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Research article

Impact of variability of reproductive ageing and rate on childhood infectious disease prevention and control: insights from stage-structured population models

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Abstract: We propose a stage-structured model of childhood infectious disease transmission dynamics, with the population demographics dynamics governed by a certain family and population planning strategy giving rise to nonlinear feedback delayed effects on the reproduction ageing and rate. We first describe the long-term aging-profile of the population by describing the pattern and stability of equilibrium of the demographic model. We also investigate the disease transmission dynamics, using the epidemic model when the population reaches the positive equilibrium (limiting equation). We establish conditions for the existence, uniqueness and global stability of the disease endemic equilibrium. We then prove the global stability of the endemic equilibrium for the original epidemic model with varying population demographics. The global stability of the endemic equilibrium allows us to examine the effects of reproduction ageing and rate, under different family planning strategies, on the childhood infectious disease transmission dynamics. We also examine demographic distribution, diseases reproductive number, infant disease rate and age distribution of disease, and as such, the work can be potentially used to inform targeted age group for optimal vaccine booster programs.

Keywords: childhood infectious diseases; stage structured dynamical model; delay differential equation; stability; nonlinear feedback on reproduction ageing and rate

1. Introduction

Family planning involves consideration of the number (including the choice of zero) and spacing of children a family wishes to have. A number of factors can impact the family planning at the individual family level. At the population level the family planning and reproduction strategy including fertility, birth age and spacing of children, may be heavily influenced by economical conditions and societal resources which can be weighted heavily by the age-distribution of the entire population. In developing countries, policies like subsidizing education raise the earning power of women and the opportunity cost of having children, consequently lowers fertility [1]. Access to contraceptives may also yield lower fertility rates. In developed countries, the proportion of retired people is increasing, adding burden on the workforce population to support pensions and social programs. Increasing high skill migration may be an effective way to increase the return to education leading to lower fertility and a greater supply of highly skilled individuals [1], thus address the aging population problem.

A well-known example of family planning and age-distribution of population being significantly regulated by political and social-economic consideration is the China's one-child policy implemented for many years. In 1973, the Chinese government issued voluntary guidelines on fertility control to encourage later marriage, longer spacing between births, and fewer births overall [2, 3]. In 1981, China's National Family Planning Commission proposed a population control policy advocating one child per couple, which was moderated in 1984, allowing most rural families a second child [2–4]. In 2002, the policy was incorporated in the Population and Family Planning Law, at the same time, a second child was permitted in some provinces if both husband and wife were from single-child families [2, 3, 5]. In 2013, the policy was relaxed to allow a second child if either spouse was from a single-child family [3, 6]. In October of 2015, the Chinese government announced a two-child policy, effective from January 1 of 2016 [7]. The new policy that allows each couple have two children was proposed in order to help address the population aging issue. It was reported that, starting from May 2018, Chinese authorities were in the process of ending the population control policies [8].

A consequence of this recent change of the centralized population control policy after a long-term implementation of one-child per family policy is the obvious increasing of the family size, and substantial heterogeneity of the reproduction age and the spacing between the first and second child in those families with two children. This generates new close contact patterns in household and community level and thus any issue relevant to these contact patterns must be revisited. The control and prevention of childhood infectious diseases preventable by vaccine, such as pertussis, is one of these critical public health issues. Taking pertussis as an example, this childhood disease can be fatal in infants but infection can be prevented in other age groups with an effective vaccine. Pertussis vaccines wane over time, so those children who are expected to have younger siblings need to take a booster vaccine if (I) the prevalence of disease in the older age group and/or groups (recalling the potential heterogeneity of spacing between two children since females in multiple age groups may consider to give birth) is expected to be high; and (II) vaccine waning make this group of these groups less protected and more susceptible to the disease.

To the best of my knowledge, there is no study on impact of family planning and the scale of density-regulated birth rate on the long-term population demographic distribution and childhood disease dynamics. However, there are a few studies which imply the impact of demographic change on infectious disease dynamics. These work include studies on demographic transition and the dynamics of measles in China [9], the influence of demographic change on spread of infectious diseases [10], the impact of demographic transition on rubella transmission dynamics in China [11], the effects of demographic change and immigration on infectious diseases in Italy [12], the effects of demographic change on disease transmission and vaccine impact in a household structured population [13], the dynamical consequences of demographic change in a model of disease transmission [14] and the impact of demographic change on the estimated future burden of hepatitis B and seasonal influenza in the Netherlands [15].

This series of studies is dedicated to developing mathematical frameworks and analyses to examine the patterns of childhood infectious disease transmission, to identify prevalence of disease in different age groups, when female in multiple age intervals are giving birth to the second children. In this first paper of the series, we start with a simple stage-structured disease transmission model, and study the impact of family planning and the scale of density-regulated birth rate on the long-term population demographic distribution and infant disease incidence prevalence. We conduct our analyses by varying three parameters: birth rate, reproduction age interval(s), and the scale of the sub-population density regulation.

In particular, we introduce a multi-stage (m-stages) stratified model, where the population is divided by age into m groups (stages) with the i-th age stage spanning the age interval of length τ_i. Assume that females in the age groups, $k, k+1, \cdots, l$-th groups give birth. For each group, we have the classical SIS epidemic-model, where the population is divided into the susceptible and the infectious. We consider the situation where the infectious period is much shorter than the period of each age stage. Therefore, the infectious individuals in the i-th age group I_i go back to the susceptible class S_i before advancing to the $(i+1)$-th age group. The flowcharts of the demographic model and epidemiological model are shown in Figures 1 and 2 respectively.

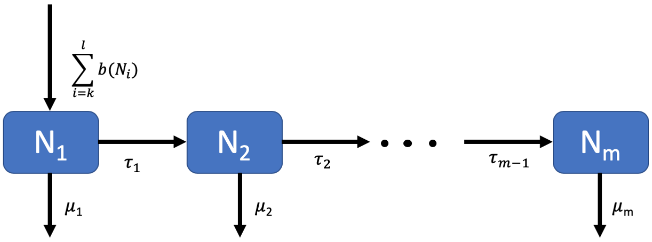

Figure 1. Flow chart of the demographic model.

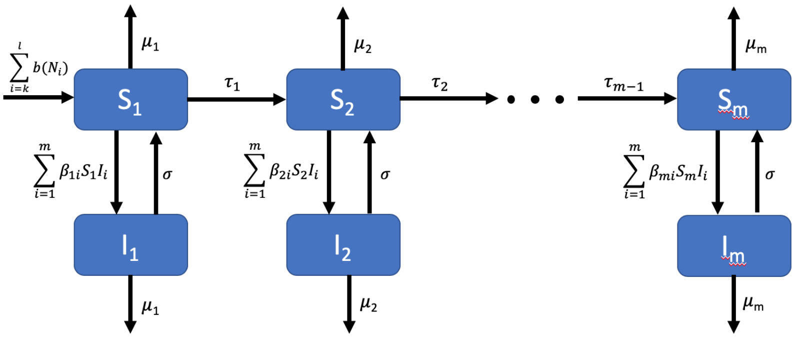

Figure 2. Flow chart of the epidemiological model.

Let $S_i(t)$ be the population of the susceptible of the ith age group, $I_i(t)$ be the population of the infectious of the ith age group. $N_i(t)$ denotes the total population of the ith age group at time t. The death rate of the ith age group is given by μ_i; σ is the recover rate; the birth rate of the ith productive group is a nonlinear function $b_i(N_i)$; β_{ij} is the transmission rate of the disease from stage j to stage i. The age-stratified epidemiological model is given by the following equations:

$$\begin{aligned}
\frac{dS_1(t)}{dt} &= \sum_{i=k}^{l} b_i(N_i(t)) - \sum_{i=k}^{l} b_i(N_i(t-\tau_1))e^{-\mu_1\tau_1} - \mu_1 S_1(t) - \sum_{i=1}^{m} \beta_{1i} S_1(t) I_i(t) + \sigma I_1(t) \\
\frac{dS_h(t)}{dt} &= \sum_{i=k}^{l} b_i(N_i(t - \sum_{j=1}^{h-1}\tau_j))e^{-\sum_{j=1}^{h-1}\mu_j\tau_j} - \sum_{i=k}^{l} b_i(N_i(t - \sum_{j=1}^{h}\tau_j))e^{-\sum_{j=1}^{h}\mu_j\tau_j} \\
&\quad - \mu_h S_h(t) - \sum_{i=1}^{m} \beta_{hi} S_h(t) I_i(t) + \sigma I_h(t) \text{ for } 1 < h < m \\
\frac{dS_m(t)}{dt} &= \sum_{i=k}^{l} b_i(N_i(t - \sum_{j=1}^{m-1}\tau_j))e^{-\sum_{j=1}^{m-1}\mu_j\tau_j} - \mu_m S_m(t) - \sum_{i=1}^{m} \beta_{mi} S_m(t) I_i(t) + \sigma I_m(t) \\
\frac{dI_h(t)}{dt} &= \sum_{i=1}^{m} \beta_{hi} S_h(t) I_i(t) - \sigma I_h(t) - \mu_h I_h(t) \text{ for } 1 \le h \le m
\end{aligned} \tag{1.1}$$

The demographic model is given by

$$\begin{aligned}
\frac{dN_1(t)}{dt} &= \sum_{i=k}^{l} b_i(N_i(t)) - \sum_{i=k}^{l} b_i(N_i(t-\tau_1))e^{-\mu_1\tau_1} - \mu_1 N_1(t) \\
\frac{dN_h(t)}{dt} &= \sum_{i=k}^{l} b_i(N_i(t - \sum_{j=1}^{h-1}\tau_j))e^{-\sum_{j=1}^{h-1}\mu_j\tau_j} - \sum_{i=k}^{l} b_i(N_i(t - \sum_{j=1}^{h}\tau_j))e^{-\sum_{j=1}^{h}\mu_j\tau_j} \\
&\quad - \mu_h N_h(t) \text{ for } 1 < h < m \\
\frac{dN_m(t)}{dt} &= \sum_{i=k}^{l} b_i(N_i(t - \sum_{j=1}^{m-1}\tau_j))e^{-\sum_{j=1}^{m-1}\mu_j\tau_j} - \mu_m N_m(t)
\end{aligned} . \tag{1.2}$$

2. The demographic model

In this section, we investigate the dynamics of the demographic model by studying the stability of equilibrium.

