Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(3): 5409–5435.
DOI: 10.3934/math.2026223
Received: 05 January 2026
Revised: 17 February 2026
Accepted: 25 February 2026
Published: 03 March 2026

Research article

On the stability and boundedness properties of operators in the Gelfand–Shilov spaces of Roumieu type

Samia Youcefi[1], **Mohammed Said Souid**[2,3], **Muath Awadalla**[4,*] **and Meraa Arab**[4]

[1] Ecole Superieure en Informatique 08-MAI-1945, Sidi Bel Abbes, Algeria
[2] Department of Economic Sciences, Ibn Khaldoun University of Tiaret, Tiaret 14000, Algeria
[3] Department of Allied Sciences, Faculty of Arts and Science, Al-Ahliyya Amman University, Amman, Jordan
[4] Department of Mathematics and Statistics, College of Science, King Faisal University, Al Ahsa 31982, Saudi Arabia

* **Correspondence:** Email: mawadalla@kfu.edu.sa.

Abstract: This article presents an extensive study of Gelfand–Shilov spaces of Roumieu type defined via sequences of positive real numbers satisfying natural growth and convexity conditions. We introduce generalized anisotropic spaces $S^{\{M\}}_{\{N\}}(\mathbb{R}^n)$ and explore their fundamental properties, including stability under differentiation, multiplication, translation, and dilatation operators. We establish boundedness results that demonstrate these spaces form differential subalgebras of the space of continuous functions. These results enrich the functional–analytic framework of ultradifferentiable spaces and provide a foundation for further studies in microlocal analysis and partial differential equations involving ultradistributions.

Keywords: function spaces; ultradifferentiable functions; anisotropic spaces; differential operators; bounded operators
Mathematics Subject Classification: 46E10, 46F05, 35S05, 46E15

1. Introduction

Gelfand–Shilov spaces constitute one of the fundamental frameworks in the theory of ultradifferentiable functions and in the analysis of partial differential equations requiring very high regularity. They provide a natural setting in which decay at infinity and growth of derivatives are controlled simultaneously, making them particularly suitable for Fourier analysis, microlocal methods, and the theory of ultradistributions [1–4]. Classical investigations mainly concern the isotropic scale $S^{\mu}_{\nu}(\mathbb{R}^n)$, where a single pair of parameters governs all variables in a uniform way [5–8].

In many analytical problems, however, different directions or different groups of variables exhibit distinct regularity and decay properties. This leads naturally to anisotropic models [9–12] in which derivatives and moments are measured by two independent weight sequences. Such situations occur, for example, when operators act with unequal strength on different variables or when propagation phenomena follow different regimes. Despite this motivation, the operator theory of anisotropic Gelfand–Shilov spaces of Roumieu type has not been developed in a systematic manner.

The purpose of the present paper is to contribute to filling this gap. We introduce and study the spaces $S_{\{N\}}^{\{M\}}(\mathbb{R}^n)$ associated with weight sequences (M_p) and (N_p) satisfying standard logarithmic convexity and moderate growth assumptions. Although the definition formally resembles that of the classical spaces, the presence of two independent sequences generates genuine additional difficulties. In particular, one can no longer reduce the analysis to a single scale, and the interaction between derivative indices and moment indices requires refined mixed estimates. Several arguments that are immediate in the isotropic theory therefore demand new combinatorial decompositions and a careful tracking of constants.

Our main objective is to study the action of fundamental operators in this anisotropic framework. We establish stability and boundedness properties for multiplication by polynomials, pointwise products, multiplier–type operators, translations, and dilatations. As a consequence, we prove that these spaces form differential subalgebras of $C(\mathbb{R}^n)$ and that the operators under consideration are continuous with respect to the natural inductive limit topology. Throughout the article, we make explicit which parts of the proofs follow classical schemes and which rely in an essential way on the interaction between the sequences (M_p) and (N_p).

Main achievements

The results obtained here provide a systematic verification that the basic transformations of analysis remain stable in anisotropic Gelfand–Shilov spaces of Roumieu type under assumptions that treat regularity and decay independently. We obtain continuity statements with bounds that explicitly reflect the two–scale structure. To the best of our knowledge, such a unified operator framework has not been previously available.

Beyond their intrinsic functional–analytic interest, these estimates form a technical basis for further developments in the theory of ultradistributions and in the study of partial differential equations where anisotropic regularity is unavoidable. Continuity of elementary operators is a fundamental ingredient for symbolic calculus, mapping properties of pseudodifferential operators, and propagation of singularities in ultraregular contexts [13–16].

The paper is organized as follows. After recalling the properties of the defining sequences and the structure of the spaces, we establish boundedness results for elementary operators and derive the corresponding algebraic stability. Special attention is devoted to the dependence of constants and to the mechanisms through which anisotropy influences the estimates.

2. Some basic concepts

Some properties and definitions are stated in this section.

We introduce the Gelfand–Shilov spaces of Roumieu type $S^{\{M\}}(\mathbb{R}^n)$, $S_{\{N\}}(\mathbb{R}^n)$, and $S_{\{N\}}^{\{M\}}(\mathbb{R}^n)$ defined by sequences of positive real numbers. First, we extend the properties of the Gelfand–Shilov spaces $S^{\mu}_{\nu}(\mathbb{R}^n)$ to the Gelfand–Shilov spaces of Roumieu type $S^{\{M\}}_{\{N\}}(\mathbb{R}^n)$. For a definition of the Gelfand–Shilov spaces of Roumieu type, we deal with sequences $\{M_p\}$ of positive real numbers that fulfill a set of general assumptions, namely:

(i) Logarithmic convexity:

$$M_0 = 1 \quad \text{and} \quad M_p^2 \le M_{p-1}M_{p+1}, \quad \forall\, p \in \mathbb{N}^*. \tag{2.1}$$

(ii) Stability with respect to differentiation and multiplication:

$$\binom{p+q}{p} M_p M_q \le M_{p+q}, \qquad \forall\, p, q \in \mathbb{N}. \tag{2.2}$$

There exists a constant $H > 0$ such that

$$M_{p+q} \le H^{p+q} M_p M_q, \qquad \forall\, p, q \in \mathbb{N}. \tag{2.3}$$

Example 2.1. *Consider the Gevrey–type sequence defined by* $M_p = (p!)^s$ *with a parameter* $s > 1$. *This sequence fulfills the assumptions in* (2.1)–(2.3). *Indeed, using Stirling's approximation, we have* $p!^s \sim p^{ps} \sim \Gamma(ps + 1)$, *which confirms that such sequences satisfy the stated properties.*

We now recall a useful property concerning any sequence $\{M_p\}$ that meets the conditions (2.1)–(2.3).

Proposition 2.2. *Let* $\{M_p\}$ *be a weight sequence satisfying condition* (2.2). *Then, there exists a constant* $C > 0$ *such that*

$$p! \le C^p M_p, \qquad \forall\, p \in \mathbb{N}. \tag{2.4}$$

Proof. We use the condition (2.2). Setting $q = 1$, we obtain

$$M_{p+1} \ge \binom{p+1}{p} M_p M_1 = (p+1) M_1 M_p, \qquad \forall\, p \in \mathbb{N}.$$

Iterating this inequality yields

$$M_p \ge p!\,(M_1)^p, \qquad \forall\, p \in \mathbb{N}.$$

Hence,

$$p! \le \left(\frac{1}{M_1}\right)^p M_p, \qquad \forall\, p \in \mathbb{N}.$$

Choosing $C = \max\{1, 1/M_1\}$ completes the proof. □

Proposition 2.3. *Let the sequence* $\{M_p\}$ *satisfy condition* (2.1). *Then, the sequence* $\{M_p^{1/p}\}$ *is increasing.*

Proof. The monotonicity of $\{M_p^{1/p}\}$ is equivalent to verifying that

$$\frac{p+1}{p} \log M_p \le \log M_{p+1}, \quad \forall\, p \in \mathbb{N}^*. \tag{2.5}$$

We establish inequality (2.5) by mathematical induction on p.

