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Research article

Analyzing the fractional Chen attractor via controlled interpolative metric contractions

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Abstract: The present study advances the idea of controlled (α, c)–interpolative metric spaces as a broader framework that extends interpolative and extended interpolative frameworks and examines their analytical properties in conjunction with Banach-type fixed-point results. The proposed metric structure is then employed to study the fractional-order Chen attractor model defined via the Atangana–Baleanu–Caputo ($\mathcal{ABC}$) derivative. Using the newly established metric framework, a suitable fixed-point operator is constructed in the Banach space $C([0, T]; \mathbb{R}^3)$, and sufficient contraction conditions ensuring the existence and uniqueness of solutions are derived. The consistency and accuracy of the proposed numerical formulation are demonstrated through simulation results, highlighting improved sensitivity and richer chaotic behavior compared to integer-order models. The obtained results confirm that the controlled interpolative metric approach provides a flexible analytical tool for exploring nonlinear fractional systems and offers new insights into the stability analysis of chaotic dynamical models such as the Chen attractor.

Keywords: controlled metric space; interpolative metric space; fractional-order Chen attractor; chaotic dynamical models; nonlinear fractional systems
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

In what follows, $\mathbb{N}$ denotes the set of all positive natural numbers. The sets of positive and nonnegative reals are written as $\mathbb{R}^+$ and $\mathbb{R}_0^+$, respectively.

Metric fixed-point theory combines a highly organized theoretical framework with practical methods that are applied in mathematics, computing, and economics. Within this theoretical framework, the Banach contraction mapping theorem, first introduced by Banach [1] in 1922, occupies a pivotal position owing to its foundational significance and versatility. This seminal work provided a robust method for establishing the existence and uniqueness of fixed-points in complete metric spaces, laying the groundwork for countless subsequent research efforts aimed at expanding and refining the understanding of this profound mapping.

Beyond its conceptual impact, the Banach contraction principle has proven to be highly practical, supporting techniques for differential equations, optimization, and more. This success has encouraged the development of generalized metric structures that weaken traditional constraints and extend fixed-point theory. These generalized spaces relax some of the classical assumptions, allowing for a broader class of mappings and facilitating the exploration of fixed-point theorems within varied contexts. In this direction, recent developments have further extended the classical framework by introducing generalized metric structures and contraction principles, thereby significantly broadening the scope and applicability of fixed-point theory [2, 3].

One of the most significant generalizations of classical metric spaces is the concept of a b-metric space, first proposed by Bakhtin [4] as a flexible framework which extends the traditional triangle inequality by a positive constant coefficient. Since then, the notion has gained considerable attention due to its ability to model a broader class of non-Euclidean structures while preserving the essential analytical properties of metric spaces. In this line, Kamran et al. [5] further generalized the concept by introducing the so-called extended b-metric spaces, providing a richer topology that accommodates mappings with weaker contractive behavior. Building upon these developments, Mlaiki et al. [6] proposed controlled metric-type spaces, which unify and extend several previous frameworks by incorporating a control function that governs the distance structure. These progressive generalizations have opened new directions for the study of fixed-point theory, stability analysis, and nonlinear mappings under generalized distance settings.

In 2023, Karapınar [7] made a significant advancement by establishing the concept of an interpolative metric space. This innovative framework not only extends the traditional metric space but also provides a way for newfound research opportunities and applications in mathematical theory.

In parallel with the evolution of metric fixed-point theory, the Chen attractor, introduced by G. Chen and Ueta [8], continues to be one of the most extensively studied models in the field of nonlinear dynamics and chaos theory. The Chen attractor has a structure similar to the Lorenz system, but it stands out with its two remarkable scroll (spiral) structures and complex topology. Recent studies have shown that the Chen attractor contains both homoclinic and heteroclinic orbits, which define the chaotic nature of the system [9, 10]. Furthermore, it has been demonstrated that the Chen attractor is topologically distinct from the Lorenz attractor and exhibits more complex dynamics [11, 12].

The Chen system possesses multiple equilibrium points, depending on the values of its parameters. The existence and stability of these equilibrium points have been investigated using Lyapunov functions and bifurcation analyses [13–15]. Additionally, recent studies have revealed that the Chen attractor also exhibits chaotic behavior in fractional and fractal–fractional derivative spaces. These works have highlighted the attractor's multiscroll and hyperchaotic characteristics and clarified the effects of fractional and fractal parameters on system dynamics [16–18].

Overall, recent research indicates that the fractional Chen system has become an important model in both theoretical and applied chaos studies.

To formalize this framework and provide a rigorous basis for subsequent analysis, we recall the concept of an (α, c)−interpolative metric (briefly $(\alpha, c)-\mathbb{IM}$), which plays a fundamental role in extending classical distance structures under weaker contractive conditions.

Definition 1.1. [7] Consider $\Game$ to be a nonempty set. The function $d_i : \Game \times \Game \to [0, \infty)$ is an $(\alpha, c)-\mathbb{IM}$ if the following statements hold:

(i_1) $d_i(\curlyvee, \curlywedge) = 0 \Leftrightarrow \curlyvee = \curlywedge$,

(i_2) $d_i(\curlyvee, \curlywedge) = d_i(\curlywedge, \curlyvee)$,

(i_3) two constants $\alpha \in (0, 1)$ and $c \geq 0$ exist such that

$$d_i(\curlyvee, \curlywedge) \leq d_i(\curlyvee, \varkappa) + d_i(\varkappa, \curlywedge) + c\left[d_i(\curlyvee, \varkappa)^{\alpha} d_i(\varkappa, \curlywedge)^{1-\alpha}\right]$$

for all $\curlyvee, \curlywedge, \varkappa \in \Game$. In this situation, the pair $(\Game, d_i)$ is named an (α, c)−interpolative metric space, which is abbreviated by $(\alpha, c)-\mathbb{IMS}$.

Remark 1.1. [19] Any metric space is an $(\alpha, c)-\mathbb{IMS}$ with $c = 0$; indeed, the metric d_i satisfies conditions (i_1)–(i_3) above.

Notice that the converse is false, as shown in Example 1.3 in [19].

Furthermore, Karapınar [7] explained the concepts of convergence, the Cauchy sequence, and completeness as well as established the Banach contraction mapping principle in the setting of $(\alpha, c)-\mathbb{IMS}$.

Karapınar and Agarwal [19], using the comparison function, demonstrated the following result.

Theorem 1.1. *Consider that* $(\Game, d_i)$ *is a complete* $(\alpha, c)-\mathbb{IMS}$*, and* $\complement : \Game \to \Game$ *is a self-mapping. Presume that a comparison function* ψ *exists such that*

$$d_i\left(\complement\curlyvee, \complement\curlywedge\right) \leq \psi\left(\Xi(\curlyvee, \curlywedge)\right),$$

$$\Xi(\curlyvee, \curlywedge) = \max\left\{d_i(\curlyvee, \curlywedge), d_i\left(\curlyvee, \complement\curlyvee\right), d_i\left(\curlywedge, \complement\curlywedge\right)\right\}$$

for all $\curlyvee, \curlywedge \in \Game$.
Consequently, there exists a unique element $u \in \Game$ *such that* $\complement u = u$.

Panda [20] introduced the following definitions, changing the above generalized triangle inequalities and analyzing the dynamics of the chaotic Rössler system.

Definition 1.2. [20] Let $\Game$ be a nonempty set and $\theta : \Game \times \Game \to [1, \infty)$. The mapping, defined as $d_i : \Game \times \Game \to [0, \infty)$, is an extended $(\alpha, c)-\mathbb{IM}$ if the statements (i_1) and (i_2) hold, as well as

(i_3') two constants $\alpha \in (0, 1)$ and $c \geq 0$ exist such that the subsequent statement is met for all $\curlyvee, \curlywedge, \varkappa \in \Game$

$$d_i(\curlyvee, \curlywedge) \leq \theta(\curlyvee, \curlywedge)\left[d_i(\curlyvee, \varkappa) + d_i(\varkappa, \curlywedge)\right] + c\left[d_i(\curlyvee, \varkappa)^{\alpha} d_i(\varkappa, \curlywedge)^{1-\alpha}\right].$$

Then, $(\Game, d_i)$ is considered as an extended $(\alpha, c)-\mathbb{IMS}$.

On the basis of these considerations, we formulate a controlled $(\alpha, c)-\mathbb{IMS}$ and derive a corresponding Banach contraction result within this class of spaces.

2. Fixed point results

In this part, we define a controlled $(\alpha, c)-\mathbb{IM}$ as a modification of the $(\alpha, c)-\mathbb{IM}$ introduced earlier. The construction uses a control function that adjusts the distance in a flexible way so that a larger class of contractive mappings can be treated. In this framework, various known metric-type spaces are encompassed, and Banach-type fixed-point theorems can be established in a more convenient and robust setting.

Definition 2.1. Let $\eth$ be a nonempty set. A mapping $\mathcal{C}_\mu : \eth \times \eth \to [0, \infty)$ is a controlled $(\alpha, c)-\mathbb{IM}$ if

(c_1) $\mathcal{C}_\mu(\curlyvee, \curlywedge) = 0$ if and only if $\curlyvee = \curlywedge$;

(c_2) $\mathcal{C}_\mu(\curlyvee, \curlywedge) = \mathcal{C}_\mu(\curlywedge, \curlyvee)$;

(c_3) there exist an $\alpha \in (0, 1)$, $c \geq 0$, and $\mu : \eth \times \eth \to [1, \infty)$ such that

$$\mathcal{C}_\mu(\curlyvee, \curlywedge) \leq \mu(\curlyvee, \varkappa)\, \mathcal{C}_\mu(\curlyvee, \varkappa) + \mu(\varkappa, \curlywedge)\, \mathcal{C}_\mu(\varkappa, \curlywedge) + c\left[\mathcal{C}_\mu(\curlyvee, \varkappa)^\alpha \mathcal{C}_\mu(\varkappa, \curlywedge)^{1-\alpha}\right]$$

for all $\curlyvee, \curlywedge, \varkappa \in \eth$.

We then call $(\eth, \mathcal{C}_\mu)$ a controlled $(\alpha, c)-\mathbb{IMS}$.

