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Research article

On Scherk-type minimal immersion in $\mathbb{R}^4$ constructed by the generalized Weierstrass–Enneper representation

Magdalena Toda and Erhan Güler*

Department of Mathematics, Texas Tech University, Lubbock, TX 79409, USA

* **Correspondence:** Email: eguler@ttu.edu.

Abstract: We introduce and study a class of Scherk-type minimal surfaces immersed in the four-dimensional Euclidean space $\mathbb{R}^4$. Motivated by the classical Scherk minimal surface in $\mathbb{R}^3$, we construct higher-codimensional analogues using the generalized Weierstrass–Enneper representation for minimal surfaces in $\mathbb{R}^4$. Explicit parametrizations are derived from holomorphic null curves in $\mathbb{C}^4$, ensuring conformality and vanishing mean curvature. The geometric properties of the resulting surface are examined through real parametrizations, orthogonal projections, and the explicit construction of an orthonormal frame for the normal bundle. Representative special cases are presented to illustrate how the geometric structure characteristic of a Scherk surface extends naturally to higher codimension.

Keywords: Scherk surface; minimal surface; $\mathbb{R}^4$; generalized Weierstrass–Enneper representation; holomorphic null curve
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1. Introduction

A minimal surface in $\mathbb{R}^3$ is a regular surface whose mean curvature vanishes identically. The study of minimal surfaces dates back to the nineteenth century and has since evolved into a central area of differential geometry, with deep connections to complex analysis and the calculus of variations. Classical foundations and geometric aspects of the theory can be found in standard references such as the monographs by Nitsche [7], Abbena et al. [1], and Dierkes et al. [2].

One of the earliest and most influential contributions to minimal surface theory was given by Scherk in 1835 [8]. Scherk constructed explicit examples of minimal surfaces, now known as Scherk surfaces, which occupy a central position in the classical theory and have served as fundamental models in the development of the subject.

A major analytical advancement in the theory was achieved through the Weierstrass–Enneper representation, which expresses minimal surfaces in $\mathbb{R}^3$ in terms of holomorphic data. This representation provides a powerful framework for the construction and classification of minimal surfaces and has inspired numerous generalizations to higher-dimensional Euclidean spaces.

In this direction, Hoffman and Osserman developed the theory of the generalized Gauss map for surfaces immersed in $\mathbb{R}^n$, establishing fundamental geometric and analytic results for higher-codimensional minimal surfaces [3,4]. Comprehensive treatments of both classical and modern aspects of minimal surface theory can be found in the surveys and monographs by Meeks and Pérez [5, 6].

More recently, Toda and Güler introduced a generalized Weierstrass–Enneper representation for minimal surfaces in $\mathbb{R}^4$ [9], providing a natural extension of classical methods to higher codimension. In the present work, Scherk-type minimal surfaces are studied within this framework. In particular, classical Scherk minimal surfaces are extended to the four-dimensional Euclidean space $\mathbb{R}^4$, and their analytic representations and geometric properties are investigated using holomorphic null curves and explicit parametrizations.

2. Scherk's minimal surface in $\mathbb{R}^3$ and its Weierstrass data

Scherk's minimal surface is one of the earliest explicit examples of a minimal surface in $\mathbb{R}^3$, introduced in 1835 [8]. Among Scherk's constructions, the singly periodic surface - commonly referred to as Scherk's second surface - occupies a central position in the classical theory of minimal surfaces.

Geometrically, Scherk's second surface can be interpreted as two orthogonal planes connected by an infinite sequence of saddle-shaped tunnels. Analytically, it admits the implicit representation

$$\sin z - \sinh x \sinh y = 0,$$

which reflects its periodic structure and characteristic saddle geometry.

An effective analytic description of Scherk's surface is provided by the classical Weierstrass–Enneper representation. In its (f, g)–form, a conformal minimal immersion $X : \Omega \to \mathbb{R}^3$ is given by

$$X(\zeta, \bar{\zeta}) = \Re \int^{\zeta} \left(\frac{1}{2} f(1 - g^2),\ \frac{i}{2} f(1 + g^2),\ f\, g \right) d\zeta,$$

where f is holomorphic and g is meromorphic on a simply connected domain $\Omega \subset \mathbb{C}$.

Proposition 2.1 (Weierstrass data for Scherk's minimal surface)**.** *Let $\Omega \subset \mathbb{C}$ be a simply connected domain and define*

$$f(\zeta) = \frac{4}{1 - \zeta^4}, \qquad g(\zeta) = i\, \zeta. \tag{2.1}$$

Then the map

$$X(\zeta, \bar{\zeta}) = \Re \int^{\zeta} \left(\frac{1}{2} f(1 - g^2),\ \frac{i}{2} f(1 + g^2),\ f\, g \right) d\zeta \tag{2.2}$$

defines a conformal minimal immersion whose image is Scherk's singly periodic minimal surface in $\mathbb{R}^3$.

The Weierstrass data (2.1) generate one fundamental domain of the surface. By exploiting the symmetries of the immersion, this domain can be extended periodically to obtain the complete Scherk surface. This formulation highlights the deep connection between minimal surface theory and complex analysis and provides a natural starting point for extending Scherk-type constructions to higher-dimensional Euclidean spaces.

3. Generalized Weierstrass–Enneper representation in $\mathbb{R}^4$

The Weierstrass–Enneper representation is a central tool in the theory of minimal surfaces, revealing a profound relationship between complex analysis and differential geometry. In the classical setting of $\mathbb{R}^3$, this representation characterizes conformal minimal immersions in terms of holomorphic data and enables explicit constructions of many fundamental examples.

When passing to higher-dimensional Euclidean spaces, and in particular to $\mathbb{R}^4$, minimal surfaces exhibit additional geometric features due to the presence of a two-dimensional normal bundle. This increase in codimension requires a suitable extension of the classical Weierstrass theory. Such an extension is provided by the generalized Weierstrass–Enneper representation, which describes conformal minimal immersions in $\mathbb{R}^4$ in terms of holomorphic null curves in $\mathbb{C}^4$. This formulation, developed by Toda and Güler [9], naturally extends the classical theory from $\mathbb{R}^3$ to higher codimension and constitutes the main analytical framework used throughout this paper.

Theorem 3.1 (Generalized Weierstrass–Enneper representation in $\mathbb{R}^4$ [9])**.** *Let $D \subset \mathbb{C}$ be a domain and let f, g, and h be holomorphic functions on D. Define a map $X\colon D \to \mathbb{R}^4$ by*

$$X(\omega, \bar{\omega}) = \operatorname{Re} \int \left(\frac{1}{2} f(1 - g^2 - h^2), \frac{i}{2} f(1 + g^2 + h^2), fg, fh \right) d\omega.$$

Then X defines a conformal minimal immersion of D into $\mathbb{R}^4$.

