AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(3): 5557–5574.
DOI: 10.3934/math.2026229
Received: 26 January 2026
Revised: 21 February 2026
Accepted: 27 February 2026
Published: 05 March 2026

Research article

On nonnegative classical solutions of coupled Euler–Poisson–Darboux–Tricomi equations with nonlinear sources

Svetlin G. Georgiev[1], **Safa M. Mirgani**[2], **Khaled Zennir**[3,*] **and Keltoum Bouhali**[3]

[1] Department of Mathematics, Sorbonne University, Paris, France; svetlingeorgiev1@gmail.com
[2] Department of Mathematics and Statistics, Imam Mohammad Ibn Saud Islamic University, Riyadh 13318, Saudi Arabia; smmmohamed@imamu.edu.sa
[3] Departement of Mathematics, College of Sciences, Qassim University, Saudi Arabia; k.zennir@qu.edu.sa, k.bouhali@qu.edu.sa

* **Correspondence:** Email: k.zennir@qu.edu.sa.

Abstract: We study the initial value problem for a class of coupled Euler–Poisson–Darboux–Tricomi equations with nonlinear source terms. Such systems arise in fluid dynamics, plasma physics, transonic flow, and wave propagation in inhomogeneous media. Using new integral representations of the solutions and appropriate fixed point theorems in Banach spaces, we establish the existence of at least one, at least two, and at least three nonnegative classical solutions under suitable conditions on the parameters and initial data. An illustrative example is provided to demonstrate the applicability of the main results.

Keywords: Euler–Poisson–Darboux–Tricomi equations; energy and industry; nonlinear equations; classical solutions; integral representations; iterative methods
Mathematics Subject Classification: 55B44, 35L52, 35G55

1. Introduction

Recent studies on Euler–Poisson–Darboux–Tricomi (EPDT) type equations have focused on both the qualitative behavior and critical phenomena for nonlinear versions of these classical models. Particular solutions of the generalized Euler-Poisson-Darboux equation

$$u_{xx} + u_{yy} + \frac{2\sigma}{x}u_x + \frac{2\beta}{y}u_y - u_{tt} - 2\gamma t^{-1}u_t = 0, x, y > 0, \tag{1.1}$$

are considered in [1]. Beyond this, research into particular or specialized solutions continued, including the construction of explicit multidimensional solutions that involve hypergeometric functions and the investigation of elliptic–hyperbolic generalized Euler–Poisson–Darboux equations, thus expanding the catalog of known solution structures for related models. Recently, these results were extended to construct a particular solution of the multidimensional generalized Euler-Poisson-Darboux equation in [2–4].

On the global existence side, a series of results refined our understanding of how damping and nonlinearity interplay for semilinear regular EPDT equations. For instance, in [5], researchers considered the following:

$$u_{tt} - t^{2m}\Delta u + \frac{\mu}{t}u_t + \frac{\nu^2}{t^2}u = |u|^p. \tag{1.2}$$

It was shown that when damping dominates the mass term and the power exponent exceeds an appropriately shifted Fujita or Strauss-type critical exponent, small initial data give rise to global-in-time solutions. Additionally, these argued that the identified critical exponent marks the threshold between global existence and finite-time blow-up for the model; see [6–8].

A noteworthy recent contribution analyzed the blow-up behavior for (1.2). In particular, the authors in [9] established the blow-up in the critical power case using a Kato–type comparison argument together with integral representations and Radon transform techniques. Additionally, they derived sharp upper lifespan estimates consistent with classical wave models in the critical regime. Moreover, they are works on more classical aspects of the equation. For instance, [10] studied a singular Euler–Poisson–Darboux problem, thereby emphasizing the nonexistence of global solutions in subcritical regimes and showing that even in the linear singular EPDT case, blow-up phenomena can occur when nonlinear powers are below critical thresholds; see [11–13].

In the context of weakly coupled systems of semilinear EPDT equations, the recent manuscript [14] investigated the critical curve separating global existence from finite-time blow-up in the (p, q) plane for nonlinearities with damping and mass interactions in the following system:

$$\begin{cases} y_{tt} - t^{2m}\Delta y + \frac{\mu_1}{t}y_t + \frac{\nu_1^2}{t^2}y &= |z|^p \\ z_{tt} - t^{2m}\Delta z + \frac{\mu_2}{t}z_t + \frac{\nu_2^2}{t^2}z &= |y|^q. \end{cases}$$

The authors identified threshold conditions (depending on spatial dimension, damping powers, and coupling parameters) that delineate regimes of small data global existence versus blow-up, and they constructed new test-function methods tailored to these coupled structures.

In this article, for $y = y(t, x), z = z(t, x),\ t > 1,\quad x \in \mathbb{R}^n$, we investigate the following initial value boundary problem

$$\begin{cases} y_{tt} - t^{2m}\Delta y + \frac{\mu_1}{t}y_t + \frac{\nu_1^2}{t^2}y &= |z|^p \\ z_{tt} - t^{2m}\Delta z + \frac{\mu_2}{t}z_t + \frac{\nu_2^2}{t^2}z &= |y|^q, \\ (y, y_t, z, z_t)(1, x) = (y_0, y_1, z_0, z_1)(x), \end{cases} \tag{1.3}$$

where

(A1) $\mu_1, \mu_2, \nu_1, \nu_2, p, q$ are nonnegative constants, $m > -1$, $y_0, z_0, y_1, z_1 \in C^2(\mathbb{R}^n)$, and $0 \leq y_0, z_0, |y_1|, |z_1| \leq B$ on $\mathbb{R}^n$. The constant B is a positive bound (or uniform bound) for the initial data. For $\mu i = \nu i = 0, m = 0$, a special case of the system equations (1.3) was considered in [15]. Recently, many papers have studied the critical exponent of the EPDT; see [16, 17].

This article is structured as follows: after presenting auxiliary results in Section 2, in Section 3, we prove the existence of at least one classical solution to problem (1.3); Sections 4 and 5 extend this analysis by establishing the existence of at least two and at least three nonnegative classical solutions, respectively; Finally, an example is given in Section 6 to illustrate the theoretical results.

2. Auxiliary results

In what follows, we assume that X is a real Banach space. Now, we recall the definition of a completely continuous operator in this space.

Definition 2.1. *Let X be a Banach space, and let $F\colon \mathcal{N} \subset X \to X$ be an operator. The operator F is said to be* compact *if, for every bounded set $B \subset \mathcal{N}$, the image $F(B)$ is relatively compact in X. Additionalyy, if F is continuous and maps bounded subsets of $\mathcal{N}$ into relatively compact subsets of X, then F is called* completely continuous.

The notion of a k-set contraction is closely related to the Kuratowski measure of non-compactness, which we recall here for completeness.

Definition 2.2. *Let Γ_X be the collection of all bounded subsets of X. The Kuratowski measure of non-compactness $\sigma\colon \Gamma_X \to \mathbb{R}^+$ is given by the following:*

$$\sigma(Y) = \inf\{\delta > 0 : \ Y \text{ can be covered by finitely many sets of diameter } \le \delta\}.$$

The main properties of the measure of non-compactness can be found in the classical literature [18].

