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Research article

A note on advection-diffusion cholera model with bacterial hyperinfectivity

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Abstract: This note gives a supplement to the recent work of Wang and Wang (2019) in the sense that: (i) for the critical case where $\mathfrak{R}_0 = 1$, cholera-free steady state is globally asymptotically stable; (ii) in a homogeneous case, the positive constant steady-state is globally asymptotically stable with additional condition when $\mathfrak{R}_0 > 1$. Our first result is achieved by proving the local asymptotic stability and global attractivity. Our second result is obtained by Lyapunov function.

Keywords: advection-diffusion cholera model; hyperinfectivity; spatial heterogeneity; basic reproduction number; global stability

1. Introduction

This note is motivated by a recent work of [1], of which a cholera dynamical model with space dependent parameters and bacterial hyperinfectivity is investigated. It is evident that risk factors for cholera are diverse and originate from multiple routes of transmission, which allow us to rely on advection-diffusion equations to describe the transport of a pathogen into host population along a theoretical river. From the standpoint of theoretical and applicable importance, in contrast to the previous studies, Wang and Wang [1] distinguishes state of V. cholerae in the water environment as $V_1(x,t)$ (hyperinfectious (HI)) and $V_2(x,t)$ (lower-infectious (LI)) vibrios to measure the infectivity of vibrios, where x and t are spatial and time variables, respectively. Recent advances on cholera dynamical model can be found in [2–7].

We first introduce the model proposed in [1], that builds up our question. Let $x = 0$ be the upstreams and L be the downstreams of the river. If there are no specific requirements, we suppose that the model equations are governed in the domain $(x,t) \in (0,L) \times (0,\infty)$, initial condition at time $t = 0$ are given for $x \in [0,L]$ and boundary condition at time $t > 0$ are given for $x = 0, L$, respectively. In [1], the following advection-diffusion equations was proposed:

$$\begin{cases} \dfrac{\partial U}{\partial t} - D_U \Delta U = \Lambda(x) - UG(x,I) - U\left[H_1(x,V_1) + H_2(x,V_2)\right] - \mu(x)U + \zeta(x)R, \\ \dfrac{\partial I}{\partial t} - D_I \Delta I = UG(x,I) + U\left[H_1(x,V_1) + H_2(x,V_2)\right] - (\mu(x) + \theta(x) + \rho(x))I, \\ \dfrac{\partial R}{\partial t} - D_R \Delta R = \rho(x)I - [\mu(x) + \zeta(x)]R, \\ \dfrac{\partial V_1}{\partial t} - D_{V_1}\Delta V_1 = -n_{V_1}\dfrac{\partial V_1}{\partial x} + \xi(x)I + B_1(x,V_1) - \delta_1(x)V_1, \\ \dfrac{\partial V_2}{\partial t} - D_{V_2}\Delta V_2 = -n_{V_2}\dfrac{\partial V_2}{\partial x} + \delta_1(x)V_1 + B_2(x,V_2) - \delta_2(x)V_2, \end{cases} \tag{1.1}$$

with initial and boundary condition

$$\begin{cases} z(x,0) = z^0(x) \geq 0,\ z = U, I, R, V_1, V_2, \text{ respectively}, \\ \dfrac{\partial z}{\partial x}(0,t) = 0,\ z = U, I, R, \text{ respectively};\ D_z\dfrac{\partial z}{\partial x}(0,t) - n_z z(0,t) = 0,\ z = V_1, V_2, \text{ respectively}, \\ \dfrac{\partial z}{\partial x}(L,t) = 0,\ z = U, I, R, V_1, V_2, \text{ respectively}. \end{cases} \tag{1.2}$$

Here U, I and R are the human densities for susceptible, infectious and recovered. D_z, $z = U, I, R, V_1, V_2$, stand for the diffusion coefficient. n_{V_1} and n_{V_2} are the convection coefficient of two states of vibrios along the river. $\Lambda(\cdot)$ represents the influx rate. The nonlinear functions $UG(\cdot, I)$, $UH_1(\cdot, V_1)$ and $UH_2(\cdot, V_2)$ stand for transmission rate among susceptible humans, infectious humans and two states of vibrios. $\mu(\cdot)$, $\delta_1(\cdot)$ and $\delta_2(\cdot)$ represent respectively the natural death rate. $\rho(\cdot)$ represents the recovery rate. $\theta(\cdot)$ is the additional death rate. $\xi(\cdot)$ represents the shedding rate of vibrios from infectious individuals, respectively. $\zeta(\cdot)$ represents the rate that recovered hosts will lose immunity. $B_1(\cdot, V_1)$ and $B_2(\cdot, V_2)$ denote the saturation growth rate of two states of vibrios in the water environment, respectively. We assume that all parameters are positive functions on $[0, L]$. Let $\mathcal{F}(\cdot, m) = G(\cdot, m)$, $H_1(\cdot, m)$, $H_2(\cdot, m)$ and $B_i(\cdot, m), i = 1, 2$, respectively. Biologically, for $m \geq 0$, $\mathcal{F}$ satisfies:

(A1) : $\mathcal{F}(\cdot, 0) = 0$, $\frac{\partial \mathcal{F}}{\partial m} > 0$ and $\frac{\partial^2 \mathcal{F}}{\partial m^2} \leq 0$;
(A2) : For $B_i(\cdot, t), i = 1, 2,$ there exists $K_i > 0$ such that $B_i(\cdot, V_i) \leq 0$ for all $V_i \geq K_i$. Further, $\frac{\partial B_i}{\partial V_i}(\cdot, 0) < \delta_i(\cdot)$.

The well-posedness of (1.1)–(1.2), that is, the existence of global solution and ultimate boundedness of solution have been confirmed (see Lemma 3.4 [1]). Further, the continuous semiflow $\Phi(t)$ induced by (1.1)–(1.2) possesses a global compact attractor $\mathcal{E}$. Clearly, $E_0 = (U^*(\cdot), 0, 0, 0, 0)$ is the a cholera-free steady state of (1.1), where $U^*(\cdot)$ satisfies

$$D_U \Delta U^*(\cdot) + \Lambda(\cdot) - \mu(\cdot)U^*(\cdot) = 0 \text{ with } \frac{\partial U^*(0)}{\partial x} = \frac{\partial U^*(L)}{\partial x} = 0. \tag{1.3}$$

