquaculture and new opportunities for companies and individuals dedicated to tilapia production. The state of Puebla has no coastline, another factor that increases the importance of tilapia production, since tilapia easily adapt to captivity. In addition, aquaculture activity is also being used in to reduce poverty and develop new sources of capital income.

Given the importance of aquaculture in Mexico, this study models the fishery problem as a Markov decision process (MDP) [2, 3], [4] with a state space related to the fish production, a set of actions involving the quantity of fish removed from the pond, a set of admissible actions that are a function of fish production, the transition law, and the utility function. In the previously approach, the main objective of a country, region, or company was to maximize the profits from its fish production, that is, profit maximization subject to a transition law. Furthermore, to obtain greater benefits from the amount of fish that can be extracted without reducing the population of fish. In this document, a discrete-type dynamic is proposed for the study of the fish population. On the other hand, in the literature on the subject, there exist references about continuous time dynamic population, for example, see [5] and [6]. The fisheries problem has been previously addressed in the context of dynamic programming [7, 8]; however, those works do not use the Euler's Equation (EE) to find the optimal solution [9]. In the present work, the optimal solution is characterized by and derived explicitly from the EE.

After modeling the fisheries problem, this study demonstrates that the dynamic programming conditions are satisfied, guaranteeing the existence of an optimal solution. In this case, each of the conditions necessary for the existence of the optimal solution is demonstrated in detail. However, this only proves the existence of an optimal policy; next, the EE is used to determine this optimal policy and the optimal value function.

Finally, the parameters are estimated using fish production statistics for the state of Puebla, Mexico. To this end, a logarithmic regression is used to approximate the utility function, following the methodology exposed in [10]. Thus, this model is used to determine the optimal policy and optimal value solution numerically and thus make comparisons with previous work.

This paper is organized as follows. First, in section materials and method is presented the statistical analysis and later, the topics of Markov decision processes and EE are commented in a general way. In section 3, the main results of the manuscript are presented, in this section the existence of optimal extraction strategies is guaranteed and they are characterized using the Euler equation. Finally, the work presents some numerical results and a discussion section of the results obtained.

2. Materials and method

This work uses the tilapia catch data from CONAPESCA [11] between 2006 and 2014 for the state of Puebla, Mexico. Figure 1 plots the weight (in kilograms) of fish caught versus total sales (in Mexican pesos) per month, over 102 months. Following CONAPESCA, the total sales data coincide with the demand for tilapia in the region. Therefore, fish demand is related to sales data.

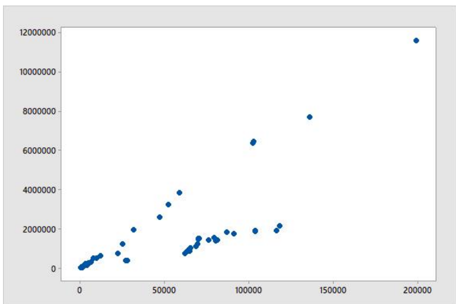

Figure 1. Amount of fish caught versus total sales per month.

In this case, the plot in Figure 1 has been adjusted to fit a logarithmic utility. Table 1 presents the results.

Table 1. Descriptive statistics of the estimation.

Parameter	Estimator	Standard error	Confidence Interval of 95%
b_1	–4,697,590	1,149,049	(–6,977,269, –2,417,909)
b_2	567,437	109,491	(350,210, 784,665)

Figure 2 shows that the data fit a logarithmic function, which is a utility function that expresses the total amount sold in Mexican pesos, $U(a) = b_1 \ln(a) - b_2$ (with conditions satisfied by a utility function [12, 13], where $b_1 := 567,437$, $b_2 := 4,697,590$, and a is the amount (in kilograms) of fish harvested. Note that, in this case, a is always a non-negative number.

