

http://www.aimspress.com/journal/MBE

MBE, 16(4): 3836–3868. DOI: 10.3934/mbe.2020214 Received: 18 March 2020 Accepted: 18 May 2020 Published: 25 May 2020

Research article

Stability analysis and Hopf bifurcation of a fractional order mathematical model with time delay for nutrient-phytoplankton-zooplankton

Ruiqing Shi*, Jianing Ren and Cuihong Wang

School of Mathematics and Computer Science, Shanxi Normal University, Linfen, 041004, China

* Correspondence: Email: shirq1979@163.com; Tel: +86-0357-2051185; Fax: +86-0357-2051185.

Abstract: In recent years, some researchers paid their attention to the interaction between toxic phytoplankton and zooplankton. Their studies showed that the mechanism of food selection in zooplankton is still immature and when different algae of the same species (toxic and nontoxic) coexist, some zooplankton may not be able to distinguish between toxic and nontoxic algae, and even show a slight preference for toxic strains. Thus, in this article, a fractional order mathematical model with time delay is constructed to describe the interaction of nutrient-phytoplankton-toxic phytoplankton-zooplankton. The main purpose of this paper is to study the influence of fractional order and time delay on the ecosystem. The sufficient conditions for the existence and local stability of each equilibrium are obtained by using fractional order stability theory. By choosing time delay as the bifurcation parameter, we find that Hopf bifurcation occurs when the time delay passes through a sequence of critical values. After that, some numerical simulations are performed to support the analytic results. At last we make some conclusion and point out some possible future work.

Keywords: fractional order; nutrient-phytoplankton-zooplankton; time delay; Hopf bifurcation; stability

1. Introduction

Plankton includes plants and animals that float freely in some fresh water bodies, and almost all aquatic life is based on plankton [1]. Aquatic ecosystems are affected by many factors, including physical and chemical signals in the environment, plankton predation and competition [2, 3]. Many scholars have carried out analysis concerning the impact of environment on the ecosystem and the treatment of sewage [4, 5, 6]. We also know how important plankton itself is to the wealth of marine ecosystems and ultimately to the planet itself. On the one hand, plankton species have positive effects on the environment, such as providing food for marine life, oxygen for animal life; on the other hand, it have harmful effects, such as economic losses to fisheries and tourism due to algae blooms [7, 8].

In recent years, different models of plankton have been established and studied, for example, model with two harmful phytoplankton [9], models with time delays [10, 11] and stochastic models [12, 13, 14]. Toxins produced by harmful phytoplankton tend to be concentrated at higher levels in the food web, as they can spread through the marine food web, affecting herbivores at higher nutrient levels, reaching fish, and through them eventually reaching marine mammals, even in seabirds [9, 11]. There is also some evidence that the occurrence of toxin-producing phytoplankton is not necessarily harmful, but rather helps maintain a stable balance of nutrient dynamics through the coexistence of all species. These results suggest that toxin-producing phytoplankton (TPP) play an important role in the growth of zooplankton populations [15].

It was shown that aquatic plant systems not only have extraordinary memories of climatic events, but also exhibit phenomenological responses based on memory [16, 17, 18]. The authors noted that environmental factors often alter the expression of chromatin in multiple responsive genes in [19]. Environment-induced chromatin markers are at certain sites and is transmitted by cell division, allowing plants to acquire memories of environmental experiences. This ensures that the plant can adapt to changes in its environment or perform better if the event occurs again. In some cases, it is passed on to the next generation, namely, epigenetic mechanisms. This mechanism is crucial for plants' stress memory and adaptation to the environment, suggesting that plants do form memory and defense mechanisms in certain environments. In addition, a large amount of zooplankton chemical signal learning and corresponding reactions have been documented for aquatic systems [20, 21]. In summary, such memory and genetic characteristics can not be neglected for plankton systems.

As we all know, fractional order derivatives are a good tool for describing the memory and genetic properties of various materials and processes. In other words, the application of fractional order dynamical systems can fully reflect some long-term memory and non-local effects. That is, fractional differential equations have an advantage over classical integer differential equations for describing such systems. In recent years, more and more researchers began to study the qualitative theory and numerical solution of fractional order biological model [22, 23, 24]. The main reason is that fractional order equations are naturally related to memory systems that exist in most biological systems [25, 26]. In addition, fractional-order derivative has also been widely studied and applied in physics[27], engineering [28], biology [29] and many other fields [30, 31, 32]. At present, there are more than six definitions of fractional derivative, among which Riemann-Liouville and Caputo derivatives are the most commonly used [33]. In the case of fractional Caputo derivative, the initial conditions are expressed by the values of the unknown function and its integer derivative with clear physical meaning [34]. So we will adapt the Caputo's definition in our paper.

The interactions between phytoplankton and zooplankton do not occur instantaneously in real ecosystems. Instead, the response of zooplankton to contacts with phytoplankton is likely to be delayed due to gestation. For example, in [35], the authors discussed Hopf bifurcation in the presence of time delay required for toxin-phytoplankton maturation. The universality of time-delay coupled system indicates its importance, applicability and practicability in a wide range of biological systems [36, 37]. In fact, time delay may change the qualitative behavior of dynamic system [38, 39]. In [40], the authors considered a fractional nutrient-phytoplankton-zooplankton system as follows

$$\begin{cases} D^{\alpha}X(t) = x_0 - aX - b_1XY + c_1Y + c_2Z, \\ D^{\alpha}Y(t) = b_2XY - \frac{d_1YZ}{e+Y} - c_3Y, \\ D^{\alpha}Z(t) = \frac{d_2YZ}{e+Y} - fYZ - c_4Z. \end{cases}$$
(1.1)

Based on the above model, we classify phytoplankton into non-toxic phytoplankton and toxic phytoplankton and put forward an improved fractional order four-dimensional ecological epidemiological model with delay. The system is established as follows:

$$\begin{aligned}
D^{\alpha}X(t) &= \Lambda - \mu X(t) - b_1 X(t) Y_1(t) - b_2 X(t) Y_2(t) + c_1 Y_1(t) + c_2 Y_2(t) + c_3 Z(t), \\
D^{\alpha}Y_1(t) &= k_1 b_1 X(t) Y_1(t) - \eta_1 Y_1(t) Z(t) - h_1 Y_1(t) Y_2(t) - \mu_1 Y_1(t), \\
D^{\alpha}Y_2(t) &= k_2 b_2 X(t) Y_2(t) - \eta_2 Y_2(t) Z(t) - h_2 Y_1(t) Y_2(t) - \mu_2 Y_2(t), \\
D^{\alpha}Z(t) &= \theta_1 \eta_1 Y_1(t - \tau) Z(t - \tau) + \theta_2 \eta_2 Y_2(t - \tau) Z(t - \tau) - \delta Y_2(t) Z(t) - \mu_3 Z(t),
\end{aligned}$$
(1.2)

subjected to the biologically feasible initial condition:

$$X(0) \ge 0, \ Y_1(t) = \phi(t) \ge 0, \ Y_2(t) = \psi(t) \ge 0, \ Z(t) = \zeta(t) \ge 0, \quad t \in [-\tau, 0],$$
(1.3)

where $\phi(t)$, $\psi(t)$ and $\zeta(t)$ are continuous function defined on $t \in [-\tau, 0]$.

The meaning of state variables and parameters are listed in Table 1; D^{α} (0 < α < 1) denotes Caputo fractional differential operator, and the model are based on the following scenarios:

(H1) X(t), $Y_1(t)$, $Y_2(t)$ and Z(t) represent nutrient population, phytoplankton population, toxic phytoplankton population and zooplankton population, respectively.

(H2)In real ecosystems, phytoplankton compete with each other for essential resources: nutrients and light. So as the model in the [41], we assume that, for nutrient X(t), phytoplankton population $Y_1(t)$ is in competition with toxic phytoplankton population $Y_2(t)$, h_1 and h_2 represent the influence on $Y_1(t)$ and $Y_2(t)$ in the competition, respectively.

(H3) Zooplankton do not grow instantaneously after consuming phytoplankton, and pregnancy of predators requires a discrete time delay τ .

(H4) Zooplankton populations feed only on phytoplankton, and only some of the dead phytoplankton and zooplankton are recycled into nutrients.

(H5) The toxic phytoplankton has both positive and negative effect on zooplankton, corresponding to the term $\theta_2 \eta_2 Y_2(t-\tau)Z(t-\tau)$ and $-\delta Y_2(t)Z(t)$ in the last equation of the system(1.2).

The above assumption (H5) is based on the result in [42]. In fact, the authors concluded that the food selection mechanism of plankton may not yet mature. When different algae of the same species (toxic and non-toxic) coexist, some zooplankton may have poor ability to select between toxic and non-toxic algae, and even show a slight preference for toxic strains.

Table 1. Description of state variables and parameters in the system (1.2).			
variables	Descriptions		
X(t)	concentration of nutrient population at time t		
$Y_1(t)$	concentration of phytoplankton population at time t		
$Y_2(t)$	concentration of toxic phytoplankton population at time t		
Z(t)	concentration of zooplankton population at time t		
Parameters	Descriptions	Default value	
Λ	Constant input of nutrient	[0.5, 3]	
b_1	Nutrient uptake rate for the phytoplankton population	[0, 3]	
b_2	Nutrient uptake rate for the toxic phytoplankton population	[0, 3]	
k_1	Nutrient-phytoplankton conversion rate	(0, 1)	
k_2	Nutrient-toxic phytoplankton conversion rate	(0, 1)	
c_1	Nutrient recycling rate after the death of phytoplankton	(0, 0.1)	
c_2	Nutrient recycling rate after the death of toxic phytoplankton	(0, 0.1)	
<i>C</i> ₃	Nutrient recycling rate after the death of zooplankton	(0, 0.7)	
η_1	Maximal zooplankton ingestion rate	(0, 3)	
η_2	Maximal zooplankton ingestion rate	(0, 2.5)	
$ heta_1$	Maximal phytoplankton-zooplankton conversion rate	(0, 0.7)	
$ heta_2$	Maximal toxic phytoplankton-zooplankton conversion rate	(0, 0.8)	
μ	Rate of nutrient loss	(0, 1.5)	
μ_1	Phytoplankton mortality rate	(0, 1)	
μ_2	Toxic phytoplankton mortality rate	(0, 1)	
μ_3	Zooplankton death rate	(0, 0.6)	
δ	Rate of zooplankton decay due to toxin producing phytoplankton	(0, 0.2)	
h_1	competition effect for phytoplankton	(0, 0.3)	
h_2	competition effect for toxic phytoplankton	(0, 0.3)	

The present paper is organized as follows. In section 2, some preliminaries are presented. In section 3, qualitative analysis of the system is performed. In section 4, some numerical examples and simulations are exploited to verify the theoretical results. In the last section, some conclusions and discussions are provided.