Definition 2.3. *Let X be a Banach space, and let $\sigma(\cdot)$ denote the Kuratowski measure of noncompactness on X. A mapping $F\colon X \to X$ is called a k-*set contraction *if there exists a constant $k \in [0, 1)$ such that*

$$\sigma(F(\mathcal{N})) \le k\,\sigma(\mathcal{N})$$

for every bounded set $\mathcal{N} \subset X$.

Obviously, every completely continuous mapping $F\colon X \to X$ is a 0-set contraction; see [19].

Remark 2.1. *Intuitively, the Kuratowski measure of noncompactness $\sigma(\mathcal{N})$ quantifies "how far" a bounded set $\mathcal{N}$ is from being compact: the smaller $\sigma(\mathcal{N})$, the closer $\mathcal{N}$ it is to being compact. A map F is called a k-*set contraction *if there exists a constant $k \in [0, 1)$ such that for every bounded set $\mathcal{N} \subset X$,*

$$\sigma(F(\mathcal{N})) \le k\,\sigma(\mathcal{N}).$$

In other words, a k-set contraction "shrinks" the noncompactness of any bounded set by at least a factor of k, which generalizes the concept of a standard contraction to possibly noncompact operators.

Our first existence result follows from the fixed point theorem stated below. Consider two operators, T and ϖ, where

$$T : \overline{\Omega} \to \mathcal{N} \quad \text{is completely continuous,} \qquad \varpi : \overline{\Omega} \to \mathcal{N} \quad \text{is a } k\text{-set contraction with } k \in [0, 1).$$

We aim to study the solvability of the following operator equation:

$$x = T(x) + \varpi(x), \quad x \in \overline{\Omega}. \tag{2.1}$$

Theorem 2.1. *[20, 21] Let E be a Banach space, $\mathcal{N}$ a closed, convex subset of E, and Ω any open subset of $\mathcal{N}$ with $0 \in \Omega$. Consider two operators, T and ϖ, such that*

$$Tx = \varepsilon x,\ x \in \overline{\Omega},\ for\ \varepsilon > 1,$$

and

$$\varpi : \overline{\Omega} \to Y,$$

where

(i) $I - \varpi : \overline{\Omega} \to \mathcal{N}$ *is continuous, compact, and*

(ii) $\{x \in \partial\Omega : x = \beta(I - \varpi)x\} = \emptyset,$ *for any* $\beta \in \left(0, \frac{1}{\varepsilon}\right)$.

Then, $\exists x^* \in \overline{\Omega}$ *such that*

$$Tx^* + \varpi x^* = x^*.$$

Remark 2.2. *The concept of a k-set contraction plays a crucial role in establishing fixed-point results for possibly noncompact operators. In particular, if $F : X \to X$ is a k-set contraction with $k \in [0, 1)$, then under suitable conditions, F admits at least one fixed point in X. This generalizes the classical Banach fixed-point theorem for contractions to a broader class of operators and is especially useful in proving the existence results for differential and integral equations where compactness cannot be directly guaranteed.*

Remark 2.3. *It is worth emphasizing that the notions of expansive mappings and k-set contractions are, in general, independent. Recall that a mapping $T : X \to X$ is called expansive if there exists a constant $\lambda > 1$ such that*

$$\|Tx - Ty\| \geq \lambda\|x - y\|, \quad \forall x, y \in X,$$

whereas T is a k-set contraction if

$$\sigma(T(\mathcal{N})) \leq k\,\sigma(\mathcal{N}), \qquad \forall\ bounded\ \mathcal{N} \subset X,$$

with $k \in [0, 1)$, and σ denoting the Kuratowski measure of noncompactness.

Definition 2.4. *Let X and $\mathcal{N}$ be real Banach spaces. A map $F : X \to \mathcal{N}$ is called expansive if there exists a constant $h > 1$ for which the following inequality*

$$\|Fa - Fb\|_{\mathcal{N}} \geq h\|a - b\|_X$$

for any $a, b \in X$.

Next, the definition of a cone in a Banach space is given.

Definition 2.5. *A closed, convex subset Λ of X is called a* cone *if*

(1) $\sigma x \in \Lambda$ *for all* $\sigma \geq 0$ *and* $x \in \Lambda$,
(2) $x, -x \in \Lambda$ *implies* $x = 0$.

Denote $\Lambda^* = \Lambda\backslash\{0\}$. Now, we state the following fixed point theorem, which will be instrumental in proving the existence of at least two nonnegative classical solutions for the initial value problem (1.3). For its proof, the reader may consult [22, 23].

Theorem 2.2. *Let Λ be a cone in a Banach space E, Γ a subset of Λ, and $\Omega_1, \Omega_2, \Omega_3$ three open bounded subsets of Λ such that*

$$\overline{\Omega}_1 \subset \overline{\Omega}_2 \subset \Omega_3 \quad \text{and} \quad 0 \in \Omega_1.$$

Assume that $T : \Gamma \to \Lambda$ is an expansive mapping, $\varpi : \overline{\Omega}_3 \to E$ is a completely continuous map, and

$$\varpi(\overline{\Omega}_3) \subset (I - T)(\Gamma).$$

Suppose that

$$(\Omega_2 \setminus \overline{\Omega}_1) \cap \Gamma \neq \emptyset, \quad (\Omega_3 \setminus \overline{\Omega}_2) \cap \Gamma \neq \emptyset;$$

there exists $u_0 \in \Lambda^*$ such that the following conditions hold:

(i) $\varpi x \neq (I - T)(x - \beta u_0)$ for all $\beta > 0$ and $x \in \partial\Omega_1 \cap (\Gamma + \beta u_0)$,
(ii) there exists $\varepsilon \geq 0$ such that $\varpi x \neq (I - T)(\beta x)$ for all $\beta \geq 1 + \varepsilon$, $x \in \partial\Omega_2$ with $\beta x \in \Gamma$,
(iii) $\varpi x \neq (I - T)(x - \beta u_0)$ for all $\beta > 0$ and $x \in \partial\Omega_3 \cap (\Gamma + \beta u_0)$.

Then, $T + \varpi$ has at least two non-zero fixed points $x_1, x_2 \in \Lambda$ such that either

$$x_1 \in \partial\Omega_2 \cap \Gamma, \quad x_2 \in (\overline{\Omega}_3 \setminus \overline{\Omega}_2) \cap \Gamma,$$

or

$$x_1 \in (\Omega_2 \setminus \Omega_1) \cap \Gamma, \quad x_2 \in (\overline{\Omega}_3 \setminus \overline{\Omega}_2) \cap \Gamma.$$

The following result will be employed to establish the existence of three nonnegative solutions to (1.3). For its proof, we follow arguments analogous to those used in [22, 23].