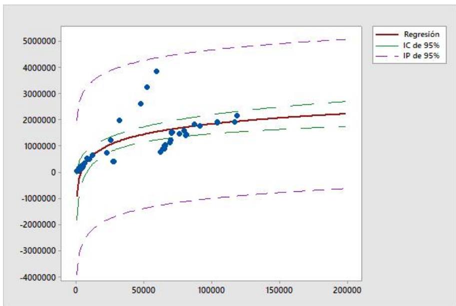

Figure 2. Logarithmic regression: kilograms of fish caught versus sales (in pesos).

In the next section, the problem of fisheries is presented in the context of an MDP, where the state space is related to fish production. The set of actions is the amount of fish extracted, and the set of admissible actions is a function of fish production. Then, the main objective of a country, region, or company is to maximize its profits, according to its fish production, that is, profit maximization subject to a transition law. To this end, consider h as the fish production function, which denotes the quantity of fish produced in a certain period (for the conditions that must satisfy a production function see [12]). In particular, let $h(x) := x^\gamma$, where $0 < \gamma < 1$ is the production factor. This class of function is widely used in fish production, see e.g., [7, 8]. Then, to solve this optimal control problem, a discrete-time version of the Euler-Lagrange equation (see chapter 6 of [14]) is deduced. In the context of MDP, this equation is simply known as EE, a deterministic version of EE can be consulted in [15] and a stochastic version for linear quadratic models is proposed in [16].

3. Results

The fishery model can be interpreted as follows. Let x_t be the amount of fish at time t, where x_t denotes the state of the system at time t. Thus, $X = [0, \infty)$ is the state space and a_t denotes the amount of fish caught. In this case, the action and the state spaces are the same, that is, $A = [0, \infty)$. The production function, $h : X \to [0, \infty)$, is given by $h(x) = x^\gamma$. This implies that the set of admissible actions, when the system is in state x, is $A(x) = [0, x^\gamma]$ for each state $x \in X$. Furthermore, $\mathbb{K} := \{(x, a) \mid x \in X, a \in A(x)\}$.

The evolution equation of the fishery process thus has the following form:

$$x_{t+1} = (x_t - a_t)^\gamma \xi_t, \tag{3.1}$$

where, for $t = 0, 1, \ldots$, $\xi_0, \xi_1, \ldots$, are nonnegative independent and identically distributed random variables, independent of the initial stock x_0, that represent the fish mortality rate, and their probability distribution function is denoted by F, that is, $F(s) = P(\xi \leq s)$ for all $s \in \mathbb{R}$ and density $\Delta \in C^2([0, \infty))$. The expected value $E[\xi] := \mu < +\infty$, and ξ represents a generic element of the sequence $\{\xi_t\}$. Finally, the utilities are represented by the following function $U : A \to \mathbb{R}$:

$$U(a) = \begin{cases} 0 & \text{if} \quad \exp(b_2/b_1) > a, \\ \\ b_1 \ln(a) - b_2 & \text{if} \quad \exp(b_2/b_1) \leq a. \end{cases}$$

Remark 3.1. *In this case, the transition law Q is determined for $x \in X$, $a \in A(x)$, and $B \in \mathfrak{B}(X)$ (where $\mathfrak{B}(X)$ is the Borel σ-algebra of X) in the following form:*

$$Q(B \mid x_t = x, a_t = a) = \int_B \omega(x, y, a) dy,$$

where the function $\omega : [0, \infty)^3 \to [0, \infty)$ is defined by

$$\omega(x, y, a) := \Delta\left(\frac{y}{(x-a)^\gamma}\right) \frac{1}{(x-a)^\gamma}$$

for $x, y \in X$ and $a \in [0, x^\gamma)$.

The MDP $M := (X, A, \{A(x) \mid x \in X\}, Q, U)$ is thus defined [3], and M is referred to as the fishery model.

