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Research article

Extinction and stationary distribution of a competition system with distributed delays and higher order coupled noises

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Abstract: A stochastic two-species competition system with saturation effect and distributed delays is formulated, in which two coupling noise sources are incorporated and every noise source has effect on two species' intrinsic growth rates in nonlinear form. By transforming the two-dimensional system with weak kernel into an equivalent four-dimensional system, sufficient conditions for extinction of two species and the existence of a stationary distribution of the positive solutions to the system are obtained. Our main results show that the two coupling noises play a significant role on the long time behavior of system.

Keywords: competitive system; coupling noises; distributed delay; extinction; stationary distribution

1. Introduction

In the last few decades, the mathematical models, which describe the effect of competition caused by limited resources on the growth of species, have been established and explored extensively [1–5]. On the other hand, time delay accompanies the whole process of species population survival and reproduction. Because of the variety of delay between different individuals of the same species or the cumulative effect of the past history of a system, it is more appropriate to establish model with distributed delay [6–8]. Cushing [9] and MacDonald [10] introduced the weak kernel and the strong kernel functions to describe the distributed delay, later on, these two kinds of kernels have been investigated by many scholars (see [11–14]).

With the idea of the weak kernel functions, we formulate the following competitive system with distributed delays

$$\begin{cases} dx(t) = x(t)\Big[r_1 - b_1 \displaystyle\int_{-\infty}^{t} \alpha_1 e^{-\alpha_1(t-s)} x(s)ds - \frac{c_1 y(t)}{1+y(t)}\Big]dt, \\ dy(t) = y(t)\Big[r_2 - b_2 \displaystyle\int_{-\infty}^{t} \alpha_2 e^{-\alpha_2(t-s)} y(s)ds - \frac{c_2 x(t)}{1+x(t)}\Big]dt, \end{cases} \tag{1.1}$$

where x and y measure the population densities of two competing species. $\alpha_i e^{-\alpha_i t}(\alpha_i > 0)$ are the weak kernel functions. The delays are the time passing between the intra-specific competition at a given instant and its effect, at a later time, on the dynamics of the two species. The coefficients r_i, b_i, c_i $(i = 1, 2)$ are positive constants, and represent the intrinsic growth rates, the intra-specific competitive rates and the inter-specific competitive rates, respectively. The term $c_1 y/(1 + y)$ (or $c_2 x/(1 + x)$) is an increasing function with respect to y (or x) and has a saturation value for large enough y (or x).

There are many related ecology and investigations of system (1.1). Especially, for system (1.1) without delays, Wang and Liu [15] analyzed the existence and global stability of almost periodic solutions of system, Li et al. [16] investigated the stability and Turing pattern of system with self- and cross-diffusion effect, Hu and Liu [17] took two coupling noise sources into account and obtained the sufficient conditions for survival results of system. On the other hand, for system (1.1) with discrete delays, Chen and Ho [18] discussed the persistence and global stability of system, Liu et al. [19] studied the existence of positive periodic solutions of system with impulsive perturbations. Obviously, it can be seen from the above literatures that they did not consider system with distributed delays. To reveal the effect of distributed delays we propose system (1.1).

Denote

$$u(t) = \int_{-\infty}^{t} \alpha_1 e^{-\alpha_1(t-s)} x(s)ds, \quad v(t) = \int_{-\infty}^{t} \alpha_2 e^{-\alpha_2(t-s)} y(s)ds. \tag{1.2}$$

By applying the linear chain technique, system (1.1) transforms into the following equivalent four-dimensional system

$$\begin{cases} dx(t) = x(t)\Big[r_1 - b_1 u(t) - \dfrac{c_1 y(t)}{1 + y(t)}\Big]dt, \\ dy(t) = y(t)\Big[r_2 - b_2 v(t) - \dfrac{c_2 x(t)}{1 + x(t)}\Big]dt, \\ du(t) = \alpha_1(x(t) - u(t))dt, \\ dv(t) = \alpha_2(y(t) - v(t))dt. \end{cases} \tag{1.3}$$

Due to environmental noise, the birth rate and other parameters involved in a system can reflect random fluctuation to some extent [20–22]. Some researches indicate that random interference cannot be ignored for competitive ecological models [23–25]. In addition, many researchers also introduced higher order perturbations into the system when the random perturbations may depend on the population's density [26, 27]. Therefore, in this paper, similarly to [17, 28] we further incorporate two noise sources in the system (1.3), that is, one noise source not only has influence on the intrinsic growth rate of one species but also on that of the other species. Strongly inspired by the above arguments, we assume that the white noises affect $r_i (i = 1, 2)$ mainly according to

$$\begin{aligned} r_1 &\to r_1 + \sigma_{11}(1 + x(t))dB_1(t) + \sigma_{12}(1 + x(t))dB_2(t), \\ r_2 &\to r_2 + \sigma_{21}(1 + y(t))dB_1(t) + \sigma_{22}(1 + y(t))dB_2(t), \end{aligned} \tag{1.4}$$

and obtain the following stochastic system which corresponds to the deterministic system (1.3)

$$\begin{cases} dx = x\Big[r_1 - b_1 u - \dfrac{c_1 y}{1 + y}\Big]dt + x(1 + x)[\sigma_{11} dB_1 + \sigma_{12} dB_2], \\ dy = y\Big[r_2 - b_2 v - \dfrac{c_2 x}{1 + x}\Big]dt + y(1 + y)[\sigma_{21} dB_1 + \sigma_{22} dB_2], \\ du = \alpha_1(x - u)dt, \\ dv = \alpha_2(y - v)dt. \end{cases} \tag{1.5}$$

Here, for convenience' sake, let $x = x(t)$, $y = y(t)$, $u = u(t)$, $v = v(t)$, and $dB_i = dB_i(t)(i = 1, 2)$. B_i are the independent standard Brownian motions defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}\}_{t\geq 0}$ satisfying the usual conditions, that is B_i are normally distributed with mean 0 and variance t (see [29]), and $\sigma^2_{ij}(i, j = 1, 2)$ denote the intensities of the white noises.

The rest of this paper is arranged as follows. Section 2 focuses on exploring the unique global positive solution to system (1.5). Sufficient conditions for extinction of the species and the existence of a stationary distribution of the positive solutions to system (1.5) are obtained in section 3 and section 4, respectively. In section 5, several examples are demonstrated to verify our results and a brief discussion is given.