Theorem 2.3. *Let Λ be a cone in a Banach space E, Γ a subset of Λ, and $\Omega_1, \Omega_2, \Omega_3$ three open bounded subsets of Λ such that $\overline{\Omega}_1 \subset \overline{\Omega}_2 \subset \Omega_3$ and $0 \in \Omega_1$. Assume that $T : \Gamma \to E$ is an expansive mapping, $\varpi : \overline{\Omega}_3 \to E$ is completely continuous, and $\varpi(\overline{\Omega}_3) \subset (I - T)(\Gamma)$.*

Suppose that $(\Omega_2 \setminus \overline{\Omega}_1) \cap \Gamma \neq \emptyset$, $(\Omega_3 \setminus \overline{\Omega}_2) \cap \Gamma \neq \emptyset$, and there exist $w_0 \in \Lambda^*$ and $\varepsilon > 0$ small enough such that the following conditions hold:

(i) *$\varpi x \neq (I - T)(\beta x)$ for all $\beta \geq 1 + \varepsilon$, $x \in \partial\Omega_1$ with $\beta x \in \Gamma$,*

(ii) *$\varpi x \neq (I - T)(x - \beta w_0)$ for all $\beta \geq 0$, $x \in \partial\Omega_2 \cap (\Gamma + \beta w_0)$,*

(iii) *$\varpi x \neq (I - T)(\beta x)$ for all $\beta \geq 1 + \varepsilon$, $x \in \partial\Omega_3$ with $\beta x \in \Gamma$.*

Then, $T + \varpi$ has at least three nontrivial fixed points $x_1, x_2, x_3 \in \Lambda$ such that

$$x_1 \in \overline{\Omega}_1 \cap \Gamma, \quad x_2 \in (\Omega_2 \setminus \overline{\Omega}_1) \cap \Gamma, \quad x_3 \in (\overline{\Omega}_3 \setminus \overline{\Omega}_2) \cap \Gamma.$$

For $w \in C^2([1, \infty), C^2(\mathbb{R}^n))$, define

$$\|w\|_1 = \max\Big\{ \sup_{1 \leq t < \infty, x \in \mathbb{R}^n} |w|, \quad \sup_{1 \leq t < \infty, x \in \mathbb{R}^n} |w_t|, \quad \sup_{1 \leq t < \infty, x \in \mathbb{R}^n} |w_{tt}|, \quad \sup_{1 \leq t < \infty, x \in \mathbb{R}^n} |w_x|, \quad \sup_{1 \leq t < \infty, x \in \mathbb{R}^n} |w_{xx}| \Big\},$$

provided it exists. Let $X = \left(C^2\left([0, \infty), C^2(\mathbb{R}^n)\right)\right)^6$ be defined by the following:

$$\|u\| = \max_{j \in \{1,2,\ldots,6\}} \|u_j\|, \quad u = (u_1, u_2, \ldots, u_6).$$

3. Existence of at least one solution

In this section, we prove the existence of at least one classical solution for problem (1.3) using the theory of k-set contractions and the fixed point theorem presented in Section 2. For $u \in X$, define the following:

$$\begin{aligned}
\varpi_1^1(u) &= t^2 u_{1tt} - t^{2(m+1)}\Delta u_1 + \mu_1 t u_{1t} + \nu_1^2 u_1 - t^2|u_2|^p,\\
\varpi_1^2(u) &= t^2 u_{2tt} - t^{2(m+1)}\Delta u_2 + \mu_2 t u_{2t} + \nu_2^2 u_2 - t^2|u_1|^q,\\
\varpi_1^3(u) &= u_1(0,x) - y_0(x),\\
\varpi_1^4(u) &= u_2(0,x) - z_0(x),\\
\varpi_1^5(u) &= u_{1t}(0,x) - y_1(x),\\
\varpi_1^6(u) &= u_{2t}(0,x) - z_1(x),\\
\varpi_1(u) &= \left(\varpi_1^1(u), \varpi_1^2(u), \dots, \varpi_1^6(u)\right), \quad 1 \le t < \infty, x \in \mathbb{R}^n.
\end{aligned}$$

Note that if $u \in X$, $u = (u_1, u_2, \dots, u_6)$ is a solution to the equation $\varpi_1(u) = 0$, $(t,x) \in [1,\infty) \times \mathbb{R}^n$, then u_1 and u_2 are solutions to the IVP (1.3).

Let

$$B_1 = \max\left\{B + B^p, B + B^q, nB, \mu_1 B, \mu_2 B, \nu_1^2 B, \nu_2^2 B\right\}.$$

Lemma 3.1. *Suppose* $(A1)$. *If* $u \in X$, $\|u\| \le B$, *then*

$$|\varpi_1^j(u)| \le B_1\left(1 + t + t^2 + t^{2(m+2)}\right), \quad (t,x) \in [1,\infty) \times \mathbb{R}^n, \quad j \in \{1, 2, \dots, 6\}.$$

Proof. We have

$$\begin{aligned}
|\varpi_1^1(u)| &= \left|t^2 u_{1tt} - t^{2(m+1)}\Delta u_1 + \mu_1 t u_{1t} + \nu_1^2 u_1 - t^2|u_2|^p\right|\\
&\le t^2|u_{1tt}| + t^{2(m+1)}|\Delta u_1| + \mu_1 t|u_{1t}| + \nu_1^2|u_1| + t^2|u_2|^p\\
&\le Bt^2 + nBt^{2(m+1)} + \mu_1 Bt + \nu_1^2 B + B^p t^2\\
&\le B_1\left(1 + t + t^2 + t^{2(m+1)}\right)\\
&\le B_1\left(1 + t + t^2 + t^{2(m+2)}\right), \quad (t,x) \in [1,\infty) \times \mathbb{R}^n,
\end{aligned}$$

and

$$\begin{aligned}
|\varpi_1^2(u)| &= \left|t^2 u_{2tt} - t^{2(m+1)}\Delta u_2 + \mu_2 t u_{2t} + \nu_2^2 u_2 - t^2|u_1|^q\right|\\
&\le t^2|u_{2tt}| + t^{2(m+1)}|\Delta u_2| + \mu_2 t|u_{2t}| + \nu_2^2|u_2| + t^2|u_1|^q\\
&\le Bt^2 + nBt^{2(m+1)} + \mu_2 Bt + \nu_2^2 B + B^q t^2\\
&\le B_1\left(1 + t + t^2 + t^{2(m+1)}\right)\\
&\le B_1\left(1 + t + t^2 + t^{2(m+2)}\right), \quad (t,x) \in [1,\infty) \times \mathbb{R}^n,
\end{aligned}$$

and

$$\begin{aligned}
|\varpi_1^3(u)| &= |u_1(0,x) - y_0(x)|\\
&\le |u_1(0,x)| + y_0(x)\\
&\le 2B\\
&\le B_1\left(1 + t + t^2 + t^{2(m+1)}\right)\\
&\le B_1\left(1 + t + t^2 + t^{2(m+2)}\right), \quad (t,x) \in [1,\infty) \times \mathbb{R}^n,
\end{aligned}$$

and

$$
\begin{aligned}
|\varpi_1^4(u)| =& |u_2(0,x) - z_0(x)| \\
\leq & |u_2(0,x)| + |z_0(x)| \\
\leq & 2B \\
\leq & B_1\left(1 + t + t^2 + t^{2(m+1)}\right) \\
\leq & B_1\left(1 + t + t^2 + t^{2(m+2)}\right), \quad (t,x) \in [1,\infty) \times \mathbb{R}^n,
\end{aligned}
$$

