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Research article

Entire product capability analysis chart with asymmetric tolerances

index S_{pa}

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Abstract: This study proposes the S_{pa} -product capability analysis chart (S_{pa} -PCAC), which can widely represent multiple process capabilities with asymmetric tolerances of Smaller-the-Better, Larger-the-Better, and Nominal-the-Best characteristics. Process capability index S_{pa} is generated based on index S_{pk} , which uses asymmetric tolerances to reasonably measure process capabilities. The interval estimates of the indices are derived to reliably assess process capabilities. The Six-Sigma-based quality-level and its connection with the process yield are introduced in the capability zone of S_{pa} -PCAC to check if the process capabilities can meet the requirements. One example of an entire product is given for application.

Keywords: product capability analysis chart; process capability index; asymmetric tolerances; capability zone; Six-Sigma

1. Introduction

The process capability analysis in the manufacturing process plays a critical role in the quality improvement program. When a process is statistically controlled, its capability can be evaluated to

manufacture products based on specifications. By means of the process capability analysis, manufacturing managers are able to perceive process capability levels. Both process capability indices (PCIs) and process capability analysis charts frequently appear in the process capability analysis [1–6]. In the multi-process environment, many experts and scholars integrate different types of PCIs to accurately and reliably develop various analysis charts which can monitor and control process capabilities.

Singhal [7] introduced a graphical chart with the first-generation process capability index (yield-based index), the Smaller-the-Better (STB) capability index, and the Larger-the-Better (LTB) capability index. The yield-based index is the minimal value of the horizontal and vertical axes, which refines the previous chart to include capability zones, instead of the expected fallout contours. The processes can fall into different performance categories (or capability zones) which easily explain the state of each process and help factory managers improve the quality of each process. Nevertheless, the yield-based index has difficulty telling the difference between the off-target process and the on-target process. Pearn and Chen [9] introduced a modified PCAC combining the process capability indices of the 2nd and 3rd generations, attempting to point out the factors that result in process failure at centering the target. Not only the second-generation but also the third-generation process capability indices were selected because of their sensitivity to process centering. Furthermore, Chen et al. [10] noted that another important problem is how to adopt asymmetric tolerances to accurately evaluate the quality of bilateral quality characteristics. According to Chen et al. [11], a Process Capability Analysis Chart (PCAC) was employed to precisely assess the potential capability and performance of an entire product possessing STB, LTB, and Nominal-the-Best (NTB) characteristics using asymmetric tolerances. Reasonably, the index of asymmetric tolerances indicates the condition of a process when the asymmetric tolerances make the process deviate from the target.

Process yield is another criterion used in manufacturing to analyze process performance. The yield-based index has difficulty mirroring a specific process yield. Huang et al. [12] proposed yield index to take over the yield-based index. The yield index of the process offers a gauge of the process yield by reason of their one-to-one relations. Further, they also proposed an entire PCAC using the process yield index to precisely demonstrate the process performance of the backlight module. Shortly thereafter, Chen et al. [13] constructed a contour plot called the control chart of the process yield index with the multi-characteristic process capability analysis (MCPCA). The process yield index is computed as a function of the accuracy index and the precision index. The accuracy index refers to the departure rate while the precision index estimates the deviation related to the tolerances. MCPCA yields timely information so that the management is able to easily discern a product's quality level and has a clear picture about which parameters need to be targeted for quality improvement.

However, there exists a crucial problem that needs to be solved. That is, most approaches are untrustworthy because the point estimate of PCI sampling is a random variable without assessing any sampling errors. Several researchers have addressed this. Pearn et al. [14] developed the third-generation index PCAC. In this model, multiple samples of the control chart urge the distributions of the estimated third-generation index and accuracy index. Precise lower confidence bounds are tallied and plotted. Pearn and Wu [15] mentioned that several PCACs have so overlooked sampling errors that capability measures and groupings seem undependable. Therefore, they proposed a decent method to transform estimated indices into lower confidence bounds as well as plot the corresponding lower confidence bounds on PCAC. Chen and Chen [16] advanced a PCAC taking the interval estimates of these indices into account to accurately assess a product's

characteristics. Chen et al. [17] came up with the process capability analysis chart. Besides, they proposed a new estimator of the second-generation index and discussed the features of the statistical estimation and hypothesis testing, in an effort to reliably evaluate the process capability. Chang et al. [18] not only considered the sample errors but also employed the mathematical programming, so that a statistical hypothesis testing model was developed to help manufacturing sections examine the outcome of their implemented improvements. Recently, Chen and Chen [19] developed an advanced PCAC by means of asymmetric tolerances, in order to appraise the process performance of T-Bar Ceiling Suspension System. Moreover, this method, considering the interval estimates of the PCIs, attempted to surmount the weakness of scrutiny or sampling errors of point estimates.