We now briefly give the basic reproduction number (BRN) of (1.1) by the method developed in [8]. Linearizing (1.1)–(1.2) at E_0 obtains below linear cooperative system for infectious compartments (with $\mathbf{u} = (I, V_1, V_2)^T$):

$$
\begin{cases}
\dfrac{\partial \mathbf{u}}{\partial t} = \mathcal{B}\mathbf{u} = (\mathbb{F}+\mathbb{B})\mathbf{u},\\
\dfrac{\partial I}{\partial x}(L,t) = \dfrac{\partial I}{\partial x}(0,t) = 0,\\
\dfrac{\partial V_i}{\partial x}(L,t) = D_{V_i}\dfrac{\partial V_i}{\partial x}(0,t) - n_{V_i}V_i(0,t) = 0,\ i=1,2,
\end{cases}
\tag{1.4}
$$

where

$$
\mathbb{F}(\cdot) = \begin{pmatrix} U^*(\cdot)G_I(\cdot,0) & U^*(\cdot)H_{1V_1}(\cdot,0) & U^*(\cdot)H_{2V_2}(\cdot,0) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

and

$$
\mathbb{B} = \begin{pmatrix} D_I\Delta + h_1 & 0 & 0 \\ \xi(\cdot) & D_1\Delta + h_2 & 0 \\ 0 & \delta_1(\cdot) & D_2\Delta + h_3 \end{pmatrix}
$$

with $h_1 = -(\mu(\cdot)+\rho(\cdot)+\theta(\cdot))$, $h_2 = -n_{V_1}\frac{\partial}{\partial x} + B_{1V_1}(\cdot,0) - \delta_1(\cdot)$ and $h_3 = -n_{V_2}\frac{\partial}{\partial x} + B_{2V_2}(\cdot,0) - \delta_2(\cdot)$.

Let $\mathbb{X} = C([0,L],\mathbb{R}^3)$ with general supreme norm

$$
\|\psi\|_{\mathbb{X}} := \max\{ \sup_{x\in[0,L]} |\psi_1(\cdot)|,\ \sup_{x\in[0,L]} |\psi_2(\cdot)|,\ \sup_{x\in[0,L]} |\psi_3(\cdot)|\ \},\ \psi = (\psi_1,\psi_2,\psi_3) \in \mathbb{X},
$$

and $\Pi(t) : \mathbb{X} \to \mathbb{X}$ (resp. $\bar{\Pi}(t) : \mathbb{X} \to \mathbb{X}$) be the solution semigroup with generator $\mathcal{B}$ (resp. $\mathbb{B}$). Then $\bar{\Pi}(t)\phi(\cdot)$ stands for the distribution by introducing initial cases $\phi(\cdot)$ over time. $\mathbb{F}(\cdot)\bar{\Pi}(t)\phi(\cdot)$ represents the distribution of new infection. Hence the next generation operator is the following positive operator on $\mathbb{X}$,

$$
\mathcal{L}(\phi)(\cdot) = \int_0^\infty \mathbb{F}(\cdot)(\bar{\Pi}(t)\phi)(\cdot)\mathrm{d}t,\ \phi \in \mathbb{X}. \tag{1.5}
$$

Substituting $\mathbf{u} = e^{\lambda t}\psi$ with $\psi = (\psi_0(\cdot),\psi_1(\cdot),\psi_2(\cdot))$ into (1.4), which allows us to study the following eigenvalue problem

$$
\begin{cases}
\lambda\psi = \mathcal{B}\psi,\\
\dfrac{\partial\psi_0(L)}{\partial x} = \dfrac{\partial\psi_0(0)}{\partial x} = 0,\\
\dfrac{\partial\psi_i(L)}{\partial x} = D_{V_i}\dfrac{\partial\psi_i(0)}{\partial x} - n_{V_i}\psi_i(0) = 0,\ i=1,2.
\end{cases}
\tag{1.6}
$$

The following result gives the expression of BRN, $\mathfrak{R}_0$, principle eigenvalue of (1.6) and the relationship between them.

(R1) : $\mathfrak{R}_0 = r(\mathcal{L})$, where $r(\mathcal{L})$ is the spectral radius of $\mathcal{L}$;
(R2) : $s(\mathcal{B})$ is the principal eigenvalue of (1.6), where $s(\mathcal{B})$ is the spectral bound of $\mathcal{B}$;
(R3) : $\mathrm{sign}(\mathfrak{R}_0 - 1) = \mathrm{sign}(s(\mathcal{B}))$.

These assertions are obvious, and also can be found in Theorem 3.5 [9] and Lemma 2.2 [8]. By using the BRN, $\mathfrak{R}_0$, the following sharp threshold dynamics of (1.1)–(1.2) was obtained.

Theorem 1.1. *Let $\mathfrak{R}_0 = r(\mathcal{L})$ and $z \in \{U,I,R,V_1,V_2\}$ where (U,I,R,V_1,V_2) is the solution of system (1.1)–(1.2) [1, Theorem 3.1], then we have*

(i) *If* $\Re_0 < 1$ *and* $\zeta(\cdot) \equiv 0$, *then* E_0 *of system (1.1)–(1.2) is globally attractive.*
(ii) *If* $\Re_0 > 1$, *for any* $z^0(\cdot) \in C([0, L], \mathbb{R}^5_+)$ *with* $I^0(\cdot) \not\equiv 0$ *or* $V_1^0(\cdot) \not\equiv 0$ *or* $V_2^0(\cdot) \not\equiv 0$, *then exists* $\sigma_* > 0$ *such that*

$$\liminf_{t\to\infty} z(\cdot, t; z^0) \geq \sigma_*, \ uniformly\ holds.$$

Furthermore, in terms of homogeneous environmental conditions, the global stability of positive equilibrium have been considered. The dependence $\Re_0$ on model parameters was shown by the analytical and numerical approaches. It comes naturally to a question: When $\Re_0 = 1$, what happens to the dynamics of E_0 for system (1.1)–(1.2)? In fact, the method used for the case of $\Re_0 > 1$(or < 1) cannot be directly applied to such a critical case. Thus, dealing with this question is the first motivation of current work. Our second motivation is inspired by [10–13], the method developed there can indeed show the dynamics of cholera-free steady state if $\Re_0 = 1$. The first result of current work reads as:

Theorem 1.2. *Let* $\Re_0 = r(\mathcal{L})$ *and* $\mathbb{Y} = C([0, L], \mathbb{R}^5)$, *then we have the following results:*

(i) *If* $\Re_0 = 1$ *and* $\zeta(\cdot) \equiv 0$, *then* E_0 *is locally asymptotically stable in* $\mathbb{Y}$.
(ii) *If* $\Re_0 = 1$ *and* $\zeta(\cdot) \equiv 0$, *then* E_0 *is globally attractive in* $\mathbb{Y}$.