2. Global positive solution

Assign

$$\mathbb{R}^d_+ = \{x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_i > 0, 1 \leq i \leq d\}.$$

Theorem 2.1. *For any given initial value* $(x(0), y(0), u(0), v(0)) \in \mathbb{R}^4_+$, *system* (1.5) *has a unique global positive solution for any* $t \geq 0$.

Proof. Obviously, for any initial value $(x(0), y(0), u(0), v(0)) \in \mathbb{R}^4_+$, there is a unique local solution $(x(t), y(t), u(t), v(t)) \in \mathbb{R}^4_+$ on $t \in [0, \tau_\varrho)$, where τ_ϱ denotes the explosion time. To complete the proof, we only need to show that $\tau_\varrho = +\infty$ a.s. Similarly to [29], we construct a nonnegative C^2–function $V : \mathbb{R}^4_+ \to \mathbb{R}_+$ satisfying

$$\liminf_{n\to+\infty,(x,y,u,v)\in\mathbb{R}^4_+\setminus O_n} V(x, y, u, v) = +\infty \;\text{ and }\; LV(x, y, u, v) \leq K$$

where $O_n = (\frac{1}{n}, n) \times (\frac{1}{n}, n) \times (\frac{1}{n}, n) \times (\frac{1}{n}, n)$ and $n > 1$ is a sufficiently large integer and K is a positive constant. Let $0 < p < 1$, we assign

$$V(x, y, u, v) = -\ln x - \ln y - \ln u - \ln v + \frac{x^p}{p} + \frac{y^p}{p} + \frac{u^2}{2\alpha_1} + \frac{v^2}{2\alpha_2}. \tag{2.1}$$

Since $-\ln z \to +\infty$ as $z \to 0$, and $\frac{z^p}{p} - \ln z \to +\infty$ as $z \to +\infty$, where $p > 0$, we can obtain that $\liminf_{n\to+\infty,(x,y,u,v)\in\mathbb{R}^4_+\setminus O_n} V(x, y, u, v) = +\infty$. In the following, we will verify that $LV(x, y, u, v) \leq K$. An application of Itô's formula shows that

$$\begin{aligned}
&LV(x, y, u, v)\\
&= -(r_1 - b_1 u - \frac{c_1 y}{1 + y}) + \frac{1}{2}(\sigma^2_{11} + \sigma^2_{12})(1 + x)^2 - (r_2 - b_2 v - \frac{c_2 x}{1 + x})\\
&\quad + \frac{1}{2}(\sigma^2_{21} + \sigma^2_{22})(1 + y)^2 - \frac{\alpha_1 x}{u} + \alpha_1 - \frac{\alpha_2 y}{v} + \alpha_2 + r_1 x^p - b_1 u x^p - \frac{c_1 y}{1 + y} x^p\\
&\quad + \frac{1}{2}(p - 1)(\sigma^2_{11} + \sigma^2_{12})x^p + (p - 1)(\sigma^2_{11} + \sigma^2_{12})x^{p+1} + \frac{1}{2}(p - 1)(\sigma^2_{11} + \sigma^2_{12})x^{p+2}\\
&\quad + r_2 y^p - b_2 v y^p - \frac{c_2 x}{1 + x} y^p + \frac{1}{2}(p - 1)(\sigma^2_{21} + \sigma^2_{22})y^p + (p - 1)(\sigma^2_{21} + \sigma^2_{22})y^{p+1}\\
&\quad + \frac{1}{2}(p - 1)(\sigma^2_{21} + \sigma^2_{22})x^{p+2} + ux - u^2 + vy - v^2
\end{aligned}$$

$$
\begin{aligned}
&\le b_1u + c_1+\frac{1}{2}(\sigma_{11}^2+\sigma_{12}^2)+(\sigma_{11}^2+\sigma_{12}^2)x+\frac{1}{2}(\sigma_{11}^2+\sigma_{12}^2)x^2 + b_2v + c_2+\frac{1}{2}(\sigma_{21}^2+\sigma_{22}^2)\\
&\quad + (\sigma_{21}^2+\sigma_{22}^2)y + \frac{1}{2}(\sigma_{21}^2+\sigma_{22}^2)y^2 + \alpha_1 + \alpha_2 + r_1x^p + \frac{1}{2}(p-1)(\sigma_{11}^2+\sigma_{12}^2)x^{p+2}\\
&\quad + r_2y^p + \frac{1}{2}(p-1)(\sigma_{21}^2 + \sigma_{22}^2)y^{p+2} + \frac{x^2}{2} - \frac{u^2}{2} + \frac{y^2}{2} - \frac{v^2}{2}\\
&\le c_1 + c_2 + \alpha_1 + \alpha_2 + \frac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2) + \frac{1}{2}(\sigma_{21}^2 + \sigma_{22}^2) + \tilde{K} := K,
\end{aligned}
\tag{2.2}
$$

where

$$
\begin{aligned}
\tilde{K} = \sup_{(x,y,u,v)\in\mathbb{R}_+^4} \Big\{ &-\frac{1}{2}(1-p)(\sigma_{11}^2 + \sigma_{12}^2)x^{p+2}+ \frac{1}{2}(1 + \sigma_{11}^2 + \sigma_{12}^2)x^2+ (\sigma_{11}^2 + \sigma_{12}^2)x + r_1x^p + r_2y^p\\
&-\frac{1}{2}(1-p)(\sigma_{21}^2 + \sigma_{22}^2)y^{p+2} + \frac{1}{2}(1 + \sigma_{21}^2 + \sigma_{22}^2)y^2 + (\sigma_{21}^2 + \sigma_{22}^2)y + b_1u - \frac{u^2}{2} + b_2v - \frac{v^2}{2}\Big\}.
\end{aligned}
$$

The rest proof is similar to Theorem 2.1 in [30] and hence we omit it here.