Proposition 2.1. *Let $(\eth, \mathcal{C}_\mu)$ be a controlled $(\alpha, c)-\mathbb{IMS}$ in the sense of Definition 2.1. If $\mu \equiv 1$, then $(\eth, \mathcal{C}_\mu)$ satisfies*

$$\mathcal{C}_\mu(\curlyvee, \curlywedge) \leq \mathcal{C}_\mu(\curlyvee, \varkappa) + \mathcal{C}_\mu(\varkappa, \curlywedge) + c\left[\mathcal{C}_\mu(\curlyvee, \varkappa)^\alpha \mathcal{C}_\mu(\varkappa, \curlywedge)^{1-\alpha}\right] \qquad (\forall \curlyvee, \curlywedge, \varkappa \in \eth).$$

Hence, the $(\eth, \mathcal{C}_\mu)$ is an $(\alpha, c)-\mathbb{IMS}$. Conversely, any $(\alpha, c)-\mathbb{IMS}$ becomes a controlled $(\alpha, c)-\mathbb{IMS}$ by choosing $\mu \equiv 1$.

Proof. Substituting $\mu \equiv 1$ into the controlled inequality yields the displayed bound. The converse is immediate by reading the same inequality with $\mu \equiv 1$. □

Proposition 2.2. *Assume $(\eth, \mathcal{C}_\mu)$ is a controlled $(\alpha, c)-\mathbb{IM}$, and a function $\theta : \eth \times \eth \to [1, \infty)$ exists such that for all $\curlyvee, \curlywedge, \varkappa \in \eth$,*

$$\mu(\curlyvee, \varkappa) = \mu(\varkappa, \curlywedge) = \theta(\curlyvee, \curlywedge).$$

Then, $(\eth, \mathcal{C}_\mu)$ satisfies

$$\mathcal{C}_\mu(\curlyvee, \curlywedge) \leq \theta(\curlyvee, \curlywedge)(\mathcal{C}_\mu(\curlyvee, \varkappa) + \mathcal{C}_\mu(\varkappa, \curlywedge)) + c\left[\mathcal{C}_\mu(\curlyvee, \varkappa)^\alpha \mathcal{C}_\mu(\varkappa, \curlywedge)^{1-\alpha}\right] \qquad (\forall \curlyvee, \curlywedge, \varkappa \in \eth),$$

that is, $(\eth, \mathcal{C}_\mu)$ is an extended $(\alpha, c)-\mathbb{IMS}$ *with moderator θ. In particular, taking $\theta \equiv 1$ recovers the classical case.*

Proof. Insert $\mu(\curlyvee, \varkappa) = \mu(\varkappa, \curlywedge) = \theta(\curlyvee, \curlywedge)$ into the controlled inequality

$$\mathcal{C}_\mu(\curlyvee, \curlywedge) \leq \theta(\curlyvee, \curlywedge)\mathcal{C}_\mu(\curlyvee, \varkappa) + \theta(\curlyvee, \curlywedge)\mathcal{C}_\mu(\varkappa, \curlywedge) + c\left[\mathcal{C}_\mu(\curlyvee, \varkappa)^\alpha \mathcal{C}_\mu(\varkappa, \curlywedge)^{1-\alpha}\right],$$

which is exactly the extended $(\alpha, c)-\mathbb{IM}$ bound. □

Remark 2.1. If instead one assumes $\mu(\curlyvee, \varkappa) = \theta_1(\curlyvee, \varkappa)$ and $\mu(\varkappa, \curlywedge) = \theta_2(\varkappa, \curlywedge)$ with $\theta_1, \theta_2 : \mho \times \mho \to [1, \infty)$ not depending on the third point, the controlled inequality reduces to

$$\mathcal{C}_\mu(\curlyvee, \curlywedge) \leq \theta_1(\curlyvee, \varkappa)\, \mathcal{C}_\mu(\curlyvee, \varkappa) + \theta_2(\varkappa, \curlywedge)\, \mathcal{C}_\mu(\varkappa, \curlywedge) + c\, [\mathcal{C}_\mu(\curlyvee, \varkappa)^\alpha \mathcal{C}_\mu(\varkappa, \curlywedge)^{1-\alpha}],$$

which is a standard two-sided "extended" variant in the literature.

In a controlled (α, c)–$\mathbb{IMS}$ $(\mho, \mathcal{C}_\mu)$, the topological structure is introduced through the notion of $\mathcal{C}_\mu$-balls. For each point $\curlyvee_0 \in \mho$ and any $\varepsilon > 0$, the corresponding $\mathcal{C}_\mu$-open and $\mathcal{C}_\mu$-closed balls are defined respectively as

$$\mathbb{B}_{\mathcal{C}_\mu}(\curlyvee_0, \varepsilon) = \{\, \curlywedge \in \mho : \mathcal{C}_\mu(\curlyvee_0, \curlywedge) < \varepsilon \,\}, \qquad \overline{\mathbb{B}}_{\mathcal{C}_\mu}(\curlyvee_0, \varepsilon) = \{\, \curlywedge \in \mho : \mathcal{C}_\mu(\curlyvee_0, \curlywedge) \leq \varepsilon \,\}.$$

Observe that $\curlyvee_0 \in \mathbb{B}_{\mathcal{C}_\mu}(\curlyvee_0, \varepsilon)$ for every $\varepsilon > 0$ and $\mathbb{B}_{\mathcal{C}_\mu}(\curlyvee_0, \varepsilon) \subseteq \overline{\mathbb{B}}_{\mathcal{C}_\mu}(\curlyvee_0, \varepsilon)$. The collection of all such open balls,

$$\mathcal{B}_{\mathcal{C}_\mu} = \{\, \mathbb{B}_{\mathcal{C}_\mu}(\curlyvee, \varepsilon) : \curlyvee \in \mho,\ \varepsilon > 0 \,\},$$

forms a basis for a topology on $\mho$, denoted by $\mathcal{T}_{\mathcal{C}_\mu}$ and referred to as the *topology induced by* $\mathcal{C}_\mu$. A subset $U \subseteq \mho$ is said to be $\mathcal{C}_\mu$-open if and only if for each $\curlyvee \in U$, there exists $\varepsilon > 0$ such that $\mathbb{B}_{\mathcal{C}_\mu}(\curlyvee, \varepsilon) \subseteq U$. The corresponding $\mathcal{C}_\mu$-closed sets are complements of $\mathcal{C}_\mu$-open sets, and the family $\mathcal{T}_{\mathcal{C}_\mu}$ satisfies the usual topological axioms that the empty set and $\mho$ itself are open. Unions of arbitrarily many open sets are open, whereas intersections of only finitely many open sets preserve openness.

This construction defines a neighborhood system $\mathcal{N}(\curlyvee) = \{\, \mathbb{B}_{\mathcal{C}_\mu}(\curlyvee, \varepsilon) : \varepsilon > 0 \,\}$ around each point $\curlyvee \in \mho$, which characterizes local proximity with respect to the generalized distance $\mathcal{C}_\mu$. In classical metric spaces, $\mathcal{T}_{\mathcal{C}_\mu}$ coincides with the standard metric topology; however, in controlled or extended interpolative settings, the function $\mathcal{C}_\mu$ may depend on additional control terms such as $\mu(\curlyvee, \curlywedge)$, and the resulting geometry may exhibit nonuniform scaling or degeneracy at infinity.

Example 2.1. Assume that $\mho$ is a set of natural numbers. Define $\mathcal{C}_\mu : \mho \times \mho \to R^+$ by $\mathcal{C}_\mu(\curlyvee, \curlywedge) = (\curlyvee - \curlywedge)^2$ and $\mu : \mho \times \mho \to [1, \infty)$ by $\mu(\curlyvee, \curlywedge) = e^{\curlyvee + \curlywedge}$. Then, $(\mho, \mathcal{C}_\mu)$ is a controlled $\left(\frac{1}{2}, 2\right)$–$\mathbb{IMS}$.

Proof. Clearly, conditions (c_1) and (c_2) hold for all $\curlyvee, \curlywedge \in \mho$.
For any $\curlyvee, \curlywedge, \varkappa \in \mho$, consider

$$\begin{aligned}
\mathcal{C}_\mu(\curlyvee, \curlywedge) &= (\curlyvee - \curlywedge)^2 \\
&= (\curlyvee - \varkappa + \varkappa - \curlywedge)^2 \\
&= (\curlyvee - \varkappa)^2 + (\varkappa - \curlywedge)^2 + 2\,(\curlyvee - \varkappa)\,(\varkappa - \curlywedge) \\
&< e^{\curlyvee + \varkappa}(\curlyvee - \varkappa)^2 + e^{\varkappa + \curlywedge}(\varkappa - \curlywedge)^2 + 2\,(\curlyvee - \varkappa)\,(\varkappa - \curlywedge) \\
&\leq \mu(\curlyvee, \varkappa)\, \mathcal{C}_\mu(\curlyvee, \varkappa) + \mu(\varkappa, \curlywedge)\, \mathcal{C}_\mu(\varkappa, \curlywedge) + 2\left[\mathcal{C}_\mu(\curlyvee, \varkappa)^{\frac{1}{2}} \mathcal{C}_\mu(\varkappa, \curlywedge)^{\frac{1}{2}}\right].
\end{aligned}$$

Thus, $(\mho, \mathcal{C}_\mu)$ is a controlled $\left(\frac{1}{2}, 2\right)$–$\mathbb{IMS}$. □

Definition 2.2. A sequence $\{\curlyvee_\flat\}$ in $\mho$ is said to be a convergent sequence in a controlled (α, c)–$\mathbb{IMS}$ $(\mho, \mathcal{C}_\mu)$ if for every $\varepsilon > 0$, there is $M = M(\epsilon) \in \mathbb{N}$ such that $\mathcal{C}_\mu(\curlyvee_\flat, \curlyvee) < \varepsilon$ for all $\flat \geq M$ and $\curlyvee \in \mho$. In this situation, $\lim_{\flat \to \infty} \curlyvee_\flat = \curlyvee$.