Base case: For $p = 1$, the inequality (2.5) follows directly from condition (2.1) after applying the logarithm function.

Inductive step: Assume that the estimate (2.5) holds for some integer $p \geq 1$. We now verify that it also remains valid for $p+1$. Using the assumption (2.1) together with the induction hypothesis, we obtain

$$\begin{aligned} 2\log M_{p+1} &\leq \log M_p + \log M_{p+2} \\ &\leq \frac{p}{p+1}\log M_{p+1} + \log M_{p+2}; \end{aligned}$$

hence,

$$2\log M_{p+1} - \frac{p}{p+1}\log M_{p+1} \leq \log M_{p+2},$$

that is

$$\frac{p+2}{p+1}\log M_{p+1} \leq \log M_{p+2},$$

which corresponds precisely to relation (2.5) for the index $p+1$. This concludes the proof. □

Notation 2.4. *For all $k \in \mathbb{N}$, we denote $m_k = M_{k+1}/M_k$.*

Proposition 2.5. *For each sequence $\left(M_p\right)$ satisfying* (2.1)*, the sequence $m_k = M_{k+1}/M_k$ is increasing and satisfies the following inequality:*

$$m_k^{j-k} \leq \frac{M_j}{M_k}, \quad \forall (j,k) \in \mathbb{N}^2. \tag{2.6}$$

Proof. The increase of the sequence m_k is deduced immediately from the condition (2.1). We have, for all $k \in \mathbb{N}$, that

$$\begin{aligned} M_{k+1}^2 \leq M_k M_{k+2} &\implies \frac{M_{k+1}}{M_k} \leq \frac{M_{k+2}}{M_{k+1}} \\ &\implies m_k \leq m_{k+1}. \end{aligned}$$

For the estimation (2.6), we distinguish two cases:

(1) If $j > k$, we can write

$$\frac{M_j}{M_k} = \frac{M_{k+1}}{M_k}\frac{M_{k+2}}{M_{k+1}}\cdots\frac{M_{j-1}}{M_{j-2}}\frac{M_j}{M_{j-1}} = m_k m_{k+1}\cdots m_{j-2}m_{j-1}.$$

Because the sequence m_k is increasing, we get

$$\frac{M_j}{M_k} \geq (m_k)^{j-k}.$$

(2) If $j \leq k$, we can write

$$\frac{M_j}{M_k} = \frac{M_j}{M_{j+1}}\frac{M_{j+1}}{M_{j+2}}\cdots\frac{M_{k-2}}{M_{k-1}}\frac{M_{k-1}}{M_k} = \frac{1}{m_j}\frac{1}{m_{j+1}}\cdots\frac{1}{m_{k-2}}\frac{1}{m_{k-1}}.$$

Also, because the sequence m_k is increasing, we have

$$\frac{M_j}{M_k} \geq \left(\frac{1}{m_{k-1}}\right)^{k-j} \geq \left(\frac{1}{m_k}\right)^{k-j} = (m_k)^{j-k},$$

completing the proof. □

There are three essential types of the Gelfand–Shilov spaces of Roumieu type: $S^{\{M\}}(\mathbb{R}^n)$, $S_{\{N\}}(\mathbb{R}^n)$, and $S^{\{M\}}_{\{N\}}(\mathbb{R}^n)$. Following this section, we investigate each of them in detail.

Definition 2.6 ([17]). *We denote by $S^{\{M\}}(\mathbb{R}^n)$ the space of functions like $\varphi \in C^\infty(\mathbb{R}^n)$ such that*

$$\exists A > 0, \forall \ell \in \mathbb{Z}^n_+, \exists C_\ell > 0, \forall \kappa \in \mathbb{Z}^n_+ : \sup_{x\in\mathbb{R}^n} \left| x^\ell D^\kappa \varphi(x) \right| \leqslant C_\ell A^{|\kappa|} M_{|\kappa|},$$

where the constant C_ℓ depends only on the functions φ and ℓ.

In order to write the space $S^{\{M\}}(\mathbb{R}^n)$ as an inductive limit of normed spaces, we consider the following family of spaces.

Definition 2.7 ([17]). *Letting $A > 0$, we denote by $S^{\{M\},A}(\mathbb{R}^n)$ the set of functions $\varphi \in C^\infty(\mathbb{R}^n)$ such that*

$$\forall \ell \in \mathbb{Z}^n_+, \exists C_\ell > 0, \forall \kappa \in \mathbb{Z}^n_+ : \sup_{x\in\mathbb{R}^n} \left| x^\ell D^\kappa \varphi(x) \right| \leqslant C_\ell A^{|\kappa|} M_{|\kappa|},$$

where the constant C_ℓ depends only on the functions φ and ℓ.

Proposition 2.8 ([18]). *The space $S^{\{M\},A}(\mathbb{R}^n)$ is a normed space under the following norm:*

$$\|\varphi\|_\ell = \sup_{x\in\mathbb{R}^n} \sup_{\kappa\in\mathbb{Z}^n_+} \frac{\left| x^\ell D^\kappa \varphi(x) \right|}{A^{|\kappa|} M_{|\kappa|}}, \quad \ell \in \mathbb{Z}^n_+.$$

Proposition 2.9 ([18]). *The space $S^{\{M\}}(\mathbb{R}^n)$ is an inductive limit of normed spaces $S^{\{M\},A}(\mathbb{R}^n)$, $A > 0$, and $S^{\{M\}}(\mathbb{R}^n) = \bigcup_{A>0} S^{\{M\},A}(\mathbb{R}^n)$.*

Definition 2.10 ([17]). *We denote by $S_{\{N\}}(\mathbb{R}^n)$ the space of functions $\varphi \in C^\infty(\mathbb{R}^n)$ such that*

$$\exists B > 0, \forall \kappa \in \mathbb{Z}^n_+, \exists C_\kappa > 0, \forall \ell \in \mathbb{Z}^n_+ : \sup_{x\in\mathbb{R}^n} \left| x^\ell D^\kappa \varphi(x) \right| \leqslant C_\kappa B^{|\ell|} N_{|\ell|},$$

where the constant C_κ depends only on the functions φ and κ.

In order to write the space $S_{\{N\}}(\mathbb{R}^n)$ as an inductive limit of normed spaces, we consider the following family of spaces.

Definition 2.11 ([17]). *Letting $B > 0$, we denote by $S_{\{N\},B}(\mathbb{R}^n)$ the set of functions $\varphi \in C^\infty(\mathbb{R}^n)$ such that*

$$\forall \kappa \in \mathbb{Z}^n_+, \exists C_\kappa > 0, \forall \ell \in \mathbb{Z}^n_+ : \sup_{x\in\mathbb{R}^n} \left| x^\ell D^\kappa \varphi(x) \right| \leqslant C_\kappa B^{|\ell|} N_{|\ell|},$$

where the constant C_κ depends only on the functions φ and κ.

