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Centro-affine dual curve pair of pre-tangent-dual and pre-normal-dual curves

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Abstract: This study examines a new transformation, called the equi-centro-transform, for curves defined within the frame of centro-affine differential geometry. By creating centro-affine dual curves of the pre-tangent-dual and pre-normal-dual curves under this transformation, the relationship between dual curve pairs is investigated geometrically and analytically. The primary objective of the study is to characterize the singularities that arise of a given curve and its centro-affine dual. Furthermore, the necessary and sufficient conditions for the formation of singularities in dual curve pairs are determined, and the behavior of singularities changing type under the transformation is analyzed in detail. Moreover, after a detailed examination of the pre-tangent dual curve and the pre-normal dual curve, these curves are compared with other related curve types to reveal their geometric relationships. At the end of the study, the theorems presented are supported with appropriate examples, the obtained theoretical results are concretized, and graphs of all examined curve types are drawn and presented visually using MATLAB.

Keywords: Centro-affine differential geometry; dual curve; equi-centro-transform; singularity theory
Mathematics Subject Classification: Primary: 53A15; Secondary: 53A04, 57R45, 58K05

1. Introduction

Classical Euclidean geometry is based on metric concepts such as distance and angle. However, affine geometry, a branch of differential geometry, abandons this metric structure and focuses solely on more general properties of objects, such as parallelism and ratio. On the other hand, centro-affine geometry is a special and extremely rich sub-branch of affine geometry, where the origin of the center occupies an absolute and privileged position. When examining a curve or surface, the positions of these objects relative to the origin and the frames established by vectors drawn from the origin become our primary tools of study. This centro-affine perspective allows us to define purely intrinsic quantities for curves, such as "centro-affine curvature", that are independent of the metric and invariant only under affine transformations [1]. Giblin and Sano defined the height and distance functions as equi-centro-affine invariant functions and studied their singularities in [2]. The results of this study provide a basis for subsequent work in the equi-centro-affine context. In [3], Li et al. investigated the singularities of parallel curves, involute curves, and pre-evolute curves of equi-centro-affine framed curves.

In the mysterious world of curves, one of the most fascinating ways to understand a curve is to examine the new curves that derive from it. In this context, pedal and contrapedal curves are among the most elegant and well-established structures in geometry. The fundamental idea is to create new geometric objects through orthogonal projections and perpendiculars from a fixed point (the pole). A pedal curve is the set of points that, when a line is drawn from every point on a given curve to the pole, are marked on that line and perpendicular to the tangent drawn at that point. In other words, a pedal curve traces the footprints of all perpendiculars from the pole to the curve. Conversely, a contrapedal curve traces the intersections of lines (normals) passing through every point on the given curve and perpendicular to the tangent at that point with parallel lines drawn to the pole. These two curves are like dynamic reflections of each other; one is based on distance and perpendicularity, the other emphasizes the concepts of parallelism and direction. This pair of curves, frequently encountered in fields such as geometric transformations, singularity theory, and differential equations, provides a powerful tool for a deep understanding of an underlying curve. Pedal and contrapedal curves have been studied in various planes and spaces. In [4], Li and Pei defined the tangent-dual and position-dual curves (pedal and contrapedal curves) in equi-centro-affine plane and gave their relations with other curves. Li et al. [5] gave the pedal curves of frontals in Euclidean space. Zhang et al. [6] considered the pedal and contrapedal curves in the equi-affine plane. Zhao et al. [7] introduced the pedal curves of mixed-type curves by using the lightcone frame in the Lorentz–Minkowski plane. Tuncer et al. [8] presented the properties of pedal and contrapedal curves via the Legendrian Frenet frame along a front. The pedal curves and B-Gauss map of non-lightlike regular curves in Minkowski 3-space were defined in [9]. Yao et al. [10] investigated the pedal curves of framed immersions in Euclidean 3-space. Li et al. [11] examined the pedals and orthotomics of frontals in the de Sitter 2-space. Kaya [12] established a characterization between pedal curves and surfaces.

The novelty of the present work lies in the formulation of a new duality framework in the equi-centro-affine plane via pre-tangent-dual and pre-normal-dual curves. Although dual and pre-dual curves have been studied in affine and centro-affine geometries, the proposed approach introduces an intrinsic construction that is independent of classical pedal or contrapedal transformations. Moreover, this study establishes explicit relationships between the singularities of pre-tangent-dual and pre-normal-dual curves and their geometric invariants, providing a unified and systematic treatment of regularity and singularity phenomena. The equivalence between the proposed curves and other related dual curve types further clarifies their geometric significance and extends existing duality theories within a broader equi-centro-affine framework. This paper is organized as follows. Section 2 recalls the basic definitions and fundamental results on equi-centro-affine curves. In Section 3, a new dual curve pair associated with pre-tangent-dual and pre-normal-dual framed curves is constructed, and their regularity and singularity properties are investigated. Section 4 discusses the relationships between pre-tangent-dual and pre-normal-dual curves and other related curve types. Finally, Section 5 presents illustrative examples that support and visualize the theoretical results.

2. Preliminaries

Let $\mathbb{R}^2$ be the affine space, then we have

$$\det(x, y) = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_1 y_2 - x_2 y_1,$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$. We denote by

$$SL(2, \mathbb{R}) = \{A \in GL(2, \mathbb{R}) \mid \det(A) = 1\}$$

the special linear group, i.e., the subgroup of $GL(2, \mathbb{R})$ consisting of all 2×2 real matrices with determinant one [13]. The equi-centro-transform refers to the subgroup of the centro-affine transformation group consisting of $SL(2, \mathbb{R})$ [14].

Consider a regular curve $\gamma : I \to \mathbb{R}^2$. In the equi-centro-affine setting, we require the non-degeneracy condition $\det(\gamma(t), \gamma'(t)) \neq 0$ for all $t \in I$, where $\gamma'(t) = \frac{d\gamma}{dt}$. The equi-centro-affine arc length is then defined as [3]

$$s(t) = \int_{t_0}^{t} det(\gamma(u), \gamma^{'}(u))\, du.$$

If γ is a regular curve with parameter s in an equi-centro-affine plane, then the following condition is satisfied [3]:

$$\det(\gamma(s), \gamma^{'}(s)) = 1.$$

Consider a regular curve $\gamma(s)$ parameterized by equi-centro-affine arc length in the equi-centro-affine plane. The set $\{\gamma(s), \gamma^{'}(s)\}$ is the equi-centro-affine Frenet frame of the curve γ. Let $t(s) = \gamma^{'}(s)$, then $\{\gamma(s), t(s)\} \in SL(2, \mathbb{R})$. The Frenet formulas under the equi-centro-transform for the curve γ are given by [3]

$$\frac{d}{ds}\begin{pmatrix} \gamma(s) \\ t(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\kappa(s) & 0 \end{pmatrix}\begin{pmatrix} \gamma(s) \\ t(s) \end{pmatrix}, \tag{2.1}$$

where $\kappa(s) = det(\gamma'(s), \gamma''(s))$ is the curvature of curve γ.