Linearizing (1.2) at the zero equilibrium gives

$$\begin{aligned}
\frac{dN_1(t)}{dt} &= \sum_{i=k}^{l} b_i'(0)N_i(t) - \sum_{i=k}^{l} b_i'(0)N_i(t-\tau_1)e^{-\mu_1\tau_1} - \mu_1 N_1(t) \\
\frac{dN_h(t)}{dt} &= \sum_{i=k}^{l} b_i'(0)N_i(t - \sum_{j=1}^{h-1}\tau_j)e^{-\sum_{j=1}^{h-1}\mu_j\tau_j} - \sum_{i=k}^{l} b_i'(0)N_i(t - \sum_{j=1}^{h}\tau_j)e^{-\sum_{j=1}^{h}\mu_j\tau_j} - \mu_h N_h(t) \text{ for } 1 < h < m \\
\frac{dN_m(t)}{dt} &= \sum_{i=k}^{l} b_i'(0)N_i(t - \sum_{j=1}^{m-1}\tau_j)e^{-\sum_{j=1}^{m-1}\mu_j\tau_j} - \mu_m N_m(t)
\end{aligned} \tag{2.1}$$

Let λ be eigenvalue of the linear system (2.1). By calculation, the characteristic equation is given by

$$\sum_{i=k}^{l} b_i'(0)\frac{(1-e^{-(\lambda+\mu_i)\tau_i})e^{-\lambda\sum_{j=1}^{i-1}\tau_j-\sum_{j=1}^{i-1}\mu_j\tau_j}}{\mu_i+\lambda}=1 \tag{2.2}$$

Now we make the following assumption:

(A1) The birth function takes the form $b_i(x) = p_i x q(x)$ where $q(x)$ is a non-negative monotone decreasing function.

Note that p_i is the maximal number of children a female in age group i could give per unit time, $q(x)$ is the function which implies the restriction of resources, so assumption (A1) reflects the ecological consideration that the reproduction is linear in x only for small densities and decreases as a consequence of intra specific competition. For example, one well known birth function which takes the form in assumption (A1) is the Ricker function $b(x) = pxe^{-qx}$.

With this assumption, we have the following theorem on local stability of the zero equilibrium.

Theorem 1. *Under assumptions (A1), if* $\sum_{i=k}^{l} p_i q(0)\frac{e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}(1-e^{-\mu_i\tau_i})}{\mu_i} > 1$, *then the zero equilibrium is unstable; if* $\sum_{i=k}^{l} p_i q(0)\frac{e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}(1-e^{-\mu_i\tau_i})}{\mu_i} < 1$, *the zero equilibrium is stable.*

Proof. Let $G(\lambda) := \sum_{i=k}^{l} b_i'(0)g_i(\lambda)$, where $g_i(\lambda) = \frac{(1-e^{-(\lambda+\mu_i)\tau_i})e^{-\lambda\sum_{j=1}^{i-1}\tau_j-\sum_{j=1}^{i-1}\mu_j\tau_j}}{\mu_i+\lambda}$. Then the characteristic equation (2.2) can be written as $G(\lambda) = 1$. Calculating the derivative gives $g_i'(\lambda) < 0$ on $(-\mu_i, \infty)$. Furthermore, $g_i(\lambda) \to 0$ as $\lambda \to \infty$ and $g_i(\lambda) \to +\infty$ as $\lambda \to -\mu_i$. From assumption (A1), we have $b_i'(0) = p_i q(0) > 0$. Therefore, $G(\lambda)$ is monotone decreasing on $(-\mu, \infty)$ where $\mu = \min\{\mu_i, i = k, k+1, ..., l\}$. Moreover, $G(\lambda) \to 0$ as $\lambda \to \infty$ and $G(\lambda) \to +\infty$ as $\lambda \to -\mu$.

If $G(0) = \sum_{i=k}^{l} p_i q(0)\frac{e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}(1-e^{-\mu_i\tau_i})}{\mu_i} > 1$, $G(\lambda) > 1$, since $G(\lambda)$ is monotone decreasing on $(-\mu, \infty)$ and $\lim_{\lambda\to\infty} G(\lambda) = 0$, the characteristic equation has a positive real root. So the zero equilibrium is unstable. If $G(0) = \sum_{i=k}^{l} p_i q(0)\frac{e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}(1-e^{-\mu_i\tau_i})}{\mu_i} < 1$, $G(\lambda) < 1$, since $G(\lambda)$ is monotone decreasing on $(-\mu, \infty)$, $\lim_{\lambda\to\infty} G(\lambda) = 0$ and $\lim_{\lambda\to-\mu} G(\lambda) = +\infty$, the characteristic equation has a positive real root. So the zero equilibrium is stable. □

Suppose that there is a positive equilibrium $(N_1^*, N_2^*, ..., N_m^*)$, then we have

$$\begin{aligned}
&\sum_{i=k}^{l} b_i(N_i^*) - \sum_{i=k}^{l} b_i(N_i^*)e^{-\mu_1\tau_1} - \mu_1 N_1^* = 0\\
&\sum_{i=k}^{l} b_i(N_i^*)e^{-\sum_{j=1}^{h-1}\mu_j\tau_j} - \sum_{i=k}^{l} b_i(N_i^*)e^{-\sum_{j=1}^{h}\mu_j\tau_j} - \mu_h N_h^* = 0 \text{ for } 1 < h < m\\
&\sum_{i=k}^{l} b_i(N_i^*)e^{-\sum_{j=1}^{m-1}\mu_j\tau_j} - \mu_m N_m^* = 0
\end{aligned} \tag{2.3}$$

From Eq (2.3) we derive

$$\begin{aligned} N_1^* &= \frac{1}{\mu_1}\sum_{i=k}^{l} b_i(N_i^*)(1-e^{-\mu_1\tau_1}) \\ N_h^* &= \frac{1}{\mu_h}\sum_{i=k}^{l} b_i(N_i^*)e^{-\sum_{j=1}^{h-1}\mu_j\tau_j}(1-e^{-\mu_h\tau_h}) \text{ for } 1<h<m \\ N_m^* &= \frac{1}{\mu_m}\sum_{i=k}^{l} b_i(N_i^*)e^{-\sum_{j=1}^{m-1}\mu_j\tau_j} \end{aligned} \tag{2.4}$$

The conditions for existence of this positive equilibrium is given in the following theorem.

Theorem 2. *Under assumption (A1) with* $\lim_{x\to\infty} q(x) = 0$*, the positive equilibrium* $(N_1^*, N_2^*, ..., N_m^*)$ *exists and is unique if* $\sum_{i=k}^{l} p_i q(0)\frac{e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}(1-e^{-\mu_i\tau_i})}{\mu_i} > 1$.

Proof. From Eq (2.4),

$$N_h^* = \frac{\mu_1 e^{-\sum_{j=1}^{h-1}\mu_j\tau_j}(1-e^{-\mu_h\tau_h})}{\mu_h(1-e^{-\mu_1\tau_1})}N_1^* \tag{2.5}$$

for $k \le h \le l$. Equation (2.5) and the first equation in (2.4) imply that

$$\frac{\mu_1 N_1^*}{1-e^{-\mu_1\tau_1}} = \sum_{i=k}^{l} b_i\Big(\frac{\mu_1 e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}(1-e^{-\mu_i\tau_i})}{\mu_i(1-e^{-\mu_1\tau_1})}N_1^*\Big) \tag{2.6}$$

From assumption (A1), $b_i(x) = p_i x q(x)$, so Eq (2.6) becomes

$$1 = \sum_{i=k}^{l} p_i \frac{e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}(1-e^{-\mu_i\tau_i})}{\mu_i} q\Big(\frac{\mu_1 e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}(1-e^{-\mu_i\tau_i})}{\mu_i(1-e^{-\mu_1\tau_1})}N_1^*\Big) \tag{2.7}$$

So the positive equilibrium exists if there exists a positive N_1^* such that Eq (2.7) holds. Now let $G(x) = \sum_{i=k}^{l} p_i \frac{e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}(1-e^{-\mu_i\tau_i})}{\mu_i} q(\frac{\mu_1 e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}(1-e^{-\mu_i\tau_i})}{\mu_i(1-e^{-\mu_1\tau_1})}x)$. Since $q(x)$ is monotone decreasing with respect to x, $G(x)$ is monotone decreasing function. Furthermore, $\lim_{x\to\infty} G(x) = 0$. So $G(x) = 1$ has a unique positive solution if and only if $G(0) > 1$,i.e., $\sum_{i=k}^{l} p_i q(0)\frac{e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}(1-e^{-\mu_i\tau_i})}{\mu_i} > 1$. □

Note that Theorems 1 and 2 imply that the positive equilibrium exists and is unique if and only if the zero equilibrium is unstable.

For the next, we study stability of this positive equilibrium.

We denote by C_m^+ the non-negative cone of the Banach space of continuous functions $C_m = \{\varphi = (\varphi_1, \varphi_2, ..., \varphi_m) : [-r, 0] \to \mathbb{R}^m continuous\}$, where $r = \max\{\tau_1, \tau_2, ..., \tau_m\}$, i.e. $C_m^+ = \{\varphi \in C_m : \varphi_i(\theta) \ge 0 \quad for \quad \theta \in [-r, 0], i = 0, 1, 2, ..., m\}$. By using the method of steps, it can be shown that for each $\varphi \in C_m^+$, there is a unique solution of (1.2) $\pi(\varphi, t) = (N_1(\varphi, t), N_2(\varphi, t), ..., N_m(\varphi, t)) \in \mathbb{R}_m^+$ through φ that is well defined and satisfies $\pi(\varphi; .)|_{[-r,0]} = \varphi$.