2. Preliminaries

For convenience, we list some of the basic definitions and lemmas of the fractional calculus. In fractional-order calculus, there are many fractional-order integration and fractional-order differentiation that have been defined, for example, the Grunwald-Letnikov (GL) definition, the Riemann-Liouville (RL) definition and the Caputo definition. Since the initial condition is the same as the form of integral differential equation, we will adopt the definition of Caputo in this paper.

Definition 2.1. [34] The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $f : \mathbb{R}^+ \to \mathbb{R}^+$

R is defined by

$${}_0D_t^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\mathrm{d}s, \ t \ge 0.$$

Based on this definition of Riemann-Liouville fractional integral, the fractional-order derivative in Riemann-Liouville sense and Caputo sense are given.

Definition 2.2. [34] The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a function $f : R^+ \to R$ is defined by

$${}_{0}^{RL}D_{t}^{\alpha}f(t) = \frac{\mathrm{d}^{k}}{\mathrm{d}t^{k}}({}_{0}D_{t}^{-(k-\alpha)}f(t)) = \frac{1}{\Gamma(k-\alpha)}\frac{\mathrm{d}^{k}}{\mathrm{d}t^{k}}\int_{0}^{t}(t-s)^{k-\alpha-1}f(s)\mathrm{d}s, \ t \ge 0,$$

where $k - 1 \le \alpha < k$, $k \in N$ and $\Gamma(\cdot)$ is the Gamma function, $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha - 1} e^{-t} dt$.

In particular, when $0 < \alpha < 1$ *, we have*

$${}_{0}^{RL}D_{t}^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{0}^{t}(t-s)^{-\alpha}f(s)\mathrm{d}s.$$

Definition 2.3. [34] The Caputo fractional derivative of order $\alpha > 0$ for a function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined by

$${}_{0}^{C}D_{t}^{-\alpha}f(t) = {}_{0}D_{t}^{-(k-\alpha)}f^{(k)}(t) = \frac{1}{\Gamma(k-\alpha)}\int_{0}^{t}(t-s)^{k-\alpha-1}f^{(k)}(s)\mathrm{d}s, \quad t \ge 0,$$

where $k - 1 \le \alpha < k$, $k \in N$ and $f^{(m)}(t)$ is the m-order derivative of f(t). In particular, when $0 < \alpha < 1$, we have

$${}_{0}^{C}D_{t}^{-\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}\frac{f'(s)}{(t-s)^{\alpha}}\mathrm{d}s.$$

Definition 2.4. [34] The two-parameter Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{i=0}^{+\infty} \frac{z^i}{\Gamma(\alpha i + \beta)}, \quad \alpha > 0, \quad \beta > 0.$$

When $\beta = 1$, the two-parameter Mittag-Leffler function becomes to the one-parameter Mittag-Leffler function, i.e.

$$E_{\alpha}(z) = E_{\alpha,1}(z) = \sum_{i=0}^{+\infty} \frac{z^i}{\Gamma(\alpha i + 1)}, \quad \alpha > 0.$$

Theorem 2.5. [43] Consider the following commensurate fractional-order system:

$$\frac{\mathrm{d}^{\alpha}x}{\mathrm{d}t^{\alpha}} = f(x), \quad x(0) = x_0,$$

with $0 < \alpha < 1$ and $x \in \mathbb{R}^n$. The equilibrium points of the above system are calculated by solving the equation: f(x) = 0. These points are locally asymptotically stable if all eigenvalues λ_i of the Jacobian matrix evaluated at the equilibrium points satisfy the inequality: $|\arg(\lambda_i)| > \frac{\alpha \pi}{2}$.

Mathematical Biosciences and Engineering

Volume 16, Issue 4, 3836–3868.

3. Qualitative analysis of the system (1.2)

3.1. The existence of equilibriums

Since the proof of the positivity and boundedness of the solution of the system(1.2) is similar to Theorem 2 and Theorem 3 in the Ref.[38], we will not prove it here.

The equilibriums of model (1.2) are obtained by solving the following algebraic system

$$\begin{cases} \Lambda - \mu X - b_1 X Y_1 - b_2 X Y_2 + c_1 Y_1 + c_2 Y_2 + c_3 Z = 0, \\ k_1 b_1 X Y_1 - \eta_1 Y_1 Z - h_1 Y_1 Y_2 - \mu_1 Y_1 = 0, \\ k_2 b_2 X Y_2 - \eta_2 Y_2 Z - h_2 Y_1 Y_2 - \mu_2 Y_2 = 0, \\ \theta_1 \eta_1 Y_1 Z + \theta_2 \eta_2 Y_2 Z - \delta Y_2 Z - \mu_3 Z = 0. \end{cases}$$
(3.1)

By simple calculation, we obtain seven equilibriums of system (1.2), namely:

$$(1)E_0 = (X^{(0)}, 0, 0, 0) \text{ with } X^{(0)} = \frac{\Lambda}{\mu}.$$

$$(2)E_1 = (X^{(1)}, Y_1^{(1)}, 0, 0) \text{ with } X^{(1)} = \frac{\mu_1}{k_1 b_1}, Y_1^{(1)} = \frac{\mu(1 - R_1)}{b_1(\frac{c_1 R_1}{b_1 X^{(0)}} - 1)}, \text{ where } R_1 = \frac{X^{(0)}}{X^{(1)}}.$$
 And the

feasibility conditions for E_1 are simplified as:

$$X^{(0)} < X^{(1)} < \frac{c_1}{b_1}$$
 or $\frac{c_1}{b_1} < X^{(1)} < X^{(0)}$.

$$(3)E_2 = (X^{(2)}, 0, Y_2^{(2)}, 0)$$
 with $X^{(2)} = \frac{\mu_2}{k_2 b_2}, Y_2^{(2)} = \frac{\mu(1-R_2)}{b_2(\frac{c_2 R_2}{b_2 X^{(0)}} - 1)}$, where $R_2 = \frac{X^{(0)}}{X^{(2)}}$. And the

feasibility conditions for E_2 are simplified as:

$$X^{(0)} < X^{(2)} < \frac{c_2}{b_2} \quad \text{or} \quad \frac{c_2}{b_2} < X^{(2)} < X^{(0)}.$$

$$^{3)}, Y_1^{(3)}, Y_2^{(3)}, 0) \text{ with } Y_1^{(3)} = \frac{k_2 b_2 X^{(3)} - \mu_2}{h_2}, Y_2^{(3)} = \frac{k_1 b_1 X^{(3)} - \mu_1}{h_1}, \text{ and } X^{(3)} \text{ is uniquely}$$

determined by the following equation:

$$a_1 X^2 + a_2 X + a_3 = 0, (3.2)$$

where

 $(4)E_3 = (X^{(4)})$

$$a_{1} = -b_{1}b_{2}(h_{1}k_{2} + h_{2}k_{1}) < 0,$$

$$a_{2} = -\mu h_{1}h_{2} + \mu_{2}b_{1}h_{1} + \mu_{1}b_{2}h_{2} + c_{1}h_{1}b_{2}k_{2} + c_{2}h_{2}k_{1}b_{1},$$

$$a_{3} = \Lambda h_{1}h_{2} - \mu_{2}c_{1}h_{1} - \mu_{1}c_{2}h_{2}.$$

If $\Lambda > \frac{\mu_2 c_1}{h_2} + \frac{\mu_1 c_2}{h_1}$, then Descartes rule of sign ensures that the above Eq.(3.2) possesses a uniquely positive root. And the feasibility conditions for E_3 are simplified as:

$$\Lambda > \frac{\mu_2 c_1}{h_2} + \frac{\mu_1 c_2}{h_1}, \quad X^{(3)} > X^{(1)} \text{ and } X^{(3)} > X^{(2)}.$$

 $(5)E_4 = (X^{(4)}, Y_1^{(4)}, 0, Z^{(4)})$ with

$$X^{(4)} = \frac{\Lambda \eta_1 + c_1 \eta_1 Y_1^{(4)} - \mu_1 c_3}{\mu \eta_1 + b_1 \eta_1 Y_1^{(4)} - k_1 b_1 c_3}, \quad Y_1^{(4)} = \frac{\mu_3}{\theta_1 \eta_1}, \quad Z^{(4)} = \frac{k_1 b_1 X^{(4)} - \mu_1}{\eta_1}.$$

Mathematical Biosciences and Engineering

Volume 16, Issue 4, 3836-3868.

The feasibility conditions of E_4 are simplified as:

$$R_3 = \frac{X^{(4)}}{X^{(1)}} > 1.$$

 $(6)E_5 = (X^{(5)}, 0, Y_2^{(5)}, Z^{(5)})$ with

$$X^{(5)} = \frac{\Lambda \eta_2 + c_2 \eta_2 Y_2^{(5)} - \mu_2 c_3}{\mu \eta_2 + b_2 \eta_2 Y_2^{(5)} - k_2 b_2 c_3}, \quad Y_2^{(5)} = \frac{\mu_3}{\theta_2 \eta_2 - \delta}, \quad Z^{(5)} = \frac{k_2 b_2 X^{(5)} - \mu_2}{\eta_2}.$$

Considering the biological background, we assume $\theta_2 \eta_2 > \delta$ is reasonable, and the feasibility conditions for E_5 are simplified as:

$$R_4 = \frac{X^{(5)}}{X^{(2)}} > 1$$

$$(7)E_6 = (X^{(6)}, Y_1^{(6)}, Y_2^{(6)}, Z^{(6)}) \text{ with}$$

$$Y_1^{(6)} = \frac{k_2 b_2 X^{(6)} - \eta_2 Z^{(6)} - \mu_2}{h_2}, \quad Y_2^{(6)} = \frac{k_1 b_1 X^{(6)} - \eta_1 Z^{(6)} - \mu_1}{h_1},$$

$$Z^{(6)} = \frac{(\theta_1 \eta_1 h_1 k_2 b_2 + \theta_2 \eta_2 h_2 k_1 b_1 - \delta h_2 k_1 b_1) X^{(6)} + \delta h_2 \mu_1 - \theta_1 \eta_1 h_1 \mu_2 - \theta_2 \eta_2 h_2 \mu_1 - \mu_3 h_1 h_2}{\eta_1 (\theta_1 \eta_2 h_1 + \theta_2 \eta_2 h_2 - \delta h_2)}. \text{ And } X^{(6)}$$
uniquely determined by the following equation:

is uniquely determined by the following equation:

$$b_1 X^2 + b_2 X + b_3 = 0, (3.3)$$

where

$$\begin{split} b_1 &= (b_1h_1\eta_2 + b_2h_2\eta_1)(\theta_1\eta_1h_1k_2b_2 + \theta_2\eta_2h_2k_1b_1 - \delta h_2k_1b_1) \\ &-b_1b_2\eta_1(h_1k_2 + h_2k_1)(\theta_1\eta_2h_1 + \theta_2\eta_2h_2 - \delta h_2), \\ b_2 &= (b_1h_1\eta_2 + b_2h_2\eta_1)(\delta h_2\mu_1 - \theta_1\eta_1h_1\mu_2 - \theta_2\eta_2h_2\mu_1 - \mu_3h_1h_2) \\ &+ (c_3h_1h_2 - c_1h_1\eta_2 - c_2h_2\eta_1)(\theta_1\eta_1h_1k_2b_2 + \theta_2\eta_2h_2k_1b_1 - \delta h_2k_1b_1) \\ &+ \eta_1(\theta_1\eta_2h_1 + \theta_2\eta_2h_2 - \delta h_2)(b_1h_1\mu_2 + b_2h_2\mu_1 + c_1h_1k_2b_2 + c_2h_2k_1b_1 - \mu h_1h_2), \\ b_3 &= (c_3h_1h_2 - c_1h_1\eta_2 - c_2h_2\eta_1)(\delta h_2\mu_1 - \theta_1\eta_1h_1\mu_2 - \theta_2\eta_2h_2\mu_1 - \mu_3h_1h_2) \\ &+ (\theta_1\eta_2h_1 + \theta_2\eta_2h_2 - \delta h_2)(\Lambda h_1h_2 - c_1h_1\mu_2 - c_2h_2\mu_1), \end{split}$$