Definition 3.2. *For each $t = 0, 1, \ldots$, define the space $\mathbb{H}_t$ of admissible histories up to time t as $\mathbb{H}_0 = X$, and $\mathbb{H}_t = \mathbb{K} \times \mathbb{H}_{t-1}$ for $t = 1, 2, \ldots$. A policy is defined as a sequence $\pi = \{\pi_t, t = 0, 1, \ldots\}$ of stochastic kernels, defined on A given $\mathbb{H}_t$ [3]. In this paper, the set of policies is denoted by Π. In particular, a stationary policy is of the form $\pi = \{f, f, \ldots\}$, where f is defined as a measurable function $f : X \to A$ such that $f(x) \in [0, x^\gamma]$ for all $x \in X$. The set of all stationary policies is denoted by $\mathbb{F}$.*

The sequence of consecutive states and corresponding actions are denoted by $\{x_t\}$ and $\{a_t\}$, respectively, and E_x^π denotes the expected value with respect to the probability measure P_x^π, which is defined on a canonical measurable space $(\Omega, \mathfrak{F})$ according to the Ionescu–Tulcea theorem [3].

For each $\pi \in \Pi$ and initial state $x \in X$, let

$$V(\pi, x) = E_x^\pi \left[\sum_{t=0}^{\infty} \alpha^t U(a_t)\right]$$

be the *total expected discounted utility* when policy π is applied, given initial state x. The constant $\alpha \in (0, 1)$ is a fixed *discount factor*.

Definition 3.3. *A policy π^* is optimal if $V(\pi^*, x) = \upsilon^*(x)$ for all $x \in X$, where*

$$\upsilon^*(x) := \sup_{\pi \in \Pi} V(\pi, x), \qquad x \in X,$$

and υ^* is called the optimal value function.

3.1. Fishery model solution

Let M be the fishery model described in the previous section. In this section, the necessary conditions to guarantee the existence of an optimal policy in the fishery model M will be proven. It is worth mentioning that the proof is carried out theoretically, so that, in the next sections, this policy can be calculated explicitly and, consequently, the optimal value function can be determined.

Lemma 3.4. *There exists a sequence* $\left\{X_j\right\}_{j\in\mathbb{N}_0}$ *(where* $\mathbb{N}_0$ *is the set of natural numbers union zero) of non-empty Borel subsets X such that* $X_j \subset X_{j+1}$ *for all* $j \in \mathbb{N}_0$ *and*

$$X = \bigcup_{j\geq 0} int\left(X_j\right),$$

where $int(X_j)$ *denotes the interior of set* X_j *for each* $j \in \mathbb{N}_0$*. In addition, for any* $j \in \mathbb{N}_0$*, let* m_j *be*

$$m_j = \sup_{x\in X_j} \sup_{a\in A(x)} U^+(a), \tag{3.2}$$

where $U^+(a) = \max\{U(a), 0\}$*. Assume that* $m_0 > 0$ *and* $m_j < \infty$ *for every* $j \in \mathbb{N}_0$ *and*

$$\alpha \limsup_{j\to\infty} \frac{m_{j+1}}{m_j} < 1. \tag{3.3}$$

Finally, for each $j \in \mathbb{N}_0$ *and* $x \in X_j$*,* $a \in A(x)$*,*

$$Q\left(X_{j+1} \mid x, a\right) = 1. \tag{3.4}$$

Proof. Let $\varepsilon > 0$ be a fixed number and define $X_j := [0, j + \varepsilon]$, for each $j \in \mathbb{N}_0$. Note that $X_j \subset X_{j+1}$ and

$$X = \bigcup_{j\geq 0} int\left(X_j\right).$$

Furthermore,

$$\begin{aligned} m_j &= \sup_{x\in X_j} \sup_{a\in A(x)} U^+(a) = \sup_{x\in\left[0,j+\varepsilon\right]} \sup_{a\in[0,x^\gamma]} b_1 \ln(a) - b_2 \\ &= \sup_{x\in\left[0,j+\varepsilon\right]} \gamma b_1 \ln(x) - b_2 \\ &= \gamma b_1 \ln(j+\varepsilon) - b_2 \end{aligned}$$

for each $j \in \mathbb{N}_0$ and $a \geq \exp(b_2/b_1)$. On the other hand, substituting m_j and m_{j+1} in Eq. (3.3) and applying L'Hôpital's rule yields

$$\alpha \limsup_{j\to\infty} \frac{m_{j+1}}{m_j} = \alpha \frac{1+\varepsilon}{2+\varepsilon} < \alpha < 1.$$