Proposition 2.12 ([18]). *The space $S_{\{N\},B}(\mathbb{R}^n)$ is a normed space by the following norm:*

$$\|\varphi\|_\kappa = \sup_{x\in\mathbb{R}^n} \sup_{\ell\in\mathbb{Z}^n_+} \frac{\left| x^\ell D^\kappa \varphi(x) \right|}{B^{|\ell|} N_{|\ell|}}, \quad \kappa \in \mathbb{Z}^n_+.$$

Proposition 2.13 ([18]). *The space $S_{\{N\}}(\mathbb{R}^n)$ is an inductive limit of normed spaces $S_{\{N\},B}(\mathbb{R}^n)$, $B > 0$, and $S_{\{N\}}(\mathbb{R}^n) = \bigcup_{B>0} S_{\{N\},B}(\mathbb{R}^n)$.*

Definition 2.14 ([17]). *We denote by $S_{\{N\}}^{\{M\}}(\mathbb{R}^n)$ the space of functions $\varphi \in C^\infty(\mathbb{R}^n)$ such that*

$$\exists C > 0, \forall \kappa, \ell \in \mathbb{Z}_+^n : \sup_{x\in\mathbb{R}^n} \left| x^\ell D^\kappa \varphi(x) \right| \leqslant C^{|\kappa|+|\ell|+1} M_{|\kappa|} N_{|\ell|},$$

where the constant C depends only on the function φ.

Remark 2.15. *It is clear that*

$$S_{\{N\}}^{\{M\}}(\mathbb{R}^n) \subset S^{\{M\}}(\mathbb{R}^n) \cap S_{\{N\}}(\mathbb{R}^n) \text{ for all sequences } M \text{ and } N.$$

In order to write the space $S_{\{N\}}^{\{M\}}(\mathbb{R}^n)$ as an inductive limit of normed spaces, we consider the following family of spaces.

Definition 2.16 ([17]). *Letting $A > 0$ and $B > 0$, we denote by $S_{\{N\},B}^{\{M\},A}(\mathbb{R}^n)$ the set of functions $\varphi \in C^\infty(\mathbb{R}^n)$ such that*

$$\exists C > 0, \forall \kappa, \ell \in \mathbb{Z}_+^n : \sup_{x\in\mathbb{R}^n} \left| x^\ell D^\kappa \varphi(x) \right| \leqslant C A^{|\kappa|} B^{|\ell|} M_{|\kappa|} N_{|\ell|}, \tag{2.7}$$

where the constant C depends only on the function φ.

Proposition 2.17 ([18]). *The space $S_{\{N\},B}^{\{M\},A}(\mathbb{R}^n)$ is a normed space by the following norm:*

$$\|\varphi\|_{A,B} = \sup_{x\in\mathbb{R}^n} \sup_{\kappa,\ell\in\mathbb{Z}_+^n} \frac{\left| x^\ell D^\kappa \varphi(x) \right|}{A^{|\kappa|} B^{|\ell|} M_{|\kappa|} N_{|\ell|}}.$$

Proposition 2.18 ([18]). *The space $S_{\{N\}}^{\{M\}}(\mathbb{R}^n)$ is an inductive limit of normed spaces $S_{\{N\},B}^{\{M\},A}(\mathbb{R}^n)$, $A, B > 0$, and $S_{\{N\}}^{\{M\}}(\mathbb{R}^n) = \bigcup_{A,B>0} S_{\{N\},B}^{\{M\},A}(\mathbb{R}^n)$.*

Remark 2.19. *The space $S_{\{N\}}^{\{M\}}(\mathbb{R}^n)$ is called anisotropic Gelfand–Shilov spaces of Roumieu type in $\mathbb{R}^n$.*

Remark 2.20. *If $M = N$, the space $S_{\{M\}}^{\{M\}}(\mathbb{R}^n)$ is called an isotropic Gelfand–Shilov space of Roumieu type in $\mathbb{R}^n$. According to the condition (2.3), we have the following definition of the space $S_{\{M\}}^{\{M\}}(\mathbb{R}^n)$.*

The space $S_{\{M\}}^{\{M\}}(\mathbb{R}^n)$ is the set of functions like $\varphi \in C^\infty(\mathbb{R}^n)$ such that

$$\exists C > 0, \forall \kappa, \ell \in \mathbb{Z}_+^n : \left\| x^\ell D^\kappa \varphi \right\|_{L^\infty(\mathbb{R}^n)} \leq C^{|\kappa|+|\ell|+1} M_{|\kappa|+|\ell|}.$$

Example 2.21. *If $M_p = p!^\mu, \mu \geq 1$ and $N_p = p!^\nu, \nu \geq 1$, then $S_{\{N\}}^{\{M\}}(\mathbb{R}^n)$ is the Gelfand–Shilov space and is denoted by $S_\nu^\mu(\mathbb{R}^n)$.*

Topological clarification. The spaces $S^{\{M\},A}(\mathbb{R}^n)$ form an increasing family of Banach spaces, and the Roumieu–type space

$$S^{\{M\}}(\mathbb{R}^n) = ind \lim_{A\to+\infty} S^{\{M\},A}(\mathbb{R}^n)$$

is a limit of Fréchet (LF)–space endowed with the inductive limit topology.

In this setting, when we say that a linear operator

$$T : S^{\{M\}}(\mathbb{R}^n) \longrightarrow S^{\{M\}}(\mathbb{R}^n)$$

is *bounded*, we mean that for every $A > 0$, there exist $A' > 0$ and $C > 0$ such that

$$T : S^{\{M\},A}(\mathbb{R}^n) \longrightarrow S^{\{M\},A'}(\mathbb{R}^n)$$

is a bounded operator between Banach spaces.

By a standard result on inductive limits of locally convex spaces, this boundedness on step spaces implies that T maps bounded sets of $S^{\{M\}}(\mathbb{R}^n)$ into bounded sets and is therefore a continuous linear operator on the (LF)–space $S^{\{M\}}(\mathbb{R}^n)$ (see, for example, [19, 20]).

Lemma 2.22 (Continuity criterion in $S^{\{M\}}_{\{N\}}(\mathbb{R}^n)$)**.** *Let T be a linear operator such that for every $A, B > 0$, there exist $A', B' > 0$, and $C > 0$ satisfying*

$$\|T\varphi\|_{A',B'} \leq C\|\varphi\|_{A,B} \quad \forall\varphi \in S^{\{M\},A}_{\{N\},B}(\mathbb{R}^n).$$

Then, T is a bounded and continuous linear operator on $S^{\{M\}}_{\{N\}}(\mathbb{R}^n)$.

3. The main results

In this section, we extend some elementary properties of the Gelfand–Shilov spaces $S^{\mu}_{\nu}(\mathbb{R}^n)$ to the Gelfand–Shilov spaces of Roumieu type $S^{\{M\}}_{\{N\}}(\mathbb{R}^n)$, which are related to bounded operators from each space to itself.

3.1. Operator of multiplication by x

We will show that for all $\sigma \in \mathbb{Z}^n_+$, the spaces $S^{\{M\}}(\mathbb{R}^n)$, $S_{\{N\}}(\mathbb{R}^n)$, and $S^{\{M\}}_{\{N\}}(\mathbb{R}^n)$ are stable under multiplication by x^σ; furthermore, the operator of multiplication by x^σ is bounded in each space.