If $b_1b_3 < 0$, then Descartes rule of sign ensures that the above Eq.(3.3) possesses a uniquely positive root. The feasibility conditions for E_6 are simplified as: $Y_1^{(6)}, Y_2^{(6)}, Z^{(6)} > 0, b_1b_3 < 0.$

Remark 3.1. (1) The necessary conditions for the existence of E_3 are $R_1 > 1$ and $R_2 > 1$.

(2) Because of the complexity of computation, we have not obtain the exact formula of positive equilibrium E_6 .

3.2. Local stability of Equilibriums

In this subsection we discuss the stability of each equilibrium when $\tau=0$.

Obviously, the eigenvalues of the Jacobian matrix of system (1.2) at equilibrium E_0 are $\lambda_1 = -\mu < -\mu$ 0, $\lambda_2 = -\mu_3 < 0$, $\lambda_3 = k_1 b_1 X^{(0)} - \mu_1$, $\lambda_4 = k_2 b_2 X^{(0)} - \mu_2$, so we get the following result.

Theorem 3.2. If $R_1 < 1$ and $R_2 < 1$, then the disease-free equilibrium E_0 is locally asymptotically stable and it is unstable if $R_1 > 1$ or $R_2 > 1$.

The Jacobian matrix of system (1.2) at equilibrium E_1 is

$$\begin{pmatrix} -\mu - b_1 Y_1^{(1)} & -b_1 X^{(1)} + c_1 & -b_2 X^{(1)} + c_2 & c_3 \\ k_1 b_1 Y_1^{(1)} & 0 & -h_1 Y_1^{(1)} & -\eta_1 Y_1^{(1)} \\ 0 & 0 & k_2 b_2 X^{(1)} - h_2 Y_1^{(1)} - \mu_2 & 0 \\ 0 & 0 & 0 & \theta_1 \eta_1 Y_1^{(1)} - \mu_3 \end{pmatrix}.$$

The characteristic equation at the equilibrium E_1 is

$$[\lambda - (k_2 b_2 X^{(1)} - h_2 Y^{(1)}_1 - \mu_2)] [\lambda + \mu_3 - \theta_1 \eta_1 Y^{(1)}_1] [\lambda^2 + (\mu + b_1 Y^{(1)}_1) \lambda + k_1 b_1 Y^{(1)}_1 (b_1 X^{(1)} - c_1)] = 0.$$
(3.4)

Thus, we get the following result.

Theorem 3.3. If $1 < R_1 < \frac{X^{(0)}b_1}{c_1}$, $R_2 < R_1$ and $\mu_3 > \theta_1\eta_1Y_1^{(1)}$, then the equilibrium E_1 is locally asymptotically stable.

The Jacobian matrix of system (1.2) at equilibrium E_2 is

$$\begin{pmatrix} -\mu - b_2 Y_2^{(2)} & -b_1 X^{(2)} + c_1 & -b_2 X^{(2)} + c_2 & c_3 \\ 0 & k_1 b_1 X^{(2)} - h_1 Y_2^{(2)} - \mu_1 & 0 & 0 \\ k_2 b_2 Y_2^{(2)} & -h_1 Y_2^{(2)} & 0 & -\eta_2 Y_2^{(2)} \\ 0 & 0 & 0 & \theta_2 \eta_2 Y_2^{(2)} - \delta Y_2^{(2)} - \mu_3 \end{pmatrix} .$$

The characteristic equation at the equilibrium E_2 is

$$[\lambda - (k_1 b_1 X^{(2)} - h_1 Y_2^{(2)} - \mu_1)] [\lambda + \delta Y_2^{(2)} + \mu_3 - \theta_2 \eta_2 Y_2^{(2)}] [\lambda^2 + (\mu + b_2 Y_2^{(2)})\lambda + k_2 b_2 Y_2^{(2)} (b_2 X^{(2)} - c_2)] = 0.$$
(3.5)

So we get the following result.

Theorem 3.4. If $1 < R_2 < \frac{X^{(0)}b_2}{c_2}$, $R_2 > R_1$ and $\mu_3 > (\theta_2\eta_2 - f)Y_2^{(2)}$, then the equilibrium E_2 is locally asymptotically stable.

The Jacobian matrix of system (1.2) at equilibrium E_3 is

$$\begin{pmatrix} -\mu - b_1 Y_1^{(3)} - b_2 Y_2^{(3)} & -b_1 X^{(3)} + c_1 & -b_2 X^{(3)} + c_2 & c_3 \\ k_1 b_1 Y_1^{(3)} & 0 & -h_1 Y_1^{(3)} & -\eta_1 Y_1^{(3)} \\ k_2 b_2 Y_2^{(3)} & -h_2 Y_2^{(3)} & 0 & -\eta_2 Y_2^{(3)} \\ 0 & 0 & 0 & \theta_1 \eta_1 Y_1^{(3)} + \theta_2 \eta_2 Y_2^{(3)} - \delta Y_2^{(3)} - \mu_3 \end{pmatrix},$$

with the characteristic equation

$$[\lambda - (\theta_1 \eta_1 Y_1^{(3)} + \theta_2 \eta_2 Y_2^{(3)} - \delta Y_2^{(3)} - \mu_3)](\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3) = 0,$$
(3.6)

where

$$\begin{array}{rcl} A_1 &=& \mu + b_1 Y_1^{(3)} + b_2 Y_2^{(3)}, \\ A_2 &=& k_1 b_1 Y_1^{(3)} (b_1 X^{(3)} - c_1) + k_2 b_2 Y_2^{(3)} (b_2 X^{(3)} - c_2) - h_1 h_2 Y_1^{(3)} Y_2^{(3)}, \\ A_3 &=& -h_1 h_2 Y_1^{(3)} Y_2^{(3)} (\mu + b_1 Y_1^{(3)} + b_2 Y_2^{(3)}) - h_1 k_2 b_2 Y_1^{(3)} Y_2^{(3)} (b_1 X^{(3)} - c_1) - h_2 k_1 b_1 Y_1^{(3)} Y_2^{(3)} (b_1 X^{(3)} - c_1). \end{array}$$

Mathematical Biosciences and Engineering

Volume 16, Issue 4, 3836-3868.

Denote D(P) denote the discriminant of a polynomial

 $Q(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3$. Then

 $\widetilde{D}(Q) = 18A_1A_2A_3 + (A_1A_2)^2 - 4A_3A_1^3 - 4A_2^3 - 27A_3^3.$ If $\mu_3 > \theta_1\eta_1Y_1^{(3)} + \theta_2\eta_2Y_2^{(3)} - \delta Y_2^{(3)}$, in order to discuss the stability of the equilibrium E_3 , we get the following result by use of the same method as in Ref [44].

Proposition 3.5. The equilibrium E_3 is asymptotically stable if one of the following conditions holds for polynomial Q and D(Q):

 $(1)D(Q) > 0, A_1 > 0, A_3 > 0 and A_1A_2 > A_3.$ $(2)D(Q) < 0, A_1 \ge 0, A_2 \ge 0, A_3 \ge 0 \text{ and } \alpha < \frac{2}{3}.$

The Jacobian matrix of system (1.2) at equilibrium E_4 is

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & 0 & m_{23} & m_{24} \\ 0 & 0 & m_{33} & 0 \\ 0 & m_{42} & m_{43} & m_{44} \end{pmatrix}$$

where

$$\begin{split} m_{11} &= -\mu - b_1 Y_1^{(4)}, & m_{12} = -b_1 X^{(4)} + c_1, & m_{13} = -b_2 X^{(4)} + c_2 \\ m_{14} &= c_3, & m_{21} = k_1 b_1 Y_1^{(4)}, & m_{23} = -h_1 Y_1^{(4)} \\ m_{24} &= -\eta_1 Y_1^{(4)}, & m_{33} = k_2 b_2 X^{(4)} - \eta_2 Z^{(4)} - h_2 Y_1^{(4)} - \mu_2, & m_{42} = \theta_1 \eta_1 Z^{(4)}, \\ m_{43} &= \theta_2 \eta_2 Z^{(4)} - \delta Z^{(4)}, & m_{44} = \theta_1 \eta_1 Y_1^{(4)} - \mu_3. \end{split}$$

The characteristic equation at the equilibrium E_4 is

$$[\lambda - (k_2 b_2 X^{(4)} - \eta_2 Z^{(4)} - h_2 Y_1^{(4)} - \mu_2)][\lambda^3 + B_1 \lambda^2 + B_2 \lambda + B_3] = 0,$$
(3.7)

where

$$B_1 = -m_{11} - m_{44},$$

$$B_2 = m_{11}m_{44} - m_{12}m_{21} - m_{24}m_{42},$$

$$B_3 = m_{11}m_{24}m_{42} + m_{12}m_{21}m_{44} - m_{14}m_{21}m_{42}.$$

Denote D(P) denote the discriminant of a polynomial

 $Q(\lambda) = \lambda^3 + B_1 \lambda^2 + B_2 \lambda + B_3$. Then

 $\widetilde{D(Q)} = 18B_1B_2B_3 + (B_1B_2)^2 - 4B_3B_1^3 - 4B_2^3 - 27B_3^3.$ If $\mu_2 > k_2b_2X^{(4)} - \eta_2Z^{(4)} - h_2Y_1^{(4)}$, in order to discuss the stability of the equilibrium E_4 , we get the following result by use of the same method as in Ref [44].