Finally, for $t = 0, 1, 2, \ldots$, if $x_t = x \in X$ and $a_t \in A(x)$, there exists $j \in \mathbb{N}_0$ such that $x \in X_j$, and, since $\xi \in (0, 1)$, thus $x_{t+1} \in X_{j+1}$ almost surely with respect to the probability measure P^π_x, that is, $Q(X_{j+1} \mid x, a) = 1$. □

Definition 3.5. *The value iteration functions are defined as*

$$V_n(x) = \max_{a\in A(x)} \{b_1 \ln(a) - b_2 + \alpha E\left[V_{n-1}((x-a)^\gamma \xi)\right]\} \tag{3.5}$$

for all $x \in X$ and $n = 1, 2, \ldots$, with $V_0(\cdot) = 0$.

Using Lemma 3.4, the proof of the following lemma can be adapted from the proof of Proposition 1 and Theorem 1 by Jaskiewicz and Nowak [17].

Lemma 3.6. *The fishery problem satisfies the following conditions.*

a) *The optimal value function $\upsilon^* : X \to \mathbb{R}$ is a measurable function and a solution of*

$$\upsilon^*(x) = \max_{a\in A(x)} \{b_1 \ln(a) - b_2 + \alpha E\left[\upsilon^*((x-a)^\gamma \xi)\right]\} \tag{3.6}$$

for each $x \in X$.

b) *There exists a selector $f^* \in \mathbb{F}$ such that, in Eq. (3.6), the maximum is attained, that is,*

$$\upsilon^*(x) = b_1 \ln\left(f^*(x)\right) - b_2 + \alpha E\left[\upsilon^*((x - f^*(x))^\gamma \xi)\right] \tag{3.7}$$

for all $x \in X$ and f^* is optimal.

c) *$V_n : X \to \mathbb{R}$ is a measurable function, $n = 1, 2, \ldots$, and $\{V_n\}$ converges pointwise to υ^*. Moreover, for each $n = 1, 2, \ldots$, there exists $f_n \in \mathbb{F}$ such that*

$$V_n(x) = b_1 \ln\left(f_n(x)\right) - b_2 + \alpha E\left[V_{n-1}((x - f_n(x))^\gamma \xi)\right]$$

for all $x \in X$, where f_n is the maximizer of the value iteration functions.

We define the functions

$$G(x, a) := b_1 \ln(a) - b_2 + \alpha E\left[\upsilon^*\left(((x-a)^\gamma)\, \xi\right)\right]$$

and

$$G^n(x, a) := b_1 \ln(a) - b_2 + \alpha E\left[V_{n-1}\left(((x-a)^\gamma)\, \xi\right)\right],$$

$n = 1, 2, \ldots$, for each $(x, a) \in \mathbb{K}$.

Theorem 3.7. *The iterative value functions satisfy the functional equation*

$$V_n'(x) = \alpha\gamma E\left[V_{n-1}\left(\left(\frac{xV_n'(x) - b_1}{V_n'(x)}\right)^\gamma \xi\right)\xi\right]\left(\frac{xV_n'(x) - b_1}{V_n'(x)}\right)^{\gamma-1} \tag{3.8}$$

for each $n \geq 2$ and $x \in int(X)$, where ξ is a generic element of the sequence $\{\xi_t\}$ (see Eq.3.1). This equation is known in the literature of MDPs as Euler equation.