Remark 3.2. *For later use we recall a standard convention: A holomorphic curve $\Psi : D \to \mathbb{C}^4$ is called a holomorphic null curve if its derivative $\Psi'(\omega)$ is null with respect to the complex bilinear extension of the Euclidean inner product, i.e., $\langle \Psi'(\omega), \Psi'(\omega) \rangle = 0$ for all $\omega \in D$. Equivalently, one may speak of a holomorphic null-curve differential when a holomorphic $\mathbb{C}^4$-valued 1-form $\Phi(\omega)\, d\omega$ satisfies $\langle \Phi(\omega), \Phi(\omega) \rangle = 0$. This terminology will be used explicitly in the Scherk-type verification below to avoid any possible misunderstanding.*

4. Scherk-type minimal surface in $\mathbb{R}^4$

Motivated by the classical theory of minimal surfaces in $\mathbb{R}^3$, we employ the generalized Weierstrass–Enneper representation to construct a class of minimal surfaces in $\mathbb{R}^4$ that naturally extends the classical Scherk surface. The guiding principle is simple: We retain the holomorphic data (f, g) that generate Scherk's minimal surface in $\mathbb{R}^3$ and introduce an additional holomorphic function h encoding the extra normal direction available in codimension two. In this way, the resulting immersion inherits the classical Scherk features while allowing genuinely new behavior in $\mathbb{R}^4$.

The minimal surface obtained in this manner will be referred to as a Scherk-type minimal surface in $\mathbb{R}^4$. This surface includes, as special cases, the classical Scherk surface contained in a three-dimensional affine subspace of $\mathbb{R}^4$, as well as genuinely four-dimensional examples that exhibit geometric phenomena absent from the classical three-dimensional theory. The terminology "Scherk type" is used here in a constructive sense to emphasize the analogy with Scherk's classical minimal surface in $\mathbb{R}^3$, without implying any classification or uniqueness statement.

4.1. Weierstrass data of the Scherk-type family in $\mathbb{R}^4$

We work with the generalized Weierstrass–Enneper representation in $\mathbb{R}^4$. Let $D \subset \mathbb{C}$ be a simply connected domain and let f, g, and h be holomorphic functions on D. Then the map

$$X(\omega, \bar{\omega}) = \Re \int_{\omega_0}^{\omega} \left(\frac{1}{2} f(1 - g^2 - h^2),\ \frac{i}{2} f(1 + g^2 + h^2),\ fg,\ fh \right) d\zeta \tag{4.1}$$

defines a conformal minimal immersion $X : D \to \mathbb{R}^4$.

Motivated by the classical Weierstrass data of Scherk's singly periodic minimal surface in $\mathbb{R}^3$, we introduce the following codimension-two extension.

Definition 4.1. *Let $\mu, \lambda \in \mathbb{C}$ be fixed parameters. The Scherk-type Weierstrass data in $\mathbb{R}^4$ are defined by*

$$f(\omega) = \frac{4\mu}{1 - \omega^4}, \qquad g(\omega) = i\omega, \qquad h(\omega) = \lambda g(\omega) = i\lambda\, \omega, \tag{4.2}$$

on a simply connected domain $D \subset \mathbb{C} \setminus \{\pm 1, \pm i\}$.

Here, f coincides with the classical Scherk factor up to the complex scaling parameter μ, while the additional term h introduces a second Gauss map component corresponding to the extra normal direction in $\mathbb{R}^4$. The parameter λ measures the interaction between the two Gauss map components (equivalently, between the last two components of the holomorphic lift), and it is precisely this parameter that controls whether the resulting surface is effectively three-dimensional or genuinely codimension two.

Theorem 4.2 (Scherk-type holomorphic null-curve differential in $\mathbb{C}^4$)**.** *Let $\mu, \lambda \in \mathbb{C}$ and define the holomorphic functions*

$$f(\omega) = \frac{4\mu}{1 - \omega^4}, \qquad g(\omega) = i\,\omega, \qquad h(\omega) = \lambda\, g(\omega) = i\lambda\, \omega,$$

on a simply connected domain $D \subset \mathbb{C} \setminus \{\pm 1, \pm i\}$. Define the $\mathbb{C}^4$-valued holomorphic map

$$\Phi(\omega) = \left(\frac{1}{2} f(1 - g^2 - h^2),\ \frac{i}{2} f(1 + g^2 + h^2),\ fg,\ fh \right).$$

Then $\Phi(\omega)\, d\omega$ is a holomorphic null-curve differential on D, i.e.,

$$\langle \Phi(\omega), \Phi(\omega) \rangle = 0 \quad \textit{for all } \omega \in D.$$

Consequently, any holomorphic primitive $F(\omega) = \int^{\omega} \Phi(\zeta)\, d\zeta$ defines a holomorphic curve whose derivative is null (hence a holomorphic null curve in the standard sense).

Proof. Let

$$f(\omega) = \frac{4\mu}{1 - \omega^4}, \qquad g(\omega) = i\,\omega, \qquad h(\omega) = \lambda g(\omega) = i\lambda\, \omega,$$

where $\mu, \lambda \in \mathbb{C}$, and let $D \subset \mathbb{C} \setminus \{\pm 1, \pm i\}$ be a simply connected domain. The functions f, g, and h are holomorphic on D. Hence, $\Phi(\omega)$ is holomorphic on D.

Since $g(\omega) = i\omega$, we have

$$g^2(\omega) = -\omega^2, \qquad h^2(\omega) = \lambda^2 g^2(\omega) = -\lambda^2\omega^2.$$

Therefore,

$$g^2 + h^2 = -(1+\lambda^2)\omega^2,$$

which implies

$$1 - g^2 - h^2 = 1 + (1+\lambda^2)\omega^2, \qquad 1 + g^2 + h^2 = 1 - (1+\lambda^2)\omega^2.$$

Substituting these expressions into $\Phi(\omega)$, we obtain

$$\Phi(\omega) = \left(\frac{2\mu(1 + (1+\lambda^2)\omega^2)}{1-\omega^4}, \ \frac{2i\mu(1 - (1+\lambda^2)\omega^2)}{1-\omega^4}, \ \frac{4i\mu\,\omega}{1-\omega^4}, \ \frac{4i\mu\lambda\,\omega}{1-\omega^4} \right).$$

Remark 4.3. *In some parts of the literature, the symbol Φ is reserved for a curve and Φ' for its derivative; here, we intentionally treat Φ as the holomorphic differential/lift entering the Weierstrass representation, so the nullity statement is $\langle \Phi, \Phi \rangle = 0$. This is precisely the condition needed for the conformal/minimal immersion $X = \Re \int \Phi \, d\omega$.*

A direct computation using the complex bilinear extension of the Euclidean inner product on $\mathbb{C}^4$ shows that

$$\langle \Phi(\omega), \Phi(\omega) \rangle = 0 \quad \text{for all } \omega \in D.$$

Hence, $\Phi(\omega)\, d\omega$ is a holomorphic null-curve differential.

Since D is simply connected, Φ admits a holomorphic primitive F with $F' = \Phi$, i.e.,

$$F(\omega) = \int^{\omega} \Phi(\zeta)\, d\zeta \subset \mathbb{C}^4.$$

By construction, $F'(\omega)$ is null for all ω, which is exactly the definition of a holomorphic null curve. This completes the proof. □

Remark 4.4. *If $\lambda = 0$, the fourth component of the lift vanishes identically, so the resulting immersion is the classical Scherk surface contained in the hyperplane $\{X_4 = 0\} \subset \mathbb{R}^4$. For $\lambda \neq 0$, the fourth component is nontrivial, and the immersion generally explores the extra codimension-two geometry of $\mathbb{R}^4$.*

Remark 4.5. *The parameter μ acts as an overall complex scaling of the holomorphic null-curve differential. Its geometric effect is to rescale the immersion (and, when μ is not real, also to rotate within the complexified model before taking real parts). For the purposes of constructing and visualizing the Scherk-type family, it is convenient to normalize $\mu = 1$. Throughout the remainder of this paper, we therefore fix*

$$\mu = 1,$$

which simplifies the formulas without changing the essential geometric phenomena governed by λ.