Definition 2.3. A sequence $\{\vee_\flat\}$ in $\backepsilon$ is said to be a Cauchy sequence in a controlled $(\alpha, c)-\mathbb{IMS}$ $(\backepsilon, \mathcal{C}_\mu)$ if for every $\varepsilon > 0$, there is $M = M(\epsilon) \in \mathbb{N}$ such that $\mathcal{C}_\mu(\vee_\flat, \vee_\hbar) < \varepsilon$ for all $\flat, \hbar \geq M$.

Definition 2.4. A controlled $(\alpha, c)-\mathbb{IMS}$ $(\backepsilon, \mathcal{C}_\mu)$ is complete if and only if every Cauchy sequence in $\backepsilon$ is convergent.

Example 2.2. Let $\backepsilon = \mathbb{R}$, and define $\mathcal{C}_\mu : \backepsilon \times \backepsilon \to [0, \infty)$ by

$$\mathcal{C}_\mu(\vee, \wedge) = \begin{cases} 0, & \vee = \wedge, \\ \dfrac{1}{1 + |\vee - \wedge|}, & \vee \neq \wedge. \end{cases}$$

Let the control function $\mu : \backepsilon \times \backepsilon \to [1, \infty)$ be

$$\mu(\vee, \wedge) = \begin{cases} 1, & \vee = \wedge, \\ \dfrac{1}{\mathcal{C}_\mu(\vee, \wedge)} = 1 + |\vee - \wedge|, & \vee \neq \wedge. \end{cases}$$

Then, for any fixed $\alpha \in (0, 1)$ and $c \geq 0$, the pair $(\backepsilon, \mathcal{C}_\mu)$ is a *controlled* $(\alpha, c)-\mathbb{IMS}$, although it is *not* an *extended* $(\alpha, c)-\mathbb{IMS}$.

Proof. The axioms (c_1)–(c_2) are immediate from the definition of $\mathcal{C}_\mu$. To verify (c_3), fix $\vee, \wedge, \varkappa \in \backepsilon$ and consider two cases.

Case A: $\varkappa \neq \vee$, *and* $\varkappa \neq \wedge$. Then, $\mathcal{C}_\mu(\vee, \varkappa), \mathcal{C}_\mu(\varkappa, \wedge) > 0$, and

$$\mu(\vee, \varkappa)\mathcal{C}_\mu(\vee, \varkappa) = 1, \qquad \mu(\varkappa, \wedge)\mathcal{C}_\mu(\varkappa, \wedge) = 1.$$

Hence,

$$\mu(\vee, \varkappa)\mathcal{C}_\mu(\vee, \varkappa) + \mu(\varkappa, \wedge)\mathcal{C}_\mu(\varkappa, \wedge) = 2 \geq 1 \geq \mathcal{C}_\mu(\vee, \wedge)$$

because $\mathcal{C}_\mu(\vee, \wedge) \leq 1$ for all $\vee, \wedge$. Therefore,

$$\mathcal{C}_\mu(\vee, \wedge) \leq \mu(\vee, \varkappa)\mathcal{C}_\mu(\vee, \varkappa) + \mu(\varkappa, \wedge)\mathcal{C}_\mu(\varkappa, \wedge) + c\, \mathcal{C}_\mu(\vee, \varkappa)^\alpha \mathcal{C}_\mu(\varkappa, \wedge)^{1-\alpha}.$$

Case B: $\varkappa = \vee$ *(the case* $\varkappa = \wedge$ *is analogous).* Then, $\mathcal{C}_\mu(\vee, \varkappa) = 0$, $\mu(\vee, \varkappa) = 1$, and

$$\mu(\varkappa, \wedge)\mathcal{C}_\mu(\varkappa, \wedge) = \mu(\vee, \wedge)\mathcal{C}_\mu(\vee, \wedge) = 1.$$

Again, $\mathcal{C}_\mu(\vee, \wedge) \leq 1$ implies the same inequality as in (c3). Thus, $(\backepsilon, \mathcal{C}_\mu)$ is a controlled $(\alpha, c)-\mathbb{IMS}$.

To see that $(\backepsilon, \mathcal{C}_\mu)$ is *not* extended, suppose by contradiction that there exists a function $\theta : \backepsilon \times \backepsilon \to [1, \infty)$ depending only on $(\vee, \wedge)$ (and not on $\varkappa$) such that, for some $\alpha \in (0, 1)$ and $c \geq 0$,

$$\mathcal{C}_\mu(\vee, \wedge) \leq \theta(\vee, \wedge)[\mathcal{C}_\mu(\vee, \varkappa) + \mathcal{C}_\mu(\varkappa, \wedge)] + c\, \mathcal{C}_\mu(\vee, \varkappa)^\alpha \mathcal{C}_\mu(\varkappa, \wedge)^{1-\alpha} \tag{2.1}$$

for all $\vee, \wedge, \varkappa \in \backepsilon$. Fix $\vee \neq \wedge$, and choose $\varkappa_\flat \in \mathbb{R}$ with $|\varkappa_\flat - \vee| = |\varkappa_\flat - \wedge| = \flat$ for $\flat \to \infty$. Then,

$$\mathcal{C}_\mu(\vee, \varkappa_\flat) = \mathcal{C}_\mu(\varkappa_\flat, \wedge) = \frac{1}{1 + \flat} \xrightarrow[\flat \to \infty]{} 0.$$

Consequently, the term appearing on the right of (2.1) fulfills

$$\theta(\curlyvee,\curlywedge)\Big[\mathcal{C}_\mu(\curlyvee,\varkappa_\flat)+\mathcal{C}_\mu(\varkappa_\flat,\curlywedge)\Big]+c\,\mathcal{C}_\mu(\curlyvee,\varkappa_\flat)^\alpha\mathcal{C}_\mu(\varkappa_\flat,\curlywedge)^{1-\alpha}=\frac{2\theta(\curlyvee,\curlywedge)+c}{1+\flat}\longrightarrow 0,$$

whereas the left-hand side remains strictly positive:

$$\mathcal{C}_\mu(\curlyvee,\curlywedge)=\frac{1}{1+|\curlyvee-\curlywedge|}>0.$$

This contradiction shows that no such θ can work for all $\varkappa$, so $(\eth,\mathcal{C}_\mu)$ fails to be an extended (α,c)–$\mathbb{IMS}$. □

Remark 2.2. Because $\mathcal{C}_\mu$ is symmetric and definite, the topology generated by the $\mathcal{C}_\mu$-balls is metrizable, for example, by the chain metric

$$d(\curlyvee,\curlywedge)=\inf\{\sum_{i=0}^{\flat-1}\mathcal{C}_\mu(u_i,u_{i+1})\}$$

over all finite chains $\curlyvee=u_0,\dots,u_\flat=\curlywedge$; hence, convergence, Cauchy sequences, and completeness are understood in the usual metric sense.

We now proceed to prove a Banach-type contraction result in the setting of a controlled (α,c)–$\mathbb{IMS}$. This result guarantees both the existence and uniqueness of a fixed-point for a suitable contraction mapping defined on $(\eth,\mathcal{C}_\mu)$.

Theorem 2.1. *Let $(\eth,\mathcal{C}_\mu)$ be a complete, controlled (α,c)–$\mathbb{IMS}$. If a mapping $\complement:\eth\to\eth$ satisfies the contractive condition*

$$\mathcal{C}_\mu(\complement\curlyvee,\complement\curlywedge)\le\eth\,\mathcal{C}_\mu(\curlyvee,\curlywedge),\qquad \textit{for all } \curlyvee,\curlywedge\in\eth, \tag{2.2}$$

where $\eth\in(0,1)$, then $\complement$ possesses a unique fixed-point in $\eth$. Moreover, for any $\curlyvee_0\in\eth$, the sequence $\{\complement^\flat\curlyvee_0\}$ converges to this fixed-point.

Proof. We begin by constructing an iterative sequence starting from an arbitrary point in $\eth$. Choose an initial element $\curlyvee_0\in\eth$ and form the sequence $\{\curlyvee_\flat\}$ by

$$\curlyvee_{\flat+1}=\complement\curlyvee_\flat,\qquad \flat\in\mathbb{N}.$$

If there exists $\flat_0\in\mathbb{N}$ such that $\curlyvee_{\flat_0}=\curlyvee_{\flat_0+1}$, then $\curlyvee_{\flat_0}$ is a fixed-point of $\complement$, and the proof is complete. Hence, without loss of generality, we assume that $\curlyvee_\flat\neq\curlyvee_{\flat+1}$ for all $\flat\in\mathbb{N}$, which implies that $\mathcal{C}_\mu(\curlyvee_\flat,\curlyvee_{\flat+1})>0$ for every $\flat\in\mathbb{N}$. From (2.2), we get

$$\mathcal{C}_\mu(\curlyvee_\flat,\curlyvee_{\flat+1})\le\eth\mathcal{C}_\mu(\curlyvee_{\flat-1},\curlyvee_\flat) \tag{2.3}$$

for all $\flat\in\mathbb{N}$.
Regarding the above, by iteration, we have

$$\mathcal{C}_\mu(\curlyvee_\flat,\curlyvee_{\flat+1})\le\eth^\flat\mathcal{C}_\mu(\curlyvee_0,\curlyvee_1) \tag{2.4}$$

for all $\flat \in \mathbb{N}$. Passing to the limit as $\flat \to \infty$ on both sides of (2.3), we attain that

$$\lim_{\flat\to\infty} \mathcal{C}_\mu(\curlyvee_\flat, \curlyvee_{\flat+1}) = 0. \tag{2.5}$$

Thus, there exists $p \in \mathbb{N}$ such that for all $\flat \geq p$,

$$\mathcal{C}_\mu(\curlyvee_\flat, \curlyvee_{\flat+1}) \leq 1. \tag{2.6}$$

Now, we will demonstrate that the sequence $\{\curlyvee_\flat\}$ is Cauchy.
With this goal in mind, we assert that

$$\lim_{\flat\to\infty} \mathcal{C}_\mu(\curlyvee_\flat, \curlyvee_{\flat+r+1}) = 0. \tag{2.7}$$