In fact, by taking integral and making substitutions, system (1.2) can be written as

$$\begin{aligned}
N_1(t) &= \int_0^{\tau_1} e^{-\mu_1\theta} \sum_{i=k}^{l} b_i(N_i(t-\theta))d\theta \\
N_h(t) &= \int_{\sum_{j=1}^{h-1}\tau_j}^{\sum_{j=1}^{h}\tau_j} \sum_{i=k}^{l} b_i(N_i(t-\theta))e^{-\sum_{j=1}^{h-1}\mu_j\tau_j-\mu_h(\theta-\sum_{j=1}^{h-1}\tau_j)}d\theta \qquad 1<h<m \\
N_m(t) &= \int_{\sum_{j=1}^{m-1}\tau_j}^{\infty} \sum_{i=k}^{l} b_i(N_i(t-\theta))e^{-\sum_{j=1}^{m-1}\mu_j\tau_j-\mu_m(\theta-\sum_{j=1}^{m-1}\tau_j)}d\theta
\end{aligned} \tag{2.8}$$

In what follows, we give a preliminary result, then we give a theorem on global stability of the positive equilibrium.

Lemma 1. *Under assumption (A1), if the birth functions* $b_i(x)$ *are bounded for* $i = k, k+1, ..., l$, *for every* $\varphi \in C_m^+$ *with* $\varphi_i(0) > 0$, $i = 1, 2, ..., m$, *the solution* $\pi(\varphi; t)$ *of* (1.2) *is bounded above for* $t > 0$.

Proof. Let $N(t) = \sum_{i=1}^{m} N_i(t)$. By adding up the m equations in Eq (1.2), we obtain

$$\begin{aligned}
\frac{dN}{dt} &= \sum_{i=k}^{l} b_i(N_i(t)) - \sum_{i=1}^{m} \mu_i N_i(t) \\
&\leq \sum_{i=k}^{l} b_i(N_i(t)) - \mu N(t)
\end{aligned}$$

where μ is the smallest death rate in the m age groups, i.e., $\mu = \min\{\mu_i, i = 1, 2, ..., m\}$. Since the birth functions $b_i(x)$ are bounded for $i = k, k+1, ..., l$, there are M_i for $i = 1, 2, ..., m$ such that $b_i(N_i(t)) \leq M_i$. Let $M = \sum_{i=k}^{l} M_i$, then $\frac{dN}{dt} \leq M - \mu N(t)$, which means that $\frac{dN}{dt} < 0$ when $N > \frac{M}{\mu}$. So N is bounded, i.e., there is $\bar{N}$ such that $N(t) \leq \bar{N}$ for $t \geq 0$. Therefore, $N_i(t) \leq \bar{N}$ for $t \geq 0$ for $i = 1, 2, ..., m$. The solution $\pi(\varphi; t)$ is bounded for $t > 0$. □

Theorem 3. *Under assumption (A1) with* $\lim_{x\to\infty} q(x) = 0$, *if* $\sum_{i=k}^{l} p_i q(0) \frac{e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}(1-e^{-\mu_i\tau_i})}{\mu_i} > 1$ *and* $\sum_{i=k}^{l} \frac{|b_i'(N_i^*)| e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}(1+e^{-\mu_i\tau_i})}{\mu_i} < 1$, *then the positive equilibrium* $(N_1^*, N_2^*, ..., N_m^*)$ *is locally stable.*

Proof. The linearized equations at the endemic equilirbium $(N_1^*, N_2^*, ..., N_m^*)$ of system (1.2) is given by

$$\begin{aligned}
\frac{dN_1(t)}{dt} &= \sum_{i=k}^{l} b_i'(N_i^*)N_i(t) - \sum_{i=k}^{l} b_i'(N_i^*)N_i(t-\tau_1)e^{-\mu_1\tau_1} - \mu_1 N_1(t) \\
\frac{dN_h(t)}{dt} &= \sum_{i=k}^{l} b_i'(N_i^*)N_i(t-\sum_{j=1}^{h-1}\tau_j)e^{-\sum_{j=1}^{h-1}\tau_j} - \sum_{i=k}^{l} b_i'(N_i^*)N_i(t-\sum_{j=1}^{h}\tau_j)e^{-\sum_{j=1}^{h}\tau_j} - \mu_h N_h(t) \text{ for } 1<h<m \\
\frac{dN_m(t)}{dt} &= \sum_{i=k}^{l} b_i'(N_i^*)N_i(t-\sum_{j=1}^{m-1}\tau_j)e^{-\sum_{j=1}^{m-1}\mu_j\tau_j} - \mu_m N_m(t)
\end{aligned} \tag{2.9}$$

Let λ be eigenvalue of the linear system (2.9). By calculation, the characteristic equation is given by

$$\sum_{i=k}^{l} b_i'(N_i^*)\frac{(1-e^{-(\lambda+\mu_i)\tau_i})e^{-\lambda\sum_{j=1}^{i-1}\tau_j-\sum_{j=1}^{i-1}\mu_j\tau_j}}{\mu_i+\lambda}=1 \tag{2.10}$$

Suppose that the characteristic equation (2.10) has an eigenvalue with non-negative real part, i.e., there exits $\lambda = x + iy$ such that $x \geq 0$, then

$$\begin{aligned}
&\left|b_i'(N_i^*)\frac{(1-e^{-(\lambda+\mu_i)\tau_i})e^{-\lambda\sum_{j=1}^{i-1}\tau_j-\sum_{j=1}^{i-1}\mu_j\tau_j}}{\mu_i+\lambda}\right|\\
=&\left|b_i'(N_i^*)\frac{(1-e^{-(x+iy+\mu_i)\tau_i})e^{-(x+iy)\sum_{j=1}^{i-1}\tau_j-\sum_{j=1}^{i-1}\mu_j\tau_j}}{\mu_i+x+iy}\right|\\
\leq&\left|b_i'(N_i^*)\right|\frac{\left|1-e^{-(x+iy+\mu_i)\tau_i}\right|\left|e^{-(x+iy)\sum_{j=1}^{i-1}\tau_j-\sum_{j=1}^{i-1}\mu_j\tau_j}\right|}{|\mu_i+x+iy|}\\
\leq&\left|b_i'(N_i^*)\right|\frac{\left|1-e^{-(x+\mu_i)\tau_i}(\cos y\tau_i-i\sin y\tau_i)\right|\left|e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}\right|}{|\mu_i+x+iy|}\\
=&\left|b_i'(N_i^*)\right|\frac{\sqrt{(1-e^{-(x+\mu_i)\tau_i}\cos y\tau_i)^2+(e^{-(x+\mu_i)\tau_i}\sin y\tau_i)^2}\left|e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}\right|}{\sqrt{(x+\mu_i)^2+y^2}}\\
\leq&\left|b_i'(N_i^*)\right|\frac{\sqrt{1+e^{-2(x+\mu_i)}-2e^{-(x+\mu_i)\tau_i}\cos y\tau_i}\left|e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}\right|}{\mu_i}\\
\leq&\left|b_i'(N_i^*)\right|\frac{\sqrt{1+e^{-2(x+\mu_i)}+2e^{-(x+\mu_i)\tau_i}}\left|e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}\right|}{\mu_i}\\
=&\left|b_i'(N_i^*)\right|\frac{(1+e^{-(x+\mu_i)\tau_i})e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}}{\mu_i}\\
\leq&\left|b_i'(N_i^*)\right|\frac{(1+e^{-\mu_i\tau_i})e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}}{\mu_i}
\end{aligned} \tag{2.11}$$

Therefore, Eq (2.10) and inequality (2.11) indicate that

$$\begin{aligned}
1=&\left|\sum_{i=k}^{l} b_i'(N_i^*)\frac{(1-e^{-(\lambda+\mu_i)\tau_i})e^{-\lambda\sum_{j=1}^{i-1}\tau_j-\sum_{j=1}^{i-1}\mu_j\tau_j}}{\mu_i+\lambda}\right|\\
\leq&\sum_{i=k}^{l}\left|b_i'(N_i^*)\frac{(1-e^{-(\lambda+\mu_i)\tau_i})e^{-\lambda\sum_{j=1}^{i-1}\tau_j-\sum_{j=1}^{i-1}\mu_j\tau_j}}{\mu_i+\lambda}\right|\\
\leq&\sum_{i=k}^{l}\left|b_i'(N_i^*)\right|\frac{(1+e^{-\mu_i\tau_i})e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}}{\mu_i}
\end{aligned} \tag{2.12}$$

which contradicts with the assumption that $\sum_{i=k}^{l}\frac{|b_i'(N_i^*)|e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}(1+e^{-\mu_i\tau_i})}{\mu_i} < 1$. So the characteristic equation (2.10) has no eigenvalue with non-negative real part, the positive equilibrium $(N_1^*, N_2^*, ..., N_m^*)$ is locally stable.

□

Theorem 4. *Under assumption (A1) with* $\lim_{x\to\infty} q(x) = 0$ *and the birth functions* $b_i(x)$ *bounded for* $i = k, k+1, ..., l$, *assume that* $\mu_h = \mu$ *and* $\tau_h = \tau$ *for some* $\mu > 0$, $\tau > 0$ *and all* $k \le h \le l$. *If* $\sum_{i=k}^{l} \frac{|b_i'(N_i^*)| e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}(1+e^{-\mu_i\tau_i})}{\mu_i} < 1$, *then the positive equilibrium* $(N_1^*, N_2^*, ..., N_m^*)$ *is globally stable. i.e.,* $\lim_{t\to\infty} \pi(\varphi; t) = (N_1^*, N_2^*, ..., N_m^*)$ *for* $\varphi \in C_m^+$ *with* $\varphi_i(0) > 0$.