4.2. Holomorphic lift and explicit primitive for the Scherk-type family

We now specialize the generalized Weierstrass–Enneper representation in $\mathbb{R}^4$ to the Scherk-type choice of holomorphic data with $\mu = 1$. Let $D \subset \mathbb{C} \setminus \{\pm 1, \pm i\}$ be simply connected and fix $\lambda \in \mathbb{C}$. We consider

$$f(\omega) = \frac{4}{1-\omega^4}, \qquad g(\omega) = i\,\omega, \qquad h(\omega) = \lambda\, g(\omega) = i\lambda\,\omega. \tag{4.3}$$

With this choice, the two Gauss map components are linearly dependent and satisfy

$$g(\omega)^2 + h(\omega)^2 = -(1+\lambda^2)\omega^2.$$

Substituting (4.3) into (4.1) yields a holomorphic lift

$$\Phi(\omega) = \big(\Phi_1(\omega), \Phi_2(\omega), \Phi_3(\omega), \Phi_4(\omega)\big) \subset \mathbb{C}^4,$$

with components

$$\Phi_1(\omega) = \frac{2(1+(1+\lambda^2)\omega^2)}{1-\omega^4}, \tag{4.4}$$

$$\Phi_2(\omega) = \frac{2i(1-(1+\lambda^2)\omega^2)}{1-\omega^4}, \tag{4.5}$$

$$\Phi_3(\omega) = \frac{4i\,\omega}{1-\omega^4}, \tag{4.6}$$

$$\Phi_4(\omega) = \frac{4i\lambda\,\omega}{1-\omega^4}. \tag{4.7}$$

In particular, we have the pointwise proportionality relation

$$\Phi_4(\omega) = \lambda\,\Phi_3(\omega), \tag{4.8}$$

which will be central in distinguishing the real and complex parameter regimes.

Since D is simply connected, Φ admits a holomorphic primitive $F = (F_1, F_2, F_3, F_4)$ with $F' = \Phi$, for instance,

$$\begin{aligned} F_1(\omega) &= \int^{\omega} \frac{2(1+(1+\lambda^2)\zeta^2)}{1-\zeta^4}\, d\zeta, \\ F_2(\omega) &= \int^{\omega} \frac{2i(1-(1+\lambda^2)\zeta^2)}{1-\zeta^4}\, d\zeta, \\ F_3(\omega) &= \int^{\omega} \frac{4i\,\zeta}{1-\zeta^4}\, d\zeta, \\ F_4(\omega) &= \int^{\omega} \frac{4i\lambda\,\zeta}{1-\zeta^4}\, d\zeta, \end{aligned} \tag{4.9}$$

where different integration constants correspond to translations in $\mathbb{R}^4$.

Theorem 4.6 (Scherk-type minimal surface in $\mathbb{R}^4$). *Let $D \subset \mathbb{C} \setminus \{\pm 1, \pm i\}$ be simply connected and let (f, g, h) be given by* (4.3)*. If F is a holomorphic primitive of the associated holomorphic null-curve differential Φ, then*

$$X(\omega, \bar{\omega}) = \Re F(\omega), \qquad \omega \in D,$$

defines a conformal minimal immersion $X : D \to \mathbb{R}^4$. The image $X(D)$ is called a Scherk-type minimal surface in $\mathbb{R}^4$.

Proof. Define

$$\Phi(\omega) = \left(\frac{1}{2} f (1 - g^2 - h^2),\ \frac{i}{2} f (1 + g^2 + h^2),\ fg,\ fh \right),$$

with (f, g, h) as in (4.3). Each component of Φ is holomorphic on D.

(a) Nullity. Using $g^2 = -\omega^2$ and $h^2 = -\lambda^2\omega^2$, we compute

$$\Phi_1^2 + \Phi_2^2 = -\frac{16(1 + \lambda^2)\omega^2}{(1 - \omega^4)^2}, \qquad \Phi_3^2 + \Phi_4^2 = \frac{16(1 + \lambda^2)\omega^2}{(1 - \omega^4)^2}.$$

Hence, $\sum_{j=1}^{4} \Phi_j^2 = 0$, so Φ is a holomorphic null-curve differential.

(b) Primitive. Since D is simply connected, Φ admits a holomorphic primitive F with $F' = \Phi$.

(c) Conformality and minimality. Let $X = \Re F$ and write $\omega = u + iv$. Then $X_u = \Re\Phi$ and $X_v = -\Im\Phi$. Nullity implies

$$\langle X_u, X_u \rangle = \langle X_v, X_v \rangle, \qquad \langle X_u, X_v \rangle = 0,$$

so the induced metric is conformal. Each component of X is harmonic; hence, the mean curvature vector vanishes identically.

(d) Immersion. To verify immersion, it suffices to show that the vector-valued holomorphic differential $\Phi(\omega)$ never vanishes on D (i.e., its components have no common zero). If $\omega \neq 0$, then

$$\Phi_3(\omega) = \frac{4i\,\omega}{1 - \omega^4} \neq 0,$$

hence $\Phi(\omega) \neq 0$. At $\omega = 0$, we have $\Phi_1(0) = 2$ and $\Phi_2(0) = 2i$, so again $\Phi(0) \neq 0$. Therefore, Φ has no zeros on D, and X is an immersion on D. □

4.3. Real parametrization

Let $\omega = u + iv$ with $(u, v) \in \mathbb{R}^2$. We recall the identities

$$\omega^2 = (u^2 - v^2) + 2iuv, \qquad \omega^4 = (u^4 - 6u^2v^2 + v^4) + i(4u^3v - 4uv^3).$$

We consider the Scherk-type Weierstrass data

$$f(\omega) = \frac{4}{1 - \omega^4}, \qquad g(\omega) = i\,\omega, \qquad h(\omega) = \lambda g(\omega) = i\lambda\,\omega.$$

A direct computation gives

$$g(\omega)^2 = -\omega^2, \qquad h(\omega)^2 = -\lambda^2\omega^2, \qquad g(\omega)^2 + h(\omega)^2 = -(1 + \lambda^2)\omega^2.$$

Substituting these expressions into the generalized Weierstrass–Enneper differential, we obtain the holomorphic lift

$$\Phi(\omega) = (\Phi_1(\omega), \Phi_2(\omega), \Phi_3(\omega), \Phi_4(\omega)),$$

where

$$\Phi_1(\omega) = \frac{1}{2} f(1 - g^2 - h^2) = \frac{2(1 + (1 + \lambda^2)\omega^2)}{1 - \omega^4}, \tag{4.10}$$

$$\Phi_2(\omega) = \frac{i}{2} f(1 + g^2 + h^2) = \frac{2i(1 - (1 + \lambda^2)\omega^2)}{1 - \omega^4}, \tag{4.11}$$

$$\Phi_3(\omega) = fg = \frac{4i\,\omega}{1 - \omega^4}, \tag{4.12}$$

$$\Phi_4(\omega) = fh = \frac{4i\lambda\,\omega}{1 - \omega^4}. \tag{4.13}$$

Since the domain is simply connected, the differential Φ admits a holomorphic primitive F satisfying $F' = \Phi$. The associated real parametrization of the Scherk-type minimal surface is defined by

$$X(u, v) = \Re F(u + iv).$$

Equivalently, the coordinate functions $X(u, v) = (X_1, X_2, X_3, X_4)$ are given by

$$\begin{aligned} X_1(u, v) &= \Re \int \frac{2(1 + (1 + \lambda^2)\omega^2)}{1 - \omega^4}\, d\omega, \\ X_2(u, v) &= \Re \int \frac{2i(1 - (1 + \lambda^2)\omega^2)}{1 - \omega^4}\, d\omega, \\ X_3(u, v) &= \Re \int \frac{4i\,\omega}{1 - \omega^4}\, d\omega, \\ X_4(u, v) &= \Re \int \frac{4i\lambda\,\omega}{1 - \omega^4}\, d\omega, \end{aligned} \tag{4.14}$$

where $\omega = u + iv$.