Reviewing the development described in aforementioned studies, each article proposed a refined tool or method which contributed valuable information that managers can use to implement quality improvement. Nonetheless, owing to rising requirements for products' quality, a single instrument cannot provide managers complete and reliable information. This is because several critical issues are difficult to be simultaneously taken into account in a model. These issues include capability evaluation of asymmetric process, process yield, process accuracy and loss, and the reliability of estimation. To provide comprehensive and reliable information managers can use, this study retains the merits of the past approaches and constructs a useful and dynamic approach via a S_{pa} product capability analysis chart (S_{pa} -PCAC). This model has four key points. First, the S_{na} -PCAC can widely represent multiple process capabilities with asymmetric tolerances consisting of STB, LTB, and NTB characteristics. A new index S_{pa} is generated based on the index of process yield, which can provide a consistent and reasonable measure for processes with asymmetric tolerances. Second, considering sampling errors, this study derives interval estimations of the used indices to reliably assess the process capabilities. Third, based on the loss function defined in the Taguchi method, the index, C_a , is incorporated into the S_{pa} -PCAC because it is adequate to assess the validity of the manufacturing process and process loss. Fourth, the quality-level of Six-Sigma and the connection with process yield are introduced in the process capability zone of S_{pa} -PCAC to check if the process capabilities can reach the preset levels. Managers are able to adopt S_{pa} -PCAC, a practical and easy-to-use tool, attempting to evaluate and enhance manufacturing processes constantly.

This study is divided into the following sections. In section 2, we not only begin with the definition of index S_{pa} but also introduce how to construct the model of S_{pa} -PCAC based on the required quality level. Next, in section 3, we derive the confidence interval of S_{pa} from the mathematical programming. In section 4, we adopt a multi-process product containing seven quality characteristics as an example to illustrate S_{pa} -PCAC. Lastly, in section 5, we summarize our results and provide the managers with reliable information in order to implement quality improvements.

2. S_{pa} product capability analysis chart (S_{pa} -PCAC)

2.1. Review of process capabilities for multiple characteristics

Kane [20] proposed two process capability indices, C_{pu} and C_{pl} , to evaluate process capabilities of STB and LTB characteristics.

$$C_{pu} = \frac{USL - \mu}{3\sigma}, C_{pl} = \frac{\mu - LSL}{3\sigma}, \qquad (1)$$

In Eq (1), USL is the upper lower specification limit while LSL is the lower specification limit; μ is the process mean, and σ is the process standard deviation. In general conditions, one-to-one mathematical relationships prevail over the process yield and C_{pu} or C_{pl} , which is expressed as % Yield = $\Phi(3C_{pi})$ for i = u, l.

Based on Chen et al. [11], in the process of NTB, index C_a is associated with the relative distance between the mean and the target. C_a is expressed as

$$C_a = 1 - \max\left\{\frac{\mu - T}{D_u}, \frac{T - \mu}{D_l}\right\},\tag{2}$$

where $D_u = USL - T$, $D_l = T - LSL$ and T is the target. Obviously, equally relative distances create the identical values for C_a . Index C_a can demonstrate process accuracy as well as process loss. In addition, Boyles [21] proposed index S_{pk} , which accurately measures the process yield for the normal process with bilateral specifications, as expressed in the following equation:

$$S_{pk} = \frac{1}{3} \Phi^{-1} \left\{ \frac{1}{2} \Phi \left(\frac{USL - \mu}{\sigma} \right) + \frac{1}{2} \Phi \left(\frac{\mu - LSL}{\sigma} \right) \right\},\tag{3}$$

where Φ is an accumulative density function of the standard normal distribution, N(0,1), while Φ^{-1} is a reversed function of Φ . Actually,

$$\frac{USL-\mu}{\sigma} = \frac{T+d-\mu}{\sigma} = \frac{d-(\mu-T)}{\sigma} = \frac{d}{\sigma} \left(1 - \frac{\mu-T}{d}\right),$$

$$\frac{\mu-LSL}{\sigma} = \frac{\mu-(T-d)}{\sigma} = \frac{d+(\mu-T)}{\sigma} = \frac{d}{\sigma} \left(1 + \frac{\mu-T}{d}\right),$$
(4)

In Eq (4), d = (USL - LSL)/2 is the distance to the upper and lower limits, indicating half the length of the specification interval. Thus, S_{pk} is rewritten as

$$S_{pk} = \frac{1}{3} \Phi^{-1} \left\{ \frac{1}{2} \Phi \left[\frac{d}{\sigma} \left(1 - \frac{\mu - T}{d} \right) \right] + \frac{1}{2} \Phi \left[\frac{d}{\sigma} \left(1 + \frac{\mu - T}{d} \right) \right] \right\}.$$
 (5)

Huang et al. [12] suggested a multi-process capability analysis chart (MPCAC) containing indices C_{pu} , C_{pl} , C_a , and $C_{ps}(S_{pk})$ as well as evaluated the integrated process capability of a multi-process product (backlight module) adopting STB, LTB, and NTB symmetric specifications respectively. On the contrary, as stated by Pearn and Shu [22], index S_{pk} cannot provide consistent and reasonable measures for processes by means of asymmetric tolerances. This study proposes a generalization index, S_{pa} , based on S_{pk} , which can apply asymmetric tolerances to processes. The index, S_{pa} , is expressed as follows:

$$S_{pa} = \frac{1}{3} \Phi^{-1} \left\{ \frac{1}{2} \Phi \left[\frac{(1-\delta)}{\theta} \right] + \frac{1}{2} \Phi \left[\frac{(1+\delta)}{\theta} \right] \right\}.$$
 (6)