Proof. Deriving the function G^n with respect to a yields

$$G^n_a(x,a) = \frac{b_1}{a} - \alpha\gamma E[V'_{n-1}((x-a)^\gamma\xi)\xi](x-a)^{\gamma-1},$$

$(x,a) \in \mathbb{K}$. Evaluating the preceding equation in the maximizer $f_n \in \mathbb{F}$ results in

$$\frac{b_1}{f_n(x)} - \alpha\gamma E[V'_{n-1}((x-f_n(x))^\gamma\xi)\xi](x-f_n(x))^{\gamma-1}. \tag{3.9}$$

Furthermore,

$$V'_n(x) = \alpha\gamma E[V'_{n-1}((x-f_n(x))^\gamma\xi)\xi](x-f_n(x))^{\gamma-1}, \tag{3.10}$$

$x \in int(X)$, and substituting Eq. (3.9) in Eq. (3.10) yields

$$\frac{b_1}{f_n(x)} = V'_n(x),$$

$x \in int(X)$. Then,

$$f_n(x) = \frac{b_1}{V'_n(x)}. \tag{3.11}$$

Finally, substituting Eq. (3.11) in Eq. (3.10) yields Eq. (3.8). □

Theorem 3.8. *The optimal value function and the optimal policy are given by*

$$\upsilon^*(x) = b_1 K \ln x + C$$

and

$$f^*(x) = (1-\alpha\gamma)x,$$

respectively, for each $x \in X$, where

$$K = \frac{1}{1-\alpha\gamma},$$

$$C = \frac{1}{1-\alpha}\left[\frac{b_1\alpha\gamma}{1-\alpha\gamma}\ln(\alpha\gamma) + b_1\ln(1-\alpha\gamma) + \frac{b_1\alpha\mu}{1-\alpha\gamma} - b_2\right].$$

Proof. First, note that

$$V_1(x) = b_1\gamma\ln x - b_2,$$

$x \in X$, it follows that

$$V'_1(x) = \frac{b_1\gamma}{x},$$

$x \in (0,+\infty)$. For $n = 2$, using Eq. (3.8) and the preceding equation, yields

$$\begin{aligned} V'_2(x) &= \alpha E\left[V'_1\left(\left(\frac{xV'_2(x)-b_1}{V'_2(x)}\right)^\gamma\xi\right)\xi\right]\gamma\left(\frac{xV'_2(x)-b_1}{V'_2(x)}\right)^{\gamma-1} \\ &= \frac{b_1\alpha\gamma^2V'_2(x)}{xV'_2(x)-b_1}. \end{aligned}$$

Therefore,

$$V_2'(x) = \frac{b_1\left(\alpha\gamma^2 + 1\right)}{x},$$

$x \in (0, +\infty)$. It will be proven by induction that, for all $n \geq 2$, $V_n'(x) = b_1 \frac{K_n}{x}$. To this end, suppose that

$$V_{n-1}'(x) = b_1 \frac{K_{n-1}}{x}, \tag{3.12}$$

$x \in (0, +\infty)$, where $K_{n-1} = \sum_{i=0}^{n-2} (\alpha\gamma)^i + \alpha^{n-2}\gamma^{n-1}$. Thus, substituting Eq. (3.12) into Eq. (3.8) yields

$$V_n'(x) = b_1 \frac{\alpha\gamma K_{n-1} V_n'(x)}{x V_n'(x) - 1},$$

from which it follows that

$$V_n'(x) = b_1 \frac{\alpha\gamma K_{n-1} + 1}{x} = b_1 \frac{K_n}{x},$$

$x \in (0, +\infty)$, where $K_n = \sum_{i=0}^{n-1} (\alpha\gamma)^i + \alpha^{n-1}\gamma^n$. Then

$$V_n(x) = b_1 K_n \ln x + C_n$$

for all $n \geq 1$, $x \in X$. Taking the limit when $n \to \infty$,

$$V_n(x) = b_1 K_n \ln x + C_n \to \upsilon^*(x) = b_1 K \ln x + C,$$

$x \in X$, where

$$K = \lim_{n\to\infty} K_n = \frac{1}{1 - \alpha\gamma}. \tag{3.13}$$

Therefore, for all $x \in X$,

$$\begin{aligned} b_1 K \ln x + C &= \max_{a \in [0, x^\gamma]} \left\{ b_1 \ln(a) - b_2 + \alpha E\left[\upsilon^*\left((x-a)^\gamma \xi\right)\right] \right\} \\ &= \max_{a \in [0, x^\gamma]} \left\{ b_1 \left[\ln(a) + \alpha\gamma K \ln(x - a) + \alpha\mu K\right] + \alpha C - b_2 \right\}. \end{aligned} \tag{3.14}$$