3. Extinction

Theorem 3.1. *If $r_1 < \frac{1}{2}(\sigma_{11}^2+\sigma_{12}^2)$, $r_2 < \frac{1}{2}(\sigma_{21}^2+\sigma_{22}^2)$, then both $x(t)$ and $y(t)$ will be extinct exponentially.*

Proof. Using Itô's formula in the first equation of system (1.5), one can show that

$$
\begin{aligned}
d\ln x(t) = \Big[&r_1 - b_1u(t) - \frac{c_1y(t)}{1+y(t)} - \frac{1}{2}\sigma_{11}^2 - \sigma_{11}^2x(t) - \frac{1}{2}\sigma_{11}^2x(t)^2 - \frac{1}{2}\sigma_{12}^2 - \sigma_{12}^2x(t) - \frac{1}{2}\sigma_{12}^2x(t)^2\Big]dt\\
&+ \sigma_{11}(1 + x(t))dB_1(t) + \sigma_{12}(1 + x(t))dB_2(t).
\end{aligned}
\tag{3.1}
$$

Integrating from 0 to t and dividing by t on both sides of the above equality lead to

$$
\begin{aligned}
\frac{\ln x(t) - \ln x(0)}{t} &= r_1 - \frac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2) - \frac{b_1}{t}\int_0^t u(s)ds - \frac{c_1}{t}\int_0^t \frac{y(s)}{1+y(s)}ds - \frac{\sigma_{11}^2}{t}\int_0^t x(s)ds\\
&\quad - \frac{\sigma_{12}^2}{t}\int_0^t x(s)ds - \frac{\sigma_{11}^2}{2t}\int_0^t x^2(s)ds - \frac{\sigma_{12}^2}{2t}\int_0^t x^2(s)ds + \frac{\sigma_{11}B_1(t)}{t}\\
&\quad + \frac{\sigma_{12}B_2(t)}{t} + \frac{\sigma_{11}\int_0^t x(s)dB_1(s)}{t} + \frac{\sigma_{12}\int_0^t x(s)dB_2(s)}{t}.
\end{aligned}
\tag{3.2}
$$

Denote $Z_i(t) = \sigma_{1i}\int_0^t x(s)dB_i(s)$, so that its quadratic variation is

$$
\langle Z_i(t), Z_i(t)\rangle = \sigma_{1i}^2\int_0^t x^2(s)ds.
$$

Using the exponential martingale inequality and the Borel-Cantell lemma, similar to [27, 31], we have that for almost all $\omega \in \Omega$, there is a random integer $l_0 = l_0(\omega)$ such that for $l \ge l_0$,

$$
Z_i(t) \le 2\ln l + \frac{1}{2}\langle Z_i(t), Z_i(t)\rangle = 2\ln l + \frac{\sigma_{1i}^2}{2}\int_0^t x^2(s)ds. \tag{3.3}
$$

For all $0 \le l-1 \le t \le l$, $l \ge l_0$, substituting (3.3) to (3.2), we have

$$\begin{aligned}\frac{\ln x(t)-\ln x(0)}{t} &\le r_1 - \frac{1}{2}(\sigma_{11}^2+\sigma_{12}^2) - \frac{b_1}{t}\int_0^t u(s)ds - \frac{c_1}{t}\int_0^t \frac{y(s)}{1+y(s)}ds \\ &\quad - \frac{\sigma_{11}^2+\sigma_{12}^2}{t}\int_0^t x(s)ds + \frac{4\ln l}{t} + \frac{\sigma_{11}B_1(t)}{t} + \frac{\sigma_{12}B_2(t)}{t} \\ &\le r_1 - \frac{1}{2}(\sigma_{11}^2+\sigma_{12}^2) + \frac{4\ln l}{l-1} + \frac{\sigma_{11}B_1(t)}{t} + \frac{\sigma_{12}B_2(t)}{t}.\end{aligned} \tag{3.4}$$

The strong law of local martingales implies that

$$\lim_{t\to+\infty}\frac{B_i(t)}{t} = 0 \;\; a.s. \;\; i = 1,2. \tag{3.5}$$

Taking the superior limit on both sides we get

$$\limsup_{t\to+\infty}\frac{\ln x(t)}{t} \le r_1 - \frac{1}{2}(\sigma_{11}^2+\sigma_{12}^2) < 0 \;\; a.s. \tag{3.6}$$

Similarly, one can obtain that

$$\limsup_{t\to+\infty}\frac{\ln y(t)}{t} \le r_2 - \frac{1}{2}(\sigma_{21}^2+\sigma_{22}^2) < 0 \;\; a.s.$$

The proof is completed.

4. Existence of stationary distribution

Consider the integral equation

$$X(t) = X(t_0) + \int_{t_0}^t f(s,X(s))ds + \sum_{r=1}^{l}\int_{t_0}^t \sigma_r(s,X(s))dB_\xi(s), \tag{4.1}$$

where $\sigma_r(s,X(s))$ and $B_\xi(s)$ are vectors.

Lemma 4.1. *[32] Suppose that the coefficients of* (4.1) *are independent of t and satisfy the following conditions for some constant N*

$$|f(s,x)-f(s,y)| + \sum_{r=1}^{l}|\sigma_r(s,x)-\sigma_r(s,y)| \le N|x-y|, \;\; |f(s,x)| + \sum_{r=1}^{l}|\sigma_r(s,x)| \le N(1+|x|) \tag{4.2}$$

in $O_R \subset \mathbb{R}_+^d$ for every $R > 0$ and there exists a nonnegative C^2−function $W(x)$ in $\mathbb{R}_+^d$ such that $LW(x) \le$ -1 outside some compact set. Then the system (4.1) exists a solution which is a stationary distribution.

Remark 4.1. *The condition (4.2) in Lemma 4.1 can be replaced by the global existence of the solution of (4.1) in view of Remark 5 in Xu [33].*

Assign

$$\beta_i = r_i - c_i - \frac{1}{2}(\sigma_{i1}^2+\sigma_{i2}^2), \;\; i = 1,2.$$

Theorem 4.1. *If $\beta_i > 0$ $(i = 1, 2)$, then there exists a positive solution $(x(t), y(t), u(t), v(t))$ of the system* (1.5) *which is a stationary Markov process.*

Proof. Since we have obtained the existence of the global positive solution of system (1.5) in Theorem 2.1, from Remark 4.1, we only need to consider a nonnegative C^2−function $W(x, y, u, v)$ and a closed set $O \subset \mathbb{R}^4_+$ satisfying

$$LW(x, y, u, v) \leq -1 \quad \text{for any} \quad (x, y, u, v) \in \mathbb{R}^4_+/O. \tag{4.3}$$