In other words, E_0 is globally asymptotically stable in $\mathbb{Y}$ when $\Re_0 = 1$ and $\zeta(\cdot) \equiv 0$. Before going into proving Theorem 1.2, we first present the known result for the critical case of $\Re_0 = 1$. $s(\mathcal{B}) = \omega(\Pi(t)) = 0$ is the principle eigenvalue of (1.6) (see (R2) and (R3)), corresponding to $s(\mathcal{B}) = 0$, there is a positive eigenvector, where $\omega(\Pi(t))$ represents the exponential growth bound. Further, $\|\Pi(t)\| \leq \mathbb{M}_0$ for some $\mathbb{M}_0 > 0$.

In [1], the authors considered the global stability of E^* by using Lyapunov functions when all the parameters are constants, where $E^* = (\tilde{U}, \tilde{I}, \tilde{V}_1, \tilde{V}_2)$ is defined as the positive constant steady state. In a special case that $\zeta(\cdot) \equiv 0$, $n_{V_i} = 0$ and $B_i(\cdot, V_i) \equiv 0$, $i = 1, 2$, in system (1.1)–(1.2), we continue to consider the following model:

$$\begin{cases} \dfrac{\partial U}{\partial t} - D_U \Delta U = \Lambda - UG(I) - U\left[H_1(V_1) + H_2(V_2)\right] - \mu U, \\ \dfrac{\partial I}{\partial t} - D_I \Delta I = UG(I) + U\left[H_1(V_1) + H_2(V_2)\right] - (\mu + \theta + \rho)I, \\ \dfrac{\partial V_1}{\partial t} - D_{V_1} \Delta V_1 = \xi I - \delta_1 V_1, \\ \dfrac{\partial V_2}{\partial t} - D_{V_2} \Delta V_2 = \delta_1 V_1 - \delta_2 V_2, \end{cases} \tag{1.7}$$

with initial and boundary condition (1.2). Similarly, if there are no specific requirements, we suppose that the model equations are governed in the domain $(x, t) \in (0, L) \times (0, \infty)$. In [11], the global stability of E^* is achieved when $G(I) = \alpha I$ and $H_i(V_i) = \beta_i \frac{V_i}{V_i + K_i}$, $i = 1, 2$. In fact, without simulation purpose, global stability of E^* with general incidence functions $G(I)$ and $H_i(V_i)$ can be achieved by the same Lyapunov function with additional condition. The second result of current work reads as:

Theorem 1.3. *Suppose that*

(A3) $\left(\frac{I}{\tilde{I}} - \frac{G(I)}{G(\tilde{I})}\right)\left(\frac{G(\tilde{I})}{G(I)} - 1\right) \leq 0$, $\left(\frac{V_i}{\tilde{V}_i} - \frac{H_i(V_i)}{H_i(\tilde{V}_i)}\right)\left(\frac{H_i(\tilde{V}_i)}{H_i(V_i)} - 1\right) \leq 0, i = 1, 2,$

holds. Then the positive steady-state solution E^* *of system (1.7) is globally asymptotically stable if* $\Re_0 > 1$.

2. Proof of Theorem 1.2

Proof of (i) of Theorem 1.2. Let $\tilde{\sigma} > 0$. Assume that initial data is around of E_0, i.e., for small $\varsigma > 0$, $\|\phi - E_0\| \le \varsigma$.

Define

$$w_1(\cdot,t) = \frac{U(\cdot,t)}{U^*(\cdot)} - 1 \text{ and } \Theta(t) = \max_{x\in[0,L]}\{w_1(\cdot,t),0\}.$$

By using the equality (1.3), we rewrite the U equation as

$$\frac{\partial w_1}{\partial t} - D_U\Delta w_1 - 2D_U\frac{\nabla U^*(\cdot)\cdot\nabla w_1}{U^*(\cdot)} + \frac{\Lambda(\cdot)}{U^*(\cdot)}w_1 = -\frac{U\Big(G(\cdot,I) + H_1(\cdot,V_1) + H_2(\cdot,V_2)\Big)}{U^*(\cdot)}.$$

Solving above equation yields

$$w_1(\cdot,t) = T_1(t)w_1^0 - \int_0^t T_1(t-s)\frac{U(\cdot,s)\Big(G(\cdot,I(\cdot,s)) + H_1(\cdot,V_1(\cdot,s)) + H_2(\cdot,V_2(\cdot,s))\Big)}{U^*(\cdot)}\mathrm{d}s,$$

where $w_1^0 = U^0/U^* - 1$, and $T_1(t)$ the positive semigroup induced by

$$D_U\Delta + 2D_U\frac{\nabla U^*(\cdot)\cdot\nabla}{U^*(\cdot)} - \frac{\Lambda(\cdot)}{U^*(\cdot)},$$

which satisfies $\|T_1(t)\| \le \mathbb{M}_1e^{-rt}$ for some $\mathbb{M}_1 > 0$ and $r > 0$. From the positivity of $T_1(t)$, we get

$$\Theta(t) \le \max_{x\in[0,L]}\{T_1(t)w_1^0, 0\} \le \|T_1(t)w_1^0\| \le \mathbb{M}_1e^{-rt}\left\|\frac{U^0}{U^*(x)} - 1\right\| \le \varsigma\frac{\mathbb{M}_1e^{-rt}}{\tilde{U}^*}, \tag{2.1}$$

where $\tilde{U}^* = \min_{x\in[0,L]} U^*(\cdot)$. Hence,

$$U(\cdot,t) - U^* = U^*\Big(\frac{U(\cdot,t)}{U^*} - 1\Big) \le \|U^*\|\Theta(t) \le \frac{\varsigma\mathbb{M}_1\|U^*\|}{\tilde{U}^*}. \tag{2.2}$$

Further by (A1), it gives

$$G(\cdot,I) \le G_I(\cdot,0)I,\ H_i(\cdot,V_i) \le H_{iV_i}(\cdot,0)V_i, \text{ and } B_i(\cdot,V_i) \le B_{iV_i}(\cdot,0)V_i. \tag{2.3}$$