3.1.1. Case of the space $S^{\{M\}}(\mathbb{R}^n)$

Theorem 3.1. *If the sequence (M_p) satisfies the condition* (2.2)*, then for all $\sigma \in \mathbb{Z}^n_+$, the operator of multiplication by x^σ is defined and bounded in each space $S^{\{M\},A}(\mathbb{R}^n)$ for every $A > 0$.*

Proof. Let $A > 0$, and let $\varphi \in S^{\{M\},A}(\mathbb{R}^n)$. By definition, we have

$$\forall\ell \in \mathbb{Z}^n_+, \exists C_\ell > 0, \forall\kappa \in \mathbb{Z}^n_+ : \sup_{x\in\mathbb{R}^n} \left|x^\ell D^\kappa\varphi(x)\right| \leqslant C_\ell A^{|\kappa|} M_{|\kappa|}.$$

By the Leibniz formula, we have

$$\begin{aligned}
\left\|x^\ell D^\kappa (x^\sigma\varphi)\right\|_{L^\infty(\mathbb{R}^n)} &= \left\|x^\ell \sum_{\gamma\leq\kappa} \binom{\kappa}{\gamma} D^\gamma(x^\sigma) D^{\kappa-\gamma}\varphi\right\|_{L^\infty(\mathbb{R}^n)} \\
&\leq \sum_{\substack{\gamma\leq\kappa\\ \gamma\leq\sigma}} \binom{\kappa}{\gamma} \frac{\sigma!}{(\sigma-\gamma)!} \left\|x^{\ell+\sigma-\gamma} D^{\kappa-\gamma}\varphi\right\|_{L^\infty(\mathbb{R}^n)} \\
&\leq \sum_{\substack{\gamma\leq\kappa\\ \gamma\leq\sigma}} \binom{\kappa}{\gamma} \frac{\sigma!}{(\sigma-\gamma)!} C_{\ell+\sigma-\gamma} A^{|\kappa-\gamma|} M_{|\kappa-\gamma|},
\end{aligned}$$

which gives with the condition (2.2)

$$\left\|x^{\ell}D^{\kappa}\left(x^{\sigma}\varphi\right)\right\|_{L^{\infty}(\mathbb{R}^{n})} \leq A^{|\kappa|}M_{|\kappa|}\sum_{\gamma\leq\sigma}\frac{\sigma!}{(\sigma-\gamma)!}C_{\ell+\sigma-\gamma}A^{-|\gamma|}\frac{1}{M_{|\gamma|}} \leq C'_{\ell}A^{|\kappa|}M_{|\kappa|}, \tag{3.1}$$

with

$$C'_{\ell} = \sum_{\gamma\leq\sigma}\frac{\sigma!}{(\sigma-\gamma)!}C_{\ell+\sigma-\gamma}A^{-|\gamma|}\frac{1}{M_{|\gamma|}}.$$

Thus, $x^{\sigma}\varphi \in S^{\{M\},A}(\mathbb{R}^{n})$.

For the boundedness of the operator of multiplication by x^{σ}, we have from (3.1) and Lemma 2.22, for all $\ell \in \mathbb{Z}_{+}^{n}$,

$$\begin{aligned}\|x^{\sigma}\varphi\|_{\ell} &= \sup_{x\in\mathbb{R}^{n}}\sup_{\kappa\in\mathbb{Z}_{+}^{n}}\frac{\left|x^{\ell}D^{\kappa}\left(x^{\sigma}\varphi(x)\right)\right|}{A^{|\kappa|}M_{|\kappa|}} \\ &\leq \sup_{x\in\mathbb{R}^{n}}\sup_{\kappa\in\mathbb{Z}_{+}^{n}}\sum_{\substack{\gamma\leq\kappa\\ \gamma\leq\sigma}}\binom{\kappa}{\gamma}\frac{\sigma!}{(\sigma-\gamma)!}\frac{\left|x^{\ell+\sigma-\gamma}D^{\kappa-\gamma}\varphi(x)\right|}{A^{|\kappa|}M_{|\kappa|}},\end{aligned}$$

and using the condition (2.2), we obtain

$$\begin{aligned}\|x^{\sigma}\varphi\|_{\ell} &\leq \sup_{x\in\mathbb{R}^{n}}\sup_{\kappa\in\mathbb{Z}_{+}^{n}}\sum_{\substack{\gamma\leq\kappa\\ \gamma\leq\sigma}}\frac{\sigma!}{(\sigma-\gamma)!}\frac{\left|x^{\ell+\sigma-\gamma}D^{\kappa-\gamma}\varphi(x)\right|}{A^{|\kappa|}M_{|\kappa-\gamma|}M_{|\gamma|}} \\ &\leq \sum_{\gamma\leq\sigma}\frac{\sigma!}{(\sigma-\gamma)!}A^{-|\gamma|}\frac{1}{M_{|\gamma|}}\|\varphi\|_{\ell+\sigma-\gamma},\end{aligned}$$

which means that the operator of multiplication by x^{σ} is bounded in $S^{\{M\},A}(\mathbb{R}^{n})$. □

Corollary 3.2. *If the sequence $\left(M_{p}\right)$ satisfies the condition* (2.2)*, then for all $\sigma \in \mathbb{Z}_{+}^{n}$, the operator of multiplication by x^{σ} is defined and bounded from the space $S^{\{M\}}(\mathbb{R}^{n})$ into itself.*

3.1.2. Case of the space $S_{\{N\}}(\mathbb{R}^{n})$

Theorem 3.3. *If the sequence $\left(N_{p}\right)$ satisfies the condition* (2.2)-(2.3) *then for all $\sigma \in \mathbb{Z}_{+}^{n}$, the operator of multiplication by x^{σ} is defined and bounded in each space $S_{\{N\},B}(\mathbb{R}^{n})$, for every $B > 0$, and transforms the space $S_{\{N\},B}(\mathbb{R}^{n})$ into the space $S_{\{N\},BH}(\mathbb{R}^{n})$, where H is the constant of the condition* (2.3)*.*

Proof. Let $B > 0$, and let $\varphi \in S_{\{N\},B}(\mathbb{R}^{n})$. By definition, we have $\forall\kappa \in \mathbb{Z}_{+}^{n}, \exists C_{\kappa} > 0, \forall\ell \in \mathbb{Z}_{+}^{n}$: $\sup_{x\in\mathbb{R}^{n}}\left|x^{\ell}D^{\kappa}\varphi(x)\right| \leqslant C_{\kappa}B^{|\ell|}N_{|\ell|}$. By the Leibniz formula, we have

$$\begin{aligned}\left\|x^{\ell}D^{\kappa}\left(x^{\sigma}\varphi\right)\right\|_{L^{\infty}(\mathbb{R}^{n})} &= \left\|x^{\ell}\sum_{\gamma\leq\kappa}\binom{\kappa}{\gamma}D^{\gamma}(x^{\sigma})D^{\kappa-\gamma}\varphi\right\|_{L^{\infty}(\mathbb{R}^{n})} \\ &\leq \sum_{\substack{\gamma\leq\kappa\\ \gamma\leq\sigma}}\binom{\kappa}{\gamma}\frac{\sigma!}{(\sigma-\gamma)!}\left\|x^{\ell+\sigma-\gamma}D^{\kappa-\gamma}\varphi\right\|_{L^{\infty}(\mathbb{R}^{n})}\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{\substack{\gamma\leq\kappa\\ \gamma\leq\sigma}} \binom{\kappa}{\gamma}\frac{\sigma!}{(\sigma-\gamma)!}C_{\kappa-\gamma}B^{|\ell+\sigma-\gamma|}N_{|\ell+\sigma-\gamma|}\\
&= \sum_{\substack{\gamma\leq\kappa\\ \gamma\leq\sigma}} \frac{\kappa!}{(\kappa-\gamma)!}\binom{\sigma}{\gamma}C_{\kappa-\gamma}B^{|\ell+\sigma-\gamma|}N_{|\ell+\sigma-\gamma|}\\
&\leq \sum_{\substack{\gamma\leq\kappa\\ \gamma\leq\sigma}} \frac{\kappa!}{(\kappa-\gamma)!}\binom{\sigma+\ell}{\gamma}C_{\kappa-\gamma}B^{|\ell+\sigma-\gamma|}N_{|\ell+\sigma-\gamma|},
\end{aligned}$$