and

$$
\begin{aligned}
|\varpi_1^5(u)| =& |u_{1t}(0,x) - y_1(x)| \\
\leq & |u_{1t}(0,x)| + |y_1(x)| \\
\leq & 2B \\
\leq & B_1\left(1 + t + t^2 + t^{2(m+1)}\right) \\
\leq & B_1\left(1 + t + t^2 + t^{2(m+2)}\right), \quad (t,x) \in [1,\infty) \times \mathbb{R}^n,
\end{aligned}
$$

and

$$
\begin{aligned}
|\varpi_1^6(u)| =& |u_{2t}(0,x) - z_1(x)| \\
\leq & |u_{2t}(0,x)| + |z_1(x)| \\
\leq & 2B \\
\leq & B_1\left(1 + t + t^2 + t^{2(m+1)}\right) \\
\leq & B_1\left(1 + t + t^2 + t^{2(m+2)}\right), \quad (t,x) \in [1,\infty) \times \mathbb{R}^n.
\end{aligned}
$$

Now, the proof is completed. □

(A2) Suppose that there exists a nonnegative function $g \in C([1,\infty) \times \mathbb{R}^n)$ and a positive constant A such that $g > 0$ on $(1,\infty) \times \left(\mathbb{R}^n \setminus \left\{\bigcup_{j=1}^n \{x_j = 0\}\right\}\right)$ and

$$2\left(1 + t + t^2 + t^3 + t^4 + t^{2(m+4)}\right)\prod_{j=1}^n\left(1 + |x_j| + x_j^2\right) \times \int_1^t \left|\int_0^x g(s,\tau)d\tau\right| ds \leq A, \quad 1 \leq t < \infty, x \in \mathbb{R}^n,$$

where

$$\int_0^x = \int_0^{x_1}\int_0^{x_2}\dots\int_0^{x_n}, \quad y = (y_1, y_2, \dots, y_n), \quad dy = dy_n dy_{n-1}\dots dy_1.$$

For $u \in X$, define the following operators:

$$
\begin{aligned}
\varpi_2^j(u) &= \int_1^t\int_0^x (t-s)^2 g(s,y)\prod_{j=1}^n (x_j - y_j)^2 \varpi_1^j(u)(s,y)dsdy, \quad j \in \{1,2,\dots,6\}, \\
\varpi_2(u) &= (\varpi_2^1(u)(t,x), \varpi_2^2(u)(t,x), \dots, \varpi_2^6(u)(t,x)), \quad (t,x) \in [1,\infty) \times \mathbb{R}^n.
\end{aligned}
$$

Lemma 3.2. *Suppose* (*A*1)*. If* $u \in X$ *and* $\|u\| \leq B$*, then*

$$\|\varpi_2 u\| \leq AB_1.$$

Proof. We have

$$
\begin{aligned}
|\varpi_2^j(u)(t,x)| =& \left| \int_1^t \int_0^x (t-s)^2 g(s,y) \prod_{j=1}^n (x_j-y_j)^2 \varpi_1^j(u)(s,y) dy ds \right| \\
\leq& \int_1^t \left| \int_0^x (t-s)^2 g(s,y) \prod_{j=1}^n (x_j-y_j)^2 |\varpi_1^j(u)(s,y)| dy \right| ds \\
\leq& B_1 \int_1^t \left| \int_0^x (t-s)^2 g(s,y) \prod_{j=1}^n (x_j-y_j)^2 \left(1+s+s^2+s^{2(m+2)}\right) dy \right| ds \\
\leq& 2B_1 \left(1+t+t^2+t^3+t^4+t^{2(m+4)}\right) \prod_{j=1}^n \left(1+|x_j|+x_j^2\right) \\
&\times \int_1^t \left| \int_0^x g(s,\tau) d\tau \right| ds \leq AB_1, \quad 1 \leq t < \infty, x \in \mathbb{R}^n, \quad j \in \{1,2,\ldots,6\},
\end{aligned}
$$

and

$$
\begin{aligned}
\left| \frac{\partial}{\partial t} \varpi_2^j(u) \right| =& 2 \left| \int_1^t \int_0^x (t-s) g(s,y) \prod_{j=1}^n (x_j-y_j)^2 \varpi_1^j(u)(s,y) dy ds \right| \\
\leq& 2 \int_1^t \left| \int_0^x (t-s) g(s,y) \prod_{j=1}^n (x_j-y_j)^2 |\varpi_1^j(u)(s,y)| dy \right| ds \\
\leq& 2B_1 \int_1^t \left| \int_0^x (t-s) g(s,y) \prod_{j=1}^n (x_j-y_j)^2 \left(1+s+s^2+s^{2(m+2)}\right) dy \right| ds \\
\leq& 2B_1 \left(1+t+t^2+t^3+t^4+t^{2(m+4)}\right) \prod_{j=1}^n \left(1+|x_j|+x_j^2\right) \times \int_1^t \left| \int_0^x g(s,\tau) d\tau \right| ds \\
\leq& AB_1, \quad 1 \leq t < \infty, x \in \mathbb{R}^n, \quad j \in \{1,2,\ldots,6\},
\end{aligned}
$$