We prove this by a straightforward induction, starting from the following limit:

$$\begin{aligned} \mathcal{C}_\mu(\curlyvee_\flat, \curlyvee_{\flat+2}) \leq\; & \mu(\curlyvee_\flat, \curlyvee_{\flat+1})\, \mathcal{C}_\mu(\curlyvee_\flat, \curlyvee_{\flat+1}) + \mu(\curlyvee_{\flat+1}, \curlyvee_{\flat+2})\, \mathcal{C}_\mu(\curlyvee_{\flat+1}, \curlyvee_{\flat+2}) \\ & + c\left[\Big(\mathcal{C}_\mu(\curlyvee_\flat, \curlyvee_{\flat+1})\Big)^\alpha \Big(\mathcal{C}_\mu(\curlyvee_{\flat+1}, \curlyvee_{\flat+2})\Big)^{1-\alpha}\right]. \end{aligned} \tag{2.8}$$

Letting $\flat \to \infty$ in the preceding inequality and combining this with (2.5), we deduce that

$$\lim_{\flat\to\infty} \mathcal{C}_\mu(\curlyvee_\flat, \curlyvee_{\flat+2}) = 0. \tag{2.9}$$

We further note that

$$\begin{aligned} \mathcal{C}_\mu(\curlyvee_\flat, \curlyvee_{\flat+3}) \leq\; & \mu(\curlyvee_\flat, \curlyvee_{\flat+2})\, \mathcal{C}_\mu(\curlyvee_\flat, \curlyvee_{\flat+2}) + \mu(\curlyvee_{\flat+2}, \curlyvee_{\flat+3})\, \mathcal{C}_\mu(\curlyvee_{\flat+2}, \curlyvee_{\flat+3}) \\ & + c\left[\Big(\mathcal{C}_\mu(\curlyvee_\flat, \curlyvee_{\flat+2})\Big)^\alpha \Big(\mathcal{C}_\mu(\curlyvee_{\flat+2}, \curlyvee_{\flat+3})\Big)^{1-\alpha}\right]. \end{aligned} \tag{2.10}$$

Taking limits in (2.5) and (2.10), we obtain that

$$\lim_{\flat\to\infty} \mathcal{C}_\mu(\curlyvee_\flat, \curlyvee_{\flat+3}) = 0. \tag{2.11}$$

Now, continuing this way, we have

$$\lim_{\flat\to\infty} \mathcal{C}_\mu(\curlyvee_\flat, \curlyvee_{\flat+r}) = 0 \tag{2.12}$$

for some $\flat \in \mathbb{N}$. From this, we arrive at

$$\begin{aligned} \mathcal{C}_\mu(\curlyvee_\flat, \curlyvee_{\flat+r+1}) \leq\; & \mu(\curlyvee_\flat, \curlyvee_{\flat+r})\, \mathcal{C}_\mu(\curlyvee_\flat, \curlyvee_{\flat+r}) + \mu(\curlyvee_{\flat+r}, \curlyvee_{\flat+r+1})\, \mathcal{C}_\mu(\curlyvee_{\flat+r}, \curlyvee_{\flat+r+1}) \\ & + c\left[\Big(\mathcal{C}_\mu(\curlyvee_\flat, \curlyvee_{\flat+r})\Big)^\alpha \Big(\mathcal{C}_\mu(\curlyvee_{\flat+r}, \curlyvee_{\flat+r+1})\Big)^{1-\alpha}\right]. \end{aligned} \tag{2.13}$$

Passing to the limit $\flat \to \infty$ in the preceding inequality, we obtain

$$\lim_{\flat\to\infty} \mathcal{C}_\mu(\curlyvee_\flat, \curlyvee_{\flat+r+1}) = 0. \tag{2.14}$$

Hence, Condition (2.7) holds. Using this fact, we observe that

$$\mathcal{C}_{\mu}\left(\curlyvee_{\flat+1},\curlyvee_{\hbar}\right)\leq 1 \tag{2.15}$$

for all $\hbar>\flat>p$ for some $p\in\mathbb{N}$.
Also, because $\alpha\in(0,1)$, we obtain that

$$\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat+1},\curlyvee_{\hbar}\right)\right)^{1-\alpha}\leq 1. \tag{2.16}$$

Using (2.6) and (2.16), for a large enough integer a,b such that $\hbar>\flat>p$, consider

$$\begin{aligned}
\mathcal{C}_{\mu}\left(\curlyvee_{\flat},\curlyvee_{\hbar}\right) &\leq \mu\left(\curlyvee_{\flat},\curlyvee_{\flat+1}\right)\mathcal{C}_{\mu}\left(\curlyvee_{\flat},\curlyvee_{\flat+1}\right)+\mu\left(\curlyvee_{\flat+1},\curlyvee_{\hbar}\right)\mathcal{C}_{\mu}\left(\curlyvee_{\flat+1},\curlyvee_{\hbar}\right)\\
&\quad +c\left[\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat},\curlyvee_{\flat+1}\right)\right)^{\alpha}\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat+1},\curlyvee_{\hbar}\right)\right)^{1-\alpha}\right]\\
&\leq \mu\left(\curlyvee_{\flat},\curlyvee_{\flat+1}\right)\mathcal{C}_{\mu}\left(\curlyvee_{\flat},\curlyvee_{\flat+1}\right)+\mu\left(\curlyvee_{\flat+1},\curlyvee_{\hbar}\right)\Big[\mu\left(\curlyvee_{\flat+1},\curlyvee_{\flat+2}\right)\mathcal{C}_{\mu}\left(\curlyvee_{\flat+1},\curlyvee_{\flat+2}\right)\\
&\quad +\mu\left(\curlyvee_{\flat+2},\curlyvee_{\hbar}\right)\mathcal{C}_{\mu}\left(\curlyvee_{\flat+2},\curlyvee_{\hbar}\right)+c\left[\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat+1},\curlyvee_{\flat+2}\right)\right)^{\alpha}\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat+2},\curlyvee_{\hbar}\right)\right)^{1-\alpha}\right]\Big]\\
&\quad +c\left[\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat},\curlyvee_{\flat+1}\right)\right)^{\alpha}\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat+1},\curlyvee_{\hbar}\right)\right)^{1-\alpha}\right]\\
&= \mu\left(\curlyvee_{\flat},\curlyvee_{\flat+1}\right)\mathcal{C}_{\mu}\left(\curlyvee_{\flat},\curlyvee_{\flat+1}\right)+\mu\left(\curlyvee_{\flat+1},\curlyvee_{\hbar}\right)\mu\left(\curlyvee_{\flat+1},\curlyvee_{\flat+2}\right)\mathcal{C}_{\mu}\left(\curlyvee_{\flat+1},\curlyvee_{\flat+2}\right)\\
&\quad +\mu\left(\curlyvee_{\flat+1},\curlyvee_{\hbar}\right)\mu\left(\curlyvee_{\flat+2},\curlyvee_{\hbar}\right)\mathcal{C}_{\mu}\left(\curlyvee_{\flat+2},\curlyvee_{\hbar}\right)\\
&\quad +\mu\left(\curlyvee_{\flat+1},\curlyvee_{\hbar}\right)c\left[\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat+1},\curlyvee_{\flat+2}\right)\right)^{\alpha}\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat+2},\curlyvee_{\hbar}\right)\right)^{1-\alpha}\right]\\
&\quad +c\left[\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat},\curlyvee_{\flat+1}\right)\right)^{\alpha}\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat+1},\curlyvee_{\hbar}\right)\right)^{1-\alpha}\right]\\
&\leq \mu\left(\curlyvee_{\flat},\curlyvee_{\flat+1}\right)\mathcal{C}_{\mu}\left(\curlyvee_{\flat},\curlyvee_{\flat+1}\right)+\mu\left(\curlyvee_{\flat+1},\curlyvee_{\hbar}\right)\mu\left(\curlyvee_{\flat+1},\curlyvee_{\flat+2}\right)\mathcal{C}_{\mu}\left(\curlyvee_{\flat+1},\curlyvee_{\flat+2}\right)\\
&\quad +\mu\left(\curlyvee_{\flat+1},\curlyvee_{\hbar}\right)\mu\left(\curlyvee_{\flat+2},\curlyvee_{\hbar}\right)\Big[\mu\left(\curlyvee_{\flat+2},\curlyvee_{\flat+3}\right)\mathcal{C}_{\mu}\left(\curlyvee_{\flat+2},\curlyvee_{\flat+3}\right)\\
&\quad +\mu\left(\curlyvee_{\flat+3},\curlyvee_{\hbar}\right)\mathcal{C}_{\mu}\left(\curlyvee_{\flat+3},\curlyvee_{\hbar}\right)+c\left[\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat+2},\curlyvee_{\flat+3}\right)\right)^{\alpha}\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat+3},\curlyvee_{\hbar}\right)\right)^{1-\alpha}\right]\Big]\\
&\quad +\mu\left(\curlyvee_{\flat+1},\curlyvee_{\hbar}\right)c\left[\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat+1},\curlyvee_{\flat+2}\right)\right)^{\alpha}\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat+2},\curlyvee_{\hbar}\right)\right)^{1-\alpha}\right]\\
&\quad +c\left[\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat},\curlyvee_{\flat+1}\right)\right)^{\alpha}\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat+1},\curlyvee_{\hbar}\right)\right)^{1-\alpha}\right]\\
&\leq \mu\left(\curlyvee_{\flat},\curlyvee_{\flat+1}\right)\eth^{\flat}\mathcal{C}_{\mu}\left(\curlyvee_{0},\curlyvee_{1}\right)+\mu\left(\curlyvee_{\flat+1},\curlyvee_{\hbar}\right)\mu\left(\curlyvee_{\flat+1},\curlyvee_{\flat+2}\right)\eth^{\flat+1}\mathcal{C}_{\mu}\left(\curlyvee_{0},\curlyvee_{1}\right)\\
&\quad +\mu\left(\curlyvee_{\flat+1},\curlyvee_{\hbar}\right)\mu\left(\curlyvee_{\flat+2},\curlyvee_{\hbar}\right)\mu\left(\curlyvee_{\flat+2},\curlyvee_{\flat+3}\right)\eth^{\flat+2}\mathcal{C}_{\mu}\left(\curlyvee_{0},\curlyvee_{1}\right)\\
&\quad +\mu\left(\curlyvee_{\flat+1},\curlyvee_{\hbar}\right)\mu\left(\curlyvee_{\flat+2},\curlyvee_{\hbar}\right)\mu\left(\curlyvee_{\flat+3},\curlyvee_{\hbar}\right)\mathcal{C}_{\mu}\left(\curlyvee_{\flat+3},\curlyvee_{\hbar}\right)\\
&\quad +\mu\left(\curlyvee_{\flat+1},\curlyvee_{\hbar}\right)\mu\left(\curlyvee_{\flat+2},\curlyvee_{\hbar}\right)c\left[\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat+2},\curlyvee_{\flat+3}\right)\right)^{\alpha}\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat+3},\curlyvee_{\hbar}\right)\right)^{1-\alpha}\right]\\
&\quad +\mu\left(\curlyvee_{\flat+1},\curlyvee_{\hbar}\right)c\left[\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat+1},\curlyvee_{\flat+2}\right)\right)^{\alpha}\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat+2},\curlyvee_{\hbar}\right)\right)^{1-\alpha}\right]\\
&\quad +c\left[\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat},\curlyvee_{\flat+1}\right)\right)^{\alpha}\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat+1},\curlyvee_{\hbar}\right)\right)^{1-\alpha}\right].
\end{aligned} \tag{2.17}$$