Thus, from hypothesis $\zeta(\cdot) \equiv 0$ and system (1.1), we know that (I, V_1, V_2) satisfies

$$\begin{cases} \dfrac{\partial I}{\partial t} \le D_I\Delta I + U^*(\cdot)G_I(\cdot,0)I + U^*(\cdot)(H_{1V_1}(\cdot,0)V_1 + H_{2V_2}(\cdot,0)V_2) - (\mu(\cdot)+\theta(\cdot)+\rho(\cdot))I \\ \qquad +U^*(\dfrac{U(\cdot)}{U^*} - 1)G_I(\cdot,0)I + U^*(\dfrac{U(\cdot)}{U^*} - 1)(H_{1V_1}(\cdot,0)V_1 + H_{2V_2}(\cdot,0)V_2), \\ \dfrac{\partial V_1}{\partial t} \le D_{V_1}\Delta V_1 - n_{V_1}\dfrac{\partial V_1}{\partial x} + \xi(\cdot)I + B_{1V_1}(\cdot,0)V_1 - \delta_1(\cdot)V_1, \\ \dfrac{\partial V_2}{\partial t} \le D_{V_2}\Delta V_2 - n_{V_2}\dfrac{\partial V_2}{\partial x} + \delta_1(\cdot)V_1 + B_{2V_2}(\cdot,0)V_2 - \delta_2(\cdot)V_2, \\ z(x,0) = z^0(x) \ge 0,\ z = I, V_1, V_2,\ \text{respectively}, \\ \dfrac{\partial I}{\partial x}(0,t) = 0;\ D_z\dfrac{\partial z}{\partial x}(0,t) - n_z z(0,t) = 0,\ z = V_1, V_2,\ \text{respectively}, \\ \dfrac{\partial z}{\partial x}(L,t) = 0,\ z = I, V_1, V_2,\ \text{respectively}. \end{cases}$$

Namely, by a zero trick, we know that $\mathbf{u}(\cdot, t)$ satisfies

$$\frac{\partial \mathbf{u}}{\partial t} \leq \mathscr{B}\mathbf{u} + (\mathcal{H}, 0, 0)^T,$$

where $\mathcal{H}(\cdot, t) = U^*(\frac{U(\cdot)}{U^*} - 1)G_I(\cdot, 0)I + U^*(\frac{U(\cdot)}{U^*} - 1)(H_{1V_1}(\cdot, 0)V_1 + H_{2V_2}(\cdot, 0)V_2)$. Hence,

$$\mathbf{u}(\cdot, t) \leq \Pi(t)\mathbf{u}^0(\cdot) + \int_0^t \Pi(t-s)\,(\mathcal{H}(\cdot, s), 0, 0)^T\,\mathrm{d}s.$$

Since R equation is decoupled from the other equations in (1.1), we only focus on the I, V_1, V_2 equations. By (2.1), we directly get

$$\begin{aligned}
\max\{\|I(\cdot, t)\|, \|V_1(\cdot, t)\|, \|V_2(\cdot, t)\|\} \leq & \mathbb{M}_0 \max\{\|I^0\|, \|V_1^0\|, \|V_2^0\|\} \\
& + \mathbb{M}_0\alpha\|U^*\| \int_0^t \Theta(s)(\|I(s)\| + \|V_1(s)\| + \|V_2(s)\|)\mathrm{d}s \\
\leq & \mathbb{M}_0\varsigma + \mathbb{M}_2\varsigma \int_0^t e^{-rs}(\|I(s)\| + \|V_1(s)\| + \|V_2(s)\|)\mathrm{d}s,
\end{aligned}$$

where $\alpha = \max\{\max\{G_I(\cdot, 0)\}, \max\{H_{1V_1}(\cdot, 0)\}, \max\{H_{2V_2}(\cdot, 0)\}\}$, $\mathbb{M}_2 = \mathbb{M}_0\mathbb{M}_1\alpha\|U^*\|/\tilde{U}^*$. This yields that

$$\|I(\cdot, t)\| + \|V_1(\cdot, t)\| + \|V_2(\cdot, t)\| \leq 3\mathbb{M}_0\varsigma + 3\mathbb{M}_2\varsigma \int_0^t e^{-rs}(\|I(\cdot, s)\| + \|V_1(\cdot, s)\| + \|V_2(\cdot, s)\|)\mathrm{d}s.$$

With the aid of Gronwall's inequality, one obtains

$$\|I(\cdot, t)\| + \|V_1(\cdot, t)\| + \|V_2(\cdot, t)\| \leq 3\mathbb{M}_0\varsigma e^{\int_0^t 3\mathbb{M}_2\varsigma e^{-rs}\mathrm{d}s} \leq 3\mathbb{M}_0\varsigma e^{\frac{3\mathbb{M}_2\varsigma}{r}}. \tag{2.4}$$

Let $\hat{U}$ be the solution of

$$\begin{cases}
\dfrac{\partial \hat{U}}{\partial t} = D_U\Delta\hat{U} + \Lambda(\cdot) - (\mu(\cdot) + K)\hat{U}, \\
\hat{U}(\cdot, 0) = U^0(\cdot), \\
\dfrac{\partial \hat{U}}{\partial x}(L, t) = \dfrac{\partial \hat{U}}{\partial x}(0, t) = 0,
\end{cases} \tag{2.5}$$

where $K = 3\alpha\mathbb{M}_0\varsigma e^{\frac{3\mathbb{M}_2\varsigma}{r}}$. Further from (2.4) and (2.3), combined with comparison argument, $U(\cdot, t) \geq \hat{U}(\cdot, t)$, $(\cdot, t) \in (0, L) \times (0, \infty)$. Let U_ς^* be the positive steady state of (2.5). By letting $\vartheta := \hat{U} - U_\varsigma^*$, it then follows that ϑ satisfies

$$\begin{cases}
\dfrac{\partial \vartheta}{\partial t} = D_U\Delta\vartheta - (\mu(\cdot) + K)\vartheta, \\
\vartheta(\cdot, 0) = U^0 - U_\varsigma^*, \\
\dfrac{\partial \vartheta}{\partial x}(L, t) = \dfrac{\partial \vartheta}{\partial x}(0, t) = 0.
\end{cases} \tag{2.6}$$

Let $T_2(t)$ be the semigroup induced by $D_U\Delta - \mu(\cdot)$ and $\mu_* = \min_{x\in[0,L]}\{\mu(x)\}$. Solving (2.6) yields

$$\vartheta(\cdot, t) = T_2(t)(U^0 - U_\varsigma^*) - \int_0^t T_2(t-s)K\vartheta(\cdot, s)\mathrm{d}s.$$

Choosing $\mathbb{M}_3 > 0$ large enough that $\|T_2(t)\| \le \mathbb{M}_3 e^{-\mu_* t}$, which produces

$$\|\vartheta(\cdot,t)\| \le \mathbb{M}_3 \|U^0 - U^*_\varsigma\| e^{-\mu_* t} + \int_0^t \mathbb{M}_3 e^{-\mu_*(t-s)} K \|\vartheta(\cdot,s)\| \mathrm{d}s.$$