Proof. Let $\{N_i(t)\}$ be a solution of Eq (1.2). Since it's bounded, we can define

$$\delta_i = \liminf_{t\to\infty} N_i(t), \gamma_i = \limsup_{t\to\infty} N_i(t)$$

Let h be such that $k \le h \le l$, i.e. N_h is a productive group. There exists a sequence $\{t_n\}$ and a sequence $\{s_n\}$ such that $\lim_{n\to\infty} N_h(t_n) = \gamma_h$ and $\lim_{n\to\infty} N_h(s_n) = \delta_h$. So there exists some $\epsilon > 0$ such that $\delta_i - \epsilon < N_i(t_n) < \gamma_i + \epsilon$ and $\delta_i - \epsilon < N_i(s_n) < \gamma_i + \epsilon$ for n large enough for all $k \le i \le l$.

From the integrated equation (2.8),

$$N_h(t_n) < \int_{\sum_{j=1}^{h-1}\tau_j}^{\sum_{j=1}^{h}\tau_j} \sum_{i=k}^{l} p_i(\gamma_i + \epsilon)q(\delta_i - \epsilon)e^{-\sum_{j=1}^{h-1}\mu_j\tau_j - \mu_h(\theta - \sum_{j=1}^{h-1}\tau_j)}d\theta$$

Let $n \to \infty$ and $\epsilon \to 0$, the inequality becomes

$$\begin{aligned}\gamma_h &\le \int_{\sum_{j=1}^{h-1}\tau_j}^{\sum_{j=1}^{h}\tau_j} \sum_{i=k}^{l} p_i\gamma_i q(\delta_i)e^{-\sum_{j=1}^{h-1}\mu_j\tau_j - \mu_h(\theta - \sum_{j=1}^{h-1}\tau_j)}d\theta \\ &= \sum_{i=k}^{l} p_i\gamma_i q(\delta_i)e^{-\sum_{j=1}^{h-1}\mu_j\tau_j + \mu_h\sum_{j=1}^{h-1}\tau_j} \int_{\sum_{j=1}^{h-1}\tau_j}^{\sum_{j=1}^{h}\tau_j} e^{-\mu_h\theta}d\theta \\ &= \sum_{i=k}^{l} p_i\gamma_i q(\delta_i)\frac{(1 - e^{-\mu_h\tau_h})e^{-\sum_{j=1}^{h-1}\mu_j\tau_j}}{\mu_h}q(\delta_i)\end{aligned} \tag{2.13}$$

Now let $A := \sum_{i=k}^{l} p_i\gamma_i q(\delta_i)$, Eq (2.13) implies that

$$A = \sum_{i=k}^{l} p_i\gamma_i q(\delta_i) \le \sum_{i=k}^{l} p_i A\frac{(1 - e^{-\mu_i\tau_i})e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}}{\mu_i}q(\delta_i)$$

which further implies that

$$\sum_{i=k}^{l} p_i q(\delta_i)\frac{(1 - e^{-\mu_i\tau_i})e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}}{\mu_i} \ge 1 \tag{2.14}$$

Let $B := \sum_{i=k}^{l} p_i N_i^* q(N_i^*)$, from Eq (2.4)

$$B = \sum_{i=k}^{l} p_i N_i^* q(N_i^*) = \sum_{i=k}^{l} p_i B\frac{(1 - e^{-\mu_i\tau_i})e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}}{\mu_i}q(N_i^*)$$

so

$$\sum_{i=k}^{l} p_i\frac{(1 - e^{-\mu_i\tau_i})e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}}{\mu_i}q(N_i^*) = 1 \tag{2.15}$$

From assumption (A1), $q(x)$ is monotone decreasing, then Eqs (2.14) and (2.15) imply that there exists h_1 such that $k \le h_1 \le l$ and $\delta_{h_1} \le N^*_{h_1}$. From the integrated equation (2.8),

$$N_h(s_n) > \int_{\sum_{j=1}^{h-1}\tau_j}^{\sum_{j=1}^{h}\tau_j} \sum_{i=k}^{l} p_i(\delta_i - \epsilon)q(\gamma_i + \epsilon)e^{-\sum_{j=1}^{h-1}\mu_j\tau_j - \mu_h(\theta - \sum_{j=1}^{h-1}\tau_j)}d\theta$$

Following similar calculation as inequalities (2.13) and (2.14), we obtain

$$\sum_{i=k}^{l} p_i q(\gamma_i)\frac{(1 - e^{-\mu_i\tau_i})e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}}{\mu_i} \le 1. \tag{2.16}$$

From assumption (A1), $q(x)$ is monotone decreasing, then Eqs (2.15) and (2.16) imply that there exists h_2 such that $k \le h_2 \le l$ and $\gamma_{h_2} \ge N^*_{h_2}$.

By substituting variables, from Eqs (2.4) and (2.8) we obtain

$$(N_k(t) - N_k^*)e^{\sum_{j=1}^{k-1}\mu_j\tau_j} = \int_{\sum_{j=1}^{k-1}\tau_j}^{\sum_{j=1}^{k}\tau_j} \sum_{i=k}^{l}(b_i(N_i(t-\theta)) - b_i(N_i^*))e^{-\mu_k(\theta - \sum_{j=1}^{k-1}\tau_j)}d\theta \tag{2.17}$$

and

$$(N_h(t) - N_h^*)e^{\sum_{j=1}^{h-1}\mu_j\tau_j} = \int_{\sum_{j=1}^{k-1}\tau_j}^{\sum_{j=1}^{k-1}\tau_j + \tau_h} \sum_{i=k}^{l}(b_i(N_i(t-\theta)) - b_i(N_i^*))e^{-\mu_h(\theta - \sum_{j=1}^{k-1}\tau_j)}d\theta \tag{2.18}$$

Since $\mu_h = \mu$ and $\tau_h = \tau$ for $k \le h \le l$, Eqs (2.17) and (2.18) imply that

$$(N_k(t) - N_k^*)e^{\sum_{j=1}^{k-1}\mu_j\tau_j} = (N_h(t) - N_h^*)e^{\sum_{j=1}^{h-1}\mu_j\tau_j} \tag{2.19}$$

for $k \le h \le l$.

Scenario 1: $\gamma_k = N_k^*$

In this scenario, since the positive equilibrium is locally stable, $\delta_k = \gamma_k = N_k^*$, and $\lim_{t\to\infty} N_k(t) = N_k^*$. From Eq (2.19), $\lim_{t\to\infty} N_h(t) = N_h^*$ for all $k \le h \le l$.

Scenario 2: $\gamma_k > N_k^*$

In this scenario, since the positive equilibrium is locally stable, $\delta_k > N_k^*$, then $N_k(t) > N_k^*$ for t large enough. From Eq (2.19), $N_{h_1}(t) > N^*_{h_1}$ for t large enough, which means that $\delta_{h_1} \ge N^*_{h_1}$. Since we have $\delta_{h_1} \le N^*_{h_1}$ from previous discussion, $\delta_{h_1} = N^*_{h_1}$. Therefore, $\delta_{h_1} = \gamma_{h_1} = N^*_{h_1}$, i.e., $\lim_{t\to\infty} N_{h_1}(t) = N^*_{h_1}$. From Eq (2.19), $\lim_{t\to\infty} N_h(t) = N_h^*$ for all $k \le h \le l$.

Scenario 3: $\gamma_k < N_k^*$

Then $\delta_k < N_k^*$ and $N_k(t) < N_k^*$ for t large enough. From Eq (2.19), $N_{h_2}(t) < N^*_{h_2}$ for t large enough, which implies that $\gamma_{h_2} \le N^*_{h_2}$. Since we have $\gamma_{h_2} \ge N^*_{h_2}$ from previous discussion, $\gamma_{h_2} = N^*_{h_2}$. i.e., $\lim_{t\to\infty} N_{h_2}(t) = N^*_{h_2}$. From Eq (2.19), $\lim_{t\to\infty} N_h(t) = N_h^*$ for all $k \le h \le l$.

From discussion above, $\lim_{t\to\infty} N_h(t) = N_h^*$ for all $k \le h \le l$. Then by the integral equation (2.8), $\lim_{t\to\infty} N_h(t) = N_h^*$ for all $1 \le h \le m$. □

3. The epidemic model

Now we focus on the epidemic model, which is an ODE system given by

$$\frac{dI_h(t)}{dt} = \sum_{i=1}^{m} \beta_{hi}(N_h(t) - I_h(t))I_i(t) - \sigma I_h(t) - \mu_h I_h(t) \text{ for } 1 \le h \le m \tag{3.1}$$

Note that Eq (3.1) is derived from the last equation in (1.1) by replacing $S_h(t)$ by $N_h(t) - I_h(t)$. Suppose that the population has reached the positive equilibrium, then system (3.1) is given by

$$\frac{dI_h(t)}{dt} = \sum_{i=1}^{m} \beta_{hi}(N_h^* - I_h(t))I_i(t) - \sigma I_h(t) - \mu_h I_h(t) \text{ for } 1 \le h \le m \tag{3.2}$$

Let $\mathbf{I} = (I_1, I_2, ..., I_m)^T$, the flow of the solution of system (3.2) with initial value $\mathbf{I^0} = (I_1^0, I_2^0, ..., I_m^0)$ is given by $\phi_t(\mathbf{I^0})$. Let $V = (0, N_1^*) \times (0, N_2^*) \times ... \times (0, N_m^*)$. We have the following conclusion

Theorem 5. *If* $\mathbf{I}_i^0 \in (0, N_i^*)$*, then* $\phi_{ti}(\mathbf{I^0}) \in (0, N_i^*)$*, i.e., V is invariant under the flow* ϕ_t*.*

Proof. Suppose that there is a smallest t_0 such that there is $j \in \{1, 2, ..., m\}$ such that $I_j(t_0) = 0$. Since $\mathbf{I}(t)$ is not constant 0, we have $\frac{dI_j(t)}{dt}|_{t=t_0} = \sum_{i=1}^{m} \beta_{hi}(N_h^* - I_h(t_0))I_i(t_0) > 0$. On the other hand, since t_0 is the smallest s.t. $I_j(t_0) = 0$, $\frac{dI_j(t)}{dt}|_{t=t_0} = \lim_{\epsilon\to 0} \frac{I_j(t_0-\epsilon)-I_j(t_0)}{-\epsilon} \le 0$, which is a contradiction. So $I_j(t) > 0$ for all $1 \le j \le m$ and all $t > 0$ with initial value in V.