Remark 1. *The condition* $\det(\gamma(t), \mu(t)) \neq 0$ *is assumed to ensure the non-degeneracy of the framed curve in general. When the condition* $\det(\gamma(t), \mu(t)) = 0$ *appears, it is imposed only at specific parameter values in order to characterize the occurrence of singularities in the associated dual constructions.*

Definition 2.1. *[3] An equi-centro-affine framed curve is a triple* $(\gamma, \mu, \nu) : I \to \mathbb{R}^2 \times SL(2, \mathbb{R})$ *of smooth functions satisfying* $\det(\gamma(t), \mu(t)) = 0$ *and* $(\mu(t), \nu(t)) \in SL(2, \mathbb{R})$ *for all* $t \in I$*. Then, the frame* $\{\mu(t), \nu(t)\}$ *is called a moving frame of* $\gamma(t)$ *.*

The moving frame of an equi-centro-affine framed curve evolves according to the following Frenet-type formulas [3]:

$$\frac{d}{dt}\begin{pmatrix} \mu(t) \\ \nu(t) \end{pmatrix} = \begin{pmatrix} l(t) & m(t) \\ n(t) & -l(t) \end{pmatrix}\begin{pmatrix} \mu(t) \\ \nu(t) \end{pmatrix}, \ \gamma(t) = \alpha(t)\mu(t), \tag{2.2}$$

where

$$l(t) = det(\mu^{'}(t), v(t)) \, , m(t) = -det(\mu^{'}(t), \mu(t)),$$
$$n(t) = det(v^{'}(t), v(t)) \, , \alpha(t) = det(\gamma(t), v(t)).$$

We define the equi-centro-affine framed curvature of (γ, μ, v) as the mapping $(l, m, n, \alpha) : I \to \mathbb{R}^4$ under the equi-centro-transform. Consider an equi-centro-affine framed curve $(\gamma, \mu, v) : I \to \mathbb{R}^2 \times SL(2, \mathbb{R})$ with associated curvature functions l, m, n and α. When $l = 0$, the moving frame $\{\mu, v\}$ is called an adapted frame of γ. Since such a frame always exists, we simplify the curvature notation from $(0, m, n, \alpha)$ to (m, n, α) [3].

Definition 2.2. *Consider an equi-centro-affine framed curve $(\gamma, \mu, v) : I \to \mathbb{R}^2 \times SL(2, \mathbb{R})$ with curvature functions (m, n, α). According to [4], we define the equi-centro-affine pre-parallel curve $\mathcal{CAP}^\lambda : I \to \mathbb{R}^2$ of (γ, μ, v) by*

$$\mathcal{CAP}^\lambda(t) = \Gamma(t) + \lambda v(t), \tag{2.3}$$

where $\lambda \in \mathbb{R} - \{0\}$ and $\Gamma(t) = \int_{t_1}^{t} \gamma(u)du$. The equi-centro-affine pre-evolute $\varepsilon : I \to \mathbb{R}^2$ of (γ, μ, v) is defined as

$$\varepsilon(t) = \Gamma(t) - \frac{\alpha(t)}{n(t)} v(t), \; n(t) \neq 0, \tag{2.4}$$

and the equi-centro-affine pre-involute $\mathcal{AI}[t_o] : I \to \mathbb{R}^2$ at $t_0 \in I$ of (γ, μ, v) is defined as

$$\mathcal{AI}[t_0](t) = \Gamma(t) - \left(\int_{t_0}^{t} \alpha(u)du \right) \mu(t). \tag{2.5}$$

Definition 2.3. *A plane curve γ is said to have a $3/2$-cusp if it is locally equivalent to the normal form $\gamma(t) = (t^2, t^3)$. Similarly, γ is said to have a $4/3$-cusp if it is locally equivalent to $\gamma(t) = (t^3, t^4)$; see, for details, [15].*

3. Centro-affine duality of framed curves with singularities

In the centro-affine plane, Li and Pei mainly focused on position-dual and tangent-dual curves, while pre-tangent-dual and pre-normal-dual curves appeared only in comparative discussions with other curve types [4]. In contrast, the present section places pre-tangent-dual and pre-normal-dual curves at the core of the analysis. Using an adapted centro-affine frame, we investigate the singularity structures of these curves in detail, which constitutes a key extension of existing results. This focus is further motivated by the observation that, in affine geometry, pedal and contrapedal curves can be interpreted as counterparts of pre-tangent-dual and pre-normal-dual curves in centro-affine geometry.

Definition 3.1. *Assume that $(\gamma, \mu, v) : I \to \mathbb{R}^2 \times SL(2, \mathbb{R})$ is a centro-affine framed planar curve with the curvature (m, n, α) and $p \in \mathbb{R}^2$ is a fixed point. Consider the line through p that is parallel position vector. The intersection of this line with the centro-affine tangent line defines a point on the plane. Then, we can give the centro-affine pre-tangent-dual curve $\mathcal{PTD}_{\gamma,p} : I \to \mathbb{R}^2$ of γ by*

$$\mathcal{PTD}_{\gamma,p}(t) = \Gamma(t) + det(p - \Gamma(t), v(t))\mu(t), \tag{3.1}$$

where $\Gamma(t) = \int_{t_1}^{t} \gamma(u)du$.

Definition 3.2. *Consider the line through p that is parallel to the centro-affine tangent line. This line intersects the line determined by the position vector at a unique point. Let's define the locus of all such intersection points as the centro-affine pre-normal-dual curve $\mathcal{PND}_{\gamma,p} : I \to \mathbb{R}^2$ of centro-affine framed curve as*

$$\mathcal{PND}_{\gamma,p}(t) = \Gamma(t) - det(p - \Gamma(t), \mu(t))v(t), \tag{3.2}$$

where $\Gamma(t) = \int_{t_1}^{t} \gamma(u)du$.

Moreover, the centro-affine pre-tangent-dual curve $\mathcal{PTD}\gamma, p$ and the centro-affine pre-normal-dual curve $\mathcal{PND}\gamma, p$ are referred to as a centro-affine dual curve pair associated with it.