One derives from system (1.5), by Itô's formula, that

$$\begin{aligned}
&L(-\ln x + \frac{b_1}{\alpha_1}u - \ln y + \frac{b_2}{\alpha_2}v) \\
&= -\Big[r_1 - b_1 u - \frac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2) - (\sigma_{11}^2 + \sigma_{12}^2)x - \frac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2)x^2 - \frac{c_1 y}{1+y}\Big] + b_1(x-u) \\
&\quad - \Big[r_2 - b_2 v - \frac{1}{2}(\sigma_{21}^2 + \sigma_{22}^2) - (\sigma_{21}^2 + \sigma_{22}^2)y - \frac{1}{2}(\sigma_{21}^2 + \sigma_{22}^2)y^2 - \frac{c_1 x}{1+x}\Big] + b_2(y-v) \\
&\leq -\beta_1 + (b_1 + \sigma_{11}^2 + \sigma_{12}^2)x + \frac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2)x^2 - \beta_2 + (b_2 + \sigma_{21}^2 + \sigma_{22}^2)y + \frac{1}{2}(\sigma_{21}^2 + \sigma_{22}^2)y^2,
\end{aligned} \tag{4.4}$$

and for the constant $0 < p < 1$, we have

$$\begin{aligned}
&L(\frac{x^p}{p} + \frac{y^p}{p} + \frac{u^2}{2\alpha_1} + \frac{v^2}{2\alpha_2} - \frac{1}{\alpha_1}\ln u - \frac{1}{\alpha_2}\ln v) \\
&= r_1 x^p - b_1 u x^p - \frac{c_1 y}{1+y}x^p + \frac{1}{2}(p-1)(\sigma_{11}^2 + \sigma_{12}^2)x^p + (p-1)(\sigma_{11}^2 + \sigma_{12}^2)x^{p+1} \\
&\quad + \frac{1}{2}(p-1)(\sigma_{11}^2 + \sigma_{12}^2)x^{p+2} + r_2 y^p - b_2 v y^p - \frac{c_2 x}{1+x}y^p + \frac{1}{2}(p-1)(\sigma_{21}^2 + \sigma_{22}^2)y^p \\
&\quad + (p-1)(\sigma_{21}^2 + \sigma_{22}^2)y^{p+1} + \frac{1}{2}(p-1)(\sigma_{21}^2 + \sigma_{22}^2)y^{p+2} + ux - u^2 + vy - v^2 - \frac{x}{u} + 1 - \frac{y}{v} + 1 \\
&\leq r_1 x^p - \frac{1}{2}(1-p)(\sigma_{11}^2 + \sigma_{12}^2)x^{p+2} + r_2 y^p - \frac{1}{2}(1-p)(\sigma_{21}^2 + \sigma_{22}^2)y^{p+2} + \frac{x^2}{2} + \frac{u^2}{2} \\
&\quad - u^2 + \frac{y^2}{2} + \frac{v^2}{2} - v^2 + 2 - \frac{x}{u} - \frac{y}{v} \\
&= -\frac{1}{2}(1-p)(\sigma_{11}^2 + \sigma_{12}^2)x^{p+2} + r_1 x^p + \frac{x^2}{2} - \frac{1}{2}(1-p)(\sigma_{21}^2 + \sigma_{22}^2)y^{p+2} + r_2 y^p + \frac{y^2}{2} \\
&\quad - \frac{u^2}{2} - \frac{v^2}{2} + 2 - \frac{x}{u} - \frac{y}{v}.
\end{aligned} \tag{4.5}$$

Define

$$\begin{aligned}
\widetilde{W}(x, y, u, v) = {} & M_1(-\ln x + \frac{b_1}{\alpha_1}u) + M_2(-\ln y + \frac{b_2}{\alpha_2}v) + \frac{x^p}{p} + \frac{y^p}{p} + \frac{u^2}{2\alpha_1} + \frac{v^2}{2\alpha_2} \\
& - \frac{1}{\alpha_1}\ln u - \frac{1}{\alpha_2}\ln v,
\end{aligned} \tag{4.6}$$

where $M_i = \frac{2}{\beta_i}\max\{2, J_i\}$, J_i $(i = 1, 2)$ are positive constants which will be determined later. Since $\widetilde{W}(x, y, u, v)$ is a continuous function, there exists a minimum point $(\hat{x}, \hat{y}, \hat{u}, \hat{v})$ in the interior of $\mathbb{R}^4_+$. Then we can choose the following C^2–function $W : \mathbb{R}^4_+ \to \mathbb{R}_+ \cup \{0\}$

$$\begin{aligned}
W(x, y, u, v) &= \widetilde{W}(x, y, u, v) - \widetilde{W}(\hat{x}, \hat{y}, \hat{u}, \hat{v}) \\
&= M_1(-\ln x + \frac{b_1}{\alpha_1}u) + M_2(-\ln y + \frac{b_2}{\alpha_2}v) + \frac{x^p}{p} + \frac{y^p}{p} + \frac{u^2}{2\alpha_1} + \frac{v^2}{2\alpha_2} \\
&\quad - \frac{1}{\alpha_1}\ln u - \frac{1}{\alpha_2}\ln v - \widetilde{W}(\hat{x}, \hat{y}, \hat{u}, \hat{v}),
\end{aligned} \tag{4.7}$$

which, together with (4.4)–(4.5), leads to

$$
\begin{aligned}
&LW(x,y,u,v)\\
&\le -M_1\beta_1+M_1(b_1+\sigma_{11}^2+\sigma_{12}^2)x+\frac{M_1}{2}(\sigma_{11}^2+\sigma_{12}^2)x^2-M_2\beta_2+M_2(b_2+\sigma_{21}^2+\sigma_{22}^2)y\\
&\quad+\frac{M_2}{2}(\sigma_{21}^2+\sigma_{22}^2)y^2-\frac{1}{2}(1-p)(\sigma_{11}^2+\sigma_{12}^2)x^{p+2}+r_1x^p+\frac{x^2}{2}-\frac{1}{2}(1-p)(\sigma_{21}^2+\sigma_{22}^2)y^{p+2}\\
&\quad+r_2y^p+\frac{y^2}{2}-\frac{u^2}{2}-\frac{v^2}{2}+2-\frac{x}{u}-\frac{y}{v}.
\end{aligned}
\tag{4.8}
$$