Each coordinate function is therefore expressed by separating real and imaginary parts after integration of rational functions in ω. The parametrization is well defined away from the isolated singularities determined by $\omega^4 = 1$, i.e., away from $\omega \in \{\pm 1, \pm i\}$.

The resulting parametrization is rational rather than polynomial, reflecting the intrinsic non-algebraic and periodic nature of Scherk's surface. When $\lambda = 0$, the fourth coordinate vanishes identically, and the surface reduces to the classical Scherk minimal surface contained in the affine subspace $\{X_4 = 0\} \subset \mathbb{R}^4$. For $\lambda \neq 0$, the fourth coordinate is present, and the immersion generally explores codimension-two geometry.

4.4. The case $\lambda \in \mathbb{R} \setminus \{0\}$

We now restrict attention to the case where λ is a nonzero real number. In this regime, the Scherk-type construction yields an immersion into $\mathbb{R}^4$ whose image is nonetheless contained in a three-dimensional affine hyperplane.

Indeed, from (4.8), we have $\Phi_4 = \lambda\Phi_3$ with $\lambda \in \mathbb{R}$. If F is any holomorphic primitive of Φ, then $F_4 - \lambda F_3$ is holomorphic with zero derivative, hence

$$F_4(\omega) = \lambda F_3(\omega) + c \quad \text{for some constant } c \in \mathbb{C}.$$

Taking real parts gives the corresponding relation for the immersion:

$$X_4(u, v) = \lambda X_3(u, v) + \Re(c).$$

After translating X in $\mathbb{R}^4$ (absorbing $\Re(c)$), the image lies in the affine hyperplane

$$\{X_4 - \lambda X_3 = 0\} \subset \mathbb{R}^4.$$

Thus, the fourth coordinate does not introduce an independent geometric direction; rather, it encodes a fixed linear inclination of the classical Scherk surface inside $\mathbb{R}^4$.

Remark 4.7. *When $\lambda = 0$, we recover the classical Scherk surface in the hyperplane $\{X_4 = 0\}$. For $\lambda \neq 0$ with $\lambda \in \mathbb{R}$, the surface is not contained in $\{X_4 = 0\}$, but the global affine constraint $X_4 - \lambda X_3 = 0$ still holds (after translation). Hence, for real λ, the Scherk-type construction produces a three-dimensional immersion of the classical Scherk surface into $\mathbb{R}^4$, rather than a genuinely codimension-two minimal surface.*

4.5. The case $\lambda = \alpha + i\beta \in \mathbb{C} \setminus \mathbb{R}$

We now consider the genuinely complex Scherk-type deformation, where

$$\lambda = \alpha + i\beta, \qquad \alpha, \beta \in \mathbb{R},\ \beta \neq 0.$$

In this setting, the fourth coordinate is no longer constrained to be a real multiple of the third coordinate, and the immersion is no longer forced into a fixed three-dimensional affine hyperplane.

Let $\omega = u + iv$. For the Scherk-type data (4.3), we have

$$\Phi_3(\omega) = \frac{4i\,\omega}{1 - \omega^4}, \qquad \Phi_4(\omega) = \frac{4i\lambda\,\omega}{1 - \omega^4},$$

and hence (at the level of primitives)

$$F_4(\omega) = \lambda F_3(\omega) + c \quad \text{for some } c \in \mathbb{C}.$$

Taking real parts yields

$$X_4(u, v) = \Re(\lambda F_3(u + iv)) + \Re(c) = \alpha\,\Re(F_3(u + iv)) - \beta\,\Im(F_3(u + iv)) + \Re(c). \tag{4.15}$$

Since $X_3 = \Re(F_3)$, the identity (4.15) becomes

$$X_4(u, v) = \alpha\,X_3(u, v) - \beta\,\Im(F_3(u + iv)) + \Re(c).$$

This formula makes the key point transparent: When $\beta \neq 0$, the fourth coordinate contains an additional independent contribution coming from $\Im(F_3)$, which cannot be absorbed into a real scalar multiple of X_3.

Geometric interpretation. Varying (α,β) with $\beta \neq 0$ produces a two-parameter family of minimal immersions whose (X_3, X_4)-behavior exhibits a genuine coupling between the real and imaginary parts of the primitive F_3. From a geometric perspective, the real part α controls a linear inclination in the fourth coordinate direction, while the imaginary part β introduces an additional component that is invisible in the real-parameter family. As $\beta \to 0$, the surfaces converge (in the sense of the defining formulas) to the real Scherk-type case discussed above.

Finally, we emphasize that the precise statement of whether the image is contained in an affine hyperplane is addressed later by the hyperplane criterion in Theorem 7.2. The present discussion explains at the level of explicit primitives why the real and complex regimes behave fundamentally differently in the (X_3, X_4) coordinates.

5. Examples

In this section, we present explicit examples of Scherk-type minimal surfaces in $\mathbb{R}^4$ obtained from the generalized Weierstrass–Enneper representation.

Example 5.1 ((u, v)-parametrization of a Scherk-type minimal surface in $\mathbb{R}^4$)**.** *We consider the Scherk-type Weierstrass data*

$$f(\omega) = \frac{4}{1-\omega^4}, \qquad g(\omega) = i\omega, \qquad h(\omega) = \lambda g(\omega),$$

and fix the complex parameter

$$\lambda = 1 + i.$$

Since $(1 + i)^2 = 2i$*, a direct computation yields*

$$g(\omega)^2 = -\omega^2, \qquad h(\omega)^2 = -2i\,\omega^2, \qquad g(\omega)^2 + h(\omega)^2 = -(1 + 2i)\omega^2.$$

Substituting these expressions into the generalized Weierstrass–Enneper representation, we obtain the holomorphic null-curve differential

$$\begin{aligned}
\Phi_1(\omega) &= \frac{2(1 + (1 + 2i)\omega^2)}{1-\omega^4}, \\
\Phi_2(\omega) &= \frac{2i(1 - (1 + 2i)\omega^2)}{1-\omega^4}, \\
\Phi_3(\omega) &= \frac{4i\,\omega}{1-\omega^4}, \\
\Phi_4(\omega) &= \frac{4i(1+i)\,\omega}{1-\omega^4}.
\end{aligned}$$