Proposition 3.3. *For a centro-affine regular curve $\gamma : I \to \mathbb{R}^2$ and a fixed point $p \in \mathbb{R}^2$, the centro-affine dual curve pair of singularity and regularity conditions coincide in centro-affine plane.*

Proof. For a centro-affine regular curve $\gamma : I \to \mathbb{R}^2$ and fixed point $p \in \mathbb{R}^2$, the choice

$$v(t) = t(t) = \frac{\gamma'(t)}{\det(\gamma(t), \gamma'(t))}$$

yields $\mu(t) = \gamma(t)$ for all $t \in I$. Using the Definition 4.2 of [4] and the Definition 3.1, we get

$$\begin{aligned} PTD_{\gamma,p}(t) &= \Gamma(t) + det(p - \Gamma(t), t(t))\gamma(t) \\ &= \Gamma(t) + det(p - \Gamma(t), v(t))\mu(t) \\ &= \mathcal{PTD}_{\gamma,p}(t). \end{aligned}$$

□

The proof for $\mathcal{PND}_{\gamma,p}$ is analogous.

Proposition 3.4. *Let $(\gamma, \mu, v) : I \to \mathbb{R}^2 \times SL(2, \mathbb{R})$ denote a centro-affine framed planar curve with centro-affine curvature (m, n, α) and let $p \in \mathbb{R}^2$ be a fixed point. Thus, $\mathcal{PTD}_{\gamma,p}$ does not depend on the choice of parametrization of (γ, μ, v).*

Proof. Assume that $(\gamma, \mu, v) : I \to \mathbb{R}^2 \times SL(2, \mathbb{R})$ and $(\tilde{\gamma}, \tilde{\mu}, \tilde{v}) : \tilde{I} \to \mathbb{R}^2 \times SL(2, \mathbb{R})$ represent the same centro-affine framed curve, related by a reparametrization $u : \tilde{I} \to I$. From the assumption, it follows that $(\tilde{\gamma}(t), \tilde{\mu}(t), \tilde{v}(t)) = (\gamma(u(t)), \mu(u(t)), v(u(t)))$. Therefore, we can express

$$\begin{aligned} \mathcal{PTD}_{\tilde{\gamma},p}(t) &= \tilde{\Gamma}(t) + det(p - \tilde{\Gamma}(t), \tilde{v}(t))\tilde{\mu}(t) \\ &= \Gamma(u(t)) - det(p - \Gamma(u(t)), v(u(t)), \mu(u(t)) \\ &= \mathcal{PTD}_{\tilde{\gamma},p}(u(t)). \end{aligned}$$

□

The proof for $\mathcal{PND}_{\gamma,p}$ is analogous.

Theorem 3.5. *Suppose that $(\gamma, \mu, v) : I \to \mathbb{R}^2 \times SL(2, \mathbb{R})$ is a centro-affine framed planar curve with centro-affine curvature (m, n, α) for a given $p \in \mathbb{R}^2$. This establishes that a point t_0 is a singular point of $\mathcal{PTD}_{\gamma,p}(t)$ if and only if p coincides with $\Gamma(t_0)$ or $m(t)$ and $n(t)$ simultaneously vanish at t_0. Here,* $\Gamma(t) = \int_{t_1}^{t} \gamma(u)du$.

Proof. Let $\mathcal{PTD}_{\gamma,p}$ be a pre-tangent-dual in the equi-centro-affine plane. Then, we can write

$$\mathcal{PTD}_{\gamma,p} = \Gamma(t) + det(p - \Gamma(t), \nu(t))\mu(t), \tag{3.3}$$

where $\Gamma(t) = \int_{t_1}^{t} \gamma(u)du$. If we differentiate (3.3), we get

$$\mathcal{PTD}'_{\gamma,p}(t) = \gamma(t) + det(-\gamma(t), \nu(t))\mu(t) + det(p - \Gamma(t), \nu'(t))\mu(t) + det(p - \Gamma(t), \nu(t))\mu'(t). \tag{3.4}$$

From (2.2), we know that

$$\begin{aligned} \mu'(t) &= l(t)\mu(t) + m(t)\nu(t), \\ \nu'(t) &= n(t)\mu(t) - l(t)\nu(t), \\ \gamma(t) &= \alpha(t)\mu(t). \end{aligned}$$

For $l = 0$, we have

$$\begin{aligned} \mu'(t) &= m(t)\nu(t), \\ \nu'(t) &= n(t)\mu(t), \\ \gamma(t) &= \alpha(t)\mu(t). \end{aligned}$$

Then, (3.4) turns to

$$\begin{aligned} \mathcal{PTD}'_{\gamma,p}(t) &= \alpha(t)\mu(t) - \alpha(t)\mu(t) + det(p - \Gamma(t), n(t)\mu(t))\mu(t) + det(p - \Gamma(t), \nu(t))m(t)\nu(t) && (3.5) \\ &= n(t)det(p - \Gamma(t), \mu(t))\mu(t) + m(t)det(p - \Gamma(t), \nu(t))\nu(t). && (3.6) \end{aligned}$$

$\mathcal{PTD}'_{\gamma,p}(t_0) = 0$ if and only if $p = \Gamma(t_0)$ or $m(t_0) = n(t_0) = 0$, where $\Gamma(t) = \int_{t_1}^{t} \gamma(u)du$. □

The proofs of the following Propositions 3.6 and 3.7 are obtained by straightforward calculations similar to those used in [16]. For the sake of brevity, the detailed proofs are omitted.

Proposition 3.6. *Suppose there exists a smooth function $h : I \to \mathbb{R}$ and a smooth mapping $\delta : I \to \mathbb{R}^2 \setminus \{0\}$ satisfying $p - \Gamma(t) = h(t)\delta(t)$. Then, the centro-affine pre-tangent-dual curve $\mathcal{PTD}_{\gamma,p}$ constitutes a centro-affine base framed curve for the singularity problem. Specifically, the triple $(\gamma, \bar{\mu}, \bar{\nu}) : I \to \mathbb{R}^2 \times SL(2, \mathbb{R})$ forms a centro-affine framed curve, characterized by centro-affine curvatures $\bar{l}$, $\bar{m}$, and $\bar{n}$, where*

$$\begin{aligned} \bar{\mu}(t) &= \frac{A(t)\mu(t) + B(t)\nu(t)}{g(t)}, \\ \bar{\nu}(t) &= \frac{-B(t)\mu(t) + A(t)\nu(t)}{g(t)}, \\ \bar{\alpha}(t) &= g(t) = \sqrt{A^2(t) + B^2(t)}, \\ \bar{l}(t) &= \frac{A(t)B(t)(m(t) + n(t))}{g(t)}, \\ \bar{m}(t) &= \frac{-B(t)A'(t) + A(t)B'(t) + A^2(t)m(t) - B^2(t)n(t)}{g(t)}, \\ \bar{n}(t) &= \frac{A'(t)B(t) - A(t)B'(t) + A^2(t)n(t) - B^2(t)m(t)}{g(t)}, \end{aligned}$$

where $A(t) = n(t)det(\delta(t), \mu(t))$ and $B(t) = m(t)det(\delta(t), \nu(t))$.