Proposition 3.6. The equilibrium E_4 is asymptotically stable if one of the following conditions holds for polynomial Q and D(Q):

 $(1)D(Q) > 0, B_1 > 0, B_3 > 0 and B_1B_2 > B_3.$ $(2)D(Q) < 0, B_1 \ge 0, B_2 \ge 0, B_3 \ge 0 \text{ and } \alpha < \frac{2}{3}.$

The Jacobian matrix of system (1.2) at equilibrium E_5 is

$$\begin{pmatrix} \hat{m}_{11} & \hat{m}_{12} & \hat{m}_{13} & \hat{m}_{14} \\ 0 & \hat{m}_{22} & 0 & 0 \\ \hat{m}_{31} & \hat{m}_{32} & 0 & \hat{m}_{34} \\ 0 & \hat{m}_{42} & \hat{m}_{43} & \hat{m}_{44} \end{pmatrix}$$

Mathematical Biosciences and Engineering

Volume 16, Issue 4, 3836-3868.

3845

where

$$\hat{m}_{11} = -\mu - b_2 Y_2^{(5)}, \qquad \hat{m}_{12} = -b_1 X^{(5)} + c_1, \qquad \hat{m}_{13} = -b_2 X^{(5)} + c_2, \\ \hat{m}_{14} = c_3, \qquad \hat{m}_{22} = k_1 b_1 X^{(5)} - \eta_1 Z^{(5)} - h_1 Y_2^{(5)} - \mu_1, \qquad \hat{m}_{31} = k_2 b_2 Y_2^{(5)} \\ \hat{m}_{32} = -h_2 Y_2^{(5)}, \qquad \hat{m}_{34} = -\eta_2 Y_2^{(5)}, \qquad \hat{m}_{42} = \theta_1 \eta_1 Z^{(5)}, \\ \hat{m}_{43} = \theta_2 \eta_2 Z^{(5)} - \delta Z^{(5)}, \qquad \hat{m}_{44} = \theta_2 \eta_2 Y_2^{(5)} - \delta Y_2^{(5)} - \mu_3.$$

The characteristic equation at the equilibrium E_5 is

$$[\lambda - (k_1 b_1 X^{(5)} - \eta_1 Z^{(5)} - h_1 Y_2^{(5)} - \mu_1)][\lambda^3 + C_1 \lambda^2 + C_2 \lambda + C_3] = 0,$$
(3.8)

where

$$C_{1} = -\hat{m}_{11} - \hat{m}_{44},$$

$$C_{2} = \hat{m}_{11}\hat{m}_{44} - \hat{m}_{13}\hat{m}_{31} - \hat{m}_{34}\hat{m}_{43},$$

$$C_{3} = \hat{m}_{11}\hat{m}_{34}\hat{m}_{43} + \hat{m}_{13}\hat{m}_{31}\hat{m}_{44} - \hat{m}_{14}\hat{m}_{31}\hat{m}_{43}$$

Denote D(P) denote the discriminant of a polynomial

 $Q(\lambda) = \lambda^3 + C_1 \lambda^2 + C_2 \lambda + C_3$. Then

 $D(Q) = 18C_1C_2C_3 + (C_1C_2)^2 - 4C_3C_1^3 - 4C_2^3 - 27C_3^3.$

If $\mu_1 > k_1 b_1 X^{(5)} - \eta_1 Z^{(5)} - h_1 Y_2^{(5)}$, in order to discuss the stability of the equilibrium E_5 , we get the following result by use of the same method as in Ref [44].

Proposition 3.7. The equilibrium E_5 is asymptotically stable if one of the following conditions holds for polynomial Q and D(Q):

(1) $D(Q) > 0, C_1 > 0, C_3 > 0$ and $C_1C_2 > C_3$. (2) $D(Q) < 0, C_1 \ge 0, C_2 \ge 0, C_3 \ge 0$ and $\alpha < \frac{2}{3}$.

Theorem 3.8. The equilibrium E_6 is locally asymptotically stable if the following conditions hold:

$$(H5) X^{(6)} > \max\left\{\frac{c_1}{b_1}, \frac{c_2}{b_2}\right\},\$$

$$(H6) \frac{k_1 b_1}{\eta_1} < \frac{k_2 b_2}{\eta_2},\$$

$$(H7) k_3 > \frac{\eta_2 (\mu + b_1 Y_1^{(6)} + b_2 Y_2^{(6)})}{c_3 b_2},\$$

$$(H8) (e_1 e_2 - e_3) e_3 - e_4 e_1^2 > 0.$$

Proof. The Jacobian matrix of system (1.2) at equilibrium E_6 is

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ 0 & b_{42} & a_{43} + b_{43} & a_{44} + b_{44} \end{pmatrix}$$

where

$$\begin{array}{ll} a_{11}=-\mu-b_1Y_1^{(6)}-b_2Y_2^{(6)}, & a_{12}=-b_1X^{(6)}+c_1, & a_{13}=-b_2X^{(6)}+c_2, \\ a_{14}=c_3, & a_{21}=k_1b_1Y_1^{(6)}, & a_{23}=-h_1Y_1^{(6)} \\ a_{24}=-\eta_1Y_1^{(6)}, & a_{31}=k_2b_2Y_2^{(6)}, & a_{32}=-h_2Y_2^{(6)} \\ a_{34}=-\eta_2Y_2^{(6)}, & a_{43}=-fZ^{(6)}, & a_{44}=-fY_2^{(6)}-\mu_3, \\ b_{42}=\theta_1\eta_1Z^{(6)}, & b_{43}=\theta_2\eta_2Z^{(6)}, & b_{44}=\theta_1\eta_1Y_1^{(6)}+\theta_2\eta_2Y_2^{(6)} \end{array}$$

Mathematical Biosciences and Engineering

Volume 16, Issue 4, 3836–3868.

The characteristic equation at the equilibrium E_6 is

$$\lambda^4 + e_1 \lambda^3 + e_2 \lambda^2 + e_3 \lambda + e_4 = 0, \tag{3.9}$$

where

$$\begin{array}{rcl} e_1 &=& -a_{11} > 0, \\ e_2 &=& -a_{13}a_{31} - a_{12}a_{21} - a_{23}a_{32} - a_{34}a_{43} - a_{24}b_{42} - a_{34}b_{43}, \\ e_3 &=& a_{11}a_{23}a_{32} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32} + a_{11}a_{34}a_{43} - a_{14}a_{31}a_{43} - a_{24}a_{32}a_{43} + a_{11}a_{24}b_{42} \\ && -a_{14}a_{21}b_{42} + a_{11}a_{34}b_{43} - a_{14}a_{31}b_{43} - a_{23}a_{34}b_{42} - a_{24}a_{32}b_{43}, \end{array}$$

$$e_4 = a_{11}a_{24}a_{32}a_{43} + a_{12}a_{21}a_{34}a_{43} - a_{12}a_{24}a_{31}a_{43} - a_{14}a_{21}a_{32}a_{43} + a_{11}a_{23}a_{34}b_{42} + a_{11}a_{24}a_{32}b_{43} + a_{12}a_{21}a_{34}b_{43} - a_{12}a_{24}a_{31}b_{42} - a_{13}a_{21}a_{34}b_{42} - a_{14}a_{21}a_{32}b_{43} - a_{14}a_{23}a_{31}b_{42}.$$

By simple calculation, if $X^{(6)} > \max\left\{\frac{c_1}{b_1}, \frac{c_2}{b_2}\right\}$, then $e_1e_2 > e_3$; if $k_3 > \frac{\eta_2(\mu + b_1Y_1^{(6)} + b_2Y_2^{(6)})}{c_3b_2}$ and

 $\frac{k_1b_1}{\eta_1} < \frac{k_2b_2}{\eta_2}$, then $e_4 > 0$. In summary, the condition of the Routh-Hurwitz criterion above is satisfied for Eq.(3.9), that is,

$$e_1 > 0, \ e_1 e_2 > e_3, \ (e_1 e_2 - e_3) e_3 - e_4 e_1^2 > 0, \ e_4 > 0,$$

hold. So, all the roots of this Eq.(3.9) have negative real part. This ends the proof.

Remark 3.9. In Theorem (3.8), (H5)-(H8) is a sufficient condition for equilibrium E_6 to be stable, and the necessary and sufficient condition for E_6 to be stable is all roots of $E_q.(3.9)$ satisfy $|\arg(\lambda_i)| > \frac{\alpha \pi}{2}$.

3.3. Hopf bifurcation

In this subsection, according to the research methods in literature [22, 38, 45], we study the Hopf bifurcation with time delay as the parameter.

we will analyze the Hopf bifurcation of E_6 when $\tau > 0$, and the characteristic equation at the equilibrium E_6 is

$$s^{4\alpha} + p_1 s^{3\alpha} + p_2 s^{2\alpha} + p_3 s^{\alpha} + p_4 + (q_1 s^{3\alpha} + q_2 s^{2\alpha} + q_3 s^{\alpha} + q_4) e^{-s\tau} = 0,$$
(3.10)

where

 $p_1 = -a_{11} - a_{44},$

 $p_2 = a_{11}a_{44} - a_{13}a_{31} - a_{12}a_{21} - a_{23}a_{32} - a_{34}a_{43},$

- $p_3 = a_{11}a_{23}a_{32} a_{12}a_{23}a_{31} a_{13}a_{21}a_{32} + a_{12}a_{21}a_{44} + a_{11}a_{34}a_{43} + a_{13}a_{31}a_{44} a_{14}a_{31}a_{43} + a_{23}a_{32}a_{44} a_{24}a_{32}a_{43},$
- $p_4 = -a_{11}a_{23}a_{32}a_{44} + a_{11}a_{24}a_{32}a_{43} + a_{12}a_{21}a_{34}a_{43} + a_{12}a_{23}a_{31}a_{44} a_{12}a_{24}a_{31}a_{43} + a_{13}a_{21}a_{32}a_{44} a_{14}a_{21}a_{32}a_{43},$

$$q_1 = -b_{44},$$

- $q_2 = a_{11}b_{44} a_{24}b_{42} a_{34}b_{43},$
- $q_3 = a_{11}a_{24}b_{42} + a_{12}a_{21}b_{44} a_{14}a_{21}b_{42} + a_{11}a_{34}b_{43} + a_{13}a_{31}b_{44} a_{14}a_{31}b_{43} + a_{23}a_{32}b_{44} a_{23}a_{34}b_{42} a_{24}a_{32}b_{43},$
- $q_4 = -a_{11}a_{23}a_{32}b_{44} + a_{11}a_{23}a_{34}b_{42} + a_{11}a_{24}a_{32}b_{43} + a_{12}a_{21}a_{34}b_{43} + a_{12}a_{23}a_{31}b_{44} a_{12}a_{24}a_{31}b_{43} + a_{13}a_{21}a_{32}b_{44} a_{13}a_{21}a_{34}b_{42} + a_{13}a_{24}a_{31}b_{42} a_{14}a_{21}a_{32}b_{43} a_{14}a_{23}a_{31}b_{42}.$

Mathematical Biosciences and Engineering

Volume 16, Issue 4, 3836–3868.