Let $\epsilon>0$ be sufficiently small such that

$$
0<\epsilon<\min\Big\{\frac{\beta_i}{4(b_i+\sigma_{i1}^2+\sigma_{i2}^2)},\Big(\frac{(1-p)(\sigma_{i1}^2+\sigma_{i2}^2)}{4(J+3)}\Big)^{\frac{1}{p+2}},\frac{1}{J+3},\Big(\frac{1}{4(J+3)}\Big)^{\frac{1}{4}}\Big\},\ i=1,2. \tag{4.9}
$$

Define the bounded closed set

$$
O^{\epsilon}=\{(x,y,u,v)\in\mathbb{R}_+^4|\epsilon\le x\le\frac{1}{\epsilon},\ \epsilon\le y\le\frac{1}{\epsilon},\ \epsilon^2\le u\le\frac{1}{\epsilon^2},\ \epsilon^2\le v\le\frac{1}{\epsilon^2}\}. \tag{4.10}
$$

Assign

$$
\begin{aligned}
&O_1^{\epsilon}=\{(x,y,u,v)\in\mathbb{R}_+^4|0<x<\epsilon\},\ \ O_2^{\epsilon}=\{(x,y,u,v)\in\mathbb{R}_+^4|x>\frac{1}{\epsilon}\},\\
&O_3^{\epsilon}=\{(x,y,u,v)\in\mathbb{R}_+^4|0<y<\epsilon\},\ \ O_4^{\epsilon}=\{(x,y,u,v)\in\mathbb{R}_+^4|y>\frac{1}{\epsilon}\},\\
&O_5^{\epsilon}=\{(x,y,u,v)\in\mathbb{R}_+^4|0<u<\epsilon^2,\ x>\epsilon,y>\epsilon\},\ \ O_6^{\epsilon}=\{(x,y,u,v)\in\mathbb{R}_+^4|u>\frac{1}{\epsilon^2}\},\\
&O_7^{\epsilon}=\{(x,y,u,v)\in\mathbb{R}_+^4|0<v<\epsilon^2,\ y>\epsilon,x>\epsilon\},\ O_8^{\epsilon}=\{(x,y,u,v)\in\mathbb{R}_+^4|v>\frac{1}{\epsilon^2}\}.
\end{aligned}
$$

Case 1. When $(x,y,u,v)\in O_1^{\epsilon}$, it follows from (4.8) that

$$
\begin{aligned}
&LW(x,y,u,v)\\
&\le-\frac{M_1\beta_1}{4}+\Big[-\frac{M_1\beta_1}{4}+M_1(b_1+\sigma_{11}^2+\sigma_{12}^2)\epsilon\Big]-\frac{1}{4}(1-p)(\sigma_{11}^2+\sigma_{12}^2)x^{p+2}+\Big[-\frac{M_1\beta_1}{2}+J_1\Big],
\end{aligned}
$$

where

$$
\begin{aligned}
J_1=\sup_{(x,y)\in\mathbb{R}_+^2}&\Big\{-\frac{1}{4}(1-p)(\sigma_{11}^2+\sigma_{12}^2)x^{p+2}+\frac{M_1}{2}(\sigma_{11}^2+\sigma_{12}^2)x^2+\frac{x^2}{2}+r_1x^p+2\\
&-\frac{1}{4}(1-p)(\sigma_{21}^2+\sigma_{22}^2)y^{p+2}+\frac{M_2}{2}(\sigma_{21}^2+\sigma_{22}^2)y^2+\frac{y^2}{2}+M_2(b_2+\sigma_{21}^2+\sigma_{22}^2)y+r_2y^p\Big\}.
\end{aligned}
$$

Since $M_1=\frac{2}{\beta_1}\max\{2,J_1\}$, one can see that $\frac{M_1\beta_1}{4}\ge 1$. Then we have from (4.9) that

$$
LW(x,y,u,v)\le\frac{-M_1\beta_1}{4}-\frac{1}{4}(1-p)(\sigma_{11}^2+\sigma_{12}^2)x^{p+2}\le\frac{-M_1\beta_1}{4}\le-1.
$$

Similarly, for any $(x,y,u,v)\in O_3^{\epsilon}$,

$$
\begin{aligned}
&LW(x,y,u,v)\\
&\le-\frac{M_2\beta_2}{4}+\Big[-\frac{M_2\beta_2}{4}+M_2(b_2+\sigma_{21}^2+\sigma_{22}^2)\epsilon\Big]-\frac{1}{4}(1-p)(\sigma_{21}^2+\sigma_{22}^2)y^{p+2}+\Big[-\frac{M_2\beta_2}{2}+J_2\Big],
\end{aligned}
$$

where

$$J_2 = \sup_{(x,y)\in\mathbb{R}_+^2}\Big\{-\frac{1}{4}(1-p)(\sigma_{11}^2+\sigma_{12}^2)x^{p+2}+\frac{M_1}{2}(\sigma_{11}^2+\sigma_{12}^2)x^2+\frac{x^2}{2}+M_1(b_1+\sigma_{11}^2+\sigma_{12}^2)x$$
$$+r_1x^p-\frac{1}{4}(1-p)(\sigma_{21}^2+\sigma_{22}^2)y^{p+2}+\frac{M_2}{2}(\sigma_{21}^2+\sigma_{22}^2)y^2+\frac{y^2}{2}+r_2y^p+2\Big\}.$$

Since $M_2=\frac{2}{\beta_2}\max\{2,J_2\}$, we have $\frac{M_2\beta_2}{4}\geq 1$. It follows from (4.9) that

$$LW(x,y,u,v)\leq\frac{-M_2\beta_2}{4}-\frac{1}{4}(1-p)(\sigma_{21}^2+\sigma_{22}^2)y^{p+2}\leq\frac{-M_2\beta_2}{4}\leq -1.$$