Proposition 3.7. *Suppose that* $(\gamma, \mu, \nu) : I \to \mathbb{R}^2 \times SL(2, \mathbb{R})$ *is a centro-affine framed planar curve with centro-affine curvature* (m, n, α). *If* t_0 *is the singular point of the pre-tangent-dual curve* $\mathcal{PTD}_{\gamma,p}$, *we have the following:*

(1) The singular point t_0 *is a centro-affine diffeomorphic* $3/2$*-cusp if and only if*

$$m(t_0)\alpha(t_0)\nu(t_0) \neq 0.$$

(2) The singular point t_0 *is a centro-affine diffeomorphic* $4/3$*-cusp if and only if one of the following is satisfied:*

$$(a)\ If\ \alpha(t_0) = 0, m(t_0)\alpha^{'}(t_0)\nu(t_0) \neq 0,$$
$$(b)\ If\ m(t_0) = 0, m(t_0)^{'}\alpha(t_0)\nu(t_0) \neq 0.$$

Theorem 3.8. *Suppose that* $(\gamma, \mu, \nu) : I \to \mathbb{R}^2 \times SL(2, \mathbb{R})$ *is a centro-affine framed planar curve with centro-affine curvature* (m, n, α) *for a given* $p \in \mathbb{R}^2$. *Thus, a point* t_0 *lies on the centro-affine pre-evolute if and only if it is a singular point of* $\mathcal{PND}_{\gamma,p}(t)$.

Proof. Let $\mathcal{PND}_{\gamma,p}$ be a pre-normal-dual curve in the equi-centro-affine plane. Then, we can write

$$\mathcal{PND}_{\gamma,p}(t) = \Gamma(t) - det(p - \Gamma(t), \mu(t))\nu(t). \tag{3.7}$$

Differentiating (3.7), we obtain

$$\begin{aligned}\mathcal{PND}^{'}_{\gamma,p}(t) &= \gamma(t) - det(-\gamma(t), \mu(t))\nu(t) - det(p - \Gamma(t), \mu^{'}(t))\nu(t) - det(p - \Gamma(t), \mu(t))\nu^{'}(t)\\ &= \gamma(t) + det(\alpha(t)\mu(t), \mu(t))\nu(t) - det(p - \Gamma(t), m(t)\nu(t))\nu(t) - det(p - \Gamma(t), \mu(t))n(t)\mu(t)\\ &= (\alpha(t) - n(t)det(p - \Gamma(t), \mu(t)))\mu(t) - m(t)det(p - \Gamma(t), \nu(t))\nu(t).\end{aligned}$$

Hence, $\mathcal{PND}$ has a singular point at $t = t_0$, if and only if

$$\alpha(t_0) - n(t_0)det(p - \Gamma(t_0), \mu(t_0)) = 0,$$
$$m(t_0)det(p - \Gamma(t_0), \nu(t_0)) = 0.$$

Since $m(t_0) \neq 0$ and $n(t_0) \neq 0$, we can write $\dfrac{\alpha(t_0)}{n(t_0)} = det(p - \Gamma(t_0), \mu(t_0))$ and $p - \Gamma(t_0) = \lambda\nu(t_0)$, where $\lambda \in \mathbb{R} - \{0\}$. Then, $p = \Gamma(t_0) - \dfrac{\alpha(t_0)}{n(t_0)}\nu(t_0) = \varepsilon(t_0)$. Therefore, t_0 is on the pre-evolute curve in the equi-centro-affine plane. □

Propositions 3.9 and 3.10 can be verified by direct computations using the definitions of the equi-centro-affine invariants, following the same line of reasoning as in [16]. For this reason, the detailed proofs are omitted.

Proposition 3.9. *Suppose that* $(\gamma, \mu, \nu) : I \to \mathbb{R}^2 \times SL(2, \mathbb{R})$ *is a centro-affine framed planar curve with centro-affine curvature* (m, n, α). *If* t_0 *is the singular point of the pre-normal-dual curve* $\mathcal{PND}_{\gamma,p}$, *we have the following:*

(1) The singular point t_0 is a centro-affine diffeomorphic $3/2$-cusp if and only if

$$m(t_0)\left(\frac{\alpha(t_0)}{n(t_0)}\right)' \neq 0.$$

(2) The singular point t_0 is a centro-affine diffeomorphic $4/3$-cusp if and only if

$$\left(\frac{\alpha(t_0)}{n(t_0)}\right)' = 0 \text{ and } \left(\frac{\alpha(t_0)}{n(t_0)}\right)'' \neq 0.$$

Proposition 3.10. *Assume there exists a smooth function $g : I \to \mathbb{R}$ and a smooth mapping $\xi : I \to \mathbb{R}^2 \setminus \{0\}$ such that $p - \Gamma(t) = k(t)\xi(t)$. Then, the centro-affine pre-tangent-dual curve $\mathcal{PND}_{\gamma,p}$ also forms a centro-affine base framed curve for the singularity problem. Specifically, the triple $(\gamma, \mu^*, \nu^*) : I \to \mathbb{R}^2 \times SL(2, \mathbb{R})$ constitutes a centro-affine framed curve, with associated centro-affine curvatures l^*, m^*, and n^*, where*

$$\begin{aligned}
\mu^*(t) &= \frac{A^*(t)\mu^*(t) + B^*(t)\nu^*(t)}{g^*(t)},\\
\nu^*(t) &= \frac{-B^*(t)\mu^*(t) + A^*(t)\nu^*(t)}{g^*(t)},\\
\alpha^*(t) &= g^*(t) = \sqrt{(A^*(t))^2 + (B^*(t))^2},\\
l^*(t) &= \frac{A^*(t)B^*(t)(m^*(t) + n^*(t))}{g^*(t)},\\
m^*(t) &= \frac{-B^*(t)(A^*)'(t) + A^*(t)(B^*)'(t) + (A^*)^2(t)m*(t) - (B^*)^2(t)n^*(t)}{g^*(t)},\\
n^*(t) &= \frac{(A^*)'(t)B(t) - A^*(t)(B^*)'(t) + (A^*)^2(t)n(t) - (A^*)^2(t)m^*(t)}{g^*(t)},
\end{aligned}$$

where

$$A^*(t) = \alpha^*(t) - n^*(t)det(\xi^*(t), \mu^*(t)), B^*(t) = -m^*(t)det(\xi^*(t), \nu^*(t)), g^*(t) = \sqrt{(A^*)^2(t) + (B^*)^2(t)}.$$

4. The relations between the pre-tangent dual curves and pre-normal-dual curves to other plane curves

Building on the singularity analysis in Section 3, this section compares pre-tangent-dual and pre-normal-dual curves with other related curve constructions.