In view of the preceding observations, we conclude that

$$
\begin{aligned}
\mathcal{C}_{\mu}\left(\curlyvee_{\flat}, \curlyvee_{\hbar}\right) \leq\ & \mathcal{C}_{\mu}\left(\curlyvee_{0}, \curlyvee_{1}\right) \sum_{p=1}^{\infty} \eth^{\flat+p-1} \mu\left(\curlyvee_{\flat+p-1}, \curlyvee_{\flat+p}\right) \prod_{j=1}^{p-1} \mu\left(\curlyvee_{\flat+j}, \curlyvee_{\hbar}\right) \\
& +c \sum_{p=1}^{\infty}\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat+p-1}, \curlyvee_{\flat+p}\right)\right)^{\alpha}\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat+p}, \curlyvee_{\hbar}\right)\right)^{1-\alpha} \prod_{j=1}^{p-1} \mu\left(\curlyvee_{\flat+j}, \curlyvee_{\hbar}\right) \\
\leq\ & \mathcal{C}_{\mu}\left(\curlyvee_{0}, \curlyvee_{1}\right) \sum_{p=1}^{\infty} \eth^{\flat+p-1} \mu\left(\curlyvee_{\flat+p-1}, \curlyvee_{\flat+p}\right) \prod_{j=1}^{p-1} \mu\left(\curlyvee_{\flat+j}, \curlyvee_{\hbar}\right) \\
& +c \mathcal{C}_{\mu}\left(\curlyvee_{0}, \curlyvee_{1}\right) \sum_{p=1}^{\infty}\left(\eth^{\flat+p-1}\right)^{\alpha}\left(\mathcal{C}_{\mu}\left(\curlyvee_{\flat+p}, \curlyvee_{\hbar}\right)\right)^{1-\alpha} \prod_{j=1}^{p-1} \mu\left(\curlyvee_{\flat+j}, \curlyvee_{\hbar}\right).
\end{aligned}
\tag{2.18}
$$

Let $\mathcal{U}_{i}=\eth^{i} \prod_{j=0}^{i-1}\left[1+\left(\eth^{j}\right)^{\alpha}\right] \mu\left(\curlyvee_{\flat+j}, \curlyvee_{\hbar}\right)$. By ratio test, $\sum_{i=0}^{\infty} \mathcal{U}_{i}$ converges because $\lim_{i \rightarrow \infty}\left|\frac{\mathcal{U}_{i+1}}{\mathcal{U}_{i}}\right|<1$ as $\eth \in(0,1)$. Hence, from (2.18), we obtain that $\{\curlyvee_{\flat}\}$ is a Cauchy sequence. Owing to fact that $(\Game, \mathcal{C}_{\mu})$ is a complete, controlled $(\alpha, c)-\mathbb{IMS}$, $\{\curlyvee_{\flat}\}$ converges to $\curlyvee' \in \Game$. The next assertion is that $\curlyvee'$ is a fixed-point of $\complement$. Note that

$$
\mathcal{C}_{\mu}\left(\curlyvee_{\flat+1}, \complement \curlyvee'\right)=\mathcal{C}_{\mu}\left(\complement \curlyvee_{\flat}, \complement \curlyvee'\right) \leq \eth \mathcal{C}_{\mu}\left(\curlyvee_{\flat}, \curlyvee'\right). \tag{2.19}
$$

Taking the limit as $\flat \rightarrow \infty$, we obtain that $\curlyvee'=\complement \curlyvee'$. Therefore, our assertion is true.
To prove uniquness, let us assume $\curlyvee'$ and $\curlywedge'$ are two fixed-points of $\complement$. Then, we get

$$
\mathcal{C}_{\mu}\left(\curlyvee', \curlywedge'\right)=\mathcal{C}_{\mu}\left(\complement \curlyvee', \complement \curlywedge'\right) \leq \eth \mathcal{C}_{\mu}\left(\curlyvee', \curlywedge'\right)<\mathcal{C}_{\mu}\left(\curlyvee', \curlywedge'\right), \tag{2.20}
$$

a contradiction. Hence, $\curlyvee'=\curlywedge'$. □

As a direct implication of Theorem 2.1, any self-mapping that satisfies the contractive condition in a complete, controlled $(\alpha, c)-\mathbb{IMS}$ ensures the existence of a stable equilibrium point, interpreted as its unique fixed-point. This theoretical foundation not only guarantees the existence and uniqueness of fixed-points but also provides a constructive tool for verifying stability in nonlinear fractional systems. The established framework bridges abstract fixed-point theory with practical models, offering an analytical means to describe steady-state behavior, convergence, and boundedness in complex dynamical environments. In the subsequent section, we employ this framework to investigate the fractional-order Chen attractor governed by the Atangana–Baleanu derivative, demonstrating the applicability and effectiveness of the proposed metric structure in analyzing chaotic fractional systems.

3. Application to fractional order Chen attractor with Atangana–Baleanu–Caputo derivative

The classical Chen chaotic system is known as the subsequent set of nonlinear differential equations as follows:

$$
\begin{aligned}
\dot{\curlyvee}(\eta) &= a(\curlywedge(\eta)-\curlyvee(\eta)), \\
\dot{\curlywedge}(\eta) &= (c-a) \curlyvee(\eta)-\curlyvee(\eta) \varkappa(\eta)+c \curlywedge(\eta), \\
\dot{\varkappa}(\eta) &= \curlyvee(\eta) \curlywedge(\eta)-b \varkappa(\eta),
\end{aligned}
$$

where a, b, and c are positive real constants, and $\eta \in[0, T],\ T>0$.

By replacing the classical first-order derivatives with the Atangana–Baleanu fractional derivative in the Caputo sense ($\mathcal{ABC}$), the system becomes

$$
\begin{aligned}
{}_{0}^{\mathcal{ABC}}\mathcal{D}_{\eta}^{\ell} \curlyvee (\eta) &= a(\curlywedge(\eta) - \curlyvee(\eta)),\\
{}_{0}^{\mathcal{ABC}}\mathcal{D}_{\eta}^{\ell} \curlywedge (\eta) &= (c-a) \curlyvee (\eta) - \curlyvee(\eta)\varkappa(\eta) + c \curlywedge (\eta),\\
{}_{0}^{\mathcal{ABC}}\mathcal{D}_{\eta}^{\ell}\varkappa (\eta) &= \curlyvee(\eta) \curlywedge (\eta) - b\varkappa(\eta),
\end{aligned}
$$

where ${}_{0}^{\mathcal{ABC}}\mathcal{D}_{\eta}^{\ell}$ denotes the Atangana–Baleanu fractional derivative of order $\ell \in (0,1)$. To simplify the notation, we define

$$
\begin{aligned}
&\mathcal{A}_1(\eta, \curlyvee, \curlywedge, \varkappa) := a\left(\curlywedge(\eta) - \curlyvee(\eta)\right),\\
&\mathcal{A}_2(\eta, \curlyvee, \curlywedge, \varkappa) := (c-a) \curlyvee (\eta) - \curlyvee(\eta)\varkappa(\eta) + c \curlywedge (\eta),\\
&\mathcal{A}_3(\eta, \curlyvee, \curlywedge, \varkappa) := \curlyvee(\eta) \curlywedge (\eta) - b\varkappa(\eta).
\end{aligned}
\tag{3.1}
$$

Then, the system becomes

$$
\begin{aligned}
{}_{0}^{\mathcal{ABC}}\mathcal{D}_{\eta}^{\ell} \curlyvee (\eta) &= \mathcal{A}_1(\eta, \curlyvee, \curlywedge, \varkappa),\\
{}_{0}^{\mathcal{ABC}}\mathcal{D}_{\eta}^{\ell} \curlywedge (\eta) &= \mathcal{A}_2(\eta, \curlyvee, \curlywedge, \varkappa),\\
{}_{0}^{\mathcal{ABC}}\mathcal{D}_{\eta}^{\ell}\varkappa(\eta) &= \mathcal{A}_3(\eta, \curlyvee, \curlywedge, \varkappa).
\end{aligned}
\tag{3.2}
$$

3.1. Application of the fixed-point operator

To analyze existence and uniqueness of solutions for the Chen system of fractional order endowed with the Atangana–Baleanu–Caputo ($\mathcal{ABC}$) derivative, we first cast the system into an equivalent integral formulation. This representation enables the application of fixed-point theory within a suitable Banach space framework.