and

$$
\begin{aligned}
\left| \frac{\partial^2}{\partial t^2} \varpi_2^j(u) \right| =& 2 \left| \int_1^t \int_0^x g(s,y) \prod_{j=1}^n (x_j-y_j)^2 \varpi_1^j(u)(s,y) dy ds \right| \\
\leq& 2 \int_1^t \left| \int_0^x g(s,y) \prod_{j=1}^n (x_j-y_j)^2 |\varpi_1^j(u)(s,y)| dy \right| ds \\
\leq& 2B_1 \int_1^t \left| \int_0^x g(s,y) \prod_{j=1}^n (x_j-y_j)^2 \left(1+s+s^2+s^{2(m+2)}\right) dy \right| ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq 2B_1\left(1+t+t^2+t^3+t^4+t^{2(m+4)}\right)\prod_{j=1}^{n}\left(1+|x_j|+x_j^2\right)\times\int_1^t\left|\int_0^x g(s,\tau)d\tau\right|ds\\
&\leq AB_1,\quad (t,x)\in[1,\infty)\times\mathbb{R}^n,\quad j\in\{1,2,\ldots,6\},
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\frac{\partial}{\partial x_k}\varpi_2^j(u)\right| &= 2\left|\int_1^t\int_0^x (t-s)^2 g(s,y)\prod_{j=1, j\neq k}^{n}(x_j-y_j)^2(x_k-y_k)\varpi_1^j(u)(s,y)dyds\right|\\
&\leq 2\int_1^t\left|\int_0^x (t-s)^2 g(s,y)\prod_{j=1, j\neq k}^{n}(x_j-y_j)^2|x_k-y_k||\varpi_1^j(u)(s,y)|dy\right|ds\\
&\leq 2B_1\int_1^t\left|\int_0^x (t-s)^2 g(s,y)\prod_{j=1, j\neq k}^{n}(x_j-y_j)^2|x_k-y_k|\left(1+s+s^2+s^{2(m+2)}\right)dy\right|ds\\
&\leq 2B_1\left(1+t+t^2+t^3+t^4+t^{2(m+4)}\right)\prod_{j=1}^{n}\left(1+|x_j|+x_j^2\right)\times\int_1^t\left|\int_0^x g(s,\tau)d\tau\right|ds\\
&\leq AB_1,\quad 1\leq t<\infty, x\in\mathbb{R}^n,\quad j\in\{1,2,\ldots,6\},\quad k\in\{1,2,\ldots,n\},
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\frac{\partial^2}{\partial x_k^2}\varpi_2^j(u)\right| &= 2\left|\int_1^t\int_0^x (t-s)^2 g(s,y)\prod_{j=1, j\neq k}^{n}(x_j-y_j)^2\varpi_1^j(u)(s,y)dyds\right|\\
&\leq 2\int_1^t\left|\int_0^x (t-s)^2 g(s,y)\prod_{j=1, j\neq k}^{n}(x_j-y_j)^2|\varpi_1^j(u)(s,y)|dy\right|ds\\
&\leq 2B_1\int_1^t\left|\int_0^x (t-s)^2 g(s,y)\prod_{j=1, j\neq k}^{n}(x_j-y_j)^2\left(1+s+s^2+s^{2(m+2)}\right)dy\right|ds\\
&\leq 2B_1\left(1+t+t^2+t^3+t^4+t^{2(m+4)}\right)\prod_{j=1}^{n}\left(1+|x_j|+x_j^2\right)\\
&\quad\times\int_1^t\left|\int_0^x g(s,\tau)d\tau\right|ds\leq AB_1, (t,x)\in[1,\infty)\times\mathbb{R}^n, j\in\{1,2,\ldots,6\}, k\in\{1,2,\ldots,n\}.
\end{aligned}
$$

Combining the estimates obtained above, we conclude that the operator satisfies the required conditions. Hence, the proof is complete. □

Lemma 3.3. *Under assumptions* (A1) *and* (A2)*, let* $u \in X$ *satisfy*

$$
\varpi_2(u) = C,\quad (t,x)\in[1,\infty)\times\mathbb{R}^n, \tag{3.1}
$$

for some constant C*; then,* u *is a solution to the problem* (1.3).

Proof. By differentiating Eq (3.1) three times with respect to t and three times with respect to x_k, for $k \in \{1, 2, \dots, n\}$, we obtain the following:

$$g\varpi_1(u) = 0, \quad (t, x) \in [1, \infty) \times \mathbb{R}^n.$$

Then,

$$\varpi_1(u) = 0, \quad (t, x) \in (1, \infty) \times \left(\mathbb{R}^n \backslash \left\{ \bigcup_{j=1}^{n} \{x_j = 0\} \right\} \right).$$

By the fact that $\varpi_1(u)(\cdot, \cdot, \dots, \cdot)$ is a continuous function, we find the following:

$$\begin{aligned} 0 &= \lim_{t \to 1} \varpi_1(u) \\ &= \varpi_1(u)(1, x) \\ &= \lim_{x_1 \to 0} \varpi_1(u)(t, x_1, x_2, \dots, x_n) \\ &= \varpi_1(u)(t, 0, x_2, \dots, x_n) \\ &\vdots \\ &= \lim_{x_n \to 0} \varpi_1(u)(t, x_1, \dots, x_{n-1}, x_n) \\ &= \varpi_1(u)(t, x_1, x_2, \dots, x_n). \end{aligned}$$

Therefore,

$$\varpi_1(u) = 0, \quad 1 \le t < \infty, x \in \mathbb{R}^n.$$

Hence, we conclude that u is indeed a solution to problem (1.3). This completes the proof. □

Below, suppose

(A3) $\varepsilon > 1$.

In the final section, we provide an illustrative example for the constants ε, A, B, and B_1. The main result of this section is stated as follows.

Theorem 3.1. *Under assumptions* (A1)–(A3)*, system* (1.3) *has at least one solution in X.*

Proof. Let $\widetilde{\mathcal{N}}$ denote the set of all equicontinuous families in X with respect to the norm $\|\cdot\|$, that is, $\widetilde{\mathcal{N}}$ consists of all families $\{y_\sigma\}_{\sigma \in A} \subset X$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ with

$$\|y_\sigma(x) - y_\sigma(y)\| < \varepsilon \quad \text{whenever } \|x - y\| < \delta, \ \forall \sigma \in A.$$

This set is fundamental in the study of compactness and the application of the Arzelà–Ascoli theorem, which ensures the relative compactness of equicontinuous and uniformly bounded families in X. additionally, let

$$\widetilde{\widetilde{\mathcal{N}}} = \left\{ u \in \widetilde{\mathcal{N}} : u \ge \frac{\|u\|}{2}, \quad (t, x) \in [1, \infty) \times \mathbb{R}^n \right\},$$

and $Y = \overline{\widetilde{\widetilde{\mathcal{N}}}}$ be the closure of $\widetilde{\mathcal{N}}$,

$$\Omega = \{u \in \mathcal{N} : \|u\| < B\}.$$

For $u \in \overline{\Omega}$ and $\varepsilon > 0$, define the following:

$$
\begin{aligned}
Tu &= \varepsilon u, \\
\varpi u &= u - \varepsilon u - \varepsilon \varpi_2(u), \quad 1 \le t < \infty, x \in \mathbb{R}^n.
\end{aligned}
$$

For $u \in \overline{\Omega}$, we have the following:

$$
\begin{aligned}
\|(I - \varpi)u\| &= \|\varepsilon u + \varepsilon \varpi_2(u)\| \\
&\le \varepsilon\|u\| + \varepsilon\|\varpi_2(u)\| \\
&\le \varepsilon B + \varepsilon A B_1.
\end{aligned}
$$

Thus, $\varpi : \overline{\Omega} \to X$ is continuous, and $(I - \varpi)(\overline{\Omega})$ resides in a compact subset of $\mathcal{N}$. Now, suppose that there is a $u \in \overline{\Omega}$ such that $\|u\| = B$ and

$$u = \beta(I - \varpi)u$$

or

$$u = \beta\varepsilon\,(u + \varpi_2(u)), \tag{3.2}$$

for some $\beta \in \left(0, \frac{1}{\varepsilon}\right)$. This argument is fundamental in establishing the existence of at least one solution to the problem under consideration. Using $\|u\| \ge \frac{B}{2}$, we obtain $u(1, x) > \frac{B}{2}$, $x \in \mathbb{R}^n$, and

$$u(1, x) = \beta\varepsilon u(1, x), \quad x \in \mathbb{R}^n,$$

whereupon $\beta\varepsilon = 1$, which is a contradiction. Thus,

$$\{u \in \overline{\Omega} : u = \beta_1(I - \varpi)u,\ \|u\| = B\} = \emptyset$$

for any $\beta_1 \in \left(0, \frac{1}{\varepsilon}\right)$. Therefore, by Theorem 2.1, $T + \varpi$ admits a fixed point $u^* \in Y$, and

$$u^* = Tu^* + \varpi u^* = \varepsilon u^* + u^* - \varepsilon u^* - \varepsilon\varpi_2(u^*), \quad (t, x) \in [1, \infty) \times \mathbb{R}^n;$$

then

$$\varpi_2(u^*) = 0, \quad 1 \le t < \infty, x \in \mathbb{R}^n.$$

From here, we deduce that u^* is a solution to problem (1.3). Moreover, by applying Lemma 3.3, it follows that u is a solution to the following equation:

$$\mathcal{F}(u) = 0,$$

where $\mathcal{F}$ denotes the operator associated with problem (1.3). Thus, the existence of u^* ensures the existence of a corresponding solution u for the original equation.