Proposition 4.1. *Assume that $(\gamma, \mu, \nu) : I \to \mathbb{R}^2 \times SL(2, \mathbb{R})$ is a centro-affine framed planar curve with centro-affine curvature (m, n, α) for a given $p \in \mathbb{R}^2$. Then, the centro-affine pre-normal-dual curve and the centro affine pre-tangent-dual curve of the centro-affine evolute ε with respect to p are identical, that is, $\mathcal{PND}_{\gamma,p}(t) = \mathcal{PTD}_{\varepsilon,p}(t)$.*

Proof. Assume that $(\gamma, \mu, \nu) : I \to \mathbb{R}^2 \times SL(2, \mathbb{R})$ is a centro-affine framed planar curve with centro-affine curvature (m, n, α) for a given $p \in \mathbb{R}^2$. Then, we can write the centro-affine evolute curve as

$$\varepsilon(t) = \Gamma(t) - \frac{\alpha(t)}{n(t)}\nu(t),$$

where $\Gamma(t) = \int_{t_1}^{t} \gamma(u)$. The centro-affine evolute ε is expressed in terms of the integral curve $\Gamma(t)$. Therefore, although the pre-normal-dual curve is defined via the integral curve Γ, the pre-tangent-dual construction can be applied directly to ε. Using (3.1), the centro-affine pre-tangent-dual curve of ε is given by $\mathcal{PTD}_{\varepsilon,p}(t) = \varepsilon(t) + det(p - \varepsilon(t), \bar{\nu}(t))\bar{\mu}(t)$ where $\bar{\nu}(t) = -\mu(t)$ and $\bar{\mu}(t) = \nu(t)$. Using this substitution, we get

$$\begin{aligned}\mathcal{PTD}_{\varepsilon,p}(t) &= \Gamma(t) - \frac{\alpha(t)}{n(t)}\nu(t) - det(p - \Gamma(t) + \frac{\alpha(t)}{n(t)}\nu(t), \mu(t))\nu(t) \\ &= \Gamma(t) - \frac{\alpha(t)}{n(t)}\nu(t) + \frac{\alpha(t)}{n(t)}\nu(t) - det(p - \Gamma(t), \mu(t))\nu(t) \\ &= \Gamma(t) - det(p - \Gamma(t), \mu(t))\nu(t) \\ &= \mathcal{PND}_{\gamma,p}(t).\end{aligned}$$

□

Proposition 4.2. *Suppose that $(\gamma, \mu, \nu) : I \to \mathbb{R}^2 \times SL(2, \mathbb{R})$ is a centro-affine framed planar curve with centro-affine curvature (m, n, α) for a given $p \in \mathbb{R}^2$. Then, the centro-affine pre-tangent-dual curve and the centro affine pre-normal-dual curve of the centro-affine pre-involute $\mathcal{AI}$ with respect to p are identical, that is, $\mathcal{PTD}_{\gamma,p}(t) = \mathcal{PND}_{\mathcal{AI},p}(t)$.*

Proof. Assume that $(\gamma, \mu, \nu) : I \to \mathbb{R}^2 \times SL(2, \mathbb{R})$ is a centro-affine framed planar curve with centro-affine curvature (m, n, α) for a given $p \in \mathbb{R}^2$. From (2.5), the centro-affine pre-involute curve is given as

$$AI[t_0](t) = \Gamma(t) - \left(\int_{t_0}^{t} \alpha(u)du\right)\mu(t).$$

Using (3.2), the centro-affine pre-normal-dual curve of pre-involute is written as

$$\mathcal{PND}_{\mathcal{AI}[t_0],p}(t) = \mathcal{AI}[t_0] - det(p - \mathcal{AI}[t_0], \bar{\mu}(t))\bar{\nu}(t)$$

where $\bar{\mu}(t) = -\nu(t)$ and $\bar{\nu}(t) = \mu(t)$. Using this substitution, we obtain

$$\begin{aligned}\mathcal{PND}_{\mathcal{AI}[t_0],p}(t) &= \Gamma(t) - \left(\int_{t_0}^{t} \alpha(u)du\right)\mu(t) + det(p - \Gamma(t) + \left(\int_{t_0}^{t} \alpha(u)du\right)\mu(t), \nu(t))\mu(t) \\ &= \Gamma(t) - \left(\int_{t_0}^{t} \alpha(u)du\right)\mu(t) + \left(\int_{t_0}^{t} \alpha(u)du\right)\mu(t) + det(p - \Gamma(t), \nu(t))\mu(t) \\ &= \mathcal{PTD}_{\gamma,p}(t).\end{aligned}$$

Consequently, $\mathcal{PTD}_{\gamma,p}(t) = \mathcal{PND}_{\mathcal{AI}[t_0],p}(t)$. □

Corollary 4.3. *Assume that $(\gamma, \mu, \nu) : I \to \mathbb{R}^2 \times SL(2, \mathbb{R})$ is a centro-affine framed planar curve with centro-affine curvature (m, n, α) for a given $p \in \mathbb{R}^2$. Then, the following are satisfied:*

(1) $\mathcal{PND}_{\varepsilon,p}(t) = \mathcal{PND}_{\mathcal{AI}[t_0],p}(t) - \dfrac{\alpha(t)}{n(t)}\nu(t)$.

(2) $\mathcal{PTD}_{\mathcal{AI}[t_0],p}(t) = \mathcal{PTD}_{\varepsilon,p}(t) - \left(\int_{t_0}^{t} \alpha(u)du\right)\mu(t)$.