Recalling (4.8), we can calculate that

$$\begin{aligned}
&LW(x,y,u,v)\\
&\leq -\frac{1}{4}(1-p)(\sigma_{11}^2+\sigma_{12}^2)x^{p+2}-\frac{1}{4}(1-p)(\sigma_{21}^2+\sigma_{22}^2)y^{p+2}-\frac{u^2}{2}-\frac{v^2}{2}-\frac{x}{u}-\frac{y}{v}+2\\
&\quad-\frac{1}{4}(1-p)(\sigma_{11}^2+\sigma_{12}^2)x^{p+2}+\frac{M_1}{2}(\sigma_{11}^2+\sigma_{12}^2)x^2+\frac{x^2}{2}+M_1(b_1+\sigma_{11}^2+\sigma_{12}^2)x+r_1x^p\\
&\quad-\frac{1}{4}(1-p)(\sigma_{21}^2+\sigma_{22}^2)y^{p+2}+\frac{M_2}{2}(\sigma_{21}^2+\sigma_{22}^2)y^2+\frac{y^2}{2}+M_2(b_2+\sigma_{21}^2+\sigma_{22}^2)y+r_2y^p\\
&\leq -\frac{1}{4}(1-p)(\sigma_{11}^2+\sigma_{12}^2)x^{p+2}-\frac{1}{4}(1-p)(\sigma_{21}^2+\sigma_{22}^2)y^{p+2}-\frac{u^2}{2}-\frac{v^2}{2}-\frac{x}{u}-\frac{y}{v}+2+J,
\end{aligned}$$

where

$$J=\sup_{(x,y)\in\mathbb{R}_+^2}\Big\{-\frac{1}{4}(1-p)(\sigma_{11}^2+\sigma_{12}^2)x^{p+2}+\frac{M_1}{2}(\sigma_{11}^2+\sigma_{12}^2)x^2+\frac{x^2}{2}+M_1(b_1+\sigma_{11}^2+\sigma_{12}^2)x$$
$$+r_1x^p-\frac{1}{4}(1-p)(\sigma_{21}^2+\sigma_{22}^2)y^{p+2}+\frac{M_2}{2}(\sigma_{21}^2+\sigma_{22}^2)y^2+\frac{y^2}{2}+M_2(b_2+\sigma_{21}^2+\sigma_{22}^2)y+r_2y^p\Big\}.$$

Case 2. When $(x,y,u,v)\in O_2^\epsilon$, by (4.9) we can derive that

$$LW(x,y,u,v)\leq 2+J-\frac{1}{4}(1-p)(\sigma_{11}^2+\sigma_{12}^2)x^{p+2}<2+J-\frac{1}{4}(1-p)(\sigma_{11}^2+\sigma_{12}^2)\epsilon^{-(p+2)}\leq -1.$$

Similar, if $(x,y,u,v)\in O_4^\epsilon$, then we have

$$LW(x,y,u,v)<2+J-\frac{1}{4}(1-p)(\sigma_{21}^2+\sigma_{22}^2)\epsilon^{-(p+2)}\leq -1.$$

Case 3. When $(x,y,u,v)\in O_5^\epsilon$ or $(x,y,u,v)\in O_7^\epsilon$, we have from (4.9) that

$$LW(x,y,u,v)\leq 2+J-\frac{x}{u}<2+J-\frac{\epsilon}{\epsilon^2}\leq -1,$$
$$LW(x,y,u,v)\leq 2+J-\frac{y}{v}<2+J-\frac{\epsilon}{\epsilon^2}\leq -1.$$

Case 4. When $(x,y,u,v)\in O_6^\epsilon$ or $(x,y,u,v)\in O_8^\epsilon$, it follows from (4.9) that

$$LW(x,y,u,v)\leq 2+J-\frac{u^2}{4}<2+J-\frac{1}{4\epsilon^4}\leq -1,$$

$$LW(x, y, u, v) \leq 2 + J - \frac{v^2}{4} < 2 + J - \frac{1}{4\epsilon^4} \leq -1.$$

The above analysis shows that there is a closed set O^ϵ defined by (4.10) such that

$$\sup_{(x,y,u,v)\in\mathbb{R}_+^4/O^\epsilon} LW(x, y, u, v) \leq -1.$$

The proof is completed.

5. Numerical simulations and discussions

To illustrate our theoretical results, we will perform several specific numerical simulations. We first fix $r_1 = 0.4$, $r_2 = 0.6$, $b_1 = 0.5$, $b_2 = 0.8$, $c_1 = 0.1$, $c_2 = 0.3$, $\alpha_1 = 0.1$, $\alpha_2 = 0.2$. Let $\sigma_{11} = \sigma_{12} = \sigma_{21} = \sigma_{22} \equiv 0$, and we find that both of two species are persistent (see Figure 1(a$_1$)–(a$_2$)). However, if $\sigma_{11} = 0.9$, $\sigma_{12} = 0.5$, $\sigma_{21} = 1$, $\sigma_{22} = 0.8$, then we have from Theorem 3.1 that $x(t)$ and $y(t)$ will be extinct exponentially (see Figure 1(b$_1$)–(b$_2$)). When we choose $\sigma_{11} = 0.1$, $\sigma_{12} = 0.15$, $\sigma_{21} = 0.3$, $\sigma_{22} = 0.25$, it follows from Theorem 4.1 that the system (1.5) has a stationary distribution (see Figure 2).