□

4. Existence of at least two solutions

Suppose

(A4). Let $m,\ r,\ L,\ and\ R_1 > 0$ such that
$$r < L < R_1 \leq B.$$

Theorem 4.1. *Under assumptions* (A1)–(A4)*, Eq* (1.3) *has at least two nonnegative solutions in X.*

Proof. Let
$$\widetilde{\mathcal{M}} = \{u \in X : u \geq 0 \quad \text{on} \quad [1,\infty)\times\mathbb{R}^n\}.$$
With Λ, we denote the set of all equi-continuous families in $\widetilde{\mathcal{M}}$. For $v \in X$, let

$$T_1(v) : \Lambda \to X, \quad \varpi_3(v) : \Lambda \to X,$$

where

$$\begin{aligned} T_1 v &= (1+m\varepsilon)v, \\ \varpi_3 v &= -\varepsilon|\varpi_2(v)| - m\varepsilon v, \end{aligned}$$

$(t,x) \in [1,\infty)\times\mathbb{R}^n$. These operators will be used to reformulate problem (1.3) into an operator equation suitable to apply fixed point theorems. Note that any fixed point $v \in X$ of the operator $T_1 + \varpi_3$ is a solution to (1.3). Let

$$\begin{aligned} \Gamma &= \mathcal{N}, \\ \Omega_1 &= \Lambda_r = \{w \in \Lambda : \|w\| < r\}, \\ \Omega_2 &= \Lambda_L = \{w \in \Lambda : \|w\| < L\}, \\ \Omega_3 &= \Lambda_{R_1} = \{w \in \Lambda : \|w\| < R_1\}. \end{aligned}$$

(1) For $w_1, w_2 \in \Gamma$, we have the following:

$$\|T_1 w_1 - T_1 w_2\| = (1+m\varepsilon)\|w_1 - w_2\|,$$

whereupon $T_1 : \Gamma \to X$ is an expansive operator with a constant $h = 1 + m\varepsilon > 1$.

(2) For $w \in \overline{\Lambda_{R_1}}$, we obtain the following:

$$\begin{aligned} \|\varpi_3 v\| &\leq \varepsilon\|\varpi_2(w)\| + m\varepsilon\|w\| \\ &\leq \varepsilon\Big(AB_1 + mR_1\Big). \end{aligned}$$

Therefore, $\varpi_3(\overline{\Lambda_{R_1}})$ is uniformly bounded. Since $\varpi_3 : \overline{\Lambda_{R_1}} \to X$ is continuous, we have that $\varpi_3(\overline{\Lambda_{R_1}})$ is equi-continuous. Consequently, $\varpi_3 : \overline{\Lambda_{R_1}} \to X$ is a 0-set contraction.

(3) Let $w_1 \in \overline{\Lambda_{R_1}}$. Set
$$w_2 = w_1 + \frac{1}{m}|\varpi_2(w_1)|.$$
We have $w_2 \geq 0$ on $[1,\infty)\times\mathbb{R}^n$. Therefore, $w_2 \in \Gamma$ and either $-\varepsilon m w_2 = -\varepsilon m w_1 - \varepsilon|\varpi_2(w_1)|$ or $(I - T_1)w_2 = -\varepsilon m w_2 = \varpi_3 w_1$. Consequently, $\varpi_3(\overline{\Lambda_{R_1}}) \subset (I - T_1)(\Gamma)$.

(1) Assume that for any $w_0 \in \Lambda^*$, there exist $\beta \geq 0$ and either $w \in \partial\Lambda_r \cap (\Gamma + \beta v_0)$ or $w \in \partial\Lambda_{R_1} \cap (\Gamma + \beta w_0)$ such that

$$\varpi_3 = (I - T_1)(w - \beta w_0).$$

Then, either

$$-\varepsilon|\varpi_2(w)| - m\varepsilon w = -m\varepsilon(w - \beta w_0)$$

or

$$-|\varpi_2(w)| = \beta m w_0.$$

This is a contradiction.

(2) Let $\varepsilon_1 = \frac{AB_1}{mL}$. Suppose that there exists a $w_1 \in \partial\Lambda_L$ and $\beta_1 \geq 1 + \varepsilon_1$ such that

$$\varpi_3 w_1 = (I - T_1)(\beta_1 w_1). \tag{4.1}$$

Moreover, either

$$-\varepsilon|\varpi_2(w_1)| - m\varepsilon w_1 = -\beta_1 m\varepsilon w_1,$$

or

$$|\varpi_2(w_1)| + m w_1 = \beta_1 m w_1.$$

From here,

$$v_1 m L = \beta_1 m\|w_1\| \leq \|\varpi_2 w_1\| + m\|w_1\| \leq AB_1 + mL$$

and

$$\beta_1 \leq 1 + \frac{AB_1}{mL},$$

which is a contradiction.

Therefore, all conditions of Theorem 2.2 are satisfied. Then, system (1.3) admits at least two solutions, v_1 and v_2, where either

$$\|v_1\| = L < \|v_2\| < R_1$$

or

$$r < \|v_1\| < L < \|v_2\| < R_1.$$

□

5. Existence of at least three solutions

Theorem 5.1. *Under hypotheses* (A1)–(A4)*, system* (1.3) *has at least three nonnegative solutions:* $u_1, u_2,$ *and* $u_3 \in X$.

Proof. (1) Suppose that there are $\beta_1 \geq 1 + \frac{2AB_1}{mr}$, $u \in \partial\Omega_1$, and $\beta_1 u \in \Gamma$ such that

$$\varpi_3(u) = (I - T_1)(\beta_1 u).$$

Then,

$$-\epsilon|\varpi_2(u)| - m\epsilon u = -m\epsilon\beta_1 u$$

or

$$|\varpi_2(u)| + mu = m\beta_1 u.$$

Hence,

$$\beta_1 mr = \beta_1 m\|u\| \le \|\varpi_2(u)\| + m\|u\| \le AB_1 + \beta_1 mr,$$

whereupon

$$\beta_1 \le 1 + \frac{AB_1}{mr},$$

which is a contradiction. Thus, condition (**i**) of Theorem 2.3 is satisfied.