which gives, with the condition (2.2)-(2.3),

$$\begin{aligned}
\left\|x^{\ell}D^{\kappa}\left(x^{\sigma}\varphi\right)\right\|_{L^{\infty}(\mathbb{R}^n)} &\leq \sum_{\substack{\gamma\leq\kappa\\ \gamma\leq\sigma}} \frac{\kappa!}{(\kappa-\gamma)!}C_{\kappa-\gamma}B^{|\ell+\sigma-\gamma|}\frac{N_{|\ell+\sigma|}}{N_{|\gamma|}}\\
&\leq (HB)^{|\ell|}N_{|\ell|}\sum_{\substack{\gamma\leq\kappa\\ \gamma\leq\sigma}} \frac{\kappa!}{(\kappa-\gamma)!}C_{\kappa-\gamma}H^{|\sigma|}B^{|\sigma-\gamma|}\frac{N_{|\sigma|}}{N_{|\gamma|}}\\
&\leq C'_{\kappa}(HB)^{|\ell|}N_{|\ell|}, \qquad (3.2)
\end{aligned}$$

where

$$C'_{\kappa} = \sum_{\substack{\gamma\leq\kappa\\ \gamma\leq\sigma}} \frac{\kappa!}{(\kappa-\gamma)!}C_{\kappa-\gamma}H^{|\sigma|}B^{|\sigma-\gamma|}\frac{N_{|\sigma|}}{N_{|\gamma|}}.$$

Thus, $x^{\sigma}\varphi \in S_{\{N\},BH}(\mathbb{R}^n)$.

For the boundedness of the operator of multiplication by x^{σ}, we have from (3.2) and Lemma 2.22 for all $\kappa \in \mathbb{Z}_+^n$,

$$\begin{aligned}
\|x^{\sigma}\varphi\|_{\kappa} &= \sup_{x\in\mathbb{R}^n}\sup_{\ell\in\mathbb{Z}_+^n} \frac{\left|x^{\ell}D^{\kappa}\left(x^{\sigma}\varphi(x)\right)\right|}{(HB)^{|\ell|}N_{|\ell|}}\\
&\leq \sup_{x\in\mathbb{R}^n}\sup_{\ell\in\mathbb{Z}_+^n}\sum_{\substack{\gamma\leq\kappa\\ \gamma\leq\sigma}} \binom{\kappa}{\gamma}\frac{\sigma!}{(\sigma-\gamma)!}\frac{\left|x^{\ell+\sigma-\gamma}D^{\kappa-\gamma}\varphi(x)\right|}{(HB)^{|\ell|}N_{|\ell|}}\\
&\leq \sup_{x\in\mathbb{R}^n}\sup_{\ell\in\mathbb{Z}_+^n}\sum_{\substack{\gamma\leq\kappa\\ \gamma\leq\sigma}} \frac{\kappa!}{(\kappa-\gamma)!}\binom{\sigma+\ell}{\gamma}\frac{\left|x^{\ell+\sigma-\gamma}D^{\kappa-\gamma}\varphi(x)\right|}{H^{-|\sigma|}B^{|\ell|}H^{|\ell+\sigma|}N_{|\ell|}}
\end{aligned}$$

and using the condition (2.2)-(2.3), we obtain

$$\begin{aligned}
\|x^{\sigma}\varphi\|_{\kappa} &\leq \sup_{x\in\mathbb{R}^n}\sup_{\ell\in\mathbb{Z}_+^n}\sum_{\substack{\gamma\leq\kappa\\ \gamma\leq\sigma}} \frac{\kappa!}{(\kappa-\gamma)!}\binom{\sigma+\ell}{\gamma}\frac{\left|x^{\ell+\sigma-\gamma}D^{\kappa-\gamma}\varphi(x)\right|}{H^{-|\sigma|}B^{|\ell|}N_{|\ell+\sigma|}}N_{|\sigma|}\\
&\leq \sup_{x\in\mathbb{R}^n}\sup_{\ell\in\mathbb{Z}_+^n}\sum_{\substack{\gamma\leq\kappa\\ \gamma\leq\sigma}} \frac{\kappa!}{(\kappa-\gamma)!}\frac{B^{-|\gamma|}}{(HB)^{-|\sigma|}}\frac{\left|x^{\ell+\sigma-\gamma}D^{\kappa-\gamma}\varphi(x)\right|}{B^{|\ell+\sigma-\gamma|}N_{|\ell+\sigma-\gamma|}}\frac{N_{|\sigma|}}{N_{|\gamma|}}\\
&\leq \sum_{\substack{\gamma\leq\kappa\\ \gamma\leq\sigma}} \frac{\kappa!}{(\kappa-\gamma)!}\frac{B^{-|\gamma|}}{(HB)^{-|\sigma|}}\frac{N_{|\sigma|}}{N_{|\gamma|}}\|\varphi\|_{\kappa-\gamma},
\end{aligned}$$

which means that the operator of multiplication by x^{σ} is bounded from $S_{\{N\},B}(\mathbb{R}^n)$ to $S_{\{N\},BH}(\mathbb{R}^n)$. □

Corollary 3.4. *If the sequence* $\left(N_p\right)$ *satisfies the condition* (2.2)-(2.3)*, then for all* $\sigma \in \mathbb{Z}_+^n$*, the operator of multiplication by* x^σ *is defined and bounded from the space* $S_{\{N\}}(\mathbb{R}^n)$ *into itself.*

3.1.3. Case of the space $S_{\{N\}}^{\{M\}}(\mathbb{R}^n)$

Theorem 3.5. *If the sequences* $\left(M_p\right)$ *and* $\left(N_p\right)$ *satisfy the condition* (2.2)-(2.3)*, then for all* $\sigma \in \mathbb{Z}_+^n$*, the operator of multiplication by* x^σ *is defined and bounded in each space* $S_{\{N\},B}^{\{M\},A}(\mathbb{R}^n)$ *for every* $A, B >$ 0*, and transforms the space* $S_{\{N\},B}^{\{M\},A}(\mathbb{R}^n)$ *into the space* $S_{\{N\},BH}^{\{M\},A}(\mathbb{R}^n)$*, where* H *is the constant of the condition* (2.3).