Choosing appropriate branches of the logarithmic functions, we use the local identities

$$\int \frac{\omega}{1-\omega^4}\,d\omega = \frac{1}{4}\log\left(\frac{1+\omega^2}{1-\omega^2}\right), \qquad \int \frac{1}{1-\omega^4}\,d\omega = \frac{1}{2}(\arctan\omega + \operatorname{artanh}\omega),$$

to obtain, up to additive constants, a holomorphic primitive $F = (F_1, F_2, F_3, F_4)$ satisfying $F' = \Phi$, namely,

$$F_1(\omega) = \arctan\omega + \operatorname{artanh}\omega + \frac{1+2i}{2}\log\left(\frac{1+\omega^2}{1-\omega^2}\right),$$

$$F_2(\omega) = i\left(\arctan\omega + \operatorname{artanh}\omega - \frac{1+2i}{2}\log\left(\frac{1+\omega^2}{1-\omega^2}\right)\right),$$

$$F_3(\omega) = i\log\left(\frac{1+\omega^2}{1-\omega^2}\right),$$

$$F_4(\omega) = (1+i)i\log\left(\frac{1+\omega^2}{1-\omega^2}\right).$$

The associated Scherk-type minimal immersion is defined by

$$X(u,v) = \Re F(u+iv), \qquad \omega = u+iv.$$

To express the immersion explicitly in real variables, we introduce the auxiliary functions

$$R(u,v) = \sqrt{\frac{(1+u^2-v^2)^2+(2uv)^2}{(1-u^2+v^2)^2+(2uv)^2}},$$

$$\Theta(u,v) = \arctan\frac{2uv}{1+u^2-v^2} + \arctan\frac{2uv}{1-u^2+v^2}.$$

These functions arise from the polar decomposition

$$\log\left(\frac{1+\omega^2}{1-\omega^2}\right) = \ln R(u,v) + i\,\Theta(u,v).$$

Moreover, the following identities hold:

$$\Re(\arctan\omega) = \frac{1}{2}\arctan\frac{2u}{1-u^2-v^2},$$

$$\Re(\operatorname{artanh}\omega) = \frac{1}{4}\ln\frac{(1+u)^2+v^2}{(1-u)^2+v^2},$$

$$\Im(\arctan\omega + \operatorname{artanh}\omega) = \frac{1}{2}\arctan\frac{2v}{1-u^2-v^2}.$$

Consequently, the Scherk-type minimal surface $X = (X_1, X_2, X_3, X_4) : \mathbb{R}^2 \to \mathbb{R}^4$ admits the explicit real parametrization

$$\begin{aligned}
X_1(u,v) &= \frac{1}{2}\arctan\frac{2u}{1-u^2-v^2} + \frac{1}{4}\ln\frac{(1+u)^2+v^2}{(1-u)^2+v^2} + \frac{1}{2}\ln R(u,v) - \Theta(u,v),\\
X_2(u,v) &= -\frac{1}{2}\arctan\frac{2v}{1-u^2-v^2} + \frac{1}{2}\Theta(u,v) + \frac{1}{2}\ln R(u,v),\\
X_3(u,v) &= -\Theta(u,v),\\
X_4(u,v) &= -\ln R(u,v) - \Theta(u,v).
\end{aligned} \tag{5.1}$$

This example defines a conformal minimal immersion into $\mathbb{R}^4$ associated with Scherk-type Weierstrass data. The logarithmic and arctangent structure reflects the classical Scherk geometry, while the imaginary part of λ yields a genuinely four-dimensional deformation.

Example 5.2 ((r,θ) form of the Scherk-type surface with $\lambda = 1 + i$)**.** *We consider the Scherk-type minimal surface corresponding to the complex parameter $\lambda = 1 + i$, whose real parametrization was obtained in* (5.1). *To analyze its angular behavior, we introduce polar coordinates*

$$u = r\cos\theta, \qquad v = r\sin\theta,$$

with $r \geq 0$ and $\theta \in \mathbb{R}$.

With this substitution, the quadratic expressions appearing in the logarithmic terms become

$$u^2 - v^2 = r^2\cos(2\theta), \qquad 2uv = r^2\sin(2\theta),$$

and consequently,

$$1 + u^2 - v^2 = 1 + r^2\cos(2\theta), \qquad 1 - u^2 + v^2 = 1 - r^2\cos(2\theta).$$

Moreover, the argument function $\Theta(u,v)$ defined in (5.1) *takes the form*

$$\Theta(r,\theta) = \arctan\left(\frac{r^2\sin(2\theta)}{1 + r^2\cos(2\theta)}\right) + \arctan\left(\frac{r^2\sin(2\theta)}{1 - r^2\cos(2\theta)}\right).$$

Similarly, the radial factor $R(u,v)$ becomes

$$R(r,\theta) = \sqrt{\frac{(1 + r^2\cos(2\theta))^2 + r^4\sin^2(2\theta)}{(1 - r^2\cos(2\theta))^2 + r^4\sin^2(2\theta)}}.$$

Substituting these expressions into the real parametrization yields the trigonometric form

$$X(r,\theta) = \begin{pmatrix} \frac{1}{2}\arctan\frac{2r\cos\theta}{1-r^2} + \frac{1}{4}\ln\frac{(1+r\cos\theta)^2 + r^2\sin^2\theta}{(1-r\cos\theta)^2 + r^2\sin^2\theta} + \frac{1}{2}\ln R(r,\theta) - \Theta(r,\theta) \\ -\frac{1}{2}\arctan\frac{2r\sin\theta}{1-r^2} + \frac{1}{2}\Theta(r,\theta) + \frac{1}{2}\ln R(r,\theta) \\ -\Theta(r,\theta) \\ -\ln R(r,\theta) - \Theta(r,\theta) \end{pmatrix}. \tag{5.2}$$

This representation highlights the angular structure of the Scherk-type minimal surface. The dependence on θ is governed by trigonometric functions of 2θ, reflecting the classical saddle geometry of Scherk surfaces, while the logarithmic terms encode the characteristic non-algebraic behavior. The imaginary part of λ produces a genuine coupling between radial and angular directions, yielding a truly four-dimensional minimal surface in $\mathbb{R}^4$.

While Figures 1 and 2 show projections of the surface in Cartesian coordinates (u,v), Figures 3 and 4 present the same surface in polar form (r,θ), where rotational features and anisotropic stretching become more transparent.

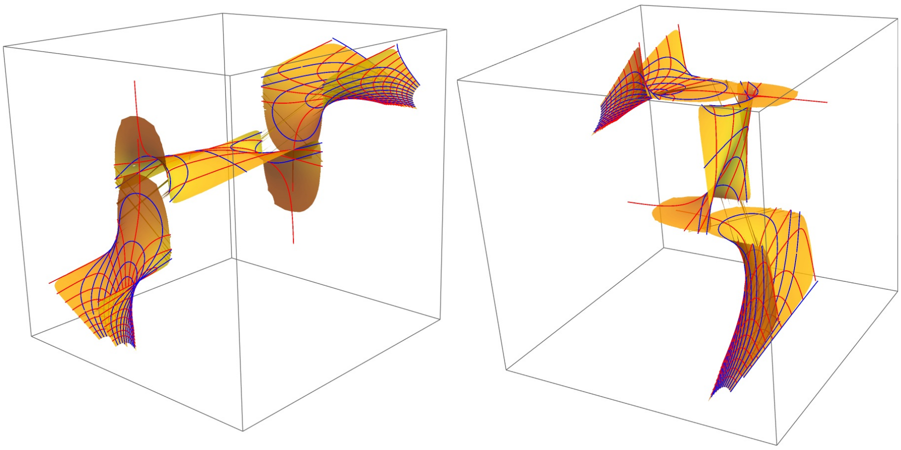

Figure 1. Projections of the Scherk-type minimal surface $X(u, v)$, defined in (5.1), into the $X_1X_2X_3$-space (left) and the $X_1X_2X_4$-space (right).