Assume that $s = i\omega = \omega(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}), \omega > 0$ is a root of Eq.(3.10). Substituting $s = i\omega$ into Eq.(3.10), one gets

$$\omega^{4\alpha}(\cos 2\alpha\pi + i\sin 2\alpha\pi) + p_1\omega^{3\alpha}(\cos \frac{3\alpha\pi}{2} + i\sin \frac{3\alpha\pi}{2}) + p_2\omega^{2\alpha}(\cos \alpha\pi + i\sin \alpha\pi) + p_3\omega^{\alpha}(\cos \frac{\alpha\pi}{2} + i\sin \frac{\alpha\pi}{2}) + p_4 + [q_1\omega^{3\alpha}(\cos \frac{3\alpha\pi}{2} + i\sin \frac{3\alpha\pi}{2}) + q_2\omega^{2\alpha}(\cos \alpha\pi + i\sin \alpha\pi) + q_3\omega^{\alpha}(\cos \frac{\alpha\pi}{2} + i\sin \frac{\alpha\pi}{2}) + q_4](\cos \omega\tau - i\sin \omega\tau) = 0.$$
(3.11)

and separating the real and imaginary parts of it, it results in

$$\begin{cases} R_2 \cos(\omega\tau) + I_2 \sin(\omega\tau) = -R_1, \\ I_2 \cos(\omega\tau) - R_2 \sin(\omega\tau) = -I_1, \end{cases}$$
(3.12)

 R_i , I_i are defined as follows:

$$\begin{aligned} R_1 &= \omega^{4\alpha} \cos 2\alpha \pi + p_1 \omega^{3\alpha} \cos \frac{3\alpha \pi}{2} + p_2 \omega^{2\alpha} \cos \alpha \pi + p_3 \omega^{\alpha} \cos \frac{\alpha \pi}{2} + p_4, \\ R_2 &= q_1 \omega^{3\alpha} \cos \frac{3\alpha \pi}{2} + q_2 \omega^{2\alpha} \cos \alpha \pi + q_3 \omega^{\alpha} \cos \frac{\alpha \pi}{2} + q_4, \\ I_1 &= \omega^{4\alpha} \sin 2\alpha \pi + p_1 \omega^{3\alpha} \sin \frac{3\alpha \pi}{2} + p_2 \omega^{2\alpha} \sin \alpha \pi + p_3 \omega^{\alpha} \sin \frac{\alpha \pi}{2}, \\ I_2 &= q_1 \omega^{3\alpha} \sin \frac{3\alpha \pi}{2} + q_2 \omega^{2\alpha} \sin \alpha \pi + q_3 \omega^{\alpha} \sin \frac{\alpha \pi}{2}. \end{aligned}$$

It can be acquired from Eq. (3.12) that

$$\begin{cases} \cos(\omega\tau) = -\frac{R_1R_2 + I_1I_2}{R_2^2 + I_2^2} = F(\omega),\\ \sin(\omega\tau) = \frac{R_2I_1 - R_1I_2}{R_2^2 + I_2^2} = G(\omega). \end{cases}$$
(3.13)

Adding the squares of the two equations of Eq.(3.12), we obtain

$$\omega^{8\alpha} + M + N = 0. \tag{3.14}$$

where *M* is a polynomial containing $\omega^{7\alpha}$, $\omega^{6\alpha}$, $\omega^{5\alpha}$, $\omega^{4\alpha}$, $\omega^{3\alpha}$, $\omega^{2\alpha}$, ω^{α} , and *N* is a constant.

Define

$$h(\omega) = \omega^{8\alpha} + M + N. \tag{3.15}$$

Suppose that N < 0. Thus, $h(\omega)$ has at least one positive root. The delay τ is regarded as a bifurcation parameter. Let $s(\omega) = \xi(\tau) + i\omega(\tau)$ be the Eq.(3.10) such that for some initial value of the bifurcation parameter τ_0 we have $\xi(\tau_0) = 0$, $\omega(\tau_0) = \omega_0$. Without loss of generality, we assume $\omega(0) > 0$. From Eq.(3.13), one can conclude

$$\tau_j = \frac{1}{\omega_0} \left[\arccos F(\omega) + 2j\pi \right], \quad j = 0, \ 1, \ 2 \cdots$$
 (3.16)

where

 $\tau_0 = \min\{\tau_j\}, \quad j = 0, \ 1, \ 2 \cdots.$

To derive the condition of the occurrence for Hopf bifurcation, we have the following Lemma.

Lemma 3.10. Assume that N < 0, then Hopf bifurcation occurs provided $h'(\omega_0) \neq 0$.

Proof. Differentiating both sides of Eq.(3.10) with respect to τ , it can be obtained that

$$(4\alpha s^{4\alpha-1} + 3\alpha p_1 s^{3\alpha-1} + 2\alpha p_2 s^{2\alpha-1} + \alpha p_3 s^{\alpha-1}) \frac{ds}{d\tau} + (3\alpha q_1 s^{3\alpha-1} + 2\alpha q_2 s^{2\alpha-1} + \alpha q_3 s^{\alpha-1}) e^{-s\tau} \frac{ds}{d\tau} + (q_1 s^{3\alpha} + q_2 s^{2\alpha} + q_3 s^{\alpha} + q_4) e^{-s\tau} \left(-\tau \frac{ds}{d\tau} - s\right) = 0.$$

Hence, one gets

$$\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^{-1} = \frac{(4\alpha s^{4\alpha-1} + 3\alpha p_1 s^{3\alpha-1} + 2\alpha p_2 s^{2\alpha-1} + \alpha p_3 s^{\alpha-1}) + (3\alpha q_1 s^{3\alpha-1} + 2\alpha q_2 s^{2\alpha-1} + \alpha q_3 s^{\alpha-1})e^{-s\tau}}{s(q_1 s^{3\alpha} + q_2 s^{2\alpha} + q_3 s^{\alpha} + q_4)e^{-s\tau}} - \frac{\tau}{s} \\ \frac{(4\alpha s^{4\alpha-1} + 3\alpha p_1 s^{3\alpha-1} + 2\alpha p_2 s^{2\alpha-1} + \alpha p_3 s^{\alpha-1})}{s(s^{4\alpha} + p_1 s^{3\alpha} + p_2 s^{2\alpha} + p_3 s^{\alpha} + p_4)} + \frac{(3\alpha q_1 s^{3\alpha-1} + 2\alpha q_2 s^{2\alpha-1} + \alpha q_3 s^{\alpha-1})}{s(q_1 s^{3\alpha} + q_2 s^{2\alpha} + q_3 s^{\alpha} + q_4)} - \frac{\tau}{s}.$$

$$(3.17)$$

Substitute $s = i\omega_0$ into Eq.(3.17), we have

$$\begin{aligned} \operatorname{Re}\left[\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^{-1}\Big|_{\tau=\tau_{0}}\right] &= \operatorname{Re}\left[-\frac{(4\alpha(i\omega_{0})^{4\alpha-1}+3\alpha p_{1}(i\omega_{0})^{3\alpha-1}+2\alpha p_{2}(i\omega_{0})^{2\alpha-1}+\alpha p_{3}(i\omega_{0})^{\alpha-1})}{(i\omega_{0})((i\omega_{0})^{4\alpha}+p_{1}(i\omega_{0})^{3\alpha}+p_{2}(i\omega_{0})^{2\alpha}+p_{3}(i\omega_{0})^{\alpha}+p_{4})}\right] \\ &+ \frac{(3\alpha q_{1}(i\omega_{0})^{3\alpha-1}+2\alpha q_{2}(i\omega_{0})^{2\alpha}+q_{3}(i\omega_{0})^{\alpha-1})}{(i\omega_{0})(q_{1}(i\omega_{0})^{3\alpha-1}+3\alpha p_{1}(i\omega_{0})^{3\alpha-1}+2\alpha p_{2}(i\omega_{0})^{2\alpha-1}+\alpha p_{3}(i\omega_{0})^{\alpha-1})}{(i\omega_{0})((i\omega_{0})^{4\alpha}+p_{1}(i\omega_{0})^{3\alpha}+p_{2}(i\omega_{0})^{2\alpha}+p_{3}(i\omega_{0})^{\alpha-1})}{(i\omega_{0})((i\omega_{0})^{4\alpha}+p_{1}(i\omega_{0})^{3\alpha}+p_{2}(i\omega_{0})^{2\alpha}+p_{3}(i\omega_{0})^{\alpha}+p_{4})}} \\ &+ \frac{(3\alpha q_{1}(i\omega_{0})^{3\alpha-1}+2\alpha q_{2}(i\omega_{0})^{2\alpha-1}+\alpha q_{3}(i\omega_{0})^{\alpha-1})}{(i\omega_{0})(q_{1}(i\omega_{0})^{3\alpha}+q_{2}(i\omega_{0})^{2\alpha}+q_{3}(i\omega_{0})^{\alpha}+q_{4})}}\right] \\ &= \frac{h'(\omega_{0})}{2\omega_{0}G}, \end{aligned}$$

where

$$G = \left(q_1 \omega_0^{3\alpha} \cos\frac{(3\alpha+1)\pi}{2} + q_2 \omega_0^{2\alpha} \cos\frac{(2\alpha+1)\pi}{2} + q_3 \omega_0^{\alpha} \cos\frac{(\alpha+1)\pi}{2}\right)^2 + \left(q_1 \omega_0^{3\alpha} \sin\frac{(3\alpha+1)\pi}{2} + q_2 \omega_0^{2\alpha} \sin\frac{(2\alpha+1)\pi}{2} + q_3 \omega_0^{\alpha} \sin\frac{(\alpha+1)\pi}{2} + q_4\right)^2.$$

then sign $\left\{\frac{\mathrm{d}\mathrm{Re}(\lambda)}{\mathrm{d}\tau}\Big|_{\tau=\tau_0}\right\} = \mathrm{sign}\left\{\mathrm{Re}\left[\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\Big|_{\tau=\tau_0}\right]\right\} = \mathrm{sign}\{h'(\omega_0)\}.$

Obviously, if $h'(\omega_0) \neq 0$ the transversality condition holds, and Hopf bifurcation occurs at $\tau = \tau_0$.