Let
$$\theta = \frac{\sigma}{d_A}$$
, $\delta = \begin{cases} \frac{\mu - T}{D_u} &, \mu \ge T \\ \frac{\mu - T}{D_l} &, \mu < T \end{cases}$, when $\begin{cases} \mu = USL \Rightarrow \delta = 1 \\ \mu = LSL \Rightarrow \delta = -1 \\ LSL < \mu < USL \Rightarrow -1 < \delta < 1 \end{cases}$ (7)

where $d_A = \min(D_u, D_l)$. Therefore, when $D_\mu = D_l = d$ (symmetric case), S_{pa} will decrease to the original index, S_{pk} . For a fixed process standard deviation - σ , S_{pa} drops more quickly as μ deviates from T and gets closer to the specification limit. On the other hand, it reduces slower when μ deviates from T and gets farther to the specification limit. The newly generalized S_{pa} is supposed to acquire its maximum at $\mu = T$ (on-target process) no matter whether the specifications are symmetric or asymmetric. It can be seen that the newly generalized S_{pa} is more suitable than the existing S_{pk} to be adopted in an asymmetric case, providing greater accuracy to monitor process potential and performance.

2.2. Development of the S_{pa} -PCAC



Figure 1. S_{pa} -PCAC.

According to the MPCAC proposed by Huang et al. [12], this study selects index S_{pa} to replace S_{pk} as well as reconstructs a S_{pa} product capability analysis chart (S_{pa} -PCAC) to assess a manufacturing process, the secondary lines standing for the values of index C_a are outlined on the S_{pa} -PCAC, in order to precisely control the process shifts and monitor the process loss. Third, the process capability zone is mapped with bold lines on S_{pa} -PCAC following the required quality level of the manufacturing process. Fourth, the interval estimates of C_{pu} , C_{pl} , and S_{pa} are drawn on

 S_{pa} -PCAC. The process capabilities reach the required level when the estimates of the confidence intervals of the PCIs are situated in the capability zone of the process. It is simple to discriminate process performance from the sites plotted on the S_{pa} -PCAC, as shown in Figure 1. Hence, the PCIs and S_{pa} -PCAC not only strengthen the limitations of previous studies but also reliably evaluate and effectively monitor process capabilities.

2.3. Capability zone

Linderman et al. [23] and Chen et al. [24] suggested that Six-Sigma is considered as a decent technique of improvement and is employed by plenty of enterprises. The number of Sigma has been regarded as a tool used by some enterprises to measure process capability. Motorola set the standard that process variability should be above or below the mean 6σ . Furthermore, Motorola stated that the process devoured by disruptions would deviate the mean from the target as much as 1.5σ [25]. As a result, based on Eq (7), when the quality level reaches k-Sigma (k σ) (i.e., $d_A = k\sigma$) and the process mean shifts at 1.5σ , then $\theta = 1/k$ and $|\delta| = 1.5/k$. The PCIs, C_{pu} , C_{pl} , and S_{pa} , are expressed as follows:

$$C_{pu} = \frac{USL - \mu}{3\sigma} = \frac{T + d - \mu}{3\sigma} = \frac{k\sigma - 1.5\sigma}{3\sigma} = \frac{k - 1.5}{3},$$

$$C_{pl} = \frac{\mu - LSL}{3\sigma} = \frac{\mu - (T - d)}{3\sigma} = \frac{k\sigma - 1.5\sigma}{3\sigma} = \frac{k - 1.5}{3},$$

$$S_{pa} = \frac{1}{3} \Phi^{-1} \left\{ \frac{1}{2} \Phi \left[\frac{(1 - \delta)}{\theta} \right] + \frac{1}{2} \Phi \left[\frac{(1 + \delta)}{\theta} \right] \right\}$$

$$= \frac{1}{3} \Phi^{-1} \left\{ \frac{1}{2} \Phi \left[\frac{(1 - 1.5/k)}{1/k} \right] + \frac{1}{2} \Phi \left[\frac{(1 + 1.5/k)}{1/k} \right] \right\}$$

$$= \frac{1}{3} \Phi^{-1} \left\{ \frac{1}{2} \Phi (k - 1.5) + \frac{1}{2} \Phi (k + 1.5) \right\}.$$
(8)

This paper defines that the process quality reaches k-Sigma quality level, then $C_{pu} \ge C_{pu}(k)$, $C_{pl} \ge C_{pl}(k)$ and $S_{pa} \ge S_{pa}(k)$. In other words, PCI(k) represents the lower limit value when the process quality reaches k-Sigma quality level. Table 1 displays the values of $C_{pu}(k)$, $C_{pl}(k)$, and $S_{pa}(k)$ for k = 3, 4, 5, 6.