Similarly, suppose that there is a smallest t_0 such that there is $j \in \{1, 2, ..., m\}$ such that $I_j(t_0) = N_j^*$, then from Eq (3.2), $\frac{dI_j}{dt}|_{t=t_0} = -\sigma I_h(t_0) - \mu_h I_h(t_0) < 0$. On the other hand, since t_0 is the smallest s.t.$I_j(t_0) = N_j^*$, $\frac{dI_j(t)}{dt}|_{t=t_0} = \lim_{\epsilon\to 0} \frac{I_j(t_0-\epsilon)-I_j(t_0)}{-\epsilon} \ge 0$, which is a contradiction. So $I_j(t) < 0$ for all $1 \le j \le m$ and all $t > 0$ with initial value in V. □

3.1. Disease-free equilibrium of the epidemic model

Equation (3.2) has $(0, 0, ..., 0)$ as the disease-free equilibrium. Linearization around this equilibrium gives the following linear system

$$\frac{dI_h(t)}{dt} = \sum_{i=1}^{m} \beta_{hi} N_h^* I_i(t) - \sigma I_h(t) - \mu_h I_h(t) \tag{3.3}$$

for $1 \le h \le m$ Following the method in [16], we get

$$F = \begin{bmatrix} \beta_{11}N_1^* & \beta_{12}N_1^* & \cdots & \beta_{1m}N_1^* \\ \beta_{21}N_2^* & \beta_{22}N_2^* & \cdots & \beta_{2m}N_2^* \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m1}N_m^* & \beta_{m2}N_m^* & \cdots & \beta_{mm}N_m^* \end{bmatrix}$$

and

$$V = \begin{bmatrix} \sigma+\mu_1 & 0 & \cdots & 0 \\ 0 & \sigma+\mu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma+\mu_m \end{bmatrix}$$

Then we have

$$FV^{-1} = \begin{bmatrix} \frac{\beta_{11}N_1^*}{\sigma+\mu_1} & \frac{\beta_{12}N_1^*}{\sigma+\mu_2} & \cdots & \frac{\beta_{1m}N_1^*}{\sigma+\mu_m} \\ \frac{\beta_{21}N_2^*}{\sigma+\mu_1} & \frac{\beta_{22}N_2^*}{\sigma+\mu_2} & \cdots & \frac{\beta_{2m}N_2^*}{\sigma+\mu_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\beta_{m1}N_m^*}{\sigma+\mu_1} & \frac{\beta_{m2}N_m^*}{\sigma+\mu_2} & \cdots & \frac{\beta_{mm}N_m^*}{\sigma+\mu_m} \end{bmatrix}$$

In particular, if we assume that $\beta_{ij} = \alpha_i\lambda_j$, it can be calculated from induction that the characteristic equation of FV^{-1} is given by $\lambda^{m-1}(\lambda - \sum_{i=1}^m \frac{\alpha_i\lambda_i N_i^*}{\sigma+\mu_i}) = 0$, then

$$R_0 = \rho(FV^{-1}) = \sum_{i=1}^m \frac{\alpha_i\lambda_i N_i^*}{\sigma + \mu_i}$$

The following theorem follows

Theorem 6. *Assume that $\beta_{ij} = \alpha_i\lambda_j$. Then when $\sum_{i=1}^m \frac{\alpha_i\lambda_i N_i^*}{\sigma+\mu_i} < 1$, the disease-free equilibrium of system (3.2) is stable; when $\sum_{i=1}^m \frac{\alpha_i\lambda_i N_i^*}{\sigma+\mu_i} > 1$, the disease-free equilibrium of system (3.2) is unstable.*

For the assumption $\beta_{ij} = \alpha_i\lambda_j$, if we assume that the population is homogeneously mixed, α_i can be interpreted as susceptibility of age group i and λ_j can be interpreted as infectivity of age group j.

3.2. Endemic equilibrium of the epidemic model

Suppose system (3.2) has a nontrivial equilibrium $(I_1^*, I_2^*, ..., I_m^*)$, by plugging in the Eq (3.2) we derive

$$\sum_{i=1}^m \beta_{hi}(N_h^* - I_h^*)I_i^* - \sigma I_h^* - \mu_h I_h^* = 0 \tag{3.4}$$

for $1 \le h \le m$. In particular, if we assume that $\beta_{ij} = \alpha_i\lambda_j$, then Eq (3.4) can be written as

$$(N_h^* - I_h^*)\sum_{i=1}^m \alpha_h\lambda_i I_i^* - \sigma I_h^* - \mu_h I_h^* = 0 \tag{3.5}$$

for $1 \le h \le m$. From Eq (3.5) we get

$$\sum_{i=1}^m \lambda_i I_i^* = \frac{(\sigma + \mu_h)I_h^*}{\alpha_h(N_h^* - I_h^*)} \tag{3.6}$$

for $1 \le h \le m$. and

$$\frac{\sigma + \mu_h}{\alpha_h(N_h^*/I_h^* - 1)} = \frac{\sigma + \mu_1}{\alpha_1(N_1^*/I_1^* - 1)}$$

Now let $M_i = \frac{N_i^*}{I_i^*} - 1$ and $l_i = \frac{\sigma+\mu_i}{\alpha_i}$, then

$$\frac{l_i}{M_i} = \frac{l_1}{M_1}$$

Plugging $I_i^* = \frac{N_i^*}{M_i+1}$ into Eq (3.6) with $h = 1$, we have

$$\sum_{i=1}^m \frac{\lambda_i N_i^*}{M_i + 1} = \frac{l_1}{M_1}$$

It follows that

$$\sum_{i=1}^{m} \frac{\lambda_i N_i^*}{l_i + l_1/M_1} = 1$$

Now define $G : (0, N_1^*) \longrightarrow \mathbb{R}$ by

$$G(x) = \sum_{i=1}^{m} \frac{\lambda_i N_i^*}{l_i + l_1/M} \tag{3.7}$$

where $M = \frac{N_1^*}{x} - 1$. Then I_1^* is a solution of $G(x) = 1$.

Note that $G(x)$ is monotone non-increasing with respect to x, $G(x) \longrightarrow 0$ as $x \longrightarrow N_1^*$ and $G(x) \longrightarrow \sum_{i=1}^{m} \frac{\lambda_i N_i^*}{l_i}$ as $x \longrightarrow 0$. So $G(x) = 1$ has a solution in $(0, N_1^*)$ if and only if $\sum_{i=1}^{m} \frac{\lambda_i N_i^*}{l_i} > 1$, and the solution is unique by monotonicity of $G(x)$.

Note that $R_0 = \sum_{i=1}^{m} \frac{\lambda_i N_i^*}{l_i}$. We conclude that

Theorem 7. *Assume that $\beta_{ij} = \alpha_i \lambda_j$. The endemic equilibrium of system (3.2) exists and is unique if and only if $R_0 = \sum_{i=1}^{m} \frac{\lambda_i N_i^*}{l_i} > 1$.*

Now we state the following theorem on global stability of the endemic equilibrium.

Theorem 8. *Assume that $\beta_{ij} = \alpha_i \lambda_j$.Let $V = (0, N_1^*) \times (0, N_2^*) \times ... \times (0, N_m^*)$. If $R_0 > 1$, then the endemic equilibrium $\{I_i^*\}$ of system* (3.2) *attracts all the forward orbits going through V.*

Proof. Let $F : V \longrightarrow \mathbb{R}^m$ be defined by $F_h(I_1, I_2, ..., I_m) = \sum_{i=1}^{m} \beta_{hi}(N_h^* - I_h)I_i - \sigma I_h - \mu_h I_h$.

It suffices to prove that

(H1) System (3.2) is cooperative, i.e., $\frac{\partial F_h}{\partial I_j} \geq 0$ for $h \neq j$.

(H2) F is irreducible in the sense that the matrix $[\frac{\partial F_h}{\partial I_j}]$ is irreducible.

(H3) Solutions of Eq (3.2) with initial value $(I_1^0, I_2^0, ..., I_m^0)$ such that $\left|I_h^0\right| \leq N_h^*$ are bounded.

Then by Theorems 1.5 and 2.4 in [17], (H1) and (H2) imply that system (3.2) doesn't have a non-constant periodic solution. By Theorem 1.1 in [17], (H1) and (H2) also imply that the solution flows of Eq (3.2) going through V have positive derivatives. Then by Theorem 4.1 in [17] and (H3) we conclude that almost all forward orbits of V converge to the endemic equilibrium $\{I_i^*\}$.

(H1) $\frac{\partial F_h}{\partial I_j} = \beta_{hj}(N_h^* - I_h) > 0$ for $h \neq j$.

(H2) $\frac{\partial F_h}{\partial I_h} = \beta_{hh}(N_h^* - I_h) - \sum_{i=1}^{m} \beta_{hi} I_i - \sigma - \mu_h$. Now let $A = [\frac{\partial F_h}{\partial I_j}]$. $A_{hj} = \frac{\partial F_h}{\partial I_j} > 0$ for $h \neq j$ by (H1). Suppose there is $1 \leq h \leq m$ such that $A_{hh} \leq 0$, $A_{hh}^2 = \sum_{i=1}^{m} A_{hi} A_{ih} > 0$. Therefore, for each pair of indices h and j, there exists a natural number n such that A_{hj}^n is positive, which implies that the matrix A is irreducible.