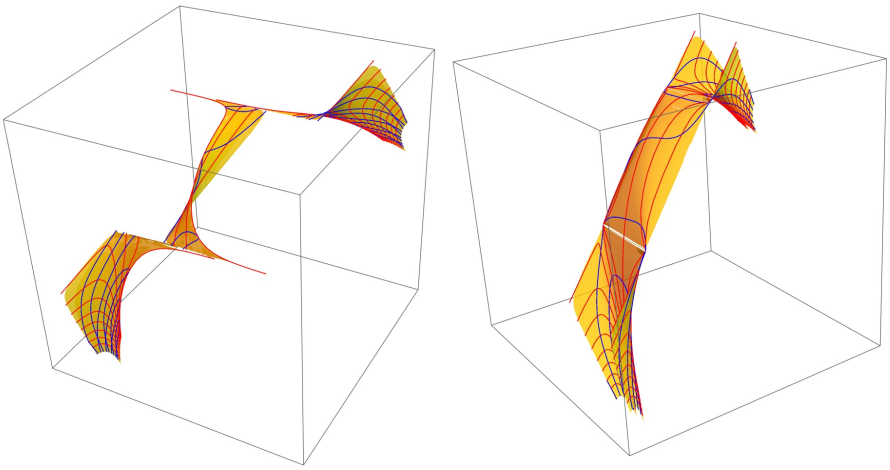

Figure 2. Projections of the Scherk-type minimal surface $X(u, v)$, defined in (5.1), into the $X_1X_3X_4$-space (left) and the $X_2X_3X_4$-space (right).

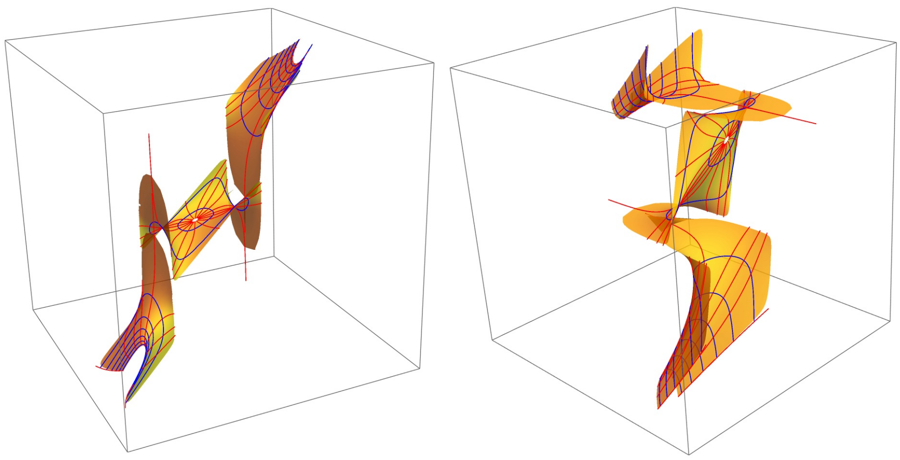

Figure 3. Projections of the Scherk-type minimal surface $X(r, \theta)$, defined in (5.2), into the $X_1X_2X_3$-space (left) and the $X_1X_2X_4$-space (right).

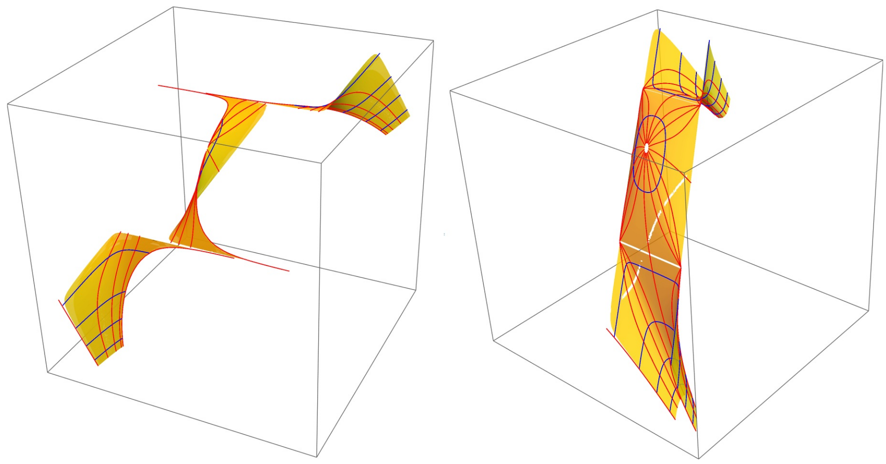

Figure 4. Projections of the Scherk-type minimal surface $X(r,\theta)$, defined in (5.2), into the $X_1X_3X_4$-space (left) and the $X_2X_3X_4$-space (right).

6. Gauss normals and the normal bundle

Purpose of this section. Although parts of the normal-bundle formalism are general for codimension-two immersions, we include this section because (i) our family is explicit and visualized, (ii) the distinction between $\lambda \in \mathbb{R}$ and $\lambda \notin \mathbb{R}$ has a sharp geometric normal-bundle interpretation, and (iii) readers often wish to compute normal frames and normal connection forms in this concrete example. Accordingly, we record here an explicit and reproducible construction specialized to our Scherk-type immersion.

For minimal surfaces in $\mathbb{R}^3$, the normal bundle is one-dimensional, and the Gauss map takes values in the unit sphere $\mathbb{S}^2$. In particular, the normal geometry is encoded by a single unit normal field. In contrast, a surface immersed in $\mathbb{R}^4$ has a two-dimensional normal bundle, and the geometry involves orthonormal normal frames, a normal connection, and the possibility of nontrivial normal twisting.

Let

$$X : D \subset \mathbb{C} \longrightarrow \mathbb{R}^4$$

be a conformal minimal immersion obtained from the generalized Weierstrass–Enneper representation associated with Scherk-type data. Write $\omega = u + iv$ and denote by $\langle \cdot, \cdot \rangle$ the Euclidean inner product on $\mathbb{R}^4$. The tangent plane at a regular point is spanned by X_u and X_v, and conformality implies

$$\langle X_u, X_u \rangle = \langle X_v, X_v \rangle, \qquad \langle X_u, X_v \rangle = 0.$$

In our setting, the immersion is constructed as the real part of a holomorphic null curve in $\mathbb{C}^4$. More precisely,

$$X(\omega, \bar{\omega}) = \Re F(\omega), \qquad F'(\omega) = \Phi(\omega), \qquad \sum_{j=1}^{4} \Phi_j(\omega)^2 = 0,$$

where $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4)$ is holomorphic on D. Decomposing Φ into real and imaginary parts gives the explicit tangent vectors

$$X_u = \Re\Phi, \qquad X_v = -\Im\Phi.$$

6.1. Tangent vectors and an explicit orthonormal normal frame

Write $\omega = u + iv$ and recall that $X(\omega, \bar{\omega}) = \Re F(\omega)$ with $F'(\omega) = \Phi(\omega)$, where $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4)$ is holomorphic and satisfies the nullity condition $\sum_{j=1}^{4} \Phi_j(\omega)^2 = 0$. Differentiating $F(u + iv)$ and taking real parts yield the standard identities

$$X_u = \Re\Phi, \qquad X_v = -\Im\Phi. \tag{6.1}$$

Consequently, the tangent plane at a regular point is spanned by X_u and X_v.