(2) Suppose that there are $\beta_1 \ge 1 + \frac{2AB_1}{mr}$, $u \in \partial\Omega_3$, and $\beta_1 u \in \Gamma$ such that $\varpi_3(u) = (I - T_1)(\beta_1 u)$. As above,

$$\beta_1 mR_1 = \beta_1 m\|u\| \le \|\varpi_2(u)\| + m\|u\| \le AB_1 + \beta_1 mR_1,$$

whereupon

$$\beta_1 \le 1 + \frac{AB_1}{mR_1} \le 1 + \frac{AB_1}{mr},$$

which is a contradiction. Thus, condition (**iii**) of Theorem 2.3 is satisfied.

(3) Suppose that for any $u_0 \in \Lambda^*$, $\exists\beta_1 \ge 0$, and $u \in \partial\Lambda_L \cap (\Gamma + \beta_1 u_0)$ such that

$$\varpi_3(u) = (I - T_1)(u - \beta_1 u_0).$$

Then,

$$-\epsilon|\varpi_2(u)| - m\epsilon u = -m\epsilon(u - \beta_1 u_0) \; or \; -|\varpi_2(u)| = \beta_1 m u_0.$$

This is a contradiction. Thus, condition (**ii**) of Theorem 2.3 is satisfied.

Now, using Theorem 2.3, it follows that system (1.3) has at least three classical solutions u_1, u_2, and u_3 that satisfy the following:

$$u_1 \in \overline{\Omega}_1 \cap \Gamma, \quad u_2 \in (\Omega_2 \setminus \overline{\Omega}_1) \cap \Gamma, \quad u_3 \in (\overline{\Omega}_3 \setminus \overline{\Omega}_2) \cap \Gamma.$$
$$u_1 \in \partial\varpi_1 \cap \Gamma, \quad u_2 \in (\Omega_2 \setminus \overline{\Omega}_1) \cap \Gamma, \quad u_3 \in (\overline{\Omega}_3) \setminus \overline{\Omega}_2) \cap \Gamma,$$

or

$$u_1 \in \Omega_1 \cap \Gamma, \quad u_2 \in (\Omega_2 \setminus \overline{\Omega}_1) \cap \Gamma, \quad u_3 \in (\overline{\Omega}_3) \setminus \overline{\Omega}_2) \cap \Gamma.$$

□

6. An example

Let

$$n = 1, y_0(x) = \frac{1}{1 + x^2}, z_0(x) = \frac{1}{1 + 2x^2 + 4x^4}, y_1(x) = \frac{1}{1 + x^8}, z_1(x) = \frac{1}{3 + 2x^6}, x \in \mathbb{R}^1,$$

$$\mu_1 = \frac{1}{3}, \mu_2 = \frac{1}{8}, \nu_1 = \frac{1}{12}, \nu_2 = \frac{1}{10}, m = \frac{1}{2}, p = \frac{1}{2}, q = \frac{1}{3},$$

and

$$R_1 = 1, \quad L = \frac{1}{4}, \quad r = \frac{1}{5}, \quad m = 10^{50}, \quad B = 1, \quad p = 2 \quad A = \frac{1}{10B_1}, \quad \varepsilon = 4.$$

Then, $B_1 = 2$, $A = \frac{1}{20}$, and $\epsilon > 1$, (i.e., ($A1$) and ($A3$) hold). Next, $0 < r < L < R_1 = B$. (i.e., ($A4$) holds). Now, we will give an example of the function g. Take the following:

$$h(\tau) = \log \frac{1 + \tau^{11}\sqrt{2} + \tau^{22}}{1 - \tau^{11}\sqrt{2} + \tau^{22}}, \quad l(\tau) = \arctan \frac{\tau^{11}\sqrt{2}}{1 - \tau^{22}}, \quad \tau \in \mathbb{R}, \quad \tau \neq \pm 1.$$

Then,

$$\begin{aligned} h'(\tau) &= \frac{22\sqrt{2}\tau^{10}(1 - \tau^{22})}{(1 - \tau^{11}\sqrt{2} + \tau^{22})(1 + \tau^{11}\sqrt{2} + \tau^{22})}, \\ l'(\tau) &= \frac{11\sqrt{2}\tau^{10}(1 + \tau^{22})}{1 + \tau^{44}}, \quad \tau \in \mathbb{R}, \quad \tau \neq \pm 1. \end{aligned}$$

Thus,

$$\begin{aligned} -\infty &< \lim_{\tau \to \pm\infty} (1 + \tau + \tau^2 + \tau^3) h(\tau) < \infty, \\ -\infty &< \lim_{\tau \to \pm\infty} (1 + \tau + \tau^2 + \tau^3) l(\tau) < \infty. \end{aligned}$$

Therefore, $\exists C_1$ such that

$$(1 + \tau + \tau^2 + \tau^3)\left(\frac{1}{44\sqrt{2}} \log \frac{1 + \tau^{11}\sqrt{2} + \tau^{22}}{1 - \tau^{11}\sqrt{2} + \tau^{22}} + \frac{1}{22\sqrt{2}} \arctan \frac{\tau^{11}\sqrt{2}}{1 - \tau^{22}} \right) \leq C_1,$$

$s \in \mathbb{R}$. By [24] (pp. 707, Integral 79), we have the following:

$$\int \frac{dz}{1 + z^4} = \frac{1}{4\sqrt{2}} \log \frac{1 + z\sqrt{2} + z^2}{1 - z\sqrt{2} + z^2} + \frac{1}{2\sqrt{2}} \arctan \frac{z\sqrt{2}}{1 - z^2}.$$

Let

$$P(s) = \frac{s^{2m+7}}{(1 + s^{8(m+4)})(1 + s + s^2)^2}, \quad s \in \mathbb{R},$$

and

$$g_1 = P(t - 1)P(x_1) \dots P(x_n), \quad 1 \leq t < \infty, x \in \mathbb{R}^n.$$

Then, $\exists C_2 > 0$, where

$$2\left(1 + t + t^2 + t^3 + t^4 + t^{2(m+4)}\right) \prod_{j=1}^{n} \left(1 + |x_j| + x_j^2\right) \times \int_1^t \left| \int_0^x g_1(s, y) dy \right| ds \leq C_2, \ 1 \leq t < \infty, x \in \mathbb{R}^n.$$

Let

$$g = \frac{A}{C_2} g_1, \quad 1 \leq t < \infty, x \in \mathbb{R}^n.$$

Thus

$$2\left(1 + t + t^2 + t^3 + t^4 + t^{2(m+4)}\right) \prod_{j=1}^{n} \left(1 + |x_j| + x_j^2\right) \times \int_1^t \left| \int_0^x g(s, \tau) d\tau \right| ds \leq A, 1 \leq t < \infty, x \in \mathbb{R}^n.$$

Therefore, all conditions of Theorems 3.1, 4.1, and 5.1 are fulfilled.

7. Conclusions

In this work, we studied the existence of classical solutions for the boundary value problem (1.3). By applying fixed point theorems in cones, we established the existence of at least one, two, and three nonnegative classical solutions under appropriate conditions.

Classical solutions are particularly important for this type of system because they guarantee the necessary smoothness and differentiability required to point wise satisfy the differential equations, rather than merely in a weak or distributional sense. This smoothness ensures that all boundary and initial conditions are strictly satisfied, which is crucial for both the theoretical analysis and practical applications, such as numerical simulations or stability studies.