(H3) It can be derived directly from Theorem 5. □

Now if we look back on the original epidemic model (3.1), we have the following Theorem from Theorem 8.

Theorem 9. *Under assumption (A1) with* $\lim_{x\to\infty} q(x) = 0$, *assume that $\beta_{ij} = \alpha_i \lambda_j$ and $\mu_h = \mu$, $\tau_h = \tau$ for some $\mu > 0$, $\tau > 0$ and all $k \leq h \leq l$.Let $V = (0, N_1^*) \times (0, N_2^*) \times ... \times (0, N_m^*)$. If $\sum_{i=k}^{l} \frac{\left|b_i'(N_i^*)\right| e^{-\sum_{j=1}^{i-1} \mu_j \tau_j} (1 + e^{-\mu_i \tau_i})}{\mu_i} < 1$ and $R_0 > 1$, then $\{I_i^*\}$ attracts all the forward orbits going through V in system* (3.1).

Proof. From Theorem 4, $\lim_{t\to\infty} N_h(t) = N_h^*$ for all $1 \leq h \leq m$. Denote $\Phi(t, s, x_0)$ the solution of system (3.1) with $x(s) = x_0$, and denote $\Theta(t, x_0)$ the solution of system (3.2) with $y(0) = x_0$. Then by Proposition 1.1 in [18], Φ is asymptotically autonomous semiflow with limit semiflow Θ. Let $O_\Phi(s, x) = \{\Phi(t, s, x) : t \geq s\}$, $x \in V$, then $O_\Phi(s, x)$ has compact closure in V since it's bounded. Let $\omega = \omega(s, x)$ which is the ω-limit set of $O_\Phi(s, x)$. By Theorem 1.8 in [18], we conclude that ω is non-empty, compact and connected, and it attracts $\Phi(t, s, x)$. Moreover, ω is invariant for the semiflow Θ and is chain recurrent for Θ.

Now suppose that $\omega \neq \{I_i^*\}$. There exists $x = (x_1, x_2, ..., x_m) \in \omega$ such that $x \neq \{I_i^*\}$. Let $\epsilon = d(x, \{I_i^*\})$. Since ω is compact, there exists $T > 0$ such that $d(\Theta(t, x_0), \{I_i^*\}) < \frac{\epsilon}{2}$ for all $x_0 \in \omega$ and $t \geq T$. By the definition of chain recurrence, there is an $(\frac{\epsilon}{2}, T)$ chain from x to x, i.e., there is a sequence $\{x = x_1, x_2, ..., x_{n+1} = x; t_1, t_2, ..., t_n\}$ for $x_i \in \omega$ and $t_i \geq T$ such that $d(\Theta(t_i, x_i), x_{i+1}) < \frac{\epsilon}{2}$. Then $d(\Theta(t_n, x_n), x) < \frac{\epsilon}{2}$, which indicates that $d(x, \{I_i^*\}) \leq d(\Theta(t_n, x_n), x) + d(\Theta(t_n, x_n), \{I_i^*\}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, which contradicts with $\epsilon = d(x, \{I_i^*\})$. Therefore, $\omega = \{I_i^*\}$, i.e., $\{I_i^*\}$ attracts all the forward orbits going through V in system (3.1). □

4. Impact of change of productive age and birth rates on infant disease rate

In this section, we assume that the birth rate of age group i is given by $b_i(N_i) = p_i N_i e^{-qN}$, we'll analyze how do demographic distribution $\frac{N_i^*}{N}$ and infant disease rate at endemic equilibrium $\frac{I_1^*}{N_1^*}$ change with birth parameters p_i, q and productive age k.

By plugging $b_i(N_i) = p_i N_i e^{-qN}$ into Eq (2.4), we have

$$\sum_{i=k}^{l} p_i N_i^* e^{-qN} = \frac{\mu_h N_h^*}{e^{-\sum_{j=1}^{h-1}\mu_j\tau_j}(1 - e^{-\mu_h\tau_h})}$$

for $1 \leq h \leq m$.

$$\frac{N_i^*}{N_h^*} = \frac{\mu_h e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}(1 - e^{-\mu_i\tau_i})}{\mu_i e^{-\sum_{j=1}^{h-1}\mu_j\tau_j}(1 - e^{-\mu_h\tau_h})}$$

for $1 < i, h < m$. Combine the above two formulas, we have

$$\sum_{i=k}^{l} p_i \frac{\mu_h e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}(1 - e^{-\mu_i\tau_i})}{\mu_i e^{-\sum_{j=1}^{h-1}\mu_j\tau_j}(1 - e^{-\mu_h\tau_h})} N_h^* e^{-qN} = \frac{\mu_h N_h^*}{e^{-\sum_{j=1}^{h-1}\mu_j\tau_j}(1 - e^{-\mu_h\tau_h})}$$

Thus

$$\sum_{i=k}^{l} \frac{p_i e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}(1 - e^{-\mu_i\tau_i})}{\mu_i} = e^{qN}$$

Solving for N gives

$$N = \frac{1}{q} In \sum_{i=k}^{l} \frac{p_i e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}(1 - e^{-\mu_i\tau_i})}{\mu_i} \tag{4.1}$$

From Eq (2.4), we define

$$Q := \frac{\mu_1 N_1^*}{1 - e^{-\mu_1\tau_1}} = \frac{\mu_h N_h^*}{e^{-\sum_{j=1}^{h-1}\mu_j\tau_j}(1 - e^{-\mu_h\tau_h})} = \frac{\mu_m N_m^*}{e^{-\sum_{j=1}^{m-1}\mu_j\tau_j}}$$

for $1 \le h \le m$. It follows that

$$\begin{aligned} N_1^* &= \frac{1-e^{-\mu_1\tau_1}}{\mu_1}Q \\ N_h^* &= \frac{e^{-\sum_{j=1}^{h-1}\mu_j\tau_j}(1-e^{-\mu_h\tau_h})}{\mu_h}Q \quad \text{for } 1 < h < m \\ N_m^* &= \frac{e^{-\sum_{j=1}^{m-1}\mu_j\tau_j}}{\mu_m}Q \end{aligned} \tag{4.2}$$

By plugging Eq (4.2) into $N = \sum_{h=1}^m N_h^*$, we have

$$N = (\frac{1-e^{-\mu_1\tau_1}}{\mu_1} + \sum_{h=1}^{m-1}\frac{e^{-\sum_{j=1}^{h-1}\mu_j\tau_j}(1-e^{-\mu_h\tau_h})}{\mu_h} + \frac{e^{-\sum_{j=1}^{m-1}\mu_j\tau_j}}{\mu_m})Q \tag{4.3}$$

Equations (4.2) and (4.3) give

$$\begin{aligned} \frac{N_1^*}{N} &= \frac{\frac{1-e^{-\mu_1\tau_1}}{\mu_1}}{\frac{1-e^{-\mu_1\tau_1}}{\mu_1} + \sum_{h=1}^{m-1}\frac{e^{-\sum_{j=1}^{h-1}\mu_j\tau_j}(1-e^{-\mu_h\tau_h})}{\mu_h} + \frac{e^{-\sum_{j=1}^{m-1}\mu_j\tau_j}}{\mu_m}} \\ \frac{N_h^*}{N} &= \frac{\frac{e^{-\sum_{j=1}^{h-1}\mu_j\tau_j}(1-e^{-\mu_h\tau_h})}{\mu_h}}{\frac{1-e^{-\mu_1\tau_1}}{\mu_1} + \sum_{h=1}^{m-1}\frac{e^{-\sum_{j=1}^{h-1}\mu_j\tau_j}(1-e^{-\mu_h\tau_h})}{\mu_h} + \frac{e^{-\sum_{j=1}^{m-1}\mu_j\tau_j}}{\mu_m}} \quad \text{for } 1 < h < m \\ \frac{N_m^*}{N} &= \frac{\frac{e^{-\sum_{j=1}^{m-1}\mu_j\tau_j}}{\mu_m}}{\frac{1-e^{-\mu_1\tau_1}}{\mu_1} + \sum_{h=1}^{m-1}\frac{e^{-\sum_{j=1}^{h-1}\mu_j\tau_j}(1-e^{-\mu_h\tau_h})}{\mu_h} + \frac{e^{-\sum_{j=1}^{m-1}\mu_j\tau_j}}{\mu_m}} \end{aligned} \tag{4.4}$$

It's obvious from Eq (4.4) that change of p_i, q and k don't make a change on the demographic distribution $\frac{N_h^*}{N}$ for $1 \le h \le m$.

Proposition 10. *Assume that the birth rate of age group i is given by $b_i(N_i) = p_iN_ie^{-qN}$. The stablized demographic distribution N_h^*/N doesn't change with birth parameters p_i, q and reproductive age k.*

Note that change of p_i, q and k does have an impact on the total number N and N_h^*, though they don't influence the ratio N_h^*/N.

It can be seen from Eq (4.1) that N increases as p_i increases, decreases as q or k increases, so does N_h^* for all $1 \le h \le m$.

Now we study the impact of changes of p_i, q and k on the infant disease rate $\frac{I_1^*}{N_1^*}$.

In the last section, we get that when $R_0 > 1$, there is a endemic equilibrium of system 3.2 where I_1^* satisfies $\sum_{i=1}^m \frac{\lambda_i\frac{N_i^*}{N}N}{l_i+\frac{l_1}{M_1}} = 1$ where $l_i = \frac{\sigma+\mu_i}{\alpha_i}$ and $M_i = \frac{N_i^*}{I_i^*} - 1$. By taking derivative with respect to q, we get

$$\sum_{i=1}^m \frac{\lambda_i\frac{N_i^*}{N}\frac{dN}{dq}(l_i+\frac{l_1}{M_1}) + \frac{l_1}{M_1^2}\frac{dM_1}{dq}\lambda_iN_i^*}{(l_i+\frac{l_1}{M_1})^2} = 0$$

So

$$\frac{dM_1}{dq} = -\sum_{i=1}^{m} \frac{\lambda_i \frac{N_i^*}{N} \frac{dN}{dq}(l_i + \frac{l_1}{M_1})}{(l_i + \frac{l_1}{M_1})^2} \Big/ \sum_{i=1}^{m} \frac{\frac{l_1}{M_1^2}\lambda_i N_i^*}{(l_i + \frac{l_1}{M_1})^2} > 0 \tag{4.5}$$

which means that $\frac{N_1^*}{I_1^*}$ increases as q increases, so $\frac{I_1^*}{N_1^*}$ decreases as q increases.