Proof. (1) Assume that $(\gamma, \mu, \nu) : I \to \mathbb{R}^2 \times SL(2, \mathbb{R})$ is a centro-affine framed planar curve with centro-affine curvature (m, n, α) for a given $p \in \mathbb{R}^2$. From (2.4), we can write the centro-affine evolute curve as

$$\varepsilon(t) = \Gamma(t) - \frac{\alpha(t)}{n(t)}\nu(t).$$

Using (3.2), the centro-affine pre-normal-dual curve of ε is given as

$$\mathcal{PND}_{\varepsilon,p}(t) = \varepsilon(t) - det(p - \varepsilon(t), \bar{\mu}(t))\bar{\nu}(t),$$

where $\bar{\mu}(t) = -\nu(t)$ and $\bar{\nu}(t) = \mu(t)$. Using this substitution, we have

$$\begin{aligned}\mathcal{PND}_{\varepsilon,p}(t) &= \Gamma(t) - \frac{\alpha(t)}{n(t)}\nu(t) - det(p - \Gamma(t) + \frac{\alpha(t)}{n(t)}\nu(t), -\nu(t))\mu(t)\\ &= \Gamma(t) - \frac{\alpha(t)}{n(t)}\nu(t) + det(p - \Gamma(t), \nu(t))\mu(t)\\ &= \mathcal{PTD}_{\gamma,p}(t) - \frac{\alpha(t)}{n(t)}\nu(t).\end{aligned}$$

From Proposition 4.2, we get

$$\mathcal{PND}_{\varepsilon,p}(t) = \mathcal{PND}_{\mathcal{AI}[t_0],p}(t) - \frac{\alpha(t)}{n(t)}\nu(t).$$

(2) Suppose that $(\gamma, \mu, \nu) : I \to \mathbb{R}^2 \times SL(2, \mathbb{R})$ is a centro-affine framed planar curve with centro-affine curvature (m, n, α) for a given $p \in \mathbb{R}^2$. From (2.5), the centro-affine pre-involute curve is given as

$$\mathcal{AI}[t_0](t) = \Gamma(t) - \left(\int_{t_0}^{t} \alpha(u)du\right)\mu(t).$$

Using (3.1), the centro-affine pre-tangent-dual curve of pre-involute is written by

$$\mathcal{PTD}_{\mathcal{AI}[t_0],p}(t) = \mathcal{AI}[t_0] + det(p - \mathcal{AI}[t_0], \bar{\nu}(t))\bar{\mu}(t),$$

where $\bar{\mu}(t) = \nu(t)$ and $\bar{\nu}(t) = -\mu(t)$. Using this substitution, we can write

$$\begin{aligned}\mathcal{PTD}_{\mathcal{AI}[t_0],p}(t) &= \Gamma(t) - \left(\int_{t_0}^{t} \alpha(u)du\right)\mu(t) - det(p - \Gamma(t) + \left(\int_{t_0}^{t} \alpha(u)du\right)\mu(t), \mu(t))\nu(t)\\ &= \Gamma(t) - det(p - \Gamma(t), \mu(t))\nu(t) - \left(\int_{t_0}^{t} \alpha(u)du\right)\mu(t)\\ &= \mathcal{PND}_{\gamma,p}(t) - \left(\int_{t_0}^{t} \alpha(u)du\right)\mu(t).\end{aligned}$$

From Proposition 4.1, we have

$$\mathcal{PTD}_{\mathcal{AI}[t_0],p}(t) = \mathcal{PTD}_{\varepsilon,p}(t) - \left(\int_{t_0}^{t} \alpha(u)du\right)\mu(t).$$

□

Proposition 4.4. *Let $(\gamma, \mu, \nu) : I \to \mathbb{R}^2 \times SL(2, \mathbb{R})$ be a centro-affine framed planar curve. Suppose the fixed point p belongs to the pre-parallel curve of γ, specifically $p = \mathcal{CAP}^{\lambda}(t_0)$. Then,*

(1) At $t = t_0$, the pre-tangent-dual curve $\mathcal{PTD}_{\gamma,p}$ coincides with Γ:

$$\mathcal{PTD}_{\gamma,p}(t_0) = \Gamma(t_0).$$

(2) At $t = t_0$, the pre-normal-dual curve $\mathcal{PND}_{\gamma,p}$ coincides with the pre-parallel curve of γ:

$$\mathcal{PND}_{\gamma,p}(t_0) = \mathcal{CAP}^{\lambda}(t_0).$$

Proof. (1) From (2.3), the equi-centro-affine pre-parallel curve is given as

$$\mathcal{CAP}^{\lambda}(t) = \Gamma(t) + \lambda\nu(t).$$

If $p = \mathcal{CAP}^{\lambda}(t_0)$, the equi-centro-affine pre-tangent-dual curve is written by

$$\begin{aligned}\mathcal{PTD}_{\gamma,p}(t_0) &= \Gamma(t_0) + det(p - \Gamma(t_0), \nu(t_0))\mu(t_0)\\ &= \Gamma(t_0) + det(\lambda\nu(t_0), \nu(t_0))\mu(t_0)\\ &= \Gamma(t_0).\end{aligned}$$

(2) Using (2.4), we can write the equi-centro-affine pre-normal-dual curve is given by

$$\begin{aligned}\mathcal{PND}_{\gamma,p}(t_0) &= \Gamma(t_0) - det(p - \Gamma(t_0), \mu(t_0))\nu(t_0))\\ &= \Gamma(t_0) - det(\lambda\nu(t_0), \mu(t_0))\nu(t_0)\\ &= \Gamma(t_0) + \lambda\nu(t_0)\\ &= \mathcal{CAP}(t_0).\end{aligned}$$

Therefore, $\mathcal{PND}_{\gamma,p}(t_0) = \mathcal{CAP}^{\lambda}(t_0)$. □

The following proposition can also be given with similar logic to the proofs above:

Proposition 4.5. *Suppose that $(\gamma, \mu, \nu) : I \to \mathbb{R}^2 \times SL(2, \mathbb{R})$ is a centro-affine framed planar curve with centro-affine curvature (m, n, α). Suppose $p = \Gamma(t_0)$ for some $t_0 \in I$, meaning p lies on the Γ. Then, we have the following result: $\mathcal{PTD}_{\gamma,p}(t_0) = \mathcal{PND}_{\gamma,p}(t_0)$.*

5. Applications

Example 5.1. *Let $\gamma(t) = (4t^3, 5t^4)$ be an equi-centro-affine plane curve. Then, γ has a centro-affine 4/3-cusp. Consider the singular point at $p = (0, 0)$ with $\gamma'(t) = (12t^2, 20t^3)$. Since* $\det(\gamma(t), \mu(t)) = 0$, *we may choose $\mu(t) = (4, 5t) \neq (0, 0)$ for all $t \in \mathbb{R}$. For $t_1 = 0$, $\Gamma(t) = (t^4, t^5)$. Furthermore, we have $det(\mu(t), \nu(t)) = 1$, then we can choose $\nu(t) = \left(t, \dfrac{1 + 5t^2}{4}\right)$. Consequently, the triple $(\gamma, \mu, \nu) : I \to \mathbb{R}^2 \times SL(2, \mathbb{R})$ forms a centro-affine framed curve under the equi-centro transform for this singularity problem. Also, we get*

$$l(t) = -5t,\ m(t) = 20,\ n(t) = \frac{1 - 5t^2}{4},\ \alpha(t) = t^3.$$

$p = (0, 0) = \Gamma(0)$ is a singular point. Then, the centro-affine dual curve pair of $\gamma(t)$ are given by

$$\begin{aligned}
\mathcal{PTD}_{\gamma,p}(t) &= \Gamma(t) + det(p - \Gamma(t), \nu(t))\mu(t) \\
&= (-t^6, -t^5 - \frac{5}{4}t^7) \\
\mathcal{PND}_{\gamma,p}(t) &= \Gamma(t) - det(p - \Gamma(t), \mu(t))\nu(t) \\
&= (t^4 + t^6, t^5 + \frac{t^5 + 5t^7}{4}).
\end{aligned}$$