Proof. Let $A > 0$ and $B > 0$, and let $\varphi \in S_{\{N\},B}^{\{M\},A}(\mathbb{R}^n)$. By definition, we have

$$\exists C > 0\ \forall \kappa, \ell \in \mathbb{Z}_+^n : \sup_{x\in\mathbb{R}^n} \left|x^\ell D^\kappa \varphi(x)\right| \leqslant CA^{|\kappa|}B^{|\ell|}M_{|\kappa|}N_{|\ell|}.$$

By the Leibniz formula, we get

$$\begin{aligned}
\left\|x^\ell D^\kappa (x^\sigma\varphi)\right\|_{L^\infty(\mathbb{R}^n)} &= \left\|x^\ell \sum_{\gamma\leq\kappa} \binom{\kappa}{\gamma} D^\gamma(x^\sigma) D^{\kappa-\gamma}\varphi\right\|_{L^\infty(\mathbb{R}^n)} \\
&\leq \sum_{\substack{\gamma\leq\kappa\\ \gamma\leq\sigma}} \binom{\kappa}{\gamma} \frac{\sigma!}{(\sigma-\gamma)!} \left\|x^{\ell+\sigma-\gamma} D^{\kappa-\gamma}\varphi\right\|_{L^\infty(\mathbb{R}^n)} \\
&\leq \sum_{\substack{\gamma\leq\kappa\\ \gamma\leq\sigma}} \binom{\kappa}{\gamma} \frac{\sigma!}{(\sigma-\gamma)!} CA^{|\kappa-\gamma|}B^{|\ell+\sigma-\gamma|}M_{|\kappa-\gamma|}N_{|\ell+\sigma-\gamma|},
\end{aligned}$$

which gives, with the condition (2.2)-(2.3),

$$\begin{aligned}
\left\|x^\ell D^\kappa (x^\sigma\varphi)\right\|_{L^\infty(\mathbb{R}^n)} &\leq CA^{|\kappa|}(HB)^{|\ell|}M_{|\kappa|}N_{|\ell|} \sum_{\gamma\leq\sigma} \frac{\sigma!}{(\sigma-\gamma)!} A^{-|\gamma|}(HB)^{|\sigma-\gamma|} \frac{N_{|\sigma-\gamma|}}{M_{|\gamma|}} \\
&\leq C' A^{|\kappa|}(HB)^{|\ell|}M_{|\kappa|}N_{|\ell|},
\end{aligned} \tag{3.3}$$

where

$$C' = \sum_{\gamma\leq\sigma} \frac{\sigma!}{(\sigma-\gamma)!} A^{-|\gamma|}(HB)^{|\sigma-\gamma|} \frac{N_{|\sigma-\gamma|}}{M_{|\gamma|}}.$$

Thus, $x^\sigma\varphi \in S_{\{N\},BH}^{\{M\},A}(\mathbb{R}^n)$.

At this point, the interaction between the two sequences becomes essential. The moderate growth properties must be applied simultaneously because neither (M_p) nor (N_p) alone is sufficient to control the sum.

Remark 3.6. *In the isotropic Gelfand–Shilov framework, the subsequent estimates are typically reduced to a single weight scale. In the present anisotropic situation, however, the indices must be distributed between the sequences* (M_p) *and* (N_p)*. This prevents a direct transfer of the classical argument and necessitates the mixed bounds derived below.*

For the boundedness of the operator of multiplication by x^σ, we have from (3.3) and Lemma 2.22,

$$\begin{aligned}\|x^\sigma\varphi\| &= \sup_{x\in\mathbb{R}^n}\sup_{\kappa,\ell\in\mathbb{Z}_+^n}\frac{\left|x^\ell D^\kappa\left(x^\sigma\varphi(x)\right)\right|}{A^{|\kappa|}\left(BH\right)^{|\ell|}M_{|\kappa|}N_{|\ell|}}\\ &\leq \sup_{x\in\mathbb{R}^n}\sup_{\kappa,\ell\in\mathbb{Z}_+^n}\sum_{\substack{\gamma\leq\kappa\\ \gamma\leq\sigma}}\binom{\kappa}{\gamma}\frac{\sigma!}{(\sigma-\gamma)!}\frac{\left|x^{\ell+\sigma-\gamma}D^{\kappa-\gamma}\varphi\right|}{A^{|\kappa|}\left(BH\right)^{|\ell|}M_{|\kappa|}N_{|\ell|}},\end{aligned}$$

and using the condition (2.2)-(2.3), we obtain

$$\begin{aligned}\|x^\sigma\varphi\| &\leq \sup_{x\in\mathbb{R}^n}\sup_{\kappa,\ell\in\mathbb{Z}_+^n}\sum_{\substack{\gamma\leq\kappa\\ \gamma\leq\sigma}}\frac{\sigma!}{(\sigma-\gamma)!}\frac{\left|x^{\ell+\sigma-\gamma}D^{\kappa-\gamma}\varphi\right|}{A^{|\kappa|}\left(BH\right)^{|\ell|}M_{|\kappa-\gamma|}M_{|\gamma|}N_{|\ell|}}\\ &= \sup_{x\in\mathbb{R}^n}\sup_{\kappa,\ell\in\mathbb{Z}_+^n}\sum_{\substack{\gamma\leq\kappa\\ \gamma\leq\sigma}}\frac{\sigma!}{(\sigma-\gamma)!}\frac{A^{|-\gamma|}\left(BH\right)^{|\sigma-\gamma|}\left|x^{\ell+\sigma-\gamma}D^{\kappa-\gamma}\varphi\right|}{A^{|\kappa-\gamma|}\left(BH\right)^{|\ell+\sigma-\gamma|}M_{|\kappa-\gamma|}M_{|\gamma|}N_{|\ell|}}\\ &\leq \sup_{x\in\mathbb{R}^n}\sup_{\kappa,\ell\in\mathbb{Z}_+^n}\sum_{\substack{\gamma\leq\kappa\\ \gamma\leq\sigma}}\frac{\sigma!}{(\sigma-\gamma)!}\frac{N_{|\sigma-\gamma|}A^{|-\gamma|}\left(BH\right)^{|\sigma-\gamma|}\left|x^{\ell+\sigma-\gamma}D^{\kappa-\gamma}\varphi\right|}{A^{|\kappa-\gamma|}B^{|\ell+\sigma-\gamma|}M_{|\kappa-\gamma|}M_{|\gamma|}N_{|\ell+\sigma-\gamma|}}\\ &\leq \sum_{\gamma\leq\sigma}\frac{\sigma!}{(\sigma-\gamma)!}A^{|-\gamma|}\left(BH\right)^{|\sigma-\gamma|}\frac{N_{|\sigma-\gamma|}}{M_{|\gamma|}}\|\varphi\|,\end{aligned}$$

which means that the operator of multiplication by x^σ is bounded from $S^{\{M\},A}_{\{N\},B}(\mathbb{R}^n)$ into $S^{\{M\},A}_{\{N\},BH}(\mathbb{R}^n)$. □

Remark 3.7. *The crucial step in the above proof is the redistribution of indices between derivative growth and spatial growth. Such a separation has no analogue in the classical isotropic case, where a single parameter governs both behaviors.*

We stress that continuity in the Roumieu setting is not automatic. The parameters defining the step spaces must be adjusted in a way that is compatible with both sequences, which explains the choice of constants.

Corollary 3.8. *If the sequences $\left(M_p\right)$ and $\left(N_p\right)$ satisfy the condition* (2.2)-(2.3)*, then for all $\sigma\in\mathbb{Z}_+^n$, the operator of multiplication by x^σ is defined and bounded from the space $S^{\{M\}}_{\{N\}}(\mathbb{R}^n)$ into itself.*

3.2. Multiplication in Gelfand–Shilov spaces of Roumieu type

We will show that each one of the spaces $S^{\{M\}}(\mathbb{R}^n)$, $S_{\{N\}}(\mathbb{R}^n)$, and $S^{\{M\}}_{\{N\}}(\mathbb{R}^n)$ are stable under multiplication by its elements; furthermore, this operator is bounded in each space.