Employing the integral representation of the Atangana–Baleanu–Caputo operator, the Chen system of fractional order can then be written as the following system of Volterra-type integral equations:

$$
\begin{aligned}
\curlyvee(\eta) - \curlyvee(0) &= \tfrac{1-\ell}{\mathcal{N}(\ell)}\mathcal{A}_1(\eta, \curlyvee, \curlywedge, \varkappa) + \tfrac{\ell}{\mathcal{N}(\ell)\Gamma(\ell)} \int_0^{\eta} (\eta-\rho)^{\ell-1}\mathcal{A}_1(\rho, \curlyvee, \curlywedge, \varkappa)\, d\rho,\\
\curlywedge(\eta) - \curlywedge(0) &= \tfrac{1-\ell}{\mathcal{N}(\ell)}\mathcal{A}_2(\eta, \curlyvee, \curlywedge, \varkappa) + \tfrac{\ell}{\mathcal{N}(\ell)\Gamma(\ell)} \int_0^{\eta} (\eta-\rho)^{\ell-1}\mathcal{A}_2(\rho, \curlyvee, \curlywedge, \varkappa)\, d\rho,\\
\varkappa(\eta) - \varkappa(0) &= \tfrac{1-\ell}{\mathcal{N}(\ell)}\mathcal{A}_3(\eta, \curlyvee, \curlywedge, \varkappa) + \tfrac{\ell}{\mathcal{N}(\ell)\Gamma(\ell)} \int_0^{\eta} (\eta-\rho)^{\ell-1}\mathcal{A}_3(\rho, \curlyvee, \curlywedge, \varkappa)\, d\rho.
\end{aligned}
$$

In this context, $\mathcal{N}(\ell)$ represents the normalization function for the $\mathcal{ABC}$ derivative, usually expressed as

$$
\mathcal{N}(\ell) = 1 - \ell + \frac{\ell}{\Gamma(\ell)}.
$$

To construct iterative approximations of the solution, a recursive formulation based on the Volterra integral representation is introduced. Let

$$
\curlyvee_0(\eta) = \curlyvee(0), \quad \curlywedge_0(\eta) = \curlywedge(0), \quad \varkappa_0(\eta) = \varkappa(0),
$$

and for $\flat \geq 1$,

$$
\begin{aligned}
\curlyvee_{\flat}(\eta) &= \frac{1-\ell}{\mathcal{N}(\ell)}\mathcal{A}_1(\eta, \curlyvee_{\flat-1}, \curlywedge_{\flat-1}, \varkappa_{\flat-1}) + \frac{\ell}{\mathcal{N}(\ell)\Gamma(\ell)}\int_0^{\eta}(\eta-\rho)^{\ell-1}\mathcal{A}_1(\rho, \curlyvee_{\flat-1}, \curlywedge_{\flat-1}, \varkappa_{\flat-1})\,d\rho,\\
\curlywedge_{\flat}(\eta) &= \frac{1-\ell}{\mathcal{N}(\ell)}\mathcal{A}_2(\eta, \curlyvee_{\flat-1}, \curlywedge_{\flat-1}, \varkappa_{\flat-1}) + \frac{\ell}{\mathcal{N}(\ell)\Gamma(\ell)}\int_0^{\eta}(\eta-\rho)^{\ell-1}\mathcal{A}_2(\rho, \curlyvee_{\flat-1}, \curlywedge_{\flat-1}, \varkappa_{\flat-1})\,d\rho,\\
\varkappa_{\flat}(\eta) &= \frac{1-\ell}{\mathcal{N}(\ell)}\mathcal{A}_3(\eta, \curlyvee_{\flat-1}, \curlywedge_{\flat-1}, \varkappa_{\flat-1}) + \frac{\ell}{\mathcal{N}(\ell)\Gamma(\ell)}\int_0^{\eta}(\eta-\rho)^{\ell-1}\mathcal{A}_3(\rho, \curlyvee_{\flat-1}, \curlywedge_{\flat-1}, \varkappa_{\flat-1})\,d\rho.
\end{aligned}
$$

This recursive formulation serves as the basis for constructing approximations that converge to the solution under suitable contractive conditions.

To analyze convergence and guarantee the existence and uniqueness of the solution, we construct an operator in a Banach space setting.

Let $\Game = C([0,T];\mathbb{R}^3)$ be the Banach space of continuous vector-valued functions $\mathbb{X}(\eta) = (\curlyvee(\eta), \curlywedge(\eta), \varkappa(\eta))$ equipped with the supremum norm.

Define the operator $\mathcal{B} : \Game \to \Game$ based on the Volterra integral form of the $\mathcal{ABC}$ fractional Chen system as

$$
\begin{aligned}
\mathcal{B}_1 \curlyvee (\eta) &= \curlyvee(0) + \frac{1-\ell}{\mathcal{N}(\ell)}\mathcal{A}_1(\eta, \curlyvee, \curlywedge, \varkappa) + \frac{\ell}{\mathcal{N}(\ell)\Gamma(\ell)}\int_0^{\eta}(\eta-\rho)^{\ell-1}\mathcal{A}_1(\rho, \curlyvee, \curlywedge, \varkappa)\,d\rho,\\
\mathcal{B}_2 \curlywedge (\eta) &= \curlywedge(0) + \frac{1-\ell}{\mathcal{N}(\ell)}\mathcal{A}_2(\eta, \curlyvee, \curlywedge, \varkappa) + \frac{\ell}{\mathcal{N}(\ell)\Gamma(\ell)}\int_0^{\eta}(\eta-\rho)^{\ell-1}\mathcal{A}_2(\rho, \curlyvee, \curlywedge, \varkappa)\,d\rho,\\
\mathcal{B}_3\varkappa(\eta) &= \varkappa(0) + \frac{1-\ell}{\mathcal{N}(\ell)}\mathcal{A}_3(\eta, \curlyvee, \curlywedge, \varkappa) + \frac{\ell}{\mathcal{N}(\ell)\Gamma(\ell)}\int_0^{\eta}(\eta-\rho)^{\ell-1}\mathcal{A}_3(\rho, \curlyvee, \curlywedge, \varkappa)\,d\rho.
\end{aligned}
$$

The combined operator is then expressed as

$$
\mathcal{B}(\mathbb{X})(\eta) = \left(\mathcal{B}_1 \curlyvee (\eta), \mathcal{B}_2 \curlywedge (\eta), \mathcal{B}_3\varkappa(\eta)\right),
$$

which will serve as the foundation for proving the existence and uniqueness of the fractional Chen system.

3.2. Metric structure and contraction condition

Let us define a generalized distance function $\mathcal{C}_{\mu} : \Game \times \Game \to \mathbb{R}^{+}$ as

$$
\mathcal{C}_{\mu}(\curlyvee, \curlywedge) = \sup_{\eta\in[0,T]} \|\curlyvee - \curlywedge\|^2 .
$$

Let $\mu(\curlyvee, \curlywedge) = \exp\left(\|\curlyvee\| + \|\curlywedge\| + 3\right)$, satisfying the conditions of a controlled $(\alpha, c)-\mathbb{IMS}$ $(\Game, \mathcal{C}_{\mu})$.

Let us assume the following inequality:

$$
\left(\frac{\eta^{\ell} + \Gamma(\ell)(1-\ell)}{\mathcal{N}(\ell)\Gamma(\ell)}\theta\right)^2 < \eth, \quad \eth \in (0,1), \tag{3.3}
$$

where $\theta^2 = \max\left(a^2, b^2, c^2\right)$.

By using the assumption and the properties

(i) $|\mathcal{A}_1(\eta, \curlyvee_1, \curlywedge, \varkappa) - \mathcal{A}_1(\eta, \curlyvee_2, \curlywedge, \varkappa)|^2 \leq a^2 |\curlyvee_1(\eta) - \curlyvee_2(\eta)|^2$,
(ii) $|\mathcal{A}_2(\eta, \curlyvee, \curlywedge_1, \varkappa) - \mathcal{A}_2(\eta, \curlyvee, \curlywedge_2, \varkappa)|^2 \leq c^2 |\curlywedge_1(\eta) - \curlywedge_2(\eta)|^2$,
(iii) $|\mathcal{A}_3(\eta, \curlyvee, \curlywedge, \varkappa_1) - \mathcal{A}_3(\eta, \curlyvee, \curlywedge, \varkappa_2)|^2 \leq b^2 |\varkappa_1(\eta) - \varkappa_2(\eta)|^2$,

we have for the first component

$$
\begin{aligned}
&|\mathcal{B}_1 \curlyvee_1 (\eta) - \mathcal{B}_1 \curlyvee_2 (\eta)|^2 \\
&= \left| \frac{1-\ell}{\mathcal{N}(\ell)} (\mathcal{A}_1(\eta, \curlyvee, \curlyvee_1, \varkappa) - \mathcal{A}_1(\eta, \curlyvee, \curlyvee_2, \varkappa)) + \frac{\ell}{\mathcal{N}(\ell)\Gamma(\ell)} \int_0^\eta (\eta - \rho)^{\ell-1} (\mathcal{A}_1(\rho, \curlyvee, \curlyvee_1, \varkappa) - \mathcal{A}_1(\rho, \curlyvee, \curlyvee_2, \varkappa))\, d\rho \right|^2 \\
&\leq \left(\frac{1-\ell}{\mathcal{N}(\ell)}\right)^2 |\mathcal{A}_1(\eta, \curlyvee, \curlyvee_1, \varkappa) - \mathcal{A}_1(\eta, \curlyvee, \curlyvee_2, \varkappa)|^2 \\
&\quad + \left(\frac{\ell}{\mathcal{N}(\ell)\Gamma(\ell)}\right)^2 |\mathcal{A}_1(\eta, \curlyvee, \curlyvee_1, \varkappa) - \mathcal{A}_1(\eta, \curlyvee, \curlyvee_2, \varkappa)|^2 \left[\int_0^\eta (\eta-\rho)^{\ell-1} d\rho\right]^2 \\
&\quad + 2\frac{\ell(1-\ell)}{\mathcal{N}(\ell)^2\Gamma(\ell)} |\mathcal{A}_1(\eta, \curlyvee, \curlyvee_1, \varkappa) - \mathcal{A}_1(\eta, \curlyvee, \curlyvee_2, \varkappa)|^2 \left[\int_0^\eta (\eta-\rho)^{\ell-1} d\rho\right] \\
&= \left[\left(\frac{1-\ell}{\mathcal{N}(\ell)}\right)^2 + \left(\frac{\eta^\ell}{\mathcal{N}(\ell)\Gamma(\ell)}\right)^2 + \frac{2\eta^\ell(1-\ell)}{\mathcal{N}(\ell)^2\Gamma(\ell)}\right] |\mathcal{A}_1(\eta, \curlyvee, \curlyvee_1, \varkappa) - \mathcal{A}_1(\eta, \curlyvee, \curlyvee_2, \varkappa)|^2 \\
&\leq \left(a\frac{\eta^\ell + \Gamma(\ell)(1-\ell)}{\mathcal{N}(\ell)}\right)^2 | \curlyvee_1 (\eta) - \curlyvee_2(\eta)|^2 \\
&\leq \eth | \curlyvee_1 (\eta) - \curlyvee_2(\eta)|^2.
\end{aligned}
$$

From the inequality stated above, we obtain

$$\mathcal{C}_\mu(\mathcal{B}_1\curlyvee_1(\eta), \mathcal{B}_1\curlyvee_2(\eta)) \leq \eth\mathcal{C}_\mu (\curlyvee_1(\eta), \curlyvee_2(\eta)).$$

Likewise, we gain

$$\mathcal{C}_\mu(\mathcal{B}_2\curlywedge_1(\eta), \mathcal{B}_2\curlywedge_2(\eta)) \leq \eth\mathcal{C}_\mu(\curlywedge_1(\eta), \curlywedge_2(\eta))$$

and

$$\mathcal{C}_\mu(\mathcal{B}_3\varkappa_1(\eta), \mathcal{B}_3\varkappa_2(\eta)) \leq \eth\mathcal{C}_\mu(\varkappa_1(\eta), \varkappa_2(\eta)).$$

Hence, Theorem 2.1 applies and guarantees a unique solution for the fractional-order Chen system for $\eta \in [0, T]$, $T > 0$.