For the curve $\mathcal{PTD}_{\gamma,p}(t)$, we may take $h(t) = -t^4$ and $\delta(t) = (1, t) \neq (0, 0)$, which satisfy $p - \Gamma(t) = h(t)\delta(t)$ for all $t \in I$, with $h(t)$ being a smooth function and $\delta(t) \neq 0$. By Proposition 3.6, the triple $(\gamma, \bar{\mu}, \bar{\nu}) : I \to \mathbb{R}^2 \times SL(2, \mathbb{R})$ forms a centro-affine framed curve, where

$$\begin{aligned}
\bar{\mu}(t) &= \left(\frac{24t}{\sqrt{25t^6 - 390t^4 + 801t^2 + 400}}, \frac{35t^2 + 5}{\sqrt{25t^6 - 390t^4 + 801t^2 + 400}} \right), \\
\bar{\nu}(t) &= \left(\frac{-80 - 79t^2 - 5t^4}{\sqrt{25t^6 - 390t^4 + 801t^2 + 400}}, \frac{-100t - 100t^3 + \frac{t - 25t^5}{4}}{\sqrt{25t^6 - 390t^4 + 801t^2 + 400}} \right), \\
\bar{\alpha}(t) &= s(t) = \frac{1}{4}\sqrt{25t^6 - 390t^4 + 801t^2 + 400}.
\end{aligned}$$

However, for the centro-affine pre-normal-dual curve, there exists no smooth function $k(t)$ and no mapping $\xi : I \to \mathbb{R}^2$ such that

$$p - \varepsilon(t) = k(t)\xi(t) = \left(\frac{-3t^4 - 5t^6}{1 - 5t^2}, \frac{-4t^5 - t^3}{1 - 5t^2} \right).$$

Hence, $\mathcal{PND}_{\gamma,p}(t)$ is not a centro-affine framed curve, and the corresponding dual curve pair is shown in Figure 1.

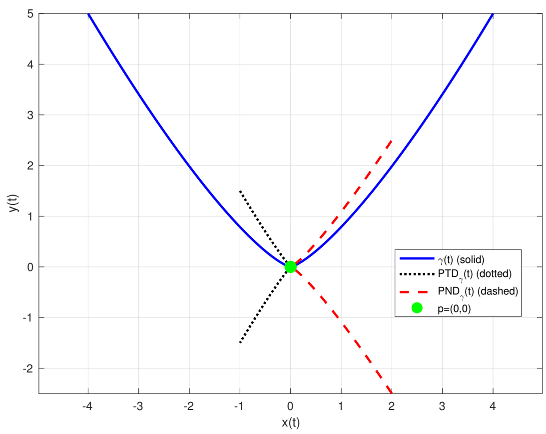

Figure 1. The curves $\gamma(t)$ (blue solid line), $\mathcal{PTD}_{\gamma,p}(t)$ (black dotted line), and $\mathcal{PND}_{\gamma,p}(t)$ (red dashed line) plotted for t around 0.

Example 5.2. *Let* $\gamma(t) = (\sin t \cos t, \cos^2 t)$ *be a equi-centro-affine framed curve. Then, we can find the*

$$\Gamma(t) = \left(\frac{\sin^2 t}{2}, \frac{t}{2} + \frac{\sin 2t}{4}\right)(t_1 = 0)$$

and $\gamma^{'}(t) = (\cos 2t, -\sin 2t)$. *Thus, we can choose* $\mu(t) = (\sin t, \cos t) \neq (0, 0)$. *Since det*$(\mu(t), \nu(t)) = 1$, *we can take* $\nu(t) = (-\cos t, \sin t)$. *The equi-centro-affine framed curvature is given as*

$$m(t) = -1, \ n(t) = 1, \ \alpha(t) = \cos t.$$

The equi-centro-affine pre-parallel curve $\mathcal{CAP}^{\lambda}$ *of* (γ, μ, ν) *is given as*

$$\begin{aligned} \mathcal{CAP}^{\lambda}(t) &= \Gamma(t) + \lambda \nu(t) \\ &= \left(\frac{\sin^2 t}{2} - \lambda \cos t, \frac{t}{2} + \frac{\sin 2t}{4} + \lambda \sin t\right). \end{aligned}$$

For $\lambda = 1$, $\mathcal{CAP}^{\lambda}(t)$ *has* $4/3$*-cusp at* $t = \pi$. *The equi-centro-affine pre-evolute* ε *of* (γ, μ, ν) *is given as*

$$\begin{aligned} \varepsilon(t) &= \Gamma(t) - \frac{\alpha(t)}{n(t)} \nu(t) \\ &= \left(\frac{\sin^2 t}{2} + \cos^2 t, \frac{t}{2} + \frac{\sin 2t}{4} - \cos t \sin t\right). \end{aligned}$$

Also, $\varepsilon(t)$ *has* $3/2$*-cusp at* $t = \pi$. *The equi-centro-affine pre-involute* $\mathcal{AI}[t_o]$ *of* (γ, μ, ν) *is given as*

$$\begin{aligned} \mathcal{AI}[t_0](t) &= \Gamma(t) - \left(\int_{t_0}^{t} \alpha(u) du\right) \mu(t) \\ &= \left(\frac{\sin^2 t}{2} - \sin^2 t + \sin t_0 \sin t, \frac{t}{2} + \frac{\sin 2t}{4} - \sin t \cos t + \sin t_0 \cos t\right). \end{aligned}$$

Additionally, $\mathcal{AI}[\pi]$ *has* $2/3$*-cusp. The above curves can be seen in Figure 2.*

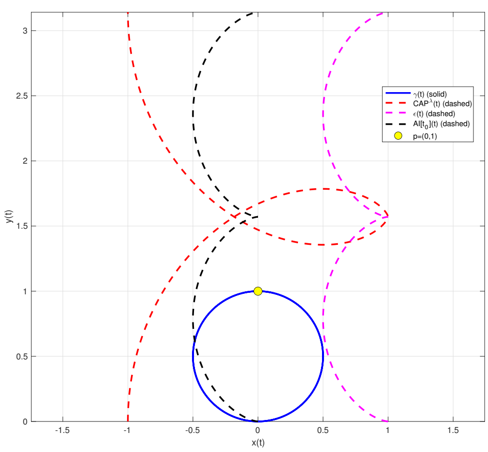

Figure 2. The curves $\gamma(t)$ (blue solid line), $\mathcal{CAP}^{\lambda}(t)$ (red dashed line), $\varepsilon(t)$ (pink dashed line), and $\mathcal{AI}(t)$ (blacked dashed line) plotted around $p = (0, 1)$.