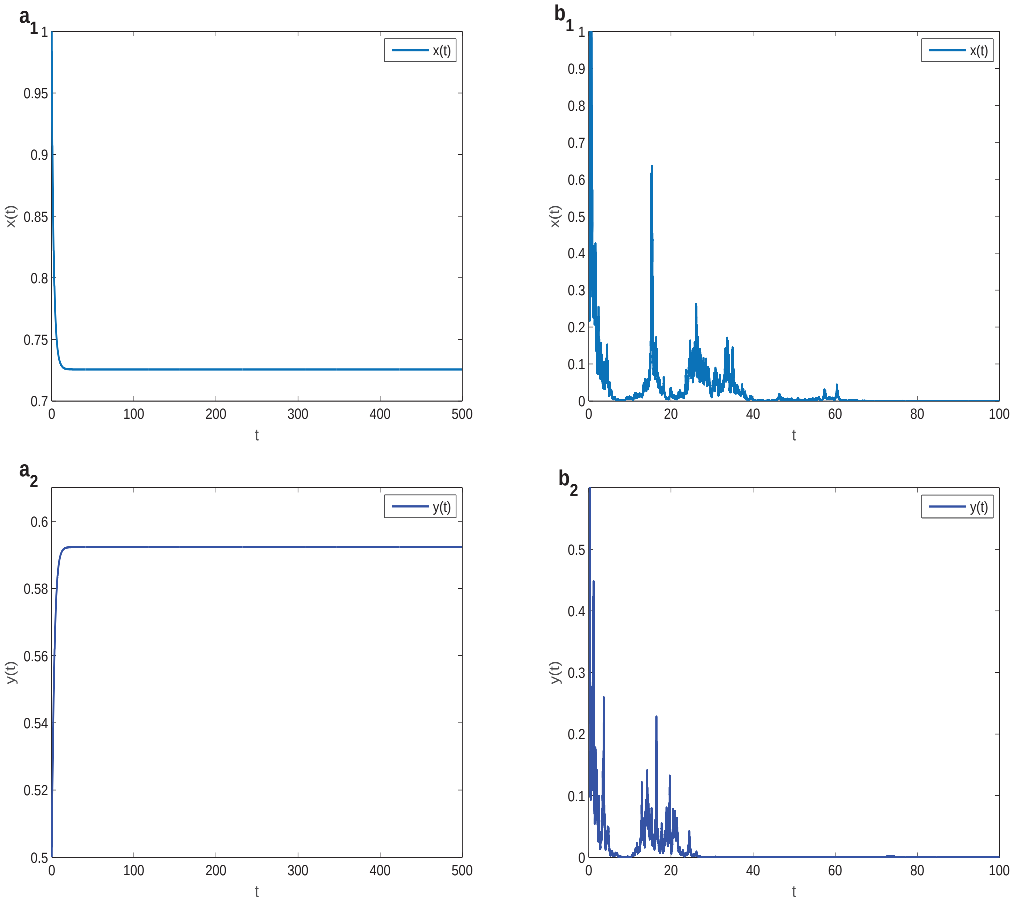

Figure 1. (a$_1$)–(a$_2$). The solution of the deterministic system (1.3); (b$_1$)–(b$_2$). Trajectory of the solution to the stochastic system (1.5), where $\sigma_{11} = 0.9$, $\sigma_{12} = 0.5$, $\sigma_{21} = 1$, $\sigma_{22} = 0.8$.

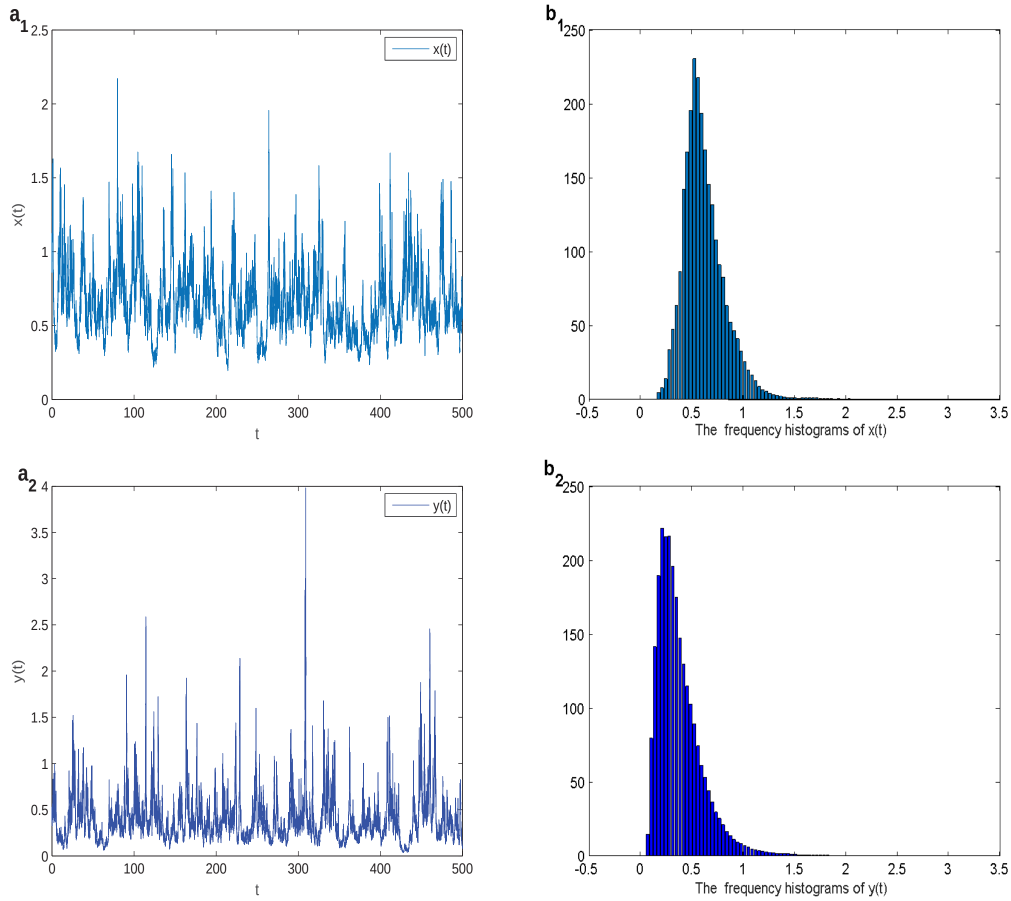

Figure 2. (a_1)–(a_2). Trajectory of the solution to the stochastic system (1.5) with $\sigma_{11} = 0.1$, $\sigma_{12} = 0.15$, $\sigma_{21} = 0.5$, $\sigma_{22} = 0.25$; (b_1)–(b_2). The frequency histograms of $x(t)$ and $y(t)$ correspond to stochastic system in Figure 2(a_1)–(a_2).

Our theoretical results and the above numerical examples reveal that the coupling noises can change the asymptotic properties of system (1.5).

(I) For ecosystems, it is important to analyze the survival of species. It can be seen from Figure 1(a_1)–(a_2) that the two competing species can survive if the competitive system is not affected by environmental noise. However, two species will go to extinction if the coupling noises are suitable large (see Figrue 1(b_1)–(b_2)).

(II) To analyze the statistic characteristic of the long-term behaviors of the sample trajectories, a useful approach is to study the stationary distribution. Figure 2 implies that the relatively small coupling noises can ensure the existence of a stationary distribution.

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Conflict of interest

The authors declare there is no conflict of interest.