Quality Level (k-Sigma)	$S_{pa}(k)$	$C_{pi}(k)$ $i = u, l$	Process Yield (%)
3-Sigma	0.61	0.50	93.319
4-Sigma	0.91	0.83	99.361
5-Sigma	1.23	1.17	99.977
6-Sigma	1.55	1.50	99.999

Table 1. The values of $C_{pu}(k)$, $C_{pl}(k)$, and $S_{pa}(k)$ and process yields.

Additionally, based on Eq (2), let $C_a = 1 - \frac{1}{a}$, where $1/a = \max\{(\mu - T)/D_u, (T - \mu)/D_l\}$. According to Chen et al. [11], the values of μ for each C_a are $[T + (1/a)*D_u]$ and $[T - (1/a)*D_l]$. The incline of the relevant secondary line is $m_1 = (a+1)/(a-1)$ as the mean is larger than the target $(\mu \ge T)$; the slope of the relevant secondary line is $m_2 = (a-1)/(a+1)$ as the mean is smaller than the target $(\mu < T)$. Owing to $m_1 \times m_2 = 1$, secondary lines $Y = m_1 X$ and $Y = m_2 X$ are symmetrical with line Y = X, as shown in Figure 1. As we can see, C_a refers to process accuracy. In general, C_a cannot be too tiny because a smaller C_a represents that the mean deviates from the target, and that will incur a great amount of process loss. For NTB quality type, when the process loss is taken into consideration (e.g., $C_a \ge 1-1/a$, and the quality level is required to reach k-Sigma, as shown in Table 1), we can define a set S_k and mark a capability zone with bold lines on S_{pa} -PCAC as follows :

Nominal-the-Best :
$$S_k = \left\{ (X, Y) \middle| S_{pa}(X, Y) \ge S_{pa}(k), \frac{a-1}{a+1} \le Y / X \le \frac{a+1}{a-1}, X \ge 0, Y \ge 0 \right\}$$
 (9)

Similarly, for the unilateral process, if the quality level reaches k-Sigma, the capability zones can be respectively plotted on the horizontal and vertical axes with bold lines according to the values of C_{pu} and C_{pl} , as depicted in Table 1. The capability zone of STB and LTB quality type can be show as following sets:

Smaller-the-Better:
$$S_u = \left\{ (X, Y) \middle| X \ge C_{pu}(k), Y = 0 \right\}$$
 (10)

Larger-the-Better:
$$S_l = \left\{ (X, Y) \middle| X = 0, Y \ge C_{pl}(k) \right\}$$
 (11)

As a result, the capability zones on S_{pa} -PCAC (the gray zones in Figure 1) can help managers check whether all process capabilities can satisfy the required quality levels. Then, the process capability can be evaluated based on the intersection of the process capability interval estimation area of each quality characteristic and the capability zone. The evaluation rules are as follows:

(1) When the intersection of the interval estimation area of the index and the gray area in Figure 1 is a nonempty set, it means that the process capability of the quality characteristic has reached the quality level and the process capability needs to be maintained.

(2) When the intersection of the interval estimation area of the index and the gray area in Figure 1 is a nonempty set and the white zone is included, it means that the process capability of the quality characteristic cannot meet the quality requirements and needs to be improved.

According to the above evaluation rules, the industries can easily determine whether the process capability of all the quality characteristics of the whole product meets the quality level requirements.

3. Interval estimates of PCIs

As mentioned in the discussion of the literature section of this paper, some limitations of the existing approaches include (1) they are unreliable since sampling errors are neglected and (2) the approaches are only applicable in case of full inspection. Point estimate is prone to resulting in uncertain sampling errors, so that it may lead to the probability of misjudgment [26–31]. To overcome these limitations, the proposed S_{pa} -PCAC takes into account the interval estimates of S_{pa} , C_{pu} , and C_{pl} , which reliably measure a product with the NTB asymmetric tolerances, LTB characteristics, and STB characteristics. This can be explained as follows. The index S_{pa} is viewed as a function of process parameters μ and σ denoted by $S_{pa}(\mu,\sigma)$. We derive the confidence interval of $S_{pa}(\mu,\sigma)$ via the following mathematical programming (MP):

$$\begin{cases} Max & S_{pa}(\mu,\sigma) \\ a_{1} \leq \mu \leq a_{2} & \text{for } US_{pa}, \\ s.t. & b_{1} \leq \sigma \leq b_{2} \end{cases} \begin{cases} Min & S_{pa}(\mu,\sigma) \\ a_{1} \leq \mu \leq a_{2} & \text{for } LS_{pa}. \\ s.t. & b_{1} \leq \sigma \leq b_{2} \end{cases}$$
(12)