The first-order condition of Eq. (3.14) is given by

$$\frac{b_1}{a} - \frac{b_1 \alpha\gamma K}{x - a} = 0,$$

and, using Eq. (3.13), this yields

$$f^*(x) = (1 - \alpha\gamma)x,$$

$x \in X$. Thus, for $x \in X$,

$$b_1 K \ln x + C = b_1(1 + \alpha\gamma K)\ln(x) + b_1\alpha\gamma K \ln(\alpha\gamma) + b_1 \ln(1 - \alpha\gamma) + b_1\alpha\mu K + \alpha C - b_2.$$

Then, $K = 1 + \alpha\gamma K = \frac{1}{1-\alpha\gamma}$ and

$$C = \frac{1}{1-\alpha}\left[\frac{b_1\alpha\gamma}{1-\alpha\gamma}\ln(\alpha\gamma) + b_1\ln(1-\alpha\gamma) + \frac{b_1\alpha\mu}{1-\alpha\gamma} - b_2\right].$$

□

3.2. Numerical results

In the example presented above, with catch data from CONAPESCA [11] between 2006 and 2014 for the state of Puebla, Mexico, the production of mojarra is obtained for different values of the production factor γ. For this numerical case, the discount factor $\alpha = 0.5$ is used and $\pi = \{a_0, a_1, a_2, \ldots, a_{102}\}$, where 102 is the number of months for which information is available between 2006 and 2014. Note that each a_i, $i = 1, 2, \ldots, 102$, corresponds to the weight of fish caught monthly, starting with $a_0 = 0$ (for more details, see CONAPESCA [11]). In the first case, the production factor is $\gamma = 0.85$, with an initial state $x_0 = 436,250$, which is the initial amount of fish; the production is thus calculated for the first and subsequent months. Using the logarithmic utility function $U(a) = b_1 \ln(a) - b_2$, $a \in A(x)$ from Section 1, one can conclude that $V_{102}(x, \pi) = 1,251,934.12$. Analogously, when the production factor is greater than $\gamma = 0.90$ but the initial stock is smaller, $x_0 = 220,000$, the value function increases to $V_{102}(x, \pi) = 1,252,077.98$ with respect to the previous case, or $\gamma = 0.95$ and $x_0 = 115,000$, such that the value function is $V_{102}(x, \pi) = 1,235,629.78$. Table 2 summarizes the results.

Table 2. Utility estimations with discount factor $\alpha = 0.5$.

γ	x_0 (kg of fish)	$V_{102}(x, \pi)$	π
0.85	436,250	1,251,934.12	$\{a_0, a_1, a_2, \ldots, a_{102}\}$
0.90	220,000	1,252,077.98	$\{a_0, a_1, a_2, \ldots, a_{102}\}$
0.95	115,000	1,235,629.78	$\{a_0, a_1, a_2, \ldots, a_{102}\}$

The following tables are similar to Table 2, but with a different discount factor.

Table 3. Utility estimations with discount factor $\alpha = 0.2$.

γ	x_0 (kg of fish)	$V_{102}(x, \pi)$	π
0.85	436,250	365,030.39	$\{a_0, a_1, a_2, \ldots, a_{102}\}$
0.90	220,000	368,347.63	$\{a_0, a_1, a_2, \ldots, a_{102}\}$
0.95	115,000	366,801.64	$\{a_0, a_1, a_2, \ldots, a_{102}\}$

Table 4. Utility estimations with discount factor $\alpha = 0.9$.