Again from the Gronwall's inequality,

$$\|\hat{U}(\cdot,t) - U^*_\varsigma\| = \|\vartheta(\cdot,t)\| \le \mathbb{M}_3 \|U^0 - U^*_\varsigma\| e^{\tilde{K}t - \mu_* t},$$

where $\tilde{K} = K\mathbb{M}_3$. Further, by letting $\varsigma > 0$ small enough that $\tilde{K} < \frac{\mu_*}{2}$, we then have

$$\|\hat{U}(\cdot,t) - U^*_\varsigma\| \le \mathbb{M}_3 \|U^0 - U^*_\varsigma\| e^{-\frac{\mu_*}{2}t}. \tag{2.7}$$

Recall that $U(\cdot,t) \ge \hat{U}(\cdot,t)$. This combined with (2.7) and a zero trick indicate that

$$\begin{aligned} U(\cdot,t) - U^* \ge \hat{U}(\cdot,t) - U^* &= \hat{U}(\cdot,t) - U^*_\varsigma + U^*_\varsigma - U^* \\ &\ge -\mathbb{M}_3 \|U^0 - U^*_\varsigma\| e^{-\frac{\mu_*}{2}t} + U^*_\varsigma - U^* \\ &\ge -\mathbb{M}_3 (\|U^0 - U^*\| + \|U^* - U^*_\varsigma\|) - \|U^*_\varsigma - U^*\| \\ &\ge -\mathbb{M}_3 \varsigma - (\mathbb{M}_3 + 1)\|U^*_\varsigma - U^*\|. \end{aligned} \tag{2.8}$$

By (2.2) and (2.8), we get

$$\|U(\cdot,t) - U^*\| \le \max\left\{ \mathbb{M}_3 \varsigma + (\mathbb{M}_3 + 1)\|U^*_\varsigma - U^*\|, \frac{\varsigma \mathbb{M}_1 \|U^*\|}{\tilde{U}^*} \right\}. \tag{2.9}$$

Consequently, by (2.4), (2.9) and $\lim_{\varsigma \to 0} U^*_\varsigma = U^*$,

$$\|U(\cdot,t) - U^*\|,\ \|I(\cdot,t)\|,\ \|V_1(\cdot,t)\| \text{ and } \|V_2(\cdot,t)\| \le \tilde{\sigma},\ \forall\, t > 0,$$

which is achieved by choosing $\varsigma = \varsigma(\tilde{\sigma}) > 0$ small enough.

Proof of (ii) of Theorem 1.2. We shall prove that $\mathcal{E} = \{E_0\}$, where $\mathcal{E}$ is a global attractor of $\Phi(t)$.

We first confirm that

- For any $\phi = (U^0, I^0, V_1^0, V_2^0) \in \mathcal{E}$, $\omega(\phi) \subset \partial\mathbb{X}_1 := \{(U, I, V_1, V_2) \in \mathbb{X}^+ : I = V_1 = V_2 = 0\}$, where $\omega(\cdot)$ is the omega limit set.

From Lemma 1 [14], we know that for any $D_P > 0$, $\Lambda(\cdot)$ and $\mu(\cdot)$ which are continuous and positive on $[0, L]$ and $P^0(\cdot) \not\equiv 0$, the following scalar reaction-diffusion equation,

$$\begin{cases} \dfrac{\partial P}{\partial t} = D_P \Delta P + \Lambda(\cdot) - \mu(\cdot)P, \\ \dfrac{\partial P}{\partial x}(L,t) = \dfrac{\partial P}{\partial x}(0,t) = 0, \\ P(\cdot,0) = P^0(\cdot), \end{cases} \tag{2.10}$$

admits a unique positive steady state $U^*(\cdot)$, which is globally asymptotically stable in $C([0,L], \mathbb{R}_+)$. It follows from the U equation of (1.1) that

$$\begin{cases} \dfrac{\partial U}{\partial t} - D_U \Delta U \le \Lambda(\cdot) - \mu(\cdot)U, \\ \dfrac{\partial U}{\partial x}(L,t) = \dfrac{\partial U}{\partial x}(0,t) = 0. \end{cases}$$

From the standard parabolic comparison theorem, we have that

$$\limsup_{t\to\infty} U(\cdot,t) \le \limsup_{t\to\infty} P(\cdot,t) = U^*(\cdot), \text{ uniformly for } x \in [0,L]. \tag{2.11}$$

Hence, we have that $U^0 \le U^*(\cdot)$. Since $\partial\mathbb{X}_1$ is invariant for $\Phi(t)$, the claim directly follow if $I^0 = V_1^0 = V_2^0 = 0$. Hence we assume that $I^0 \neq 0$ or $V_1^0 \neq 0$ or $V_2^0 \neq 0$. By Lemma 3.5 [1], we know that $\tilde{\mathbf{u}}(\cdot,t) > 0$, where $\tilde{\mathbf{u}} = U, I, R, V_1, V_2$, respectively. Hence, U satisfies that

$$\begin{cases} \dfrac{\partial U}{\partial t} - D_U \Delta U < \Lambda(\cdot) - \mu(\cdot)U, \\ \dfrac{\partial U}{\partial x}(L,t) = \dfrac{\partial U}{\partial x}(0,t) = 0, \\ U(\cdot,0) \le U^*, \end{cases}$$

By the comparison principal, we must have $U(\cdot,t) < U^*(\cdot)$ uniformly holds.

Inspired by [10, 12], let

$$\epsilon(t;\phi) := \inf\{\tilde{\epsilon} \in \mathbb{R} : I(\cdot,t) \le \tilde{\epsilon}\psi_0,\ V_1(\cdot,t) \le \tilde{\epsilon}\psi_1 \text{ and } V_2(\cdot,t) \le \tilde{\epsilon}\psi_2\}.$$

Then $\epsilon(t;\phi) > 0$, $t > 0$. We next prove the strictly decreasing property of $\epsilon(t;\phi)$. In fact, let us fix $t_1 > 0$ and define

$$\bar{I}(\cdot,t) = \epsilon(t_1;\phi)\psi_0,\ \bar{V}_1(\cdot,t) = \epsilon(t_1;\phi)\psi_1 \text{ and } \bar{V}_2(\cdot,t) = \epsilon(t_1;\phi)\psi_2, \text{ for } t \ge t_1.$$