Theorem 3.11. Suppose that (H5)-(H8) and N < 0 hold, then the positive equilibrium E_6 of system (1.2) is asymptotically stable when $\tau \in [0, \tau_0)$, $h'(\omega_0) < 0$ and unstable when $\tau > \tau_0$, $h'(\omega_0) > 0$. When $\tau = \tau_0$, $h'(\omega_0) \neq 0$ a Hopf bifurcation occurs, that is a family of periodic solutions bifurcates from E_6 as τ passes through the critical value τ_0 .

4. Numerical simulations

In this section, some numerical examples are presented to verify the theoretical results. The simulation are based on Adama-Bashforth-Moulton predictor-corrector scheme [46].

Example 1. For the following set of parameters: $\Lambda = 1$, $b_1 = 0.3$, $b_2 = 0.25$, $c_1 = 0.06$, $c_2 = 0.06$, $c_3 = 0.06$, $k_1 = 0.7$, $k_2 = 0.7$, $\eta_1 = 2.1$, $\eta_2 = 0.2$, $\mu = 1$, $\mu_1 = 0.5$, $\mu_2 = 0.3$, $\mu_3 = 0.3$, $\theta_1 = 0.6$, $\theta_2 = 0.7$, $\delta = 0.1$, $h_1 = 0.2$, $h_2 = 0.1$.

In this case $R_1 = 0.42 < 1$, $R_2 = 0.5833 < 1$. From Figure 1, we can see that the equilibrium $E_0 = (1, 0, 0, 0)$ is stable for different values of α and different sets of initial values: [X(0), Y₁(0), Y₂(0), Z(0)] = [0.9, 0.2, 0.2], [1.2, 0.5, 0.3, 0.6].

Example 2. For the following set of parameters: $\Lambda = 0.5$, $b_1 = 2.5$, $b_2 = 0.3$, $c_1 = 0.01$, $c_2 = 0.01$, $c_3 = 0.06$, $k_1 = 0.5$, $k_2 = 0.7$, $\eta_1 = 0.2$, $\eta_2 = 2.1$, $\mu = 1$, $\mu_1 = 0.3$, $\mu_2 = 0.5$, $\mu_3 = 0.3$, $\theta_1 = 0.5$, $\theta_2 = 0.1$, $\delta = 0.01$, $h_1 = 0.2$, $h_2 = 0.1$.

In this case, $1 < R_1 = 2.0833 < \frac{X^{(0)}b_1}{c_1}$, $R_2 = 0.21 < R_1$ and $\mu_3 > \theta_1 \eta_1 Y_1^{(1)}$. From Figure 2, we can see that the equilibrium $E_1 = (0.24, 0.4407, 0, 0)$ is stable for different values of α and different sets of initial values: $[X(0), Y_1(0), Y_2(0), Z(0)] = [0.1, 0.6, 0.5, 0.2], [0.2, 0.5, 0.8, 0.6].$

Example 3. For the following set of parameters: $\Lambda = 0.5$, $b_1 = 0.3$, $b_2 = 2.5$, $c_1 = 0.01$, $c_2 = 0.01$, $c_3 = 0.06$, $k_1 = 0.5$, $k_2 = 0.6$, $\eta_1 = 2.1$, $\eta_2 = 0.2$, $\mu = 1$, $\mu_1 = 0.3$, $\mu_2 = 0.5$, $\mu_3 = 0.5$, $\theta_1 = 0.1$, $\theta_2 = 0.1$, $\delta = 0.01$, $h_1 = 0.2$, $h_2 = 0.1$.

In this case, $1 < R_2 = 1.5 < \frac{X^{(0)}b_2}{c_2}$, $R_2 > R_1$ and $\mu_3 > (\theta_2\eta_2 - f)Y_2^{(2)}$. From Figure 3, we can see that the equilibrium $E_2 = (0.3333, 0, 0.2024, 0)$ is stable for different values of α and different sets of initial values: $[X(0), Y_1(0), Y_2(0), Z(0)] = [0.2, 0.1, 0.5, 0.2], [0.1, 0.05, 0.3, 0.6].$

Remark 4.1. The above three examples corresponding to the following ecological interpretation.

(1) Figure 1 indicates that if $R_1 < 1$, $R_2 < 1$, then the phytoplankton can not survive and the zooplankton will also die out. However, this phenomenon usually does not happen in the real world.

(2) Figure 2 indicates that if $R_1 > 1$, $R_1 > R_2$, then the non-toxic phytoplankton will win the competition between phytoplankton and toxic phytoplankton, while the zooplankton will die out due to excessive mortality.

(3) Figure 3 indicates that if $R_2 > 1$, $R_2 > R_1$, then the toxic phytoplankton will win the competition between phytoplankton and toxic phytoplankton, while the zooplankton will die out due to excessive mortality.

Example 4. For the following set of parameters: $\Lambda = 1.4$, $b_1 = 2.8$, $b_2 = 2.3$, $c_1 = 0.01$, $c_2 = 0.01$, $c_3 = 0.06$, $k_1 = 0.95$, $k_2 = 0.1$, $\eta_1 = 2.1$, $\eta_2 = 0.26$, $\mu = 0.2$, $\mu_1 = 0.2$, $\mu_2 = 0.8$, $\mu_3 = 0.5$, $\theta_1 = 0.2$, $\theta_2 = 0.5$, $\delta = 0.01$, $h_1 = 0.1$, $h_2 = 0.1$.

In this case, simple calculation indicates that the sufficient condition (2) of Proposition 3.6 is satisfied. From Figure 4 we can see that the equilibrium $E_4 = (0.4067, 1.1905, 0, 0.4199)$ is stable for different values of α and different sets of initial values: $[X(0), Y_1(0), Y_2(0), Z(0)] = [0.2, 0.3, 0.5, 0.2], [0.3, 0.4, 0.8, 0.6].$

Example 5. For the following set of parameters: $\Lambda = 1.4$, $b_1 = 2.3$, $b_2 = 2.5$, $c_1 = 0.01$, $c_2 = 0.01$, $c_3 = 0.06$, $k_1 = 0.1$, $k_2 = 0.95$, $\eta_1 = 0.26$, $\eta_2 = 2.1$, $\mu = 0.2$, $\mu_1 = 0.8$, $\mu_2 = 0.2$, $\mu_3 = 0.5$, $\theta_1 = 0.5$, $\theta_2 = 0.2$, $\delta = 0.01$, $h_1 = 0.1$, $h_2 = 0.1$.

In this case, simple calculation indicates that the sufficient condition (2) of Proposition 3.7 is satisfied. From Figure 5 we can see that the equilibrium $E_5 = (0.4422, 0, 1.2195, 0.4048)$ is stable for different values of α and different sets of initial values: $[X(0), Y_1(0), Y_2(0), Z(0)] = [0.2, 0.5, 0.3, 0.2], [0.3, 0.8, 0.4, 0.6].$

Remark 4.2. The above examples corresponding to the following ecological interpretation.

(1) Figure 4 indicates that if the toxic phytoplankton is less competitive than the non-toxic phytoplankton, then the toxic phytoplankton will die out.

(2) Figure 5 indicates that if the toxic phytoplankton win the competition between non-toxic phytoplankton and toxic phytoplankton, then the zooplankton may still survive under certain conditions, that is, nutrients, toxic phytoplankton and zooplankton may theoretically coexist. However, this phenomenon usually does not appear in real world. (3) From Figures 1–5 we can see that as the value of α decreases, the steady speed becomes slow for each equilibrium. This indicates that the value of α has obvious effects on the dynamical behaviors of the system.

Example 6. For the following set of parameters: $\alpha = 0.8$, $\Lambda = 1.4$, $b_1 = 0.32$, $b_2 = 0.54$, $c_1 = 0.06$, $c_2 = 0.08$, $c_3 = 0.6$, $k_1 = 0.7$, $k_2 = 0.6$, $\eta_1 = 1.8$, $\eta_2 = 0.6$, $\mu = 0.2$, $\mu_1 = 0.4$, $\mu_2 = 0.9$, $\mu_3 = 0.5$, $\theta_2 = 0.5$, $\delta = 0.1$, $h_1 = 0.1$, $h_2 = 0.1$.

In this example, we will consider the influence of toxic, i.e., θ_1 . Here, we choose $\theta_1 = 0, 0.6$, with the initial conditions: $[X(0), Y_1(0), Y_2(0), Z(0)] = [0.2, 0.2, 0.3, 0.2], [0.8, 0.8, 0.6, 0.8].$

Example 7. In this example, we will consider the influence of α .

(1) For the following set of parameters: $\Lambda = 1.4$, $b_1 = 2.3$, $b_2 = 2.5$, $c_1 = 0.01$, $c_2 = 0.01$, $c_3 = 0.06$, $k_1 = 0.1$, $k_2 = 0.95$, $\eta_1 = 0.26$, $\eta_2 = 2.1$, $\mu = 0.2$, $\mu_1 = 0.8$, $\mu_2 = 0.2$, $\mu_3 = 0.5$, $\theta_1 = 0.5$, $\theta_2 = 0.2$, $\delta = 0.01$, $h_1 = 0.1$, $h_2 = 0.1$, with the initial conditions: [X(0), Y_1(0), Y_2(0), Z(0)] = [0.2, 0.5, 0.3, 0.2].

(2) For the following set of parameters: $\Lambda = 1.4$, $b_1 = 0.32$, $b_2 = 0.54$, $c_1 = 0.06$, $c_2 = 0.08$, $c_3 = 0.6$, $k_1 = 0.7$, $k_2 = 0.6$, $\eta_1 = 1.8$, $\eta_2 = 0.6$, $\mu = 0.2$, $\mu_1 = 0.4$, $\mu_2 = 0.9$, $\mu_3 = 0.5$, $\theta_1 = 0.6$, $\theta_2 = 0.5$, $\delta = 0.1$, $h_1 = 0.1$, $h_2 = 0.1$, with the initial conditions: [X(0), Y₁(0), Y₂(0), Z(0)] = [0.5, 0.5, 0.5, 0.2].

Remark 4.3. (1) From Figure 7, we can see that if the value of α is relatively big(i.e., $\alpha=1$, 0.8), then the equilibrium E_5 is locally stable; if the value of α is relatively small(i.e., $\alpha=0.4$), then the equilibrium is unstable, and oscillation may occur.

(2) From Figure 8, we can see that if the value of α is relatively big(i.e., $\alpha=1$, 0.7), then the equilibrium E_6 is locally stable; if the value of α is relatively small(i.e., $\alpha=0.1$), then the equilibrium E_0 is locally stable.

(3) Figure 7 and Figure 8 indicate that if the value of α is relatively small, the system will be destabilized.