Remark 3.1. Consider the sufficient contraction condition (3.3),

$$q(T) = \left(\frac{T^\ell + \Gamma(\ell)(1-\ell)}{N(\ell)\Gamma(\ell)}\,\theta\right)^2 < 1, \qquad N(\ell) = 1 - \ell + \frac{\ell}{\Gamma(\ell)}.$$

Note that the minimal value of $q(T)$ is attained in the limit $T \to 0^+$ and is given by

$$q_{\min} = \lim_{T\to 0^+} q(T) = \left(\frac{1-\ell}{N(\ell)}\,\theta\right)^2.$$

Consequently, if

$$\frac{1-\ell}{N(\ell)} \geq \frac{1}{\theta},$$

then $q_{\min} \geq 1$, and hence, the above contraction condition cannot be satisfied for any $T > 0$. Therefore, the inequality

$$\frac{1-\ell}{N(\ell)} < \frac{1}{\theta} \tag{3.4}$$

is a necessary feasibility requirement for the existence of a time interval $[0, T]$ on which the contraction-based existence and uniqueness result holds.

4. Numerical investigation of Chen attractors

The contraction condition established in Section 3 guarantees the local well-posedness of the $\mathcal{ABC}$-fractional Chen system. In this section, we complement the theoretical analysis with a geometric interpretation of the admissibility condition and corresponding numerical simulations.

4.1. Admissibility structure in the fractional parameter space

Recall that (3.4) can be rewritten in the equivalent form

$$q(\ell, \theta) = \theta \frac{1-\ell}{\mathcal{N}(\ell)} < 1,$$

where $\mathcal{N}(\ell) = 1 - \ell + \frac{\ell}{\Gamma(\ell)}$. Thus, the admissible region corresponds to the sublevel set $q(\ell, \theta) < 1$.

Figure 1 illustrates the admissibility condition for the four parameter configurations reported in [21]. For each fixed parameter set $[a, b, c]$, the intersection of $\frac{1-\ell}{\mathcal{N}(\ell)}$ with $\frac{1}{\theta}$ determines a unique critical fractional threshold ℓ^*. The region $\ell > \ell^*$ corresponds to the contraction-admissible regime.

To provide a global geometric perspective, Figure 2 depicts the boundary curve defined by

$$q(\ell, \theta) = 1$$

in the (ℓ, θ)-plane. This curve separates the admissible region ($q < 1$) from the inadmissible regime ($q > 1$). The marked points indicate the critical pairs (ℓ^*, θ) associated with the parameter sets considered. The existence of a unique intersection for each fixed θ confirms that the admissibility threshold is well-defined.

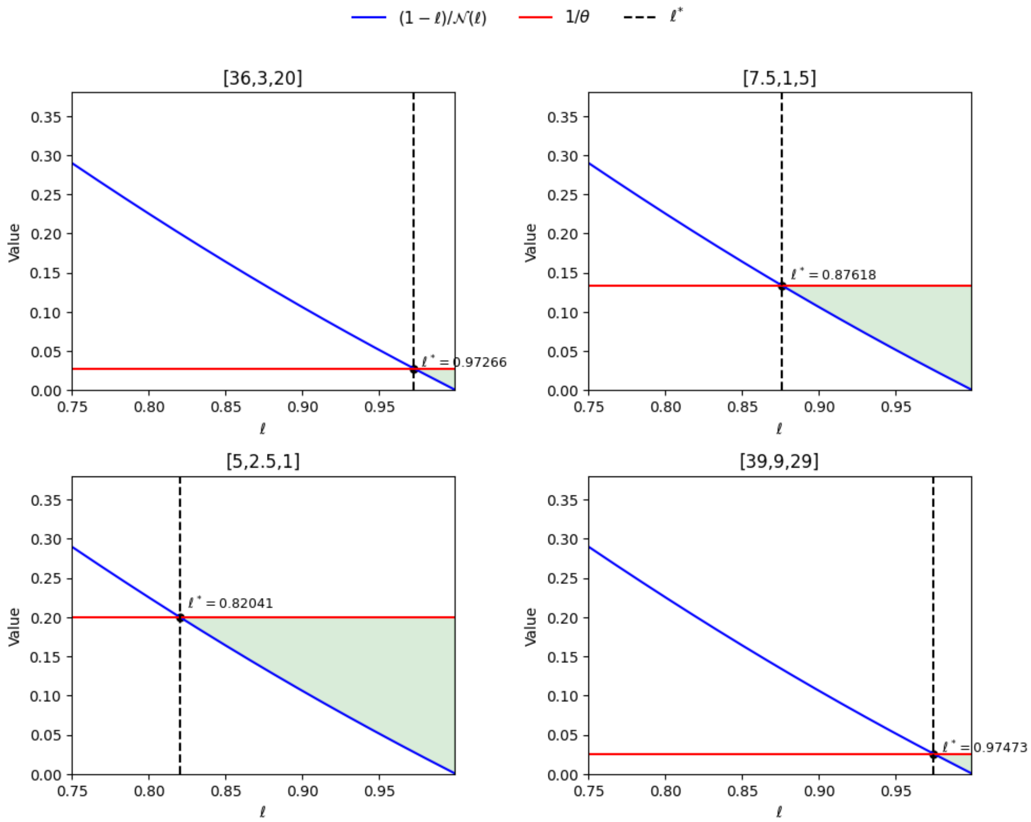

Figure 1. Comparison of admissibility regions for the four parameter sets examined in the section. The blue curves represent $\frac{1-\ell}{\mathcal{N}(\ell)}$, the red horizontal lines correspond to $\frac{1}{\theta}$, and the dashed vertical lines indicate the critical thresholds ℓ^*. The shaded regions ($\ell > \ell^*$) denote the contraction-admissible regimes determined by inequality (3.4).

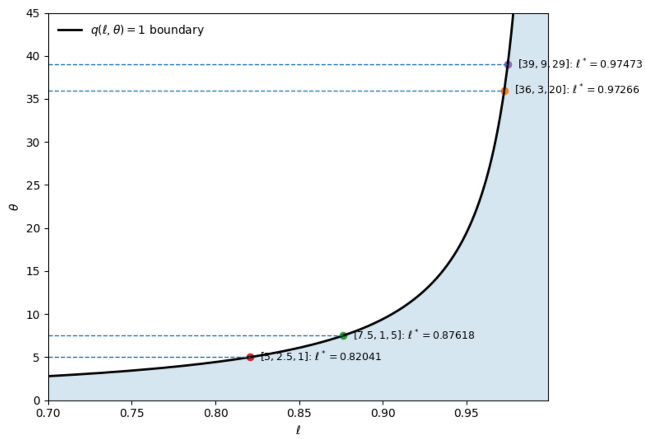

Figure 2. Admissibility map induced by the function $q(\ell, \theta) = \theta\frac{1-\ell}{\mathcal{N}(\ell)}$. The boundary curve corresponds to $q(\ell, \theta) = 1$. The shaded region ($q < 1$) indicates the contraction-admissible regime. Marked points show the critical pairs (ℓ^*, θ) associated with the parameter configurations considered.

4.2. *Fractional-order dependence of phase–space dynamics*

We now turn to numerical simulations of the fractional Chen system under the Atangana–Baleanu–Caputo operator. The phase portraits presented below correspond to parameter pairs (ℓ, θ) lying within the admissible region identified in Figure 2 for $[a, b, c] = [7.5, 1, 5]$. For the parameter configuration, fractional orders satisfying $\ell > \ell^*$ are selected.

Figures 3–5 compare phase–space projections of the fractional Chen system for the fixed parameter set $[a, b, c] = [7.5, 1, 5]$ at $\ell \in \{0.97, 0.93, 0.89\}$. Across all three cases, the trajectories remain bounded and exhibit a qualitatively similar double-lobe attractor structure in both the planar projections and the 3D representation. Although the overall geometry is preserved, modest variations in the spread and density of the trajectories can be observed among different fractional orders, most notably in the x–y projection. These observations suggest that changing ℓ influences quantitative features of the attractor, such as the extent and distribution of trajectories, without altering its qualitative phase–space organization for the considered simulation horizon.