Specifically, Theorems 2.2 and 2.3 provide conditions under which multiple nontrivial solutions exist, the highlighting the rich structure and multiplicity of solutions for nonlinear systems in Banach spaces. The presented example illustrated how the constants in the system can be chosen to meet the hypotheses of our theorems.

Overall, the results contribute to a deeper understanding of the solution structure of nonlinear boundary value problems and provide a robust framework to investigate the qualitative behavior of classical solutions in complex systems as in [25, 26].

Author contributions

Svetlin G. Georgiev: Writing-original draft; Safa M. Mirgani: Writing-review and editing; Khaled Zennir: Supervision, Investigation, Methodology; Keltoum Bouhali: Writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (Grant number IMSIU-DDRSP2603).

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. R. B. Seilkhanova, A. Hasanov, Particular solutions of generalized Euler-Poisson-Darboux equation, *Elect. J. Diff. Equ.*, **2015** (2015), 1–10.

2. A. R. Ryskan, Z. O. Arzikulov, T. G. Ergashev, Particular solutions of multidimensional generalized Euler-Poisson-Darboux equations of Elliptic-Hyperbolic type, *J. Math. Mech. Comput. Sci.*, **121** (2024), 76–88. https://doi.org/10.26577/JMMCS202412118

3. M. F. Ben Hassen, M. Hamouda, M. A. Hamza, Blow-up result for a weakly coupled system of two Euler-Poisson-Darboux-Tricomi equations with time derivative nonlinearity, *Math. Meth. Appl. Sci.*, **48** (2025), 12989–13000. https://doi.org/10.1002/mma.11078

4. M. D'Abbicco, Small data solutions for the Euler-Poisson-Darboux equation with a power nonlinearity, *J. Diff. Equ*, **286** (2021), 531–556. https://doi.org/10.1016/j.jde.2021.03.033

5. Y. Li , F. Guo, Global existence for the semilinear regular Euler-Poisson-Darboux-Tricomi equation, *J. Diff. Equ.*, **431** (2025), 113211. https://doi.org/10.1016/j.jde.2025.02.082

6. K. Zennir, S. Bousserhane Reda, T. Miyasita, S. G. Georgiev, K. Bouhali, Finite time blow up of solutions for coupled system of wave equations with nonlinear memory terms, *Nonl. Anal.: Real World Appl.*, **91** (2026), 104604. https://doi.org/10.1016/j.nonrwa.2026.104604

7. A. Palmieri, Lifespan estimates for local solutions to the semilinear wave equation in Einstein-de Sitter spacetime, *Appl. Anal.*, **102** (2023), 3577–3608. https://doi.org/10.1080/00036811.2022.2088529

8. K. Tsutaya, Y. Wakasugi, On heatlike lifespan of solutions of semilinear wave equationsin Friedmann-Lemaıtre-Robertson-Walker space time, *J. Math. Anal. Appl.*, **500** (2021), 125133. https://doi.org/10.1016/j.jmaa.2021.125133

9. N. A. Lai, A. Palmieri, H. Takamura, A blow-up result for the semilinear Euler-Poisson-Darboux-Tricomi equation with critical power nonlinearity, *J. Math. Anal. Appl.*, **553** (2025), 129835. https://doi.org/10.1016/j.jmaa.2025.129835

10. M. Fan, N. A. Lai, H. Takamura, Nonexistence of global solutions to the Euler–Poisson–Darboux equation in Rn: Subcritical case, *Nonl. Anal.*, **256** (2025), 113781. https://doi.org/10.1016/j.na.2025.113781

11. K. Zennir, Results on existence for generalized *nD* Navier-Stokes equations, *Open Math.*, **17** (2019), 1652–1679. https://doi.org/10.1515/math-2019-0137

12. S. G. Georgiev, A. Boukarou, K. Zennir, K. Bouhali, Classical solutions with new topological properties for n-dimensional physical models: Schrödinger and wave equations, *J. Nonl. Convex. Anal.*, **26** (2025), 2723–2740.

13. K. Tsutaya, Y. Wakasugi, Blow up of solutions of semilinear wave equations in Friedmann-Lemaıtre-Robertson-Walker spacetime, *J. Math. Phys.*, **61** (2020), 091503. https://doi.org/10.1063/1.5139301

14. Y. Li, F. Guo, Critical curve for weakly coupled system of semilinear Euler-Poisson-Darboux-Tricomi equations, preprint paper, 2025. https://doi.org/10.48550/arXiv.2511.08084

15. L. N. Lyakhov, Y. N. Bulatov, Fundamental solution of B-hyperbolic equation with weakly negative parameters, *Diff Equ.*, **61** (2025), 1633–1647. https://doi.org/10.7868/S3034503025120041

16. A. Palmieri, On the critical exponent for the semilinear Euler-Poisson-Darboux Tricomi equation with power nonlinearity, *J. Diff. Equ.*, **437** (2025), 113309. https://doi.org/10.1016/j.jde.2025.113309

17. Y. Li, T. He, Y. Qiao, Nonexistence of global solutions for a Euler-Poisson Darboux-Tricomi equation with combined nonlinearity, *Appl. Anal.*, **104** (2025), 1611–1634. https://doi.org/10.1080/00036811.2024.2431615

18. K. Deimling, *Nonlinear Functional Analysis*, Berlin: Springer-Verlag, 1985.

19. P. Drabek, J. Milota, *Methods in Nonlinear Analysis, Applications to Differential Equations*, Birkhäuser: Springer, 2007.

20. S. Georgiev, A. Kheloufi, K. Mebarki, Existence of classical solutions for a class of impulsive Hamilton-Jacobi equations, *Palestine J. Math.*, **13** (2024), 1084–1087.

21. K. Khemmar, K. Mebarki, S. Georgiev, Existence of solutions for a class of boundary value problems for weighted $p(t)$-Laplacian impulsive systems, *Filomat*, **38** (2024), 7563–7577.

22. S. Djebali, K. Mebarki, Fixed point index for expansive perturbation of k-set contraction mappings, *Top. Meth. Nonli. Anal.*, **54** (2019), 613–640.

23. M. Mouhous, S. Georgiev, K. Mebarki, Existence of solutions for a class of first order boundary value problems, *Arch. Math.*, **58** (2022), 141–158. http://doi.org/10.5817/AM2022-3-141

24. A. Polyanin, A. Manzhirov, *Handbook of Integral Equations*, Boca Raton: CRC Press, 1998.

25. A. Palmieri, Blow-up results for semilinear damped wave equations in Ein stein–de Sitter spacetime, *Z. Angew. Math. Phys.*, **72** (2021), 64. https://doi.org/10.1007/s00033-021-01494-x

26. K. Tsutaya, Y. Wakasugi, Blow-up of solutions of semilinear wave equations in accelerated expanding Friedmann-Lemaıtre-Robertson-Walker spacetime, *Rev. Math. Phys.*, **34** (2022), 2250003. https://doi.org/10.1142/S0129055X22500039

AIMS Press