Example 8. For the following set of parameters: $\alpha = 0.89$, $\Lambda = 1.4$, $b_1 = 0.32$, $b_2 = 0.54$, $c_1 = 0.06$, $c_2 = 0.08$, $c_3 = 0.6$, $k_1 = 0.7$, $k_2 = 0.6$, $\eta_1 = 1.8$, $\eta_2 = 0.6$, $\mu = 0.2$, $\mu_1 = 0.4$, $\mu_2 = 0.9$, $\mu_3 = 0.5$, $\theta_1 = 0.6$, $\theta_2 = 0.5$, $\delta = 0.1$, $h_1 = 0.1$, $h_2 = 0.1$.

In this example, the equilibrium $E_6 = (3.2026, 0.4114, 0.2784, 0.1608)$, and our main aim is to study the effect of time delay on the stability of the system.

(1) From Figure 9, we can see that if $\tau = 0$, then the equilibrium E_6 is stable with different sets of initial value: $[X(0), Y_1(0), Y_2(0), Z(0)] = [0.5, 0.5, 0.5, 0.2], [2.8, 1.8, 3.3, 1.3].$

(2) From Figure 10, we will see that if $\tau = 3 < \tau_0 \approx 4.4671$, then the equilibrium E_6 is stable with different sets of initial value: $[X(0), Y_1(0), Y_2(0), Z(0)] = [0.5, 0.5, 0.5, 0.2], [2.8, 1.8, 3.3, 1.3].$

(3) From Figure 11, we can see that if $\tau = 4.2 < \tau_0 \approx 4.4671$, then the equilibrium E_6 is stable with different sets of initial value: $[X(0), Y_1(0), Y_2(0), Z(0)] = [0.5, 0.5, 0.5, 0.2], [2.8, 1.8, 3.3, 1.3].$

(4) From Figure 12, we will see that if $\tau = 5 > \tau_0 \approx 4.4671$, then periodic oscillation occurs, and the equilibrium E_6 will lose its stability and periodic solutions appear through Hopf bifurcation, with different sets of initial value: $[X(0), Y_1(0), Y_2(0), Z(0)] = [0.5, 0.5, 0.5, 0.2], [0.8, 0.8, 0.3, 0.3].$

(5) From Figure 13, we can see that if $\tau = 6 > \tau_0 \approx 4.4671$, then similar periodic oscillation occurs as that in Figure 12, but with larger amplitude.

Remark 4.4. From the above example, we can see that the time delay has an effect of destabilizing the equilibrium E_6 . In other words, the larger the value of time delay is, the more possible that the equilibrium E_6 lose its stability.



Figure 1. (a)-(d) are the time series of the system (1.2), which show that the equilibrium E_0 is stable for different values of α (α =1, 0.9) when τ =0; the red and '-' and the blue and '-. -. ' line represents dynamics with initial value [0.9, 0.2, 0.2, 0.2]; the yellow and '...' and the green and '--' line represents dynamics with initial value [1.2, 0.5, 0.3, 0.6].



Figure 2. (a)-(d) are the time series of the system (1.2), which show that the equilibrium E_1 is stable for different values of α (α =1, 0.9) when τ =0; the red and '-' and the blue and '-. -. ' line represents dynamics with initial value [0.1, 0.6, 0.5, 0.2]; the yellow and '...' and the green and '--' line represents dynamics with initial value [0.2, 0.5, 0.8, 0.6].



Figure 3. (a)-(d) are the time series of the system (1.2), which show that the equilibrium E_2 is stable for different values of α (α =1, 0.9) when τ =0; the red and '-' and the blue and '-. -. ' line represents dynamics with initial value [0.2, 0.1, 0.5, 0.2]; the yellow and '...' and the green and '--' line represents dynamics with initial value [0.1, 0.05, 0.3, 0.6].



Figure 4. (a)-(d) are the time series of the system (1.2), which show that the equilibrium E_4 is stable for different values of α (α =1, 0.95) when τ =0; the red and '-' and the blue and '-.-..' line represents dynamics with initial value [0.2, 0.3, 0.5, 0.2]; the yellow and '...' and the green and '--' line represents dynamics with initial value [0.3, 0.4, 0.8, 0.6].



Figure 5. (a)-(d) are the time series of the system (1.2), which show that the equilibrium E_5 is stable for different values of α (α =1, 0.8) when τ =0; the red and '-' and the blue and '-.-.' line represents dynamics with initial value [0.2, 0.5, 0.3, 0.2]; the yellow and '...' and the green and '--' line represents dynamics with initial value [0.3, 0.8, 0.4, 0.6].



Figure 6. (a)-(d) are the time series of the system (1.2), which show the influence of θ_1 . (The initial conditions: [0.2, 0.2, 0.3, 0.2], [0.8, 0.8, 0.6, 0.8]).



Figure 7. (a)-(d) are the time series of the system (1.2), which show the influence of α . (The initial conditions: [0.2, 0.5, 0.3, 0.2]).



Figure 8. (a)-(d) are the time series of the system (1.2), which show the influence of α . (The initial conditions: [0.5, 0.5, 0.5, 0.2]).



Figure 9. (a)-(d) are the time series of the system (1.2), which show that the equilibrium E_6 is stable for $\alpha = 0.89$ and $\tau = 0$, the blue and '-' line represents dynamics with initial value [0.5, 0.5, 0.5, 0.2];the red and '-. -.' line represents dynamics with initial value [0.8, 0.8, 0.3, 0.3].



Figure 10. (a)-(d) are the time series of the system (1.2), which show that the equilibrium E_6 is stable for $\alpha = 0.89$ and $\tau = 3 < \tau_0 \approx 4.4671$, the blue and '-' line represents dynamics with initial value [0.5, 0.5, 0.5, 0.2];the red and '-. -.' line represents dynamics with initial value [0.8, 0.8, 0.3, 0.3].



Figure 11. (a)-(d) are the time series of the system (1.2), which show that the equilibrium E_6 is stable for $\alpha = 0.89$ and $\tau = 4.2 < \tau_0 \approx 4.4671$, the blue and '-' line represents dynamics with initial value [0.5, 0.5, 0.5, 0.2]; the red and '-. -.' line represents dynamics with initial value [0.8, 0.8, 0.3, 0.3].



Figure 12. (a)-(d) are time series of the system (1.2), which show that the equilibrium E_6 is unstable, and periodic oscillation occurs, for $\alpha = 0.89$ and $\tau = 5 > \tau_0 \approx 4.4671$. The blue and '-' line represents the dynamics with initial value [0.5, 0.5, 0.5, 0.2]; while the red and '-. -. -.' line represents the dynamics with initial value [0.8, 0.8, 0.3, 0.3].



Figure 13. (a)-(d) are the time series of the system (1.2), which show that the equilibrium E_6 is unstable, and periodic oscillation occurs, for $\alpha = 0.89$ and $\tau = 6 > \tau_0 \approx 4.4671$. The blue and '-' line represents the dynamics with initial value [0.5, 0.5, 0.5, 0.2]; while the red and '-. -. -.' line represents the dynamics with initial value [0.8, 0.8, 0.3, 0.3].

Fractional order α	Critical frequency ω_0	Bifurcation point $ au_0$
0.60	0.0221	49.8601
0.65	0.0318	30.9419
0.70	0.0436	20.0341
0.75	0.0576	13.3431
0.80	0.0740	8.9972
0.85	0.0927	6.1119
0.90	0.1140	4.1175
0.95	0.1378	2.7095
1.00	0.1642	1.6926

Table 2. The effect of α on the	e values ω_0 , τ_0 in system(1.2).
---	--



Figure 14. Illustration of bifurcation τ_0 versus fractional order α for system (1.2). The bifurcation points are becoming smaller and smaller as the value of α increase.

5. Discussion

In this paper, a fractional-order mathematical model is constructed to describe the active of nutrientphytoplankton-toxic phytoplankton.

Through qualitative analysis, we get the following results.

 \diamond We figure out the sufficient conditions for the existence and local stability of E_0 , E_1 , E_2 , E_3 , E_4 , E_5 , E_6 for $\tau = 0$.

 \diamond By using time delay as a bifurcation parameter, the existence of Hopf bifurcation is analyzed in detail. We find that if $\tau < \tau_0$, then the equilibrium E_6 is locally stable; while it is unstable if $\tau > \tau_0$ and Hopf bifurcation may occur near τ_0 .

Through numerical simulation we get the following results.

 \diamond Figure 1 shows the stability of equilibrium E_0 for different values of α .

 \diamond Figure 2 shows the stability of equilibrium E_1 for different values of α .

 \diamond Figure 3 shows the stability of equilibrium E_2 for different values of α .

 \diamond Figure 4 shows the stability of equilibrium E_4 for different values of α .

 \diamond Figure 5 shows the stability of equilibrium E_5 for different values of α .

 \diamond Figure 6 shows the effect of parameter θ_1 on the system (1.2).

 \diamond Figure 7 and Figure 8 indicate that the value of α is closely relate to the stability of each equilibrium. The stability of each equilibrium becomes weaker as the value of α decreases.

♦ Figures 9–11 show that E_6 is stable if $\tau \in [0, \tau_0)$; Figures 12–13 show that E_6 is unstable if $\tau > \tau_0$ and periodic oscillation may occur; Figures 9–13 indicate that the impact of τ on the dynamics of the system is crucial.

 \diamond Table 2 and Figure 14 show that the value of τ_0 arises as the value of α increases.

Remark 5.1. When $\tau = 0$, $Y_1 = Y_2$ and $h_1 = h_2 = 0$, then system (1.2) will degenerated to the model in [40].

In the model of this paper, phytoplankton is divided into two class, non-toxic phytoplankton and toxic phytoplankton. Some experimental data suggested that some zooplankton are capable of selecting for nontoxic phytoplankton, a mechanism that allows toxic phytoplankton to coexist with nontoxic phytoplankton [47]. This is also consistent with the results in Figure 9. Although some zooplankton have the ability to distinguish between toxic and non-toxic plants, some other experiments have shown that some zooplankton might not be able to distinguish between toxic and nontoxic algae, even shows a slight preference for toxic strains [48, 49]. We can find from Figure 6 that once non-toxic phytoplankton become scarce, the zooplankton start to eat the toxic phytoplankton, even if the toxicity is weak, the zooplankton may become extinct. This is dangerous for the ecosystem.

Remark 5.2. Any ecosystem depends on the natural environment, and in the real natural environment there are more or less physical and chemical signals that interact with the ecosystem. This motivate us to consider stochastic effects to the ecosystem, that is, white noise should be included into the system. We leave this as our next work.

Authors contributions

Each of the authors, Ruiqing Shi, Jianing Ren, Cuihong Wang contributed to each part of this work equally and read and approved the final version of the manuscript.

Acknowledgements

This work is partly supported by National Natural Science Foundation of China (No. 61907027). The authors would like to thank the anonymous reviewers for their helpful comments, which improved the quality of this paper greatly.

Conflict of interest

The authors declare that they have no financial or non-financial competing interests.