γ	x_0 (kg of fish)	$V_{102}(x, \pi)$	π
0.85	436,250	7,616,999.13	$\{a_0, a_1, a_2, \ldots, a_{102}\}$
0.90	220,000	8,857,481.60	$\{a_0, a_1, a_2, \ldots, a_{102}\}$
0.95	115,000	12,433,010.56	$\{a_0, a_1, a_2, \ldots, a_{102}\}$

According to Tables 2, 3, and 4, note that if α increases, then $V_{102}(x, \pi)$ increases. This means that as the discount factor increases, profits from fish sales will increase.

The following table shows a comparison between the value functions when the initial state is the same and the production factor is different.

Table 5. Utility estimations with discount factor $\alpha = 0.5$.

γ	x_0 (kg of fish)	$V_{102}(x, \pi)$	π
0.85	436,250	1,251,934.12	$\{a_0, a_1, a_2, \ldots, a_{102}\}$
0.90	436,250	1,663,042.04	$\{a_0, a_1, a_2, \ldots, a_{102}\}$
0.95	436,250	2,119,445.87	$\{a_0, a_1, a_2, \ldots, a_{102}\}$

In this case, when the initial state is the same for each production factor, observe that V increases as the production factor increases. That is, if production increases, then profits will also increase.

4. Discussions

Table 6 presents the results for the optimal value function and optimal policy given in Theorem 3.8. In this case, the same initial values and production factors are used.

Table 6. Utility estimations for the optimal policy when $\alpha = 0.5$.

γ	x_0 (kg of fish)	$\upsilon^*(x)$	$f^*(x)$
0.85	436,250	$2,074,213.42$	$250,843.75$
0.90	220,000	$1,876,290.76$	$121,000.00$
0.95	115,000	$1,703,773.25$	$60,375.00$

Note that, in the three cases presented, υ^* evaluated in the initial state is always greater than the respective cases find in the last section. Furthermore, with this optimal policy, not higher profits are obtained, but also the fish population is in equilibrium. Additionally, a production factor equal to 0.85 increases profits by 66%, whereas production factors of 0.90 and 0.95 increase profits by 50% and 38%, respectively.

Table 7. Utility estimations for the optimal policy when $\alpha = 0.2$.

γ	x_0 (kg of fish)	$\upsilon^*(x)$	$f^*(x)$
0.85	436,250	$2,616,400.34$	$362,087.50$
0.90	220,000	$2,232,771.20$	$180,400.00$
0.95	115,000	$1,865,407.39$	$93,150.00$

Unexploited fish stocks are in equilibrium (zero population growth), the causes of increment only have to balance the losses produced by natural mortality. However, once over-fishing begins, more fish is then caught than the population can replace through natural reproduction, consequently, the population level is reduced. If this reduction is not drastic, less than the initial level, and if the exploitation remains stable and below f^* levels given in Tables 6 and 7, the most probable is that the fish stock increases or a decrease in natural mortality will allow the number of fish remain stable.

Following the above, the conclusion is that the fish stock will remain in equilibrium as long as there is a reasonable fish exploitation, in this sense, it is proposed to follow the strategy f^* as shown in Tables 6 and 7. Thereby, it is guaranteed that both the population size and the catch amounts are kept at optimal levels, guaranteeing a sustainable performance. In other words, when f^* is applied, the fish stock and captures remain at a level that allows the population to fully develop its natural growth capacity. Additionally, from an economic point of view, the optimal profit is obtained. Conversely, when the fishing is greater than f^*, the stock can be reduced to lower levels than the initial state x_0, which would reduce the natural capacity for population growth. In this case, there is a period of over-fishing that excessively reduces the stock and can lead to the collapse of the fishery.

5. Conclusions

This work models the fisheries problem using CONAPESCA's mojarra extraction data from 2006 to 2014 for the state of Puebla, Mexico. Furthermore, concludes that the utility function is logarithmic. This fisheries model is studied as a MDP, allowing for the demonstration of the existence of an optimal policy. Using a dynamic programming equation and a version of the EE for the iterative value functions, the optimal value function and optimal policy are calculated for the case of an infinite horizon. This study shows how to obtain greater profits from the quantity of fish that can be extracted without reducing its population.