Remark on $i\Phi$. Although one can form the real and imaginary parts of $i\Phi$, these do not produce any additional independent directions. Indeed,

$$i\Phi = i\,\Re\Phi - \Im\Phi,$$

hence

$$\Re(i\Phi) = -\Im\Phi = X_v, \qquad \Im(i\Phi) = \Re\Phi = X_u.$$

Thus, $i\Phi$ simply repackages the same tangent basis and provides no normal information.

An explicit orthonormal tangent frame. Let

$$E = \langle X_u, X_u \rangle = \langle X_v, X_v \rangle$$

be the conformal factor. Define

$$e_1 = \frac{X_u}{\sqrt{E}}, \qquad e_2 = \frac{X_v}{\sqrt{E}}.$$

Then (e_1, e_2) is a local orthonormal tangent frame.

A concrete orthonormal normal frame by projection. Let $P^\perp$ denote the orthogonal projection onto the normal space, i.e.,

$$P^\perp(Y) = Y - \langle Y, e_1 \rangle e_1 - \langle Y, e_2 \rangle e_2 \qquad (Y \in \mathbb{R}^4).$$

Fix two constant vectors in $\mathbb{R}^4$, for example,

$$\mathbf{e}_3 = (0, 0, 1, 0), \qquad \mathbf{e}_4 = (0, 0, 0, 1).$$

Define

$$\widetilde{N}_1 = P^\perp(\mathbf{e}_3), \qquad N_1 = \frac{\widetilde{N}_1}{\|\widetilde{N}_1\|} \quad \text{whenever } \widetilde{N}_1 \neq 0,$$

and then

$$\widetilde{N}_2 = P^\perp(\mathbf{e}_4) - \langle P^\perp(\mathbf{e}_4), N_1 \rangle N_1, \qquad N_2 = \frac{\widetilde{N}_2}{\|\widetilde{N}_2\|} \quad \text{whenever } \widetilde{N}_2 \neq 0.$$

By construction, (N_1, N_2) is an orthonormal frame of the normal bundle on the open set where $\widetilde{N}_1$ and $\widetilde{N}_2$ are nonzero.

At points where $\widetilde{N}_1 = 0$ (respectively $\widetilde{N}_2 = 0$), one replaces $\mathbf{e}_3$ or $\mathbf{e}_4$ by another fixed constant vector not lying in the tangent plane at that point; since the normal space is two-dimensional, such a choice can always be made locally.

Remark 6.1. *The normal frame (N_1, N_2) is not canonical: Any other orthonormal normal frame differs by a pointwise rotation in the normal plane. However, the construction above is explicit and reproducible, and it is sufficient for describing the normal connection and normal-bundle geometry in later computations and for interpreting the visualizations.*

6.2. Normal connection

Let ∇ denote the Euclidean connection on $\mathbb{R}^4$ and let $\nabla^\perp$ be its orthogonal projection to the normal bundle. With respect to an orthonormal normal frame (N_1, N_2), the normal connection is encoded by a single real one-form ω_{12} on D defined by

$$\nabla^\perp N_1 = \omega_{12} N_2, \qquad \nabla^\perp N_2 = -\omega_{12} N_1.$$

Equivalently,

$$\omega_{12} = \langle \nabla N_1, N_2 \rangle.$$

A normal frame is parallel if and only if $\omega_{12} \equiv 0$ in that gauge. More generally, under a rotation of the normal frame by an angle function θ, the one-form transforms as $\omega_{12} \mapsto \omega_{12} + d\theta$.

6.3. Real versus complex parameter regimes

For the Scherk-type family studied in this paper, the Weierstrass data are

$$f(\omega) = \frac{4}{1 - \omega^4}, \qquad g(\omega) = i\omega, \qquad h(\omega) = \lambda g(\omega), \qquad \lambda \in \mathbb{C},$$

and the last two components of the holomorphic lift satisfy the pointwise relation

$$\Phi_4(\omega) = \lambda\, \Phi_3(\omega).$$

This relation has a direct and sharp geometric consequence.

6.4. The case $\lambda \in \mathbb{R}$

Assume that λ is real. Then the identity $\Phi_4 = \lambda\Phi_3$ implies that for any holomorphic primitive F of Φ, we have $F_4 = \lambda F_3 + c$ for a constant $c \in \mathbb{C}$. Taking real parts yields

$$X_4 = \lambda X_3 + \Re(c).$$

After a translation in $\mathbb{R}^4$, the image therefore lies in the affine hyperplane $\{X_4 - \lambda X_3 = 0\}$.

Geometrically, the surface is contained in a three-dimensional Euclidean subspace. One normal direction is the fixed unit normal to this hyperplane, and the remaining normal direction is the classical unit normal of the Scherk surface within the hyperplane.

6.5. Complex parameter case

Now assume that $\lambda = \alpha + i\beta$ with $\beta \neq 0$. In this regime, the fourth coordinate is no longer globally constrained by the third one. Accordingly, the immersion is not forced into the affine hyperplane $\{X_4 - \lambda X_3 = \text{const}\}$ that arises in the real-parameter case; in particular, one should expect genuinely codimension-two behavior (as made precise later by the hyperplane criterion in Theorem 7.2).

From the viewpoint of normal geometry, this means that there is no decomposition of the normal bundle into a fixed ambient normal direction plus a classical normal within a hyperplane. Consequently, the normal bundle cannot in general be reduced to the three-dimensional situation, and one should expect nontrivial normal twisting.

6.6. *Interpretation of the visualizations*

The plotted surfaces represent projections of the same intrinsically conformal surface into different coordinate subspaces of $\mathbb{R}^4$. Because the induced metric weights different coordinate directions in an anisotropic manner, distinct projections emphasize different geometric features. In particular, the coordinates (X_3, X_4) are amplified by the factor $1 + |\lambda|^2$, while the interaction between (X_1, X_2) and (X_3, X_4) is governed by the complex quantity $1 + \lambda^2$.

Consequently, the apparent twisting, stretching, and asymmetry observed in the figures are not artifacts of plotting or projection, but faithful representations of the underlying intrinsic geometry dictated by the metric. The visualizations thus provide geometric intuition for the analytic results established earlier, especially the distinction between the real and complex parameter regimes and the emergence of genuinely four-dimensional geometric behavior.

7. Symmetries, fundamental domain, and the hyperplane criterion

In this section, we record basic symmetries of the Scherk-type data and isolate a natural fundamental domain for both analysis and visualization. We also give a sharp criterion for when the image is contained in an affine hyperplane of $\mathbb{R}^4$, i.e., when the construction fails to be genuinely codimension two.

7.1. *Domain symmetries induced by the Weierstrass data*

Throughout, we use the Scherk-type Weierstrass data

$$f(\omega) = \frac{4}{1-\omega^4}, \qquad g(\omega) = i\omega, \qquad h(\omega) = i\lambda\omega,$$

on a simply connected domain $D \subset \mathbb{C} \setminus \{\pm 1, \pm i\}$, and we write $X(\omega, \bar{\omega}) = \Re F(\omega)$ for a holomorphic primitive $F' = \Phi$.

Proposition 7.1 (Dihedral symmetry of the data)**.** *The Scherk-type differential $\Phi(\omega)$ satisfies:*

(1) $\Phi(-\omega) = -\Phi(\omega)$*;*

(2) $\Phi(i\omega) = \mathcal{R}\,\Phi(\omega)$*, where $\mathcal{R}$ is a constant real orthogonal transformation of $\mathbb{R}^4$ depending only on the chosen ordering of coordinates.*

Consequently, up to rigid motions of $\mathbb{R}^4$, the immersion X is invariant under the dihedral group generated by $\omega \mapsto -\omega$ and $\omega \mapsto i\omega$.