References

- 1. T. Saha, M. Bandyopadhyay, Dynamical analysis of toxin producing phytoplankton-zooplankton interactions, *Nonlinear Anal. Real. World Appl.*, **10** (2009), 314–332.
- 2. O. A. Chichigina, A. A. Dubkov, D. Valenti, B. Spagnolo, Stability in a system subject to noise with regulated periodicity, *Phys. Rev. E*, **84** (2011), 021134.
- 3. A. L. Barbera, B. Spagnolo, Spatio-temporal patterns in population dynamics, *Phys. A*, **314** (2002), 120–124.

- 4. D. Valenti, L. Tranchina, M. Brai, A. Caruso, C. Cosentino, B. Spagnolo, Environmental metal pollution considered as noise: Effects on the spatial distribution of benthic foraminifera in two coastal marine areas of Sicily (Southern Italy), *Ecol. Model.*, **213** (2008), 449–462.
- 5. H. Zhang, T. Zhang, The stationary distribution of a microorganism flocculation model with stochastic perturbation, *Appl. Math. Lett.*, **103** (2020), 106217.
- 6. T. Zhang, N. Gao, T. Wang, H. Liu, Z. Jiang, Global dynamics of a model for treating microorganisms in sewage by periodically adding microbial flocculants, *Math. Biosci. Eng.*, **17** (2020), 179–201.
- 7. M. Chen, M. Fan, R. Liu, X. Wang, X. Yuan, H. Zhu, The dynamics of temperature and light on the growth of phytoplankton, *J. Theor. Biol.*, **385** (2015), 8–19.
- 8. Y. Sekerci, S. Petrovskii, Mathematical modelling of plankton-oxygen dynamics under the climate change, *B Math. Biol.*, **77** (2015), 2325–2353.
- 9. J. Zhao, J. Wei, Stability and bifurcation in a two harmful phytoplankton-zooplankton system, *Chaos Soliton. Fract.*, **39** (2009), 1395–1409.
- 10. S. Abdallah, Stability and persistence in plankton models with distributed delays, *Chaos Soliton*. *Fract.*, **17** (2003), 879–884.
- 11. R. R. Sarkar, B. Mukhopadhyay, R. Bhattacharyya, S. Banerjee, Time lags can control algal bloom in two harmful phytoplankton-zooplankton system, *Appl. Math. Comput.*, **186** (2007), 445–459.
- G. Denaro, D. Valenti, A. L. Cognata, B. Spagnolo, A. Bonanno, G. Basilone, et al., Spatiotemporal behaviour of the deep chlorophyll maximum in Mediterranean Sea: development of a stochastic model for picophytoplankton dynamics, *Ecol. Complex.*, **13** (2013), 21–34.
- 13. G. Denaro, D. Valenti, B. Spagnolo, G. Basilone, S. Mazzola, S. W. Zgozi, et al., Dynamics of two picophytoplankton groups in mediterranean sea: Analysis of the deep chlorophyll maximum by a stochastic advection-reaction-diffusion model, *PLoS One*, **8** (2013), e66765.
- 14. D. Huang, H. Wang, J. Feng, Z. Zhu, Hopf bifurcation of the stochastic model on HAB nonlinear stochastic dynamics, *Chaos Soliton. Fract.*, **27** (2006), 1072–1079.
- 15. J. Chattopadhyay, R. R. Sarkar, S. Pal, Mathematical modelling of harmful algal blooms supported by experimental findings, *Ecol. Complex.*, **1** (2004), 225–235.
- 16. M. Javidi, B. Ahmad, Dynamic analysis of time fractional order phytoplankton-toxic phytoplankton-zooplankton system, *Ecol. Model.*, **138** (2015), 8–18.
- 17. D. Straile, Meteorological forcing of plankton dynamics in a large and deep continental European lake, *Oecologia*, **122** (2000), 44–50.
- 18. R. W. Black, L. B. Slobodkin, What is cyclomorphosis? Freshwater Biol., 18 (1987), 373–378.
- 19. Y. He, Z. Li, Epigenetic environmental memories in plants: Establishment, maintenance, and reprogramming, *Trends Genet.*, **34** (2018), 1–11.
- 20. S. I. Dodson, T. A. Crowl, B. L. Peckarsky, L. B. Kats, A. P. Covich, J. M. Culp, Non-visual communication in freshwater benthos: an overview, *J. N. Am. Benthol. Soc.*, **13** (1994), 268–282.
- 21. D. P. Chivers, R. J. F. Smith, Chemical alarm signalling in aquatic predator-prey systems: A review and prospectus, *Ecoscience*, **5** (1998), 338–352.

- 22. C. Huang, J. Cao, M. Xiao, A. Alsaedi, F. E. Alsaadi, Controlling bifurcation in a delayed fractional predator-prey system with incommensurate orders, *Appl. Math. Comput.*, **293** (2017), 293–310.
- 23. E. Ahmed, A. M. A. El-Sayed, H. A. A. El-Saka, Equilibrium points, stability and numerical solutions of fractional-order predator-prey and rabies models, *J. Math. Anal. Appl.*, **325** (2007), 542–553.
- 24. F. A. Rihan, S. Lakshmanan, A. H. Hashish, R. Rakkiyappan, E. Ahmed, Fractional-order delayed predator-prey systems with Holling type-II functional response, *Nonlinear Dynam.*, **80** (2015), 777–789.
- 25. V. E. Tarasov, V. V. Tarasova, Macroeconomic models with long dynamic memory: fractional calculus approach, *Appl. Math. Comput.*, **338** (2018), 466–486.
- 26. A. Khan, J. F. Gómez-Aguilar, T. Abdeljawad, H. Khan, Stability and numerical simulation of a fractional order plant-nectar-pollinator model, *Alex. Eng. J.*, **59** (2019), 49-59.
- 27. C. I. Muresan, C. Ionescu, S. Folea, R. D. Keyser, Fractional order control of unstable processes: the magnetic levitation study case, *Nonlinear Dynam.*, **80** (2014), 1761–1772.
- 28. M. D. Ortigueira, *Fractional calculus for scientists and engineers*, Springer Netherlands, Berlin, 2011.
- 29. D. Copot, R. D. Keyser, E. Derom, M. Ortigueira, C. M. Ionescu, Reducing bias in fractional order impedance estimation for lung function evaluation, *Biomed. Signal Process*, **39** (2018), 74–80.
- 30. G. S. F. Frederico, D. F. M. Torres, Fractional conservation laws in optimal control theory, *Nonlinear Dynam.*, **53** (2008), 215–222.
- R. L. Magin, O. Abdullah, D. Baleanu, X. J. Zhou, Anomalous diffusion expressed through fractional order differential operators in the Bloch-Torrey equation, *J. Magn. Reson.*, **190** (2008), 255–270.
- 32. E. H. Doha, A. H. Bhrawy, S. S. Ezz-Eldien, A new Jacobi operational matrix: an application for solving fractional differential equations, *Appl. Math. Model.*, **36** (2012), 4931–4943.
- 33. M. S. Asl, M. Javidi, An improved PC scheme for nonlinear fractional differential equations: Error and stability analysis, *J. Comput. Appl. Math.*, **324** (2017), 101–117.
- 34. I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Academic Press, San Diego, Calif, USA, 1999.
- 35. S. Gakkhar, A. Singh, Effects of delay and seasonality on toxin producing phytoplanktonzooplankton system, *Int. J. Biomath.*, **5** (2012), 1–21.
- 36. N. Juneja, K. Agnihotri, H. Kaur, Effect of delay on globally stable prey-predator system, *Chaos Soliton. Fract.*, **111** (2018), 146–156.
- 37. T. Zhang, J. Liu, Z. Teng, Stability of Hopf bifurcation of a delayed SIRS epidemic model with stage structure, *Nonlinear Anal. Real. World Appl.*, **11** (2010), 293–306.
- 38. Z. Wang, Y. Xie, J. Lu, Y. Li, Stability and bifurcation of a delayed generalized fractional-order prey-predator model with interspecific competition, *Appl. Math. Comput.*, **347** (2019), 360–369.

- 39. Z. Wang, X. Wang, Y. Li, X. Huang, Stability and Hopf bifurcation of fractional-order complexvalued single neuron model with time delay, *Int. J. Bifurcat. Chaos*, **27** (2017), 1–13.
- 40. M. S. Asl, M. Javidi, Novel algorithms to estimate nonlinear FDEs: applied to fractional order nutrient-phytoplankton-zooplankton system, *J. Comput. Appl. Math.*, **339** (2018), 193–207.
- 41. Z. Chen, Z. Tian, S. Zhang, C. Wei, The stationary distribution and ergodicity of a stochastic phytoplankton-zooplankton model with toxin-producing phytoplankton under regime switching, *Phys. A*, **537** (2020), 122728.
- A. Barreiro, C. Guisande, I. Maneiro, A. R. Vergara, I. Riveiro, P. Iglesias, Zooplankton interactions with toxic phytoplankton: Some implications for food web studies and algal defence strategies of feeding selectivity behaviour, toxin dilution and phytoplankton population diversity, *Acta Oecol.*, **32** (2007), 279–290.
- 43. I. Petras, *Fractional-Order nonlinear systems: Modeling, analysis and simulation*, HEP/Springer, London, 2011.
- 44. E. Ahmed, A. M. A. El-Sayed, H. A. A. El-Saka, On some Routh-Hurwitz conditions for fractional order differential equations and their applications in Lorenz, Rössler, Chua and Chen systems, *Phys. Lett. A*, **358** (2006), 1–4.
- 45. B. Tao, M. Xiao, Q. Sun, J. Cao, Hopf bifurcation analysis of a delayed fractional-order genetic regulatory network model, *Neurocomputing*, **275** (2017), 677–686.
- 46. S. Bhalekar, V. Daftardar-Gejji, A predictor-corrector scheme for solving nonlinear delay differential equations of fractional order, *Fract. Calc. Appl. Anal.*, **1** (2011), 1–9.
- C. Guisande, M. Frangopulos, I. Maneiro, A. R. Vergara, I. Riveiro, Ecological advantages of toxin production by the dinoflagellate Alexandrium minutum under phosphorous limitation, *Mar. Ecol. Prog. Ser.*, 225 (2002), 169–176.
- 48. N. Turriff, J. A. Runge, A. D. Cembella, Toxin accumulation and feeding behaviour of the planktonic copepod Calanus jinmarchicus exposed to the red-tide dinoflagellate Alexandrium excavatum, *Mar. Biol.*, **123** (1995), 55–64.
- 49. M. Schultz, T. Kiorboe, Active prey selection in two pelagic copepods feeding on potentially toxic and non-toxic dinoflagellates, *J. Plankton Res.*, **31** (2009), 553–561.



© 2020 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)