3.2.1. Case of the space $S^{\{M\}}(\mathbb{R}^n)$

Theorem 3.9. *Supposing that the sequence $\left(M_p\right)$ satisfies* (2.2)*, then*

$$\forall f\in S^{\{M\},A_1}(\mathbb{R}^n),\forall g\in S^{\{M\},A_2}(\mathbb{R}^n): fg\in S^{\{M\},A_1+A_2}(\mathbb{R}^n).$$

Moreover, the operator of multiplication is bounded from the space $S^{\{M\},A_1}(\mathbb{R}^n)\times S^{\{M\},A_2}(\mathbb{R}^n)$ into the space $S^{\{M\},A_1+A_2}(\mathbb{R}^n)$.

Proof. Let $f \in S^{\{M\},A_1}(\mathbb{R}^n)$, $g \in S^{\{M\},A_2}(\mathbb{R}^n)$. Then,

$$\forall \ell \in \mathbb{Z}_+^n,\ \exists C_\ell > 0,\ \forall \kappa \in \mathbb{Z}_+^n : \left\| x^\ell D^\kappa f \right\|_{L^\infty(\mathbb{R}^n)} \leq C_\ell A_1^{|\kappa|} M_{|\kappa|}$$

and

$$\forall \ell \in \mathbb{Z}_+^n,\ \exists C'_\ell > 0,\ \forall \kappa \in \mathbb{Z}_+^n : \left\| x^\ell D^\kappa g \right\|_{L^\infty(\mathbb{R}^n)} \leq C'_\ell A_2^{|\kappa|} M_{|\kappa|}.$$

So, by the Leibniz formula, we obtain

$$\begin{aligned} \left\| x^\ell D^\kappa (fg) \right\|_{L^\infty(\mathbb{R}^n)} &= \left\| x^\ell \sum_{\gamma \leq \kappa} \binom{\kappa}{\gamma} D^\gamma f D^{\kappa-\gamma} g \right\|_{L^\infty(\mathbb{R}^n)} \\ &\leq \sum_{\gamma \leq \kappa} \binom{\kappa}{\gamma} \left\| x^\ell D^\gamma f \right\|_{L^\infty(\mathbb{R}^n)} \left\| D^{\kappa-\gamma} g \right\|_{L^\infty(\mathbb{R}^n)} \\ &\leq \sum_{\gamma \leq \kappa} \binom{\kappa}{\gamma} C_\ell C'_0 A_1^{|\gamma|} A_2^{|\kappa-\gamma|} M_{|\gamma|} M_{|\kappa-\gamma|}, \end{aligned}$$

which gives, with the condition (2.2),

$$\left\| x^\ell D^\kappa (fg) \right\|_{L^\infty(\mathbb{R}^n)} \leq M_{|\kappa|} \sum_{\gamma \leq \kappa} C_\ell C'_0 A_1^{|\gamma|} A_2^{|\kappa-\gamma|} \leq C''_\ell \left(A_1 + A_2\right)^{|\kappa|} M_{|\kappa|}. \tag{3.4}$$

Thus, $fg \in S^{\{M\},A_1+A_2}(\mathbb{R}^n)$.

Furthermore, from (3.4) and Lemma 2.22, we have for all $\ell \in \mathbb{Z}_+^n$, $f \in S^{\{M\},A_1}(\mathbb{R}^n)$, and $g \in S^{\{M\},A_2}(\mathbb{R}^n)$,

$$\|fg\|_\ell = \sup_{x \in \mathbb{R}^n} \sup_{\kappa \in \mathbb{Z}_+^n} \frac{\left| x^\ell D^\kappa (fg)(x) \right|}{(A_1 + A_2)^{|\kappa|} M_{|\kappa|}} \leq \|f\|_\ell \|g\|_0,$$

which means that the operator of multiplication transforms every bounded set of the space $S^{\{M\},A_1}(\mathbb{R}^n) \times S^{\{M\},A_2}(\mathbb{R}^n)$ in a bounded set of the space $S^{\{M\},A_1+A_2}(\mathbb{R}^n)$. □

Corollary 3.10. *If the sequence $\left(M_p\right)$ satisfies* (2.2)*, then for all $A > 0$, the space $S^{\{M\},A}(\mathbb{R}^n)$ is transformed by multiplication of its elements into the space $S^{\{M\},2A}(\mathbb{R}^n)$, furthermore this operator is bounded.*

Corollary 3.11. *If the sequence $\left(M_p\right)$ satisfies* (2.2)*, then the space $S^{\{M\}}(\mathbb{R}^n)$ is stable by the multiplication of its elements; furthermore, this operator is bounded.*

3.2.2. Case of the space $S_{\{N\}}(\mathbb{R}^n)$

Theorem 3.12. *Let $B_1, B_2 > 0$. For every*

$$f \in S_{\{N\},B_1}(\mathbb{R}^n) \quad \text{and} \quad g \in S_{\{N\},B_2}(\mathbb{R}^n)$$

the pointwise product fg belongs to $S_{\{N\},\min(B_1,B_2)}(\mathbb{R}^n)$.

Moreover, the multiplication operator is bounded from $S_{\{N\},B_1}(\mathbb{R}^n) \times S_{\{N\},B_2}(\mathbb{R}^n)$ into $S_{\{N\},\min(B_1,B_2)}(\mathbb{R}^n)$.

Proof. Let $f \in S_{\{N\},B_1}(\mathbb{R}^n), g \in S_{\{N\},B_2}(\mathbb{R}^n)$. Then,

$$\forall \kappa \in \mathbb{Z}_+^n,\ \exists C_\kappa > 0, \forall \ell \in \mathbb{Z}_+^n : \left\| x^\ell D^\kappa f \right\|_{L^\infty(\mathbb{R}^n)} \le C_\kappa B_1^{|\ell|} N_{|\ell|}$$

and

$$\forall \kappa \in \mathbb{Z}_+^n,\ \exists C'_\kappa > 0, \forall \ell \in \mathbb{Z}_+^n : \left\| x^\ell D^\kappa g \right\|_{L^\infty(\mathbb{R}^n)} \le C'_\kappa B_2^{|\ell|} N_{|\ell|}.$$

So, by the Leibniz formula, we obtain

$$\begin{aligned} \left\| x^\ell D^\kappa (fg) \right\|_{L^\infty(\mathbb{R}^n)} &= \left\| x^\ell \sum_{\gamma \le \kappa} \binom{\kappa}{\gamma} D^\gamma f D^{\kappa-\gamma} g \right\|_{L^\infty(\mathbb{R}^n)} \\ &\le \sum_{\gamma \le \kappa} \binom{\kappa}{\gamma} \left\| x^\ell D^\gamma f D^{\kappa-\gamma} g \right\|_{L^\infty(\mathbb{R}^n)} \\ &\le \sum_{\gamma \le \kappa} \binom{\kappa}{\gamma} C_\gamma C'_{\kappa-\gamma} \min\left(B_1^{|\ell|}, B_2^{|\ell|}\right) N_{|\ell|}, \end{aligned}$$

which gives

$$\left\| x^\ell D^\kappa (fg) \right\|_{L^\infty(\mathbb{R}^n)} \le C''_\kappa \min(B_1, B_2)^{|\ell|} N_{|\ell|}.$$

Thus, $fg \in S_{\{N\},\min(B_1,B_2)}(\mathbb{R}^n)$.