Example 5.3. *Let* $\gamma(t) = (-\cos t, -\sin t)$ *be a equi-centro-affine space curve. Then,* $\Gamma(t) = (-\sin t, \cos t)$ *for* $t_1 = 0$. *Since* $\det(\gamma(t), \mu(t)) = 0$, *we can choose* $\mu(t) = (-\cos t, -\sin t)$. *Additionally,* $\det(\mu(t), \nu(t)) = 1$, *then we may choose* $\nu(t) = (\sin t, -\cos t)$. *The equi-centro-affine framed curvatures are given by*

$$m(t) = 1,\ n(t) = -1,\ \alpha(t) = 1.$$

Then, the equi-centro-affine dual curve pair are

$$\begin{aligned}\mathcal{PTD}_{\gamma,p}(t) &= \Gamma(t) + \det(p - \Gamma(t), \nu(t))\mu(t)\\ &= (-\sin t - \cos^2 t, \cos t - \cos t \sin t)\\ \mathcal{PND}_{\gamma,p}(t) &= \Gamma(t) - \det(p - \Gamma(t), \mu(t))\nu(t)\\ &= (-\sin^2 t, \sin t \cos t).\end{aligned}$$

The curve γ *is a regular curve but* $\mathcal{PTD}_{\gamma,p}(t)$ *has a singular point at* $p = (-1, 0)$. *See Figure 3.*

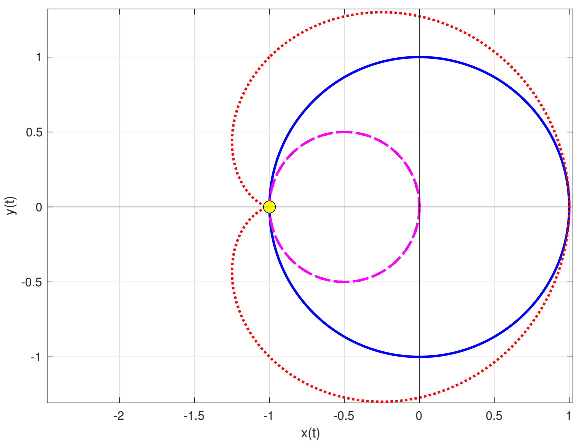

Figure 3. The curves $\gamma(t)$ (blue solid line), $\mathcal{PTD}_{\gamma,p}(t)$ (red dotted line), and $\mathcal{PND}_{\gamma,p}(t)$ (pink dashed line) plotted around $p = (-1, 0)$.

If we take

$$p = \left(-\frac{1}{2}, 0\right),$$

the equi-centro-affine dual curve pair are given as

$$\begin{aligned}\mathcal{PTD}_{\gamma,p}(t) &= \Gamma(t) + \det(p - \Gamma(t), \nu(t))\mu(t)\\ &= (-\sin t - \frac{1}{2}\cos^2 t, \cos t - \frac{1}{2}\cos t \sin t)\\ \mathcal{PND}_{\gamma,p}(t) &= \Gamma(t) - \det(p - \Gamma(t), \mu(t))\nu(t)\\ &= (-\frac{1}{2}\sin^2 t, \frac{1}{2}\sin t \cos t).\end{aligned}$$

There is no singular point on the curves γ, $\mathcal{PTD}$, *and* $\mathcal{PND}$. *See Figure 4.*

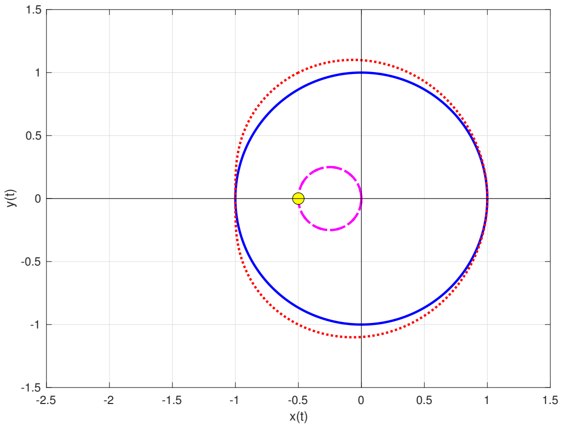

Figure 4. The curves $\gamma(t)$ (blue solid line), $\mathcal{PTD}_{\gamma,p}(t)$ (red dotted line), and $\mathcal{PND}_{\gamma,p}(t)$ (pink dashed line) plotted around $p = (-\frac{1}{2}, 0)$.

6. Conclusions

This study examines the geometric structures of $\mathcal{PTD}$ and $\mathcal{PND}$ in an equi-centro-affine frame and defines a new dual curve pair for these curves. The regularity and singularity properties of the resulting dual structures are investigated, clearly demonstrating the conditions under which they are well-defined and the situations in which they produce singularities. These analyses provide important insights into how the concept of duality can be expanded in equi-centro-affine geometry.

In comparison with existing studies, the methodological approach adopted in this paper differs in a fundamental way. While [16] introduces affine dual curve pairs through pedal and contrapedal constructions, the centro-affine setting naturally gives rise to the pre-tangent-dual and pre-normal-dual curves, which can be regarded as counterparts of pedal and contrapedal curves, respectively. From this perspective, the present study is motivated by transferring and extending the idea of affine pedal duality to the equi-centro-affine setting, where the duality is constructed intrinsically via pre-tangent and pre-normal structures rather than classical pedal-type transformations. Furthermore, the study compares $\mathcal{PTD}$ and $\mathcal{PND}$ with other related curve types, revealing their equivalence within a broader geometric context.

Overall, this study makes a significant contribution to dual curve theory in equi-centro-affine geometry, providing a strong foundation for a deeper understanding of the mathematical structure of $\mathcal{PTD}$ and $\mathcal{PND}$. It is anticipated that the results obtained will shed light on future research aimed at developing the concept of duality in different geometric frame.

Use of Generative-AI tools declaration

The author declares that no artificial intelligence (AI) tools were used in the preparation of this article.

Conflict of interest

The author declares no conflict of interest.

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