By $U(\cdot,t) < U^*(\cdot)$ and $\bar{\mathbf{u}} = (\bar{I}, \bar{V}_1, \bar{V}_2)^T$, we have

$$\begin{cases} \dfrac{\partial \bar{\mathbf{u}}}{\partial t} \ge \mathcal{B}_1 \bar{\mathbf{u}}, \\ \bar{\mathbf{u}}(\cdot,t_1) \ge \mathbf{u}(\cdot,t_1), \\ \dfrac{\partial \bar{I}}{\partial x}(L,t) = \dfrac{\partial \bar{I}}{\partial x}(0,t) = 0, \\ \dfrac{\partial \bar{V}_i}{\partial x}(L,t) = D_{V_i}\dfrac{\partial \bar{V}_i}{\partial x}(0,t) - n_{V_i}\bar{V}_i(0,t) = 0,\ i = 1,2, \end{cases}$$

where

$$\mathcal{B}_1 := \begin{pmatrix} D_I\Delta + \tilde{h}_1 & \alpha U & \alpha U \\ \xi(\cdot) & D_1\Delta + \tilde{h}_2 & 0 \\ 0 & \delta_1(\cdot) & D_2\Delta + \tilde{h}_3 \end{pmatrix}$$

with $\tilde{h}_1 = -(\mu(\cdot) + \rho(\cdot) + \theta(\cdot)) + \alpha U$, $\tilde{h}_2 = -n_{V_1}\frac{\partial}{\partial x} + B_{1V_1}(\cdot,0) - \delta_1(\cdot)$ and $\tilde{h}_3 = -n_{V_2}\frac{\partial}{\partial x} + B_{2V_2}(\cdot,0) - \delta_2(\cdot)$.

Hence, for all $(x,t) \in (0,L) \times (t_1,\infty)$,

$$\bar{\mathbf{u}}(\cdot,t) \ge \mathbf{u}(\cdot,t),\ \forall(\cdot,t) \in (0,L) \times (t_1,\infty),$$

by the comparison principle. Further,

$$\epsilon(t_1;\phi)\psi_0 = \bar{I}(\cdot,t) > I(\cdot,t),\ \epsilon(t_1;\phi)\psi_1 = \bar{V}_1(\cdot,t) > V_1(\cdot,t) \text{ and } \epsilon(t_1;\phi)\psi_2 = \bar{V}_2(\cdot,t) > V_2(\cdot,t).$$

Due to the arbitraryness of $t_1 > 0$, the strictly decreasing property of $\epsilon(t;\phi)$ directly follows.

Denote by $\epsilon_* = \lim_{t\to\infty} \epsilon(t;\phi)$. In fact, by setting $Q = (Q_2, Q_3, Q_4) \in \omega(\phi)$. It follows that there exists $\{t_k\}$ with $t_k \to \infty$ such that $\Phi(t_k)\phi \to Q$. By the following equality,

$$\lim_{t_k\to\infty} \Phi(t + t_k)\phi = \Phi(t) \lim_{t_k\to\infty} \Phi(t_k)\phi = \Phi(t)Q,$$

we directly get $\epsilon(t;Q) = \epsilon_*$, $\forall\ t \geq 0$. If $Q_2 \neq 0$ or $Q_3 \neq 0$ or $Q_4 \neq 0$, repeat the above procedures if necessary, one can obtain the strictly decreasing property of $\epsilon(t;Q)$, which leads to the contradict with $\epsilon(t;Q) = \epsilon_*$. Consequently, $Q_2 = Q_3 = Q_4 = 0$ and $\mathbf{u} \to \mathbf{0}$ as $t \to \infty$. Further, $U(\cdot,t) \to U^*(\cdot)$ as $t \to \infty$.

We next confirm that $\mathcal{E} = \{E_0\}$. From the discussions above, $\{E_0\}$ is globally attractive in $\partial\mathbb{X}_1$. Further, $\{E_0\}$ forms the only compact invariant subset in $\partial\mathbb{X}_1$. Then, $\omega(\phi) \subset \partial\mathbb{X}_1$ for any $\phi \in \mathcal{E}$, which leads to $\omega(\phi) = \{E_0\}$. By Lemma 3.4 [1], we know that $\mathcal{E}$ is compact invariant in $C([0,L],\mathbb{R}^5)$. This combined with Lemma 3.11 [12] indicate that $\mathcal{E} = \{E_0\}$. This proves Theorem 1.2.

Remark 1. Theorem 1.2 still holds if the nonlinear incidence functions $UG(\cdot,I)$, $UH_1(\cdot,V_1)$ and $UH_2(\cdot,V_2)$ are replaced by general nonlinear incidence $G(\cdot,U,I)$, $H_1(\cdot,U,V_1)$ and $H_2(\cdot,U,V_2)$.

3. Proof of Theorem 1.3

For any positive solution $(U(\cdot,t), I(\cdot,t), V_1(\cdot,t), V_2(\cdot,t))$ of (1.7), from the proof of Theorem 3.1 (i) [1], we know that $\frac{U}{\tilde{U}}$, $\frac{I}{\tilde{I}}$, $\frac{V_1}{\tilde{V}_1}$ and $\frac{V_2}{\tilde{V}_2}$ are bounded and bounded away from zero. Inspired by [15–17], one consider the following Lyapunov function:

$$W(t) := \int_\Omega L(U(\cdot,t), I(\cdot,t), V_1(\cdot,t), V_2(\cdot,t))\mathrm{d}x,$$

where

$$L := L(U,I,V_1,V_2) = a_0\left(U - \tilde{U} - \tilde{U}\ln\frac{U}{\tilde{U}}\right) + a_0\left(I - \tilde{I} - \tilde{I}\ln\frac{I}{\tilde{I}}\right) + \sum_{i=1}^{2} a_i\left(V_i - \tilde{V}_i - \tilde{V}_i\ln\frac{V_i}{\tilde{V}_i}\right)$$

and

$$\begin{cases} a_0 = \delta_1\tilde{V}_1, \\ a_1 = \tilde{U}\left[H_1(\tilde{V}_1) + H_2(\tilde{V}_2)\right]\dfrac{\delta_1\tilde{V}_1}{\xi\tilde{I}}, \\ a_2 = \tilde{U}H_2(\tilde{V}_2). \end{cases} \tag{3.1}$$

For convenience, we assume

$$\begin{aligned}
\mathcal{G}^U &:= \mathcal{G}^U(U,I,V_1,V_2) = \Lambda - UG(I) - U\left[H_1(V_1) + H_2(V_2)\right] - \mu U, \\
\mathcal{G}^I &:= \mathcal{G}^I(U,I,V_1,V_2) = \ UG(I) + U\left[H_1(V_1) + H_2(V_2)\right] - (\mu + \theta + \rho)I, \\
\mathcal{G}_1 &:= \mathcal{G}_1(I,V_1) = \xi I - \delta_1 V_1, \\
\mathcal{G}_2 &:= \mathcal{G}_2(V_1,V_2) = \delta_1 V_1 - \delta_2 V_2.
\end{aligned}$$