The upper limit of confidence interval (denoted by US_{pa}) and lower limit of confidence interval (denoted by LS_{pa}) are computed separately in accordance with the maximum and minimum values of the objective function $S_{pa}(\mu, \sigma)$ under the constraints $a_1 \le \mu \le a_2$ and $b_1 \le \sigma \le b_2$. (a_1, a_2) and (b_1, b_2) are the confidence intervals of μ and σ , respectively. These two constraints compose the feasible solution area (FSA) of the MP model, that is, $FSA(\mu, \sigma) = \{a_1 \le \mu \le a_2, b_1 \le \sigma \le b_2\}$. It is obvious to see that the maximum value of $S_{pa}(\mu, \sigma)$ must be at the minimum value b_1 of σ while the minimum value of $S_{pa}(\mu, \sigma)$ must be at the minimum value b_1 of σ while the minimum value of $S_{pa}(\mu, \sigma)$ must be at the maximum value b_2 . $S_{pa}(\mu, \sigma)$ is re-expressed as follows based on the Eqs (6) and (7). Let $S_{pa}(\mu, \sigma) = f(\mu, \sigma)$ when $\mu \ge T$, and $S_{pa}(\mu, \sigma) = g(\mu, \sigma)$ when $\mu < T$.

$$S_{pa}(\mu,\sigma) = \begin{cases} f(\mu,\sigma) = \frac{1}{3} \Phi^{-1} \left\{ \frac{1}{2} \Phi \left[\frac{d_A}{\sigma} \left(1 - \frac{\mu - T}{D_u} \right) \right] + \frac{1}{2} \Phi \left[\frac{d_A}{\sigma} \left(1 + \frac{\mu - T}{D_u} \right) \right] \right\}, \mu \ge T \\ g(\mu,\sigma) = \frac{1}{3} \Phi^{-1} \left\{ \frac{1}{2} \Phi \left[\frac{d_A}{\sigma} \left(1 - \frac{\mu - T}{D_l} \right) \right] + \frac{1}{2} \Phi \left[\frac{d_A}{\sigma} \left(1 + \frac{\mu - T}{D_l} \right) \right] \right\}, \mu < T \end{cases}$$
(13)

We can draw the figure of $S_{pa}(\mu, \sigma)$ under the fixed σ , as shown in Figure 2.



Figure 2. Relationship between S_{pa} and process mean μ .

In Figure 2, we can see the optimal solution of $S_{pa}(\mu,\sigma)$ is dependent on the relative position between the interval (a_1,a_2) and the process target T. In order to clearly explain the relevance of the position between interval (a_1,a_2) and the process target T, this study classifies three cases as follows.

Case I : $a_1 \le T \le a_2$ is composed of two situations:

(1) When μ is located in (T, a_2) (i.e. $T \le \mu \le a_2$), we know $S_{pa}(\mu, \sigma) = f(\mu, \sigma)$. Therefore, the upper limit of confidence $US_{pa} = \max\{f(\mu, b_1) | T \le \mu \le a_2\}$. Since $f(\mu, b_1)$ strictly decreases as μ increases, we can find $US_{pa} = f(T, b_1)$. Similarly, we have $LS_{pa} = \min\{f(\mu, b_2) | T \le \mu \le a_2\} = f(a_2, b_2)$.

(2) When μ is located in (a_1,T) (i.e. $a_1 \le \mu \le T$), we see that $S_{pa}(\mu,\sigma) = g(\mu,\sigma)$. Therefore, $US_{pa} = \max\{g(\mu,b_1)|a_1 \le \mu \le T\}$. Since $g(\mu,b_1)$ strictly increases as μ increases, we have $US_{pa} = g(T,b_1)$. Similarly, the $LS_{pa} = \min\{g(\mu,b_2)|a_1 \le \mu \le T\} = g(a_1,b_2)$.

Case II : $T < a_1 < a_2$

$$\begin{split} S_{pa}(\mu,\sigma) & \text{ is equal to } f(\mu,\sigma) & \text{ because of } \mu > T & \text{. We can find} \\ US_{pa} &= \max\left\{f\left(\mu,b_{1}\right) \middle| T < a_{1} \leq \mu \leq a_{2}\right\} & f\left(\mu,b_{1}\right) & \text{ strictly decreases as } \mu \text{ increases such that} \\ US_{pa} &= f\left(a_{1},b_{1}\right) & \text{ Similarly, we have } LS_{pa} &= \min\left\{f\left(\mu,b_{2}\right) \middle| T < a_{1} \leq \mu \leq a_{2}\right\} = f\left(a_{2},b_{2}\right). \end{split}$$

Case III : $a_1 < a_2 < T$

$$\begin{split} S_{pa}(\mu,\sigma) & \text{ is equal to } g(\mu,\sigma) & \text{ because of } \mu < T & \text{. We can find} \\ US_{pa} &= \max\left\{g\left(\mu,b_{1}\right) \middle| a_{1} \leq \mu \leq a_{2} < T\right\} & g\left(\mu,b_{1}\right) & \text{ strictly increases as } \mu & \text{ increases such that} \\ US_{pa} &= g\left(a_{2},b_{1}\right) & \text{ Similarly, we obtain } LS_{pa} &= \min\left\{g(\mu,b_{2}) \middle| a_{1} \leq \mu \leq a_{2} < T\right\} = g(a_{1},b_{2}) & \text{.} \end{split}$$