Similarly, we have

$$\frac{dM_1}{dp_i} = -\sum_{i=1}^{m} \frac{\lambda_i \frac{N_i^*}{N} \frac{dN}{dp_i}(l_i + \frac{l_1}{M_1})}{(l_i + \frac{l_1}{M_1})^2} \Big/ \sum_{i=1}^{m} \frac{\frac{l_1}{M_1^2}\lambda_i N_i^*}{(l_i + \frac{l_1}{M_1})^2} < 0 \tag{4.6}$$

which means that $\frac{N_1^*}{I_1^*}$ decreases as p_i increases, so $\frac{I_1^*}{N_1^*}$ increases as p_i increases.

If k gets larger, N gets smaller as discussed above, $\frac{N_i^*}{N}$ doesn't change, since we have $\sum_{i=1}^{m} \frac{\lambda_i \frac{N_i^*}{N} N}{l_i + \frac{l_1}{M_1}} = 1$, $\frac{l_1}{M_1}$ decreases thus $M_1 = \frac{N_1^*}{I_1^*}$ increases. So $\frac{I_1^*}{N_1^*}$ gets smaller. By the same argument, if k gets smaller, $\frac{I_1^*}{N_1^*}$ gets larger.

In conclusion, we have

Proposition 11. *Assume that the birth rate of age group i is given by* $b_i(N_i) = p_i N_i e^{-qN}$. *With all the other parameters fixed, the infant disease rate at endemic equilibrium* $\frac{I_1^*}{N_1^*}$ *increases as birth rate* p_i *increases, and decreases as the productive age k or q increases.*

5. Impact of change of productive age and birth rates on disease distribution

In this section, we assume that $R_0 > 1$ and the birth rate of age group i is given by $b_i(N_i) = p_i N_i e^{-qN}$, we'll analyze how does disease distribution $\frac{I_i^*}{I^*}$ at endemic equilibrium change with birth parameters p_i and q.

From Eq (3.6) we define $H := \frac{\alpha_h (N_h^*/I_h^* - 1)}{\sigma + \mu_h}$ for any $1 \le h \le m$, then

$$I_h^* = \frac{\alpha_h}{(\sigma + \mu_h)H + \alpha_h} \frac{N_h^*}{N} N \tag{5.1}$$

and

$$I^* = \sum_{j=1}^{m} I_j^* = \sum_{j=1}^{m} \frac{\alpha_j}{(\sigma + \mu_j)H + \alpha_j} \frac{N_j^*}{N} N \tag{5.2}$$

Therefore,

$$\frac{I_h^*}{I^*} = \frac{\frac{\alpha_h}{(\sigma+\mu_h)H+\alpha_h} \frac{N_h^*}{N}}{\sum_{j=1}^{m} \frac{\alpha_j}{(\sigma+\mu_j)H+\alpha_j} \frac{N_j^*}{N}} \tag{5.3}$$

Let $Q_h := \sum_{j=1}^{m} \frac{\alpha_j(\sigma+\mu_j)}{((\sigma+\mu_j)H+\alpha_j)^2} \frac{N_j^*}{N} - \frac{\sigma+\mu_h}{(\sigma+\mu_h)H+\alpha_h} \sum_{j=1}^{m} \frac{\alpha_j}{(\sigma+\mu_j)H+\alpha_j} \frac{N_j^*}{N}$

Proposition 12. *Assume that* $R_0 > 1$ *and the birth rate of age group i is given by* $b_i(N_i) = p_i N_i e^{-qN}$. *With all the other parameters fixed, how* $\frac{I_h^*}{I^*}$ *changes with q,* p_i *and k depends on the sign of* Q_h: $\frac{I_h^*}{I^*}$

increases as q increases if $Q_h > 0$ *and decreases as q increases if* $Q_h < 0$*;* $\frac{I_h^*}{I^*}$ *decreases as* p_i *increases if* $Q_h > 0$ *and increases as* p_i *increases if* $Q_h < 0$*. In particular, if* $1 \leq h \leq m$ *is such that* $\frac{\alpha_h}{\sigma+\alpha_h} < \frac{\alpha_j}{\sigma+\alpha_j}$ *for all* $j \neq h$*,* $\frac{I_h^*}{I^*}$ *decreases as q increases and increases as* p_i *increases; if* $1 \leq h \leq m$ *is such that* $\frac{\alpha_h}{\sigma+\alpha_h} > \frac{\alpha_j}{\sigma+\alpha_j}$ *for all* $j \neq h$*,* $\frac{I_h^*}{I^*}$ *increases as q increases and decreases as* p_i *increases.*

Proof. From Eq (5.3) and the conclusion we get that $\frac{N_j^*}{N}$ doesn't change with q, p_i or k

$$\begin{aligned}
&d(\frac{I_h^*}{I^*})/dq\\
=&\frac{d(\frac{\alpha_h}{(\sigma+\mu_h)H+\alpha_h}\frac{N_h^*}{N})/dq\sum_{j=1}^m\frac{\alpha_j}{(\sigma+\mu_j)H+\alpha_j}\frac{N_j^*}{N} - d(\sum_{j=1}^m\frac{\alpha_j}{(\sigma+\mu_j)H+\alpha_j}\frac{N_j^*}{N})/dq\frac{\alpha_h}{(\sigma+\mu_h)H+\alpha_h}\frac{N_h^*}{N}}{(\sum_{j=1}^m\frac{\alpha_j}{(\sigma+\mu_j)H+\alpha_j}\frac{N_j^*}{N})^2}\\
=&\frac{-\frac{\alpha_h}{((\sigma+\mu_h)H+\alpha_h)^2}\frac{N_h^*}{N}(\sigma+\mu_h)\frac{dH}{dq}\sum_{j=1}^m\frac{\alpha_j}{(\sigma+\mu_j)H+\alpha_j}\frac{N_j^*}{N} + \sum_{j=1}^m\frac{\alpha_j}{((\sigma+\mu_j)H+\alpha_j)^2}\frac{N_j^*}{N}(\sigma+\mu_j)\frac{dH}{dq}\frac{\alpha_h}{(\sigma+\mu_h)H+\alpha_h}\frac{N_h^*}{N}}{(\sum_{j=1}^m\frac{\alpha_j}{(\sigma+\mu_j)H+\alpha_j}\frac{N_j^*}{N})^2}\\
=&\frac{(\sum_{j=1}^m\frac{\alpha_j(\sigma+\mu_j)}{((\sigma+\mu_j)H+\alpha_j)^2}\frac{N_j^*}{N} - \frac{\sigma+\mu_h}{(\sigma+\mu_h)H+\alpha_h}\sum_{j=1}^m\frac{\alpha_j}{(\sigma+\mu_j)H+\alpha_j}\frac{N_j^*}{N})\frac{\alpha_h}{(\sigma+\mu_h)H+\alpha_h}\frac{dH}{dq}\frac{N_h^*}{N}}{(\sum_{j=1}^m\frac{\alpha_j}{(\sigma+\mu_j)H+\alpha_j}\frac{N_j^*}{N})^2}\\
=&\frac{Q_h\frac{\alpha_h}{(\sigma+\mu_h)H+\alpha_h}\frac{dH}{dq}\frac{N_h^*}{N}}{(\sum_{j=1}^m\frac{\alpha_j}{(\sigma+\mu_j)H+\alpha_j}\frac{N_j^*}{N})^2}
\end{aligned}$$

From last section $d(\frac{N_h^*}{I_h^*})/dq > 0$, which implies that $\frac{dH}{dq} > 0$. So we have $d(\frac{I_h^*}{I^*})/dq > 0$ if $Q_h > 0$ and $d(\frac{I_h^*}{I^*})/dq < 0$ if $Q_h < 0$, which means that $\frac{I_h^*}{I^*}$ increases as q increases if $Q_h > 0$ and decreases as q increases if $Q_h < 0$. In particular, if $1 \leq h \leq m$ is such that $\frac{\alpha_h}{\sigma+\alpha_h} < \frac{\alpha_j}{\sigma+\alpha_j}$ for all $j \neq h$, then $\frac{\sigma+\mu_h}{(\sigma+\mu_h)H+\alpha_h} > \frac{\sigma+\mu_j}{(\sigma+\mu_j)H+\alpha_j}$, which implies that $Q_h < 0$ thus $d(\frac{I_h^*}{I^*})/dq < 0$, $\frac{I_h^*}{I^*}$ decreases as q increases; if $1 \leq h \leq m$ is such that $\frac{\alpha_h}{\sigma+\alpha_h} > \frac{\alpha_j}{\sigma+\alpha_j}$ for all $j \neq h$, then $\frac{\sigma+\mu_h}{(\sigma+\mu_h)H+\alpha_h} < \frac{\sigma+\mu_j}{(\sigma+\mu_j)H+\alpha_j}$, which implies that $Q_h > 0$ thus $d(\frac{I_h^*}{I^*})/dq > 0$, $\frac{I_h^*}{I^*}$ increases as q increases.