Then

$$
\begin{aligned}
\frac{\mathrm{d}W(t)}{\mathrm{d}t} &= \int_\Omega (L_U U_t + L_I I_t + L_{V_1}(V_1)_t + L_{V_2}(V_2)_t)\mathrm{d}x \\
&= \int_\Omega \left[a_0\left(1-\frac{\tilde{U}}{U}\right)(D_U\Delta U) + a_0\left(1-\frac{\tilde{I}}{I}\right)(D_I\Delta I) + \sum_{i=1}^{2} a_i\left(1-\frac{\tilde{V}_i}{V_i}\right)(D_{V_i}\Delta V_i)\right]\mathrm{d}x \\
&\quad + \int_\Omega \left[a_0\left(1-\frac{\tilde{U}}{U}\right)\mathcal{G}^U + a_0\left(1-\frac{\tilde{I}}{I}\right)\mathcal{G}^I + \sum_{i=1}^{2} a_i\left(1-\frac{\tilde{V}_i}{V_i}\right)\mathcal{G}_i\right]\mathrm{d}x.
\end{aligned}
$$

Combining integration by parts with the boundary conditions, one can obtain

$$
\begin{aligned}
&\int_\Omega \left[a_0\left(1-\frac{\tilde{U}}{U}\right)(D_U\Delta U) + a_0\left(1-\frac{\tilde{I}}{I}\right)(D_I\Delta I) + \sum_{i=1}^{2} a_i\left(1-\frac{\tilde{V}_i}{V_i}\right)(D_{V_i}\Delta V_i)\right]\mathrm{d}x \\
=& -\int_\Omega \left[a_0 D_U\frac{\tilde{U}}{U^2}|\nabla U|^2 + a_0 D_I\frac{\tilde{I}}{I^2}|\nabla I|^2 + \sum_{i=1}^{2} a_i D_{V_i}\frac{\tilde{V}_i}{V_i^2}|\nabla V_i|^2\right]\mathrm{d}x \le 0.
\end{aligned}
\tag{3.2}
$$

Next we shall show that

$$
\mathrm{J} := a_0\left(1-\frac{\tilde{U}}{U}\right)\mathcal{G}^U + a_0\left(1-\frac{\tilde{I}}{I}\right)\mathcal{G}^I + \sum_{i=1}^{2} a_i\left(1-\frac{\tilde{V}_i}{V_i}\right)\mathcal{G}_i \le 0.
$$

In view of

$$
\begin{cases}
0 = \Lambda - \tilde{U}G(\tilde{I}) - \tilde{U}\left[H_1(\tilde{V}_1) + H_2(\tilde{V}_2)\right] - \mu\tilde{U}, \\
0 = \tilde{U}G(\tilde{I}) + \tilde{U}\left[H_1(\tilde{V}_1) + H_2(\tilde{V}_2)\right] - (\mu+\theta+\rho)\tilde{I}, \\
0 = \xi\tilde{I} - \delta_1\tilde{V}_1, \\
0 = \delta_1\tilde{V}_1 - \delta_2\tilde{V}_2,
\end{cases}
$$

we have

$$
\begin{aligned}
&\Lambda = \tilde{U}G(\tilde{I}) + \tilde{U}\left[H_1(\tilde{V}_1) + H_2(\tilde{V}_2)\right] + \mu\tilde{U}, \\
&\mu+\theta+\rho = \frac{\tilde{U}G(\tilde{I}) + \tilde{U}\left[H_1(\tilde{V}_1) + H_2(\tilde{V}_2)\right]}{\tilde{I}}, \\
&\delta_1 = \frac{\xi\tilde{I}}{\tilde{V}_1} \text{ and } \delta_2 = \frac{\delta_1\tilde{V}_1}{\tilde{V}_2}.
\end{aligned}
$$

It follows from direct calculation that

$$
\begin{aligned}
\mathrm{J} :=& a_0\left(1-\frac{\tilde{U}}{U}\right)\mathcal{G}^U + a_0\left(1-\frac{\tilde{I}}{I}\right)\mathcal{G}^I + \sum_{i=1}^{2} a_i\left(1-\frac{\tilde{V}_i}{V_i}\right)\mathcal{G}_i \\
=& a_0\left[-\mu U\left(1-\frac{\tilde{U}}{U}\right)^2 + \tilde{U}G(\tilde{I})\left(2-\frac{\tilde{U}}{U}-\frac{I}{\tilde{I}}+\frac{G(I)}{G(\tilde{I})}-\frac{\tilde{I}UG(I)}{I\tilde{U}G(\tilde{I})}\right)\right. \\
&\left. + \sum_{i=1}^{2}\tilde{U}H_i(\tilde{V}_i)\left(2-\frac{\tilde{U}}{U}-\frac{I}{\tilde{I}}+\frac{H_i(V_i)}{H_i(\tilde{V}_i)}-\frac{\tilde{I}UH_i(V_i)}{I\tilde{U}H_i(\tilde{V}_i)}\right)\right] \\
&+ a_1\xi\tilde{I}\left(1+\frac{I}{\tilde{I}}-\frac{V_1}{\tilde{V}_1}-\frac{\tilde{V}_1 I}{V_1\tilde{I}}\right) + a_2\delta_1\tilde{V}_1\left(1+\frac{V_1}{\tilde{V}_1}-\frac{\tilde{V}_2 V_1}{V_2\tilde{V}_1}-\frac{V_2}{\tilde{V}_2}\right).
\end{aligned}
\tag{3.3}
$$

Note that

$$
\begin{aligned}
&2-\frac{\tilde U}{U}-\frac{I}{\tilde I}+\frac{G(I)}{G(\tilde I)}-\frac{\tilde I U G(I)}{I\tilde U G(\tilde I)}\\
=&\left[2-\frac{\tilde U}{U}-\frac{I}{\tilde I}-\frac{\tilde I U G(I)}{I\tilde U G(\tilde I)}+1-\frac{IG(\tilde I)}{\tilde I G(I)}+\frac{I}{\tilde I}\right]-\left(\frac{G(I)}{G(\tilde I)}-\frac{I}{\tilde I}\right)\left(\frac{G(\tilde I)}{G(I)}-1\right)\\
\le&3-\frac{\tilde U}{U}-\frac{\tilde I U G(I)}{I\tilde U G(\tilde I)}-\frac{IG(\tilde I)}{\tilde I G(I)}\le 0.
\end{aligned}
\tag{3.4}
$$