The above MP model is calculated based on the sample data of the in-control process. In the short term, the XS control chart can be employed to monitor whether the process is stable; after the

process is stable, the chart data can be controlled to conduct a process capability evaluation. In the long term, the stable process capability can be evaluated by simple random sampling, in order to solve the problem of cost and timeliness [32–35]. Let X_1, \ldots, X_n be a random sample obtained from an in-control process with the normal distribution. A natural estimate of $S_{pa}(\mu, \sigma)$ may be received from replacing (1) the unrevealed process mean μ with the sample mean $\overline{X} = (X_1 + X_2 + \cdots + X_n)/n$ and (2) the unrevealed process standard deviation σ with the sample standard deviation $S = \left[\sum_{i=1}^{n} (X_i - \overline{X})^2/(n-1)\right]^{1/2}$, respectively. Hence, the MP model may be re-expressed as shown in Eqs (12) and (13):

$$\begin{cases} Max & S_{pa}(\overline{X}, S) \\ a_{1} \leq \overline{X} \leq a_{2} \text{ for } U\hat{S}_{pa}, \\ s.t. & b_{1} \leq S \leq b_{2} \end{cases} \qquad \begin{cases} Min & S_{pa}(\overline{X}, S) \\ a_{1} \leq \overline{X} \leq a_{2} \text{ for } L\hat{S}_{pa}. \\ s.t. & b_{1} \leq S \leq b_{2} \end{cases}$$
(14)

The estimates of upper and lower limits of confidence interval are denoted by $L\hat{S}_{pa}$ and $U\hat{S}_{pa}$ respectively.

$$S_{pa}(\overline{X},S) = \begin{cases} f'(\overline{X},S) = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left[\frac{d_A}{S}\left(1-\frac{\overline{X}-T}{D_u}\right)\right] + \frac{1}{2}\Phi\left[\frac{d_A}{S}\left(1+\frac{\overline{X}-T}{D_u}\right)\right]\right\}, \overline{X} \ge T\\ g'(\overline{X},S) = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left[\frac{d_A}{S}\left(1-\frac{\overline{X}-T}{D_l}\right)\right] + \frac{1}{2}\Phi\left[\frac{d_A}{S}\left(1+\frac{\overline{X}-T}{D_l}\right)\right]\right\}, \overline{X} < T \end{cases}$$
(15)

For a normal process, the $(1-\alpha/2)\%$ confidence interval estimator of process μ is $(\overline{X} - t_{\alpha_4'}(n-1) \cdot S/\sqrt{n}, \overline{X} + t_{\alpha_4'}(n-1) \cdot S/\sqrt{n})$ and $(1-\alpha/2)\%$ confidence interval estimator of process σ is $(\sqrt{(n-1)S^2/\chi^2_{\alpha/4}(n-1)}, \sqrt{(n-1)S^2/\chi^2_{1-\alpha/4}(n-1)})$, where t(n-1) is distributed as a Student's *t*-distribution with n-1 degrees of freedom while $\chi^2(n-1)$ is distributed as a chi-square distribution with n-1 degrees of freedom. We can derive the $(1-\alpha)\%$ confidence interval estimator of the $FSA(\mu,\sigma)$ according to the Cartesian product:

$$FSA(\mu,\sigma) = \{ [\hat{a}_{1} \le \mu \le \hat{a}_{2}] \cap [\hat{b}_{1} \le \sigma \le \hat{b}_{2}] \}$$

$$= \left\{ \left[\overline{X} - \frac{t_{\frac{\alpha}{4}}(n-1) \cdot S}{\sqrt{n}} \le \mu \le \overline{X} + \frac{t_{\frac{\alpha}{4}}(n-1) \cdot S}{\sqrt{n}} \right] \cap \left[\sqrt{\frac{(n-1)S^{2}}{\chi^{2}_{\frac{\alpha}{4}}(n-1)}} \le \sigma \le \sqrt{\frac{(n-1)S^{2}}{\chi^{2}_{\frac{1-\alpha}{4}}(n-1)}} \right] \right\}.$$
(16)

That is, the estimators of (a_1, a_2) and (b_1, b_2) are expressed as

$$\therefore P\left\{\left[\overline{X} - \frac{t_{\frac{n}{4}}(n-1) \cdot S}{\sqrt{n}} \le \mu \le \overline{X} + \frac{t_{\frac{n}{4}}(n-1) \cdot S}{\sqrt{n}}\right] \cap \left[\sqrt{\frac{(n-1)S^2}{\chi^2_{\frac{n}{4}}(n-1)}} \le \sigma \le \sqrt{\frac{(n-1)S^2}{\chi^2_{\frac{1-\alpha}{4}}(n-1)}}\right]\right\}$$
(17)

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$$\geq 1 - P\left\{ \left[\overline{X} + \frac{t_{\frac{\alpha}{4}}(n-1) \cdot S}{\sqrt{n}} < \mu \right] \cup \left[\overline{X} - \frac{t_{\frac{\alpha}{4}}(n-1) \cdot S}{\sqrt{n}} > \mu \right] \right\}$$
$$- P\left\{ \left[\sqrt{\frac{(n-1)S^2}{\chi^2_{1-\frac{\alpha}{4}}(n-1)}} < \sigma \right] \cup \left[\sqrt{\frac{(n-1)S^2}{\chi^2_{\frac{\alpha}{4}}(n-1)}} > \sigma \right] \right\} = 1 - \frac{\alpha}{2} - \frac{\alpha}{2} = 1 - \alpha.$$