Subsequently, particular values are assigned to the production factor to obtain numerical results for which the value function is calculated for the state of Puebla. The utility function is calculated with data available from CONAPESCA. An analysis is then presented that compares the optimal value function to increase utility, applying the optimal policy found in this work. The study concludes both theoretically and numerically that the optimal policy is indeed the best strategy to follow to maximize profits.

Conflict of interest

The authors declare that they have no potential conflicts of interest.

References

1. D. Ngom, A. Iggidir, A. Guiro, A. Ouahbi, An observer for a nonlinear age-structured model of a harvested fish population, *Math. Biosci. Eng.*, **5** (2008), 337–354.
2. D. P. Bertsekas, Dynamic Programming and Stochastic Control, 1^{st} edition, Academic Press, New York, 1976.
3. O. Hernández-Lerma, J. B. Lasserre, Discrete-Time Markov Control Processes: Basic Optimality Criteria, 1^{st} edition, Springer-Verlag, New York, 1996.
4. M. L. Puterman, Markov Decision Processes: Discrete Stochastic Dynamic, 1^{st} edition, Wiley-Interscience, California, 2005.
5. B. Liao, Q. Liu, X. Wang, K. Zhang, J. Zhang, K. H. Memon, M. A. Kalhoro, Asymptotic behaviour of the n-species stochastic Gilpin-Ayala Cooperative model, *Stoch. Environ. Res. Risk Assess*, **30** (2015), 39-45.

6. B. Liao, X. Shan, C. Zhou, Y. Han, Y. Chen, Q. Liu, A dynamic energy budget–integral projection model (DEB-IPM) to predict population-level dynamics based on individual data: A case study using the small and rapidly reproducing species engraulis japonicus, *Mar. Freshwater Res.*, **71** (2019), 461–468.

7. D. Levhari, L. J. Mirman, The great fish war: An example using a dynamic Cournot-Nash solution, *Bell J. Econ.*, **11** (1980), 322–334.

8. L. J. Mirman, Dynamic models of fishing: A heuristic approach, *Control Theory Math. Econ.*, (1979), 39–73.

9. H. Cruz-Suárez, R. Montes-de-Oca, An envelope theorem and some applications to discounted Markov decision processes, *Math. Methods Oper. Res.*, **67** (2008), 299–321.

10. Y. K. Kwan, G. C. Chow, Estimating economic effects of political povements in China, *J. Comp. Econ.*, **23** (1996), 192–208.

11. Información Estadística por Especie y Entidad. March 10th, 2020, de Gobierno de México Sitio web, CONAPESCA. (2006-2014). https://www.conapesca.gob.mx/wb/cona/informacion_estadistica_por_especie_y_entidad

12. H. Cruz-Suárez, R. Montes-de-Oca, G. Zacarías, A consumption-investment problem modelled as a discounted Markov decision process, *Kybernetika (Prague)*, **67** (2011), 909–929.

13. A. de la Fuente, Mathematical Methods and Models for Economists, 1^{st} edition, Cambridge University Press, 2000.

14. J. L. Troutman, Variational calculus and optimal control: optimization with elementary convexity, 2^{nd} edition, Springer Science and Business Media, 2012.

15. H. Cruz-Suárez, R. Montes-de-Oca, Discounted Markov control processes induced by deterministic systems, *Kybernetika*, **42(6)** (2006), 647–664.

16. H. Cruz-Suárez, G. Zacarías-Espinoza, V. Vázquez-Guevara, A version of the Euler equation in discounted Markov decision processes, *J. Appl. Math.*, (2012), 1–16.

17. A. Jaskiewicz, A. S. Nowak, Discounted dynamic programming with unbounded returns: Application to economic models, *J. Math. Anal. Appl.*, **378** (2011), 450–462.