Meanwhile, with the help of $1-x\le -\ln x$ for all $x>0$, one see that

$$
\begin{aligned}
&2-\frac{\tilde U}{U}-\frac{I}{\tilde I}+\frac{H_i(V_i)}{H_i(\tilde V_i)}-\frac{\tilde I U H_i(V_i)}{I\tilde U H_i(\tilde V_i)}\\
=&\left[2-\frac{\tilde U}{U}-\frac{I}{\tilde I}-\frac{\tilde I U H_i(V_i)}{I\tilde U H_i(\tilde V_i)}+1-\frac{V_iH_i(\tilde V_i)}{\tilde V_i H_i(V_i)}+\frac{V_i}{\tilde V_i}\right]-\left(\frac{H_i(V_i)}{H_i(\tilde V_i)}-\frac{V_i}{\tilde V_i}\right)\left(\frac{H_i(\tilde V_i)}{H_i(V_i)}-1\right)\\
\le&3-\frac{\tilde U}{U}-\frac{I}{\tilde I}-\frac{\tilde I U H_i(V_i)}{I\tilde U H_i(\tilde V_i)}-\frac{V_iH_i(\tilde V_i)}{\tilde V_i H_i(V_i)}+\frac{V_i}{\tilde V_i}\\
=&\left(\frac{V_i}{\tilde V_i}-\frac{I}{\tilde I}\right)+\left(1-\frac{\tilde U}{U}\right)+\left(1-\frac{V_iH_i(\tilde V_i)}{\tilde V_i H_i(V_i)}\right)+\left(1-\frac{\tilde I U H_i(V_i)}{I\tilde U H_i(\tilde V_i)}\right)\\
\le&\left(\frac{V_i}{\tilde V_i}-\frac{I}{\tilde I}\right)-\ln\frac{\tilde U}{U}-\ln\left(\frac{V_iH_i(\tilde V_i)}{\tilde V_i H_i(V_i)}\right)-\ln\left(\frac{\tilde I U H_i(V_i)}{I\tilde U H_i(\tilde V_i)}\right)\\
=&\left(\frac{V_i}{\tilde V_i}-\ln\frac{V_i}{\tilde V_i}\right)-\left(\frac{I}{\tilde I}-\ln\frac{I}{\tilde I}\right).
\end{aligned}
\tag{3.5}
$$

Likewise, one can show that

$$
\begin{aligned}
&1+\frac{I}{\tilde I}-\frac{V_1}{\tilde V_1}-\frac{\tilde V_1 I}{V_1\tilde I}\le\left(\frac{I}{\tilde I}-\ln\frac{I}{\tilde I}\right)-\left(\frac{V_1}{\tilde V_1}-\ln\frac{V_1}{\tilde V_1}\right),\\
&1+\frac{V_1}{\tilde V_1}-\frac{\tilde V_2 V_1}{V_2\tilde V_1}-\frac{V_2}{\tilde V_2}\le\left(\frac{V_1}{\tilde V_1}-\ln\frac{V_1}{\tilde V_1}\right)-\left(\frac{V_2}{\tilde V_2}-\ln\frac{V_2}{\tilde V_2}\right).
\end{aligned}
\tag{3.6}
$$

Applying (3.1), (3.4), (3.5) and (3.6) to (3.3), we have

$$
\begin{aligned}
\mathrm{J}\le&a_0\sum_{i=1}^{2}\tilde U H_i(\tilde V_i)\left[\left(\frac{V_i}{\tilde V_i}-\ln\frac{V_i}{\tilde V_i}\right)-\left(\frac{I}{\tilde I}-\ln\frac{I}{\tilde I}\right)\right]+a_1\xi\tilde I\left[\left(\frac{I}{\tilde I}-\ln\frac{I}{\tilde I}\right)-\left(\frac{V_1}{\tilde V_1}-\ln\frac{V_1}{\tilde V_1}\right)\right]\\
&+a_2\delta_1\tilde V_1\left[\left(\frac{V_1}{\tilde V_1}-\ln\frac{V_1}{\tilde V_1}\right)-\left(\frac{V_2}{\tilde V_2}-\ln\frac{V_2}{\tilde V_2}\right)\right]=0.
\end{aligned}
$$

Hence, by means of the selected constants a_0, a_1 and a_2 in (3.1), $\mathrm{J}\le 0$. In addition, if $\mathrm{J}=0$, one can find a constant κ such that

$$U=\tilde U,\ I=\kappa\tilde I,\ V_1=\kappa\tilde V_1 \text{ and } V_2=\kappa\tilde V_2.$$

Adding the U equation to I equation of the system (1.7) causes

$$\Lambda-\mu\tilde U-(\mu+\theta+\rho)\kappa\tilde I=0,$$

and hence, $\kappa = 1$. Hence,

$$\int_\Omega \left[a_0 \left(1 - \frac{\tilde{U}}{U}\right) \mathcal{G}^U + a_0 \left(1 - \frac{\tilde{I}}{I}\right) \mathcal{G}^I + \sum_{i=1}^{2} a_i \left(1 - \frac{\tilde{V}_i}{V_i}\right) \mathcal{G}_i \right] \mathrm{d}x \leq 0. \tag{3.7}$$

In view of (3.2) and (3.7), we can obtain $\frac{\mathrm{d}W(t)}{\mathrm{d}t} \leq 0$, and one also knows that the largest invariant subset $\mathcal{A} := \{(U, I, V_1, V_2) : \frac{\mathrm{d}W(t)}{\mathrm{d}t} = 0\}$ be constituted by just one singleton $\{E^*\}$. From section 9.9 [18] and the LaSalle's Invariance Principle, the proof is complete.

Remark 2. We can still give the corresponding hypothesis

(A4) $\left(\frac{I}{\tilde{I}} - \frac{\tilde{U}G(U,I)}{UG(\tilde{U},\tilde{I})}\right)\left(\frac{UG(\tilde{U},\tilde{I})}{\tilde{U}G(U,I)} - 1\right) \leq 0, \left(\frac{V_i}{\tilde{V}_i} - \frac{\tilde{U}H_i(U,V_i)}{UH_i(\tilde{U},\tilde{V}_i)}\right)\left(\frac{UH_i(\tilde{U},\tilde{V}_i)}{\tilde{U}H_i(U,V_i)} - 1\right) \leq 0, i = 1, 2,$

and prove the global stability of positive equilibrium E^* of system (1.7) when $UG(\cdot, I)$,$UH_1(\cdot, V_1)$ and $UH_2(\cdot, V_2)$ are replaced by general nonlinear incidences $G(\cdot, U, I)$, $H_1(\cdot, U, V_1)$ and $H_2(\cdot, U, V_2)$.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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