They result in

$$(\hat{a}_{1}, \hat{a}_{2}) = (\overline{X} - t_{\frac{\alpha}{4}}(n-1) \cdot S / \sqrt{n}, \overline{X} + t_{\frac{\alpha}{4}}(n-1) \cdot S / \sqrt{n}),$$

$$(\hat{b}_{1}, \hat{b}_{2}) = \left(\sqrt{(n-1)S^{2} / \chi^{2}_{\frac{\alpha}{4}}(n-1)}, \sqrt{(n-1)S^{2} / \chi^{2}_{\frac{1-\alpha}{4}}(n-1)}\right).$$
(18)

Further, we can obtain the estimates of upper and lower limits of confidence interval, $L\hat{S}_{pa}$ and $U\hat{S}_{pa}$, by following the analyses described in cases I~III. The solutions of the MP model based on the sample data are illustrated in Table 2.

Solutions	Case I.(1)	Case I.(2)	Case II	Case III
	$\hat{a}_1 \le T \le \overline{X} \le \hat{a}_2$	$\hat{a}_1 \leq \overline{X} \leq T \leq \hat{a}_2$	$T < \hat{a}_1 \le \overline{X} \le \hat{a}_2$	$\hat{a}_1 \le \overline{X} \le \hat{a}_2 < T$
$L\hat{S}_{pa}$	$f'(\hat{a}_2,\hat{b}_2)$	$g'(\hat{a}_1,\hat{b}_2)$	$f'(\hat{a}_2,\hat{b}_2)$	$g'(\hat{a}_1,\hat{b}_2)$
$U \hat{S}_{pa}$	$f'(T, \hat{b_1})$	$g'(T, \hat{b_1})$	$f'(\hat{a}_1,\hat{b}_1)$	$g'(\hat{a}_2,\hat{b}_1)$

Table 2. The estimates of confidence interval of S_{pa} based on the sample data.

For unilateral process specifications, indices C_{pu} and C_{pl} are applied to assess process capability. The $100(1-\alpha)\%$ lower and upper confidence limits, C_l and C_u for C_{pu} and C_{pl} , satisfy $P(C_l \le C_{pi} \le C_u) = 1 - \alpha$, for $i \in \{l, u\}$. That is,

$$P(C_{l} \le C_{pi}) = 1 - \frac{\alpha}{2}, P(C_{u} \le C_{pi}) = \frac{\alpha}{2}, \text{ for } i \in \{l, u\}.$$
(19)

Based on Chou et al. [36] as well as Pearn and Chen [37], the uniformly minimum-variance unbiased estimators (UMVUEs) for indices C_{pu} and C_{pl} are viewed as \tilde{C}_{pu} and \tilde{C}_{pl} respectively. The estimators are displayed as follows:

$$\widetilde{C}_{pu} = b_f \cdot \hat{C}_{pu}, \text{ and } \widetilde{C}_{pl} = b_f \cdot \hat{C}_{pl},$$
(20)

And

$$\hat{C}_{pu} = \frac{USL - \overline{X}}{3S}, \hat{C}_{pl} = \frac{\overline{X} - LSL}{3S},$$
(21)

where $b_f = [2/f]^{1/2} \Gamma[f/2] / \Gamma[(f-1)/2]$, and f = n-1 is regarded as a correction factor. Estimator $3\sqrt{n}\widetilde{C}_{pi}/b_f$ for $i \in \{\mu, l\}$ is distributed as non-center *t* distribution $t'(n-1;\delta)$ with n-1 degrees of freedom and non-center parameter $\delta = 3\sqrt{n}C_{pi}$. The $100(1-\alpha)\%$ lower and upper confidence limits are seen below:

$$C_{l} = \frac{b_{f}}{3\sqrt{n}} t'_{1-\alpha/2} (n-1; 3\sqrt{n}\widetilde{C}_{pi}), C_{u} = \frac{b_{f}}{3\sqrt{n}} t'_{\alpha/2} (n-1; 3\sqrt{n}\widetilde{C}_{pi}), \text{ for } i \in \{l, u\}.$$
(22)

4. Illustrative example and discussion

To illustrate the application of S_{pa} -PCAC, this study takes a multi-process product with seven quality characteristics as an example. Assume the sample data of each quality characteristic is obtained from an independent process with normal distribution. Under the condition that α -risk is 0.05 and the sample size is 30, process specifications and confidence intervals of capability indices for seven process characteristics are presented in Table 3.

Quality Characteristics	LSL	Т	USL	\overline{X}	S	(\hat{a}_1,\hat{a}_2)	(\hat{b}_1,\hat{b}_2)	$(L\hat{S}_{pa},U\hat{S}_{pa})$	(C_l, C_u)	\hat{C}_a
N1	1.140	1.146	1.150	1.146	0.001	(1.145, 1.146)	(0.001,0.001)	(0.875,1.724)		0.917
N2	3.400	3.500	3.600	3.510	0.020	(3.500, 3.520)	(0.015,0.028)	(1.022,2.159)		0.900
N3	51.000	52.000	53.000	51.700	0.350	(51.530,51.870)	(0.270,0.492)	(0.489,1.136)		0.700
L1	7.500			8.000	0.120				(1.031,1.792)	
L2	6.500			7.800	0.500				(0.626,1.137)	
S1			8.000	6.000	1.000				(0.467,0.890)	
S2			30.000	25.000	1.400				(0.878,1.542)	

Table 3. Calculations for the process capabilities of a multi-process product.