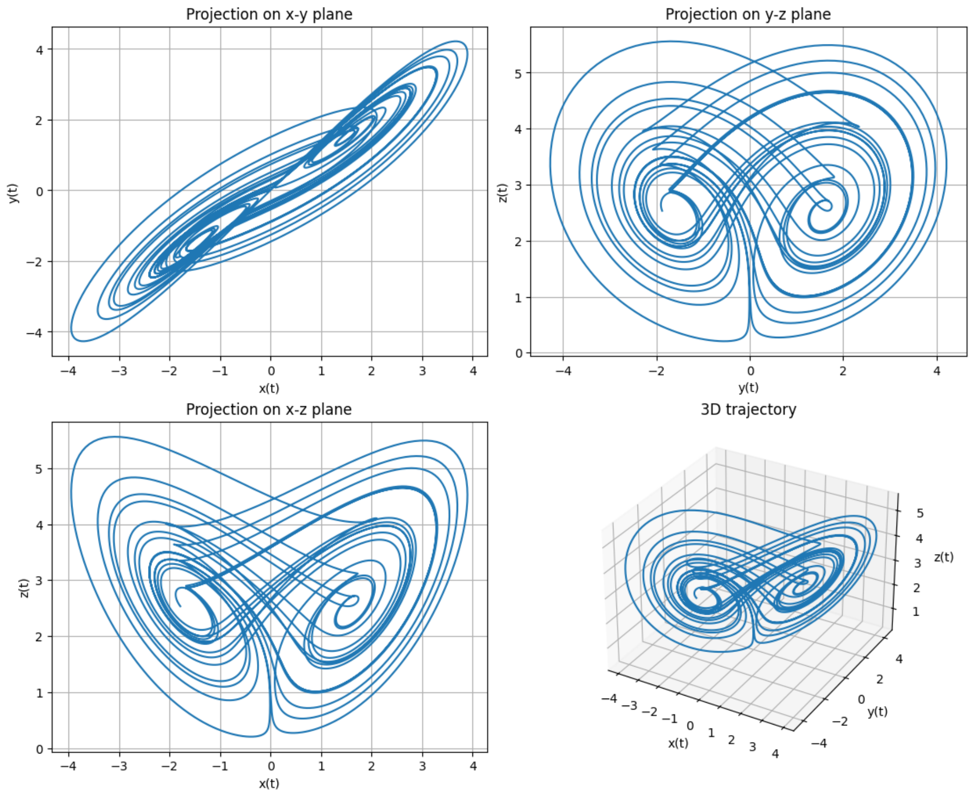

Figure 3. Phase–space projections and 3D trajectory of the fractional Chen system for $[a, b, c] = [7.5, 1, 5]$ with $\ell = 0.97$. The attractor exhibits a bounded double-lobe structure in the x–y, y–z, and x–z projections.

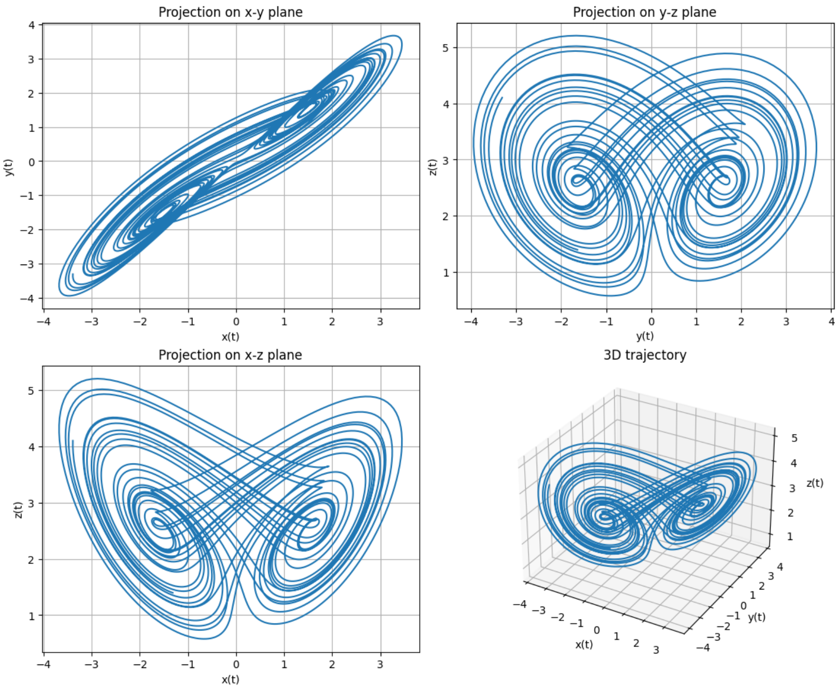

Figure 4. Phase–space projections and 3D trajectory of the fractional Chen system for $[a, b, c] = [7.5, 1, 5]$ with $\ell = 0.93$. Compared with $\ell = 0.97$, the overall double-lobe structure is preserved, but modest differences in the spread of trajectories are observed in the planar projections.

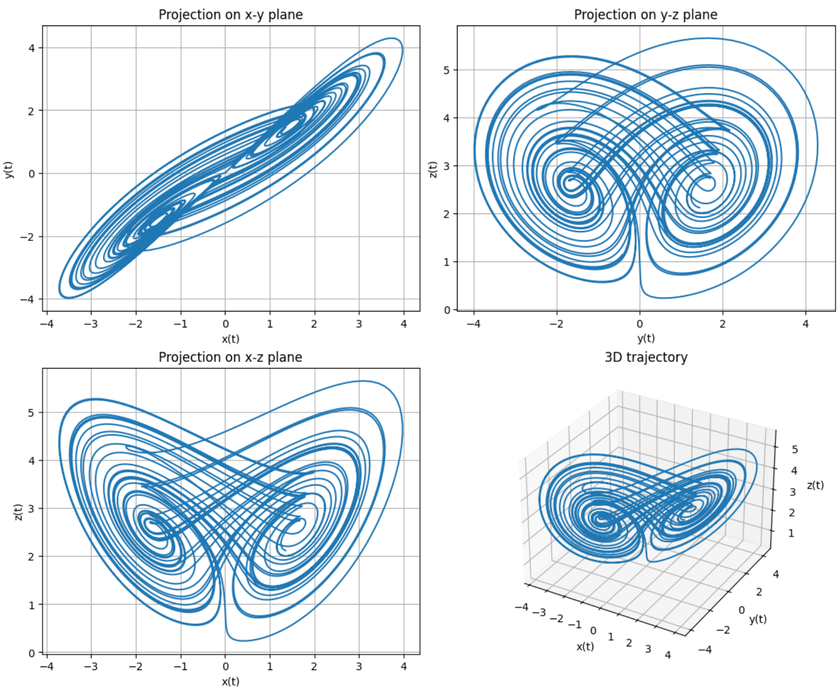

Figure 5. Phase–space projections and 3D trajectory of the fractional Chen system for $[a, b, c] = [7.5, 1, 5]$ with $\ell = 0.89$. The trajectories remain bounded and display a qualitatively similar double-lobe attractor structure to the cases $\ell = 0.97$ and $\ell = 0.93$.

Table 1 reports quantitative measures of trajectory spread and the root mean square (RMS) norm for different fractional orders. The coordinate ranges and maximum absolute values exhibit nonmonotonic variation across ℓ, indicating that the geometric extent of the attractor does not follow a simple expansion or contraction pattern as ℓ decreases.

In contrast, the RMS norm shows a mild but consistent increasing trend from $\ell = 0.99$ to $\ell = 0.89$. This suggests that although the spatial spread of the trajectories fluctuates, the overall average magnitude of the state vector gradually increases as the fractional order decreases within the considered interval.

It is important to emphasize that (3.4) does not guarantee chaotic behavior; rather, it ensures the applicability of the contraction-based fixed-point framework used to establish the existence and uniqueness of solutions. The consistency between the admissible region and the numerically observed complex attractor structures supports the structural coherence of the proposed theoretical framework.

Table 1. Trajectory spread and RMS norm for $[a, b, c] = [7.5, 1, 5]$.

ℓ	max $\lvert x \rvert$	max $\lvert y \rvert$	max $\lvert z \rvert$	Range(x)	Range(y)	Range(z)	RMS
0.99	3.743147	4.032949	5.270235	7.010016	7.500415	4.812148	3.468958
0.97	3.943751	4.271878	5.558882	7.839901	8.485431	5.355216	3.466735
0.95	3.560868	3.809937	5.034915	7.001617	7.477135	4.333248	3.474909
0.93	3.677529	3.945061	5.206100	7.126644	7.619473	4.630279	3.490600
0.91	3.806090	4.092369	5.395258	7.493246	8.044057	4.959909	3.503479
0.89	3.973712	4.292174	5.650197	7.689084	8.272345	5.415275	3.513803

5. Conclusions

The main goal of this study is to introduce the concept of a *controlled* $(\alpha, c)-\mathbb{IMS}$, which generalizes both $(\alpha, c)-\mathbb{IMS}$ and extended $(\alpha, c)-\mathbb{IMS}$ frameworks by incorporating a control function into the structure. This novel setting provides a flexible analytical foundation for exploring contraction mappings and fixed-point results under weaker assumptions than those required in classical metric spaces. The Banach-type fixed-point theorem demonstrates the internal consistency and applicability of the proposed space.

To illustrate its effectiveness, the obtained theoretical results were applied to the fractional-order Chen attractor, which involves the Atangana–Baleanu–Caputo derivative. The application confirmed that the controlled $(\alpha, c)-\mathbb{IM}$ framework not only ensures the existence and uniqueness of equilibrium solutions but also offers a constructive analytical approach for studying the stability and chaotic dynamics of nonlinear fractional systems. This represents, to the best of our knowledge, the first integration of this new metric concept with the Chen system, highlighting both its originality and its potential for broader use in fractional dynamical analysis.

Open problems and future directions

Future research directions may include extending the proposed metric structure to stochastic or fuzzy settings as well as exploring the structure's implications in other chaotic systems such as the Lorenz and Rössler attractors. Another promising line of investigation lies in the development of numerical algorithms that leverage the controlled interpolative framework to achieve improved convergence in fractional differential equation solvers. Overall, the results of this work demonstrate that the controlled $(\alpha, c)-\mathbb{IMS}$ provides a fertile ground for both theoretical advancements and real-world applications in nonlinear analysis.

Author contributions

E. Girgin: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Resources, Data Curation, Writing–Original Draft Preparation, Writing–Review and Editing, Visualization; S. Erman: Methodology, Formal analysis, Investigation, Writing–Review and Editing, Visualization, Supervision; A. Büyükkaya: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Resources; M.Öztürk; Data Curation, Writing–riginal Draft Preparation, Writing–Review and Editing, Visualization, Supervision, Project Administration, Funding Acquisition. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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