Proof. The identity $\Phi(-\omega) = -\Phi(\omega)$ follows immediately from the explicit formulas

$$\Phi_1(\omega) = \frac{2(1 + (1+\lambda^2)\omega^2)}{1-\omega^4}, \quad \Phi_2(\omega) = \frac{2i(1 - (1+\lambda^2)\omega^2)}{1-\omega^4},$$
$$\Phi_3(\omega) = \frac{4i\omega}{1-\omega^4}, \quad \Phi_4(\omega) = \frac{4i\lambda\omega}{1-\omega^4},$$

since $\omega \mapsto -\omega$ fixes ω^2 and ω^4 and changes the sign of ω.

For $\omega \mapsto i\omega$, we have $\omega^2 \mapsto -\omega^2$ and $\omega^4 \mapsto \omega^4$. Thus, Φ_1 and Φ_2 are interchanged up to multiplication by $\pm i$, while Φ_3 and Φ_4 acquire constant phase factors. Taking real parts after integration produces a rigid motion of the image, represented by a constant orthogonal transformation $\mathcal{R}$ in $\mathbb{R}^4$. □

7.2. A practical fundamental domain for visualization

The poles of f at $\omega \in \{\pm 1, \pm i\}$ correspond to Scherk-type ends. For numerical plots, it is natural to work on a square domain, avoiding these poles, for instance,

$$Q_\varepsilon = \{u + iv : \ |u| \le 1 - \varepsilon, \ |v| \le 1 - \varepsilon\}, \qquad 0 < \varepsilon \ll 1,$$

and then we use Proposition 7.1 to extend the plotted patch by symmetry. This reduces the risk of visual artifacts near the ends and makes the periodic/saddle structure more transparent.

7.3. Hyperplane criterion and genuine codimension two

We next isolate exactly when the image of X is contained in an affine hyperplane.

Theorem 7.2 (Hyperplane criterion)**.** *Let X be a Scherk-type minimal immersion associated with parameter $\lambda = \alpha + i\beta \in \mathbb{C}$. Then*

(1) If $\beta = 0$ (i.e., $\lambda \in \mathbb{R}$), the image of X is contained in the affine hyperplane $\{x_4 - \lambda x_3 = 0\} \subset \mathbb{R}^4$.
(2) If $\beta \neq 0$, the image of X is not contained in any affine hyperplane of $\mathbb{R}^4$. In particular, the immersion is genuinely codimension two.

Proof. If $\lambda \in \mathbb{R}$, then $\Phi_4(\omega) = \lambda\,\Phi_3(\omega)$ for all ω, hence for any primitive F, we have $F_4 = \lambda F_3 + c$ for a constant $c \in \mathbb{C}$. Taking real parts yields $X_4 = \lambda X_3 + \Re(c)$, so after a translation, the image lies in $\{x_4 - \lambda x_3 = 0\}$.

Assume now that $\Im(\lambda) = \beta \neq 0$. Suppose for contradiction that the image is contained in an affine hyperplane, so there exist constants $a \in \mathbb{R}^4$, $a \neq 0$, and $b \in \mathbb{R}$ such that $\langle a, X\rangle \equiv b$ on D. Differentiating gives $\langle a, X_u\rangle = \langle a, X_v\rangle \equiv 0$. Equivalently, $\langle a, \Re\Phi\rangle = \langle a, \Im\Phi\rangle \equiv 0$, hence $\langle a, \Phi\rangle \equiv 0$ as a holomorphic function.

Writing $a = (a_1, a_2, a_3, a_4)$ and using $\Phi_4 = \lambda\Phi_3$, the identity $\langle a, \Phi\rangle = 0$ becomes

$$a_1\Phi_1 + a_2\Phi_2 + (a_3 + a_4\lambda)\Phi_3 \equiv 0.$$

Now use the parity of the explicit formulas: $\Phi_1(\omega)$ and $\Phi_2(\omega)$ are even in ω, whereas $\Phi_3(\omega)$ is odd. Taking the odd part forces

$$(a_3 + a_4\lambda)\Phi_3(\omega) \equiv 0,$$

hence $a_3 + a_4\lambda = 0$. The remaining even part gives $a_1\Phi_1 + a_2\Phi_2 \equiv 0$. But Φ_1 and Φ_2 are not constant multiples of one another as holomorphic functions (their ratio depends on ω^2), so this forces $a_1 = a_2 = 0$. Thus, $a \neq 0$ implies $a_4 \neq 0$, and $\lambda = -a_3/a_4 \in \mathbb{R}$, contradicting $\Im(\lambda) \neq 0$. Hence, no affine hyperplane contains the image when $\beta \neq 0$. □

8. Applications

The Scherk-type minimal surface constructed in this paper constitutes an explicit and flexible example of a conformal minimal immersion in $\mathbb{R}^4$. Its description via the generalized Weierstrass–Enneper representation leads to closed-form parametrizations, making it possible to examine both local and global geometric properties directly and transparently.

One immediate application is the study of minimal surfaces in codimension two. While the classical Scherk surface in $\mathbb{R}^3$ is well understood, explicit examples in $\mathbb{R}^4$ remain comparatively scarce. The Scherk-type surface presented here provides a concrete model that preserves the characteristic saddle geometry while exhibiting genuinely four-dimensional features.

The dependence on the complex parameter λ yields an explicit deformation within the class of minimal immersions. Real values of λ correspond to surfaces effectively constrained to three-dimensional subspaces, whereas complex values give rise to immersions with nontrivial behavior in the additional normal direction. This clear distinction offers useful insight into the increased geometric freedom available in a higher codimension.

A further application concerns the geometry of the normal bundle. As discussed earlier, the normal bundle is geometrically simpler in the real-parameter case, while genuinely codimension-two behavior emerges when λ is complex. The Scherk-type surface thus provides an explicit example illustrating how normal-bundle phenomena arise naturally in $\mathbb{R}^4$.

Finally, the explicit nature of these parametrizations makes the surface amenable to visualization and computational experiments. More generally, the construction can be adapted to other ambient geometries, suggesting the existence of Scherk-type minimal surfaces beyond the Euclidean setting.

9. Conclusions

In this paper, we constructed a class of Scherk-type minimal surfaces in $\mathbb{R}^4$ using the generalized Weierstrass–Enneper representation. By extending the classical Weierstrass data of the Scherk surface in $\mathbb{R}^3$, we obtained explicit parametrizations arising from holomorphic null curves in $\mathbb{C}^4$. The resulting family depends on a complex parameter λ, whose real and complex regimes lead to different geometric behaviors: real values correspond to surfaces contained in affine hyperplanes of $\mathbb{R}^4$, while complex values yield genuinely four-dimensional minimal immersions.

Author contributions

Magdalena Toda and Erhan Güler: Establishment of theoretical research; Magdalena Toda and Erhan Güler: Design and interpretation of the manuscript, theoretical analysis, drafted the manuscript, revision; provided critical feedback to the manuscript, revision; Magdalena Toda and Erhan Güler: Provided suggestions for the manuscript; Magdalena Toda and Erhan Güler: Revision, translation, proofreading of the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

AI-based language assistance (such as Grammarly) was used solely to improve clarity, grammar, and presentation. All mathematical ideas, results, and proofs are entirely the authors' own. The authors assume full responsibility for the content and accuracy of the manuscript.

Conflict of interest

The authors declare no conflicts of interest.

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