The required quality level of the product should be at least 4-Sigma and the process yield should be no less than 99.38%. Based on Eqs (9)–(11) with 4-sigma quality level, the capability zone can be show as follows:

Nominal-the-Best :
$$S_k = \left\{ (X, Y) \middle| S_{pa}(X, Y) \ge 0.91, \frac{3}{5} \le Y / X \le \frac{5}{3}, X \ge 0, Y \ge 0 \right\}$$

Smaller-the-Better : $S_u = \{(X, Y) | X \ge 0.83, Y = 0\}$

Larger-the-Better : $S_l = \{(X, Y) | X = 0, Y \ge 0.83\}$

As listed in Table 1, the corresponding indices are $\hat{S}_{pa} = 0.91$, $\hat{C}_{pu} = 0.83$, and $\hat{C}_{pl} = 0.83$. Meanwhile, both process accuracy and process loss are considered. Index C_a is required to achieve 0.75. Thus, the bold lines of the capable zones and the confidence intervals of corresponding indices can be plotted on the S_{pa} -PCAC, as shown in Figure 3.



Figure 3. Capability zone.

Based on the evaluation rules and considering the above seven quality characteristics of the multi-process product, the process capabilities are illustrated as follows:

(1) Quality characteristics N_1, N_2 , and N_3 belong to the NTB type with bilateral specifications. The 95% confidence interval of N_2 is situated in the capability zone. The process capability of quality characteristic N_2 is capable. N_1 and N_3 are not entirely situated in the process capability zone, which indicates that the process capabilities of N_1 and N_3 do not reach the 4-Sigma requirement. Thus, it is necessary to take action to improve their process quality, but N_3 in particular should identify all assignable causes to increase the quality level. That can be done by moving the mean of the process to its target and decreasing its variation.

(2) Quality characteristics S_1 and S_2 fall into the STB type with unilateral specifications. Quality characteristic S_2 reaches the required 4-Sigma level, whereas the confidence interval of S_1 is not situated in the capability zone on the X-axis. Therefore, it is necessary to take action to uplift the process capabilities.

(3) Quality characteristics L_1 and L_2 are categorized into the LTB type with unilateral specifications. The confidence interval of L_2 is not situated on the bold line of the capability zone on the *Y*-Axis. To enhance the process quality, it is necessary to reduce the variation of the process or deviate the mean from the lower specification limit. As for quality characteristic L_1 , it currently meets the 4-Sigma requirement, but it should be continuously monitored.

5. Conclusions

To monitor the process capabilities of a multi-process product, managers have to obtain sufficient and reliable information in order to implement quality improvements. This study attempts to improve several insufficiencies to enable managers to make such informed decisions. First, this study reconstructed an efficient method via the S_{pa} -PCAC, which not only represented multiple process capabilities with asymmetric bilateral specifications by PCIs S_{pa} and C_a but also used PCIs C_{pu} and C_{pl} to assess multiple process capabilities categorized into the STB and LTB types displayed on the horizontal and vertical axes respectively. This study selected index S_{pa} , superior to index S_{pk} , especially when evaluating process capabilities with asymmetric bilateral specifications. Second, as stated by Cheng [26] as well as Pearn and Chen [37], a large degree of uncertainty or risk needs to be taken into account regarding capability assessments for sampling errors. To reliably evaluate the quality of an entire product consisting of multiple characteristics, the proposed S_{pa} -PCAC considered the interval estimates of the used indices. Third, based on the loss function stated in the Taguchi method [38-42], it can be learned that as long as the mean is closer to the target, the quality will be the better and the loss will be less. Index C_a was incorporated into the S_{pa} -PCAC because it is adequate to assess the accuracy of the manufacturing process. Lastly, the quality-level of Six-Sigma and the connection with process yield were introduced in the capability zone of S_{pa} -PCAC to check whether the process capabilities could reach the preset level. The S_{pa} -PCAC explained the multi-characteristics process capabilities and distinguished the process accuracy from the locations of the interval estimates of PCIs. Furthermore, with respect to the unsatisfactory processes, actions for quality improvement were taken to enhance the entire process capabilities. Consequently, our proposed model is a valuable tool for managers to improve their manufacturing processes. Finally, when the process distribution is abnormal, the model proposed in this study is not applicable. This is the research limitation of this paper. The confidence interval derived by the mathematical programming method in this article is easier than the traditional confidence interval, so it is more suitable for the industry which emphasizes timeliness. As to how much higher than $100(1-\alpha)$ % it should be, it is a more complicated issue. It is recommended to refer to the method proposed Chen et al. [34] for discussion. This study will list it as a future research topic.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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