References

1. A. J. Lotka, *Elements of Mathematical Biology*, Dover, (1924), 167–194.
2. V. Volterra, Leçons sur la théorie mathématique de la lutte pour la vie, *Bull. Amer. Math. Soc.*, **42** (1936), 304–305.
3. F. J. Ayala, M. E. Gilpin, J. G. Enrenfeld, Competition between species: theoretical models and experimental tests, *Theoret. Population Biol.*, **4** (1973), 331–356.
4. S. Ahmad, On the Nonautonomous Volterra-Lotka Competition Equations, *Proc. Amer. Math. Soc.*, **117** (1993), 199–204.
5. M. L. Zeeman, Extinction in competitive Lotka-Volterra systems, *Proc. Amer. Math. Soc.*, **123** (1995), 87–96.
6. J. Y. Wang, Z. S. Feng, A non-autonomous competitive system with stage structure and distributed delays, *Proc. Roy. Soc. Edinburgh Sect. A*, **140** (2010), 1061–1080.
7. F. M. de Oca, L. Perez, Extinction in nonautonomous competitive Lotka-Volterra systems with infinite delay, *Nonlinear Anal.*, **75** (2012), 758–768.
8. Z. Li, M. A. Han, F. D. Chen, Influence of feedback controls on an autonomous Lotka-Volterra competitive system with infinite delays, *Nonlinear Anal. Real World Appl.*, **14** (2013), 402–413.
9. J. M. Cushing, Integrodifferential equations and delay models in population dynamics, in *Lecture Notes in Biomathematics*, Springer Science & Business Media, (2013).
10. N. Macdonald, Time Lags in Biological Models, in *Lecture Notes in Biomathematics*, Springer Science & Business Media, (2013).
11. X. H. Wang, H. H. Liu, C. L. Xu, Hopf bifurcations in a predator-prey system of population allelopathy with a discrete delay and a distributed delay, *Nonlinear Dynam.*, **69** (2012), 2155–2167.
12. C. H. Zhang, X. P. Yan, G. H. Cui, Hopf bifurcations in a predator-prey system with a discrete delay and a distributed delay, *Nonlinear Anal. Real World Appl.*, **11** (2010), 4141–4153.
13. Q. Liu, D. Q. Jiang, Stationary distribution and extinction of a stochastic predator-prey model with distributed delay, *Appl. Math. Lett.*, **78** (2018), 79–87.
14. W. J. Zuo, D. Q. Jiang, X. G. Sun, T. Hayat, A. Alsaedi, Long-time behaviors of a stochastic cooperative Lotka-Volterra system with distributed delay, *Phys. A*, **506** (2018), 542–559.
15. Q. L. Wang, Z. J. Liu, Z. X. Li, R. A. Cheke, Existence and global asymptotic stability of positive almost periodic solutions of a two-species competitive system, *Int. J. Biomath.*, **7** (2014), 1450040.
16. Q. Li, Z. J. Liu, S. L. Yuan, Cross-diffusion induced Turing instability for a competition model with saturation effect, *Appl. Math. Comput.*, **347** (2019), 64–77.
17. J. Hu, Z. J. Liu, Incorportating coupling noises into a nonlinear competitive system with saturation effect, *Int. J. Biomath.*, **13** (2020), 2050012.

18. H. C. Chen, C. P. Ho, Persistence and global stability on competition system with time-delay, *Tunghai Sci.*, **5** (2003), 71–99.

19. Z. J. Liu, R. H. Tan, Y. P. Chen, Modeling and analysis of a delayed competitive system with impulsive perturbations, *Rocky Mountain J. Math.*, **38** (2008), 1505–1523.

20. R. M. May, *Stability and Complexity in Model Ecosystem*, Princeton University Press, (2001).

21. X. Y. Li, X. R. Mao, Population dynamical behavior of non-autonomous Lotka-Volterra competitive system with random perturbation, *Discrete Contin. Dyn. Syst.*, **24** (2009), 523–545.

22. F. Y. Wei, C. J. Wang, Survival analysis of a single-species population model with fluctuations and migrations between patches, *Appl. Math. Model.*, **81** (2020), 113–127.

23. A. Caruso, M. E. Gargano, D. Valenti, A. Fiasconaro, B. Spagnolo, Cyclic Fluctuations, Climatic Changes and Role of Noise in Planktonic Foraminifera in the Mediterranean Sea, *Fluc. Noise Lett.*, **5** (2005), 349–355.

24. A. Giuffrida, D. Valenti, G. Ziino, B. Spagnolo, A. Panebianco, A stochastic interspecific competition model to predict the behaviour of Listeria monocytogenes in the fermentation process of a traditional Sicilian salami, *Eur. Food Res. Technol.*, **228** (2009), 767–775.

25. D. Valenti, G. Denaro, A. La Cognata, B. Spagnolo, A. Bonanno, G. Basilone, et al., Picophytoplankton dynamics in noisy marine environment, *Acta Phys. Pol.*, **43** (2012), 1227–1240.

26. Z. W. Cao, W. Feng, X. D. Wen, L. Zu, Stationary distribution of a stochastic predator-prey model with distributed delay and higher order perturbations, *Phys. A*, **521** (2019), 467–475.

27. Q. Liu, D. Q. Jiang, T. Hayat, A. Alsaedi, Long-time behavior of a stochastic logistic equation with distributed delay and nonlinear perturbation, *Phys. A*, **508** (2018), 289–304.

28. X. Q. Liu, S. M. Zhong, L. J. Xiang, Asymptotic properties of a stochastic predator-prey model with Bedding-DeAngelis functional response, *J. Appl. Math. Comput.*, **8** (2014), 171–174.

29. X. R. Mao, *Stochastic Differential Equations and their Applications*, Horwood Publ, (1997).

30. A. Bahar, X. R. Mao, Stochastic delay Lotka-Volterra model, *J. Math. Anal. Appl.*, **292** (2004), 364–380.

31. C. Lu, X. H. Ding, Persistence and extinction of a stochastic logistic model with delays and impulsive perturbation, *Acta Math. Sci. Ser. B*, **34** (2014), 1551–1570.

32. R Khasminskii, *Stochastic Stability of Differential Equations*, Springer Science & Business Media, (2011).

33. D. Y. Xu, Y. M. Huang, Z. G. Yang, Existence theorems for periodic Markov process and stochastic functional differential equations, *Discrete Contin. Dyn. Syst.*, **24** (2009), 1005–1023.

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