Similarly, we have

$$d(\frac{I_h^*}{I^*})/dp_i = \frac{Q_h\frac{\alpha_h}{(\sigma+\mu_h)H+\alpha_h}\frac{dH}{dp_i}\frac{N_h^*}{N}}{(\sum_{j=1}^m\frac{\alpha_j}{(\sigma+\mu_j)H+\alpha_j}\frac{N_j^*}{N})^2}$$

From last section we have $d(\frac{N_h^*}{I_h^*})/dp_i < 0$, which implies that $\frac{dH}{dp_i} < 0$. So we have $d(\frac{I_h^*}{I^*})/dp_i < 0$ if $Q_h > 0$ and $d(\frac{I_h^*}{I^*})/dp_i > 0$ if $Q_h < 0$, which means that $\frac{I_h^*}{I^*}$ decreases as p_i increases if $Q_h > 0$ and increases as p_i increases if $Q_h < 0$. In particular, if $1 \leq h \leq m$ is such that $\frac{\alpha_h}{\sigma+\alpha_h} < \frac{\alpha_j}{\sigma+\alpha_j}$ for all $j \neq h$, then $\frac{\sigma+\mu_h}{(\sigma+\mu_h)H+\alpha_h} > \frac{\sigma+\mu_j}{(\sigma+\mu_j)H+\alpha_j}$, which implies that $Q_h < 0$ thus $d(\frac{I_h^*}{I^*})/dp_i > 0$, $\frac{I_h^*}{I^*}$ increases as p_i increases; if $1 \leq h \leq m$ is such that $\frac{\alpha_h}{\sigma+\alpha_h} > \frac{\alpha_j}{\sigma+\alpha_j}$ for all $j \neq h$, then $\frac{\sigma+\mu_h}{(\sigma+\mu_h)H+\alpha_h} < \frac{\sigma+\mu_j}{(\sigma+\mu_j)H+\alpha_j}$, which implies that $Q_h > 0$ thus $d(\frac{I_h^*}{I^*})/dp_i < 0$, $\frac{I_h^*}{I^*}$ decreases as p_i increases.

□

6. Family planning strategies

In this section, we study how family planning strategies influence demographic distribution at equilibrium, basic reproduction number and infant disease rate.

For simplicity, we assume that there are only two productive age groups, the kth and $(k+1)$th group. We also assume that the birth function takes a more general form $b_i(N_i) = p_i q(N_k + N_{k+1})N_i$ for $i = k, k+1$, where q is a decreasing function. Since the maximal children each female has per unit time in age group i, given by p_i, are dependent on each other, more precisely, $p_k + p_{k+1} = b$ for some constant b, which is the maximal children each female has per unit time, we assume that $p_k = b\alpha$ and $p_{k+1} = b(1-\alpha)$, then α indicates the tendency to have children at an earlier age. We study how α influence demographic distribution at equilibrium, basic reproduction number and infant disease rate.

6.1. Impact of family planning strategy on demographic distribution at equilibrium

From Eq (2.4), we have

$$\begin{aligned}
N_1^* &= \frac{1}{\mu_1}(b\alpha q(N_k^* + N_{k+1}^*)N_k^* + b(1-\alpha)q(N_k^* + N_{k+1}^*)N_{k+1}^*)(1 - e^{-\mu_1\tau_1}) \\
N_h^* &= \frac{1}{\mu_h}(b\alpha q(N_k^* + N_{k+1}^*)N_k^* + b(1-\alpha)q(N_k^* + N_{k+1}^*)N_{k+1}^*)e^{-\sum_{j=1}^{h-1}\mu_j\tau_j}(1 - e^{-\mu_h\tau_h}) \text{ for } 1 < h < m \qquad (6.1) \\
N_m^* &= \frac{1}{\mu_m}(b\alpha q(N_k^* + N_{k+1}^*)N_k^* + b(1-\alpha)q(N_k^* + N_{k+1}^*)N_{k+1}^*)e^{-\sum_{j=1}^{m-1}\mu_j\tau_j}
\end{aligned}$$

Further calculation gives

$$Q = \frac{\mu_1 N_1^*}{1 - e^{-\mu_1\tau_1}} = \frac{\mu_h N_h^*}{e^{-\sum_{j=1}^{h-1}\mu_j\tau_j}(1 - e^{-\mu_h\tau_h})} = \frac{\mu_m N_m^*}{e^{-\sum_{j=1}^{m-1}\mu_j\tau_j}}$$

where

$$Q = b\alpha q(N_k^* + N_{k+1}^*)N_k^* + b(1-\alpha)q(N_k^* + N_{k+1}^*)N_{k+1}^*$$

This implies that α doesn't influence the demographic distribution at equilibrium.

6.2. Impact of family planning strategy on the basic reproduction number and infant disease rate

Define

$$a_{i,k} = \frac{\mu_k e^{-\sum_{j=1}^{i-1}\mu_j\tau_j}(1 - e^{-\mu_i\tau_i})}{\mu_i e^{-\sum_{j=1}^{k-1}\mu_j\tau_j}(1 - e^{-\mu_k\tau_k})}$$

for $1 < i, k < m$. By plugging $b_i(N_i) = p_i q(N_k + N_{k+1})N_i$ into the equation for N_k^* in Eq (6.1), we obtain

$$q((1 + a_{k+1,k})N_k^*) = \frac{\mu_k}{(b\alpha + b(1-\alpha)a_{k+1,k})e^{-\sum_{j=1}^{k-1}}(1 - e^{-\mu_k\tau_k})} \qquad (6.2)$$

Define $F(\alpha) := b\alpha + b(1-\alpha)a_{k+1,k}$, then $F'(\alpha) = b(1 - a_{k+1,k})$. From Eq (6.2), we have the following conclusions: If $a_{k+1,k} > 1$, then $F'(\alpha) < 0$, q is monotone increasing with respect to α, thus N_k^* is monotone decreasing with respect to α; if $a_{k+1,k} < 1$, then $F'(\alpha) > 0$, q is monotone decreasing with respect to α, thus N_k^* is monotone increasing with respect to α; if $a_{k+1,k} = 1$, then $F'(\alpha) = 0$, q doesn't change with α, thus N_k^* doesn't change with α.

Note that the basic reproduction number is given by

$$R_0 = \sum_{i=1}^{m} \frac{\alpha_i \lambda_i N_i^*}{\sigma + \mu_i} = \sum_{i=1}^{m} \frac{\alpha_i \lambda_i \frac{N_i^*}{N_k^*}}{\sigma + \mu_i} N_k^* \tag{6.3}$$

So we have the following proposition.

Proposition 13. *If $a_{k+1,k} > 1$, R_0 decreases as α increases; if $a_{k+1,k} < 1$, R_0 increases as α increases; if $a_{k+1,k} = 1$, R_0 doesn't change with α.*

Proposition 13 implies that if $a_{k+1,k} > 1$, the basic reproduction number decreases when more people are having children at an early age, if $a_{k+1,k} < 1$, the basic reproduction number increases when more people are having children at an early age, if $a_{k+1,k} = 1$, the basic reproduction number doesn't depend on tendency on birth age.

From previous calculation, we have

$$\sum_{i=1}^{m} \frac{\lambda_i \frac{N_i^*}{N_k^*} N_{k^*}}{l_i + \frac{l_1}{M_1}} = 1 \tag{6.4}$$

where $l_i = \frac{\sigma + \mu_i}{\alpha_i}$, $M_1 = \frac{N_1^*}{I_1^*} - 1$. It implies that $\frac{I_1^*}{N_1^*}$ is monotone increasing with respect to N_k^*. So we have the following

Proposition 14. *If $a_{k+1,k} > 1$, $\frac{I_1^*}{N_1^*}$ decreases as α increases; if $a_{k+1,k} < 1$, $\frac{I_1^*}{N_1^*}$ increases as α increases; if $a_{k+1,k} = 1$, $\frac{I_1^*}{N_1^*}$ doesn't change with α.*

Proposition 14 implies that if $a_{k+1,k} > 1$, the infant disease rate decreases when more people are having children at an early age, if $a_{k+1,k} < 1$, the infant disease rate increases when more people are having children at an early age, if $a_{k+1,k} = 1$, the infant disease rate doesn't depend on tendency of birth age.

7. Conclusions

We proposed a stage-structured model of childhood infectious disease transmission dynamics. The population demographics dynamics is governed by a certain family and population planning strategy which gives rise to nonlinear feedback delayed effects on the reproduction ageing and rate.

The long-term aging-profile of the population is described by the pattern and stability of equilibrium of the demographic model. For this demographic model, conditions on the birth functions and death rate were given to guarantee the existence and stability of the positive equilibrium. This implies conditions on birth function and age dependent death rate to reach a stable population. We also investigate the disease transmission dynamics, using the epidemic model when the population reaches the positive equilibrium (limiting equation). We establish conditions for the existence, uniqueness and global stability of the disease endemic equilibrium and prove the global stability of the endemic equilibrium for the original epidemic model with varying population demographics.

Birth function, age-dependent death rate, recover rate and transmission coefficients are all involved in these conditions.

The global stability of the endemic equilibrium allows us to examine the effects of reproduction ageing and rate, under different family planning strategies, on the childhood infectious disease transmission dynamics. We find that increasing birth rate increases the infant disease rate and reproduction aging decreases the infant disease rate. We also find that reproduction ageing and rate doesn't change the demographic distribution at equilibrium.

We investigate impacts of family planning strategies on demographic distribution at equilibrium, basic reproduction number for childhood disease and infant disease rate. We find the conditions under which planning to have a child at an early age helps to decrease/increase the basic reproduction number and infant diseases rate. We also examine demographic distribution, diseases reproductive number, infant disease rate and age distribution of disease.

For original contributions, the model we propose is new as it is stage structured and the growth through age stages is described by time delay leading to nonlinear feedback, the idea of studying the impact of population policy and family planning strategy on disease transmission dynamics is also novel. This model can be modified to fit specific childhood diseases for specific purposes. For example, it can be modified to study the impact of China's second-child policy on pertussis transmission dynamics by incorporating more compartments to distinguish children from one-child and two-children families. The work can also be potentially used to inform targeted age group for optimal vaccine booster programs.

Conflict of interest

The authors declare no conflict of interest.

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