Furthermore, we see that the operator of multiplication transforms every bounded set of the space $S_{\{N\},B_1}(\mathbb{R}^n) \times S_{\{N\},B_2}(\mathbb{R}^n)$ in a bounded set of the space $S_{\{N\},\min(B_1,B_2)}(\mathbb{R}^n)$. □

Corollary 3.13. *If the sequence $\left(N_p\right)$ satisfies* (2.3)*, then for all $B > 0$, the space $S_{\{N\},B}(\mathbb{R}^n)$ is stable by the multiplication of its elements; furthermore, this operator is bounded.*

Corollary 3.14. *If the sequence $\left(N_p\right)$ satisfies* (2.3)*, then the space $S_{\{N\}}(\mathbb{R}^n)$ is stable by the multiplication of its elements; furthermore, this operator is bounded.*

3.2.3. Case of the space $S_{\{N\}}^{\{M\}}(\mathbb{R}^n)$

From the results concerning the cases of the spaces $S^{\{M\}}(\mathbb{R}^n)$ and $S_{\{N\}}(\mathbb{R}^n)$, we deduce the following result.

Theorem 3.15. *Suppose that the sequences $\left(M_p\right)$ and $\left(N_p\right)$ satisfy* (2.2)-(2.3)*, then*

$$\forall f \in S_{\{N\},B_1}^{\{M\},A_1}(\mathbb{R}^n), \forall g \in S_{\{N\},B_2}^{\{M\},A_2}(\mathbb{R}^n) : fg \in S_{\{N\},\min(B_1,B_2)}^{\{M\},A_1+A_2}(\mathbb{R}^n).$$

Moreover the operator of multiplication is bounded from the space $S_{\{N\},B_1}^{\{M\},A_1}(\mathbb{R}^n) \times S_{\{N\},B_2}^{\{M\},A_2}(\mathbb{R}^n)$ into the space $S_{\{N\},\min(B_1,B_2)}^{\{M\},A_1+A_2}(\mathbb{R}^n)$.

Corollary 3.16. *If the sequences $\left(M_p\right)$ and $\left(N_p\right)$ satisfy* (2.2)-(2.3)*, then for all $A, B > 0$, the space $S_{\{N\},B}^{\{M\},A}(\mathbb{R}^n)$ is transformed by the multiplication of its elements to the space $S_{\{N\},B}^{\{M\},2A}(\mathbb{R}^n)$; furthermore, this operator is bounded.*

Corollary 3.17. *If the sequences $\left(M_p\right)$ and $\left(N_p\right)$ satisfy* (2.2)-(2.3)*, then the space $S_{\{N\}}^{\{M\}}(\mathbb{R}^n)$ is stable by the multiplication of its elements; furthermore, this operator is bounded.*

3.3. Multipliers of Gelfand–Shilov spaces of Roumieu type

For the stability of the Gelfand–Shilov spaces of Roumieu type under multiplication by infinitely differentiable functions, we will impose specific conditions on these functions for each type of Gelfand–Shilov spaces of Roumieu type.

3.3.1. Case of the space $S^{\{M\}}(\mathbb{R}^n)$

Theorem 3.18. *Let f be an infinitely differentiable function satisfying the following condition:*

$$\exists h \in \mathbb{Z}_+^n, \exists C_0, A_0 > 0, \forall \kappa \in \mathbb{Z}_+^n, \forall x \in \mathbb{R}^n : |D^\kappa f(x)| \leqslant C_0 A_0^{|\kappa|} M_{|\kappa|} \left(1 + |x^h|\right). \tag{3.5}$$

Then, the operator of multiplication by the function f is bounded from $S^{\{M\},A}(\mathbb{R}^n)$ to $S^{\{M\},A+A_0}(\mathbb{R}^n)$.

Proof. Letting $A > 0$ and $\varphi \in S^{\{M\},A}(\mathbb{R}^n)$, we have

$$\forall \ell \in \mathbb{Z}_+^n, \exists C_\ell > 0, \forall \kappa \in \mathbb{Z}_+^n : \sup_{x\in\mathbb{R}^n} \left|x^\ell D^\kappa \varphi(x)\right| \leqslant C_\ell A^{|\kappa|} M_{|\kappa|};$$

hence, by the Leibniz formula and the inequality (3.5), we obtain

$$\begin{aligned} \left|x^\ell D^\kappa (f(x)\varphi(x))\right| &\leqslant \left|x^\ell\right| \sum_{\gamma\leq\kappa} \binom{\kappa}{\gamma} |D^\gamma f(x)| \left|D^{\kappa-\gamma}\varphi(x)\right| \\ &\leqslant \sum_{\gamma\leq\kappa} \binom{\kappa}{\gamma} C_0 A_0^{|\gamma|} M_{|\gamma|} \left(|x^\ell| + |x^{\ell+h}|\right) \left|D^{|\kappa-\gamma|}\varphi(x)\right| \\ &\leqslant \sum_{\gamma\leq\kappa} \binom{\kappa}{\gamma} C_0 A_0^{|\gamma|} M_{|\gamma|} \left(C_\ell + C_{\ell+h}\right) A^{|\kappa-\gamma|} M_{|\kappa-\gamma|}, \end{aligned}$$

which gives, with the condition (2.2),

$$\begin{aligned} \left|x^\ell D^\kappa (f(x)\varphi(x))\right| &\leqslant C_0 \left(C_\ell + C_{\ell+h}\right) M_{|\kappa|} \sum_{\gamma\leq\kappa} A_0^{|\gamma|} A^{|\kappa-\gamma|} \\ &\leq C'_\ell M_{|\kappa|} (A + A_0)^{|\kappa|}. \end{aligned}$$

Thus, $f\varphi \in S^{\{M\},A+A_0}(\mathbb{R}^n)$.

For the proof of the boundedness of the operator of multiplication by the function f, we have from the above inequality for all $\ell \in \mathbb{Z}_+^n$

$$\|f\varphi\|_\ell = \sup_{x\in\mathbb{R}^n} \sup_{\kappa\in\mathbb{Z}_+^n} \frac{\left|x^\ell D^\kappa (f\varphi)(x)\right|}{(A + A_0)^{|\kappa|} M_{|\kappa|}} \leqslant C_0 \left(C_\ell + C_{\ell+h}\right)$$

where C_ℓ is such that

$$\sup_{x\in\mathbb{R}^n} \sup_{\kappa\in\mathbb{Z}_+^n} \frac{\left|x^\ell D^\kappa \varphi(x)\right|}{A^{|\kappa|} M_{|\kappa|}} \leqslant C_\ell,$$

which means that the operator of multiplication by any function f satisfying (3.5) is bounded from $S^{\{M\},A}(\mathbb{R}^n)$ to $S^{\{M\},A+A_0}(\mathbb{R}^n)$. □

Corollary 3.19. *The space $S^{\{M\}}(\mathbb{R}^n)$ is stable by multiplication by any function f satisfying* (3.5)*, and the operator of multiplication by the function f is bounded in the space $S^{\{M\}}(\mathbb{R}^n)$.*

3.3.2. Case of the space $S_{\{N\}}(\mathbb{R}^n)$

Theorem 3.20. *Let f be an infinitely differentiable function satisfying the following condition:*

$$\forall \kappa \in \mathbb{Z}_+^n, \exists C_\kappa > 0 : |D^\kappa f(x)| \leq C_\kappa,\ \forall x \in \mathbb{R}^n. \tag{3.6}$$

Then, for all $B > 0$, the operator of multiplication by the function f is bounded from $S_{\{N\},B}(\mathbb{R}^n)$ into itself.

Proof. Let $B > 0$, and let $\varphi \in S_{\{N\},B}(\mathbb{R}^n)$ so that

$$\forall \kappa \in \mathbb{Z}_+^n, \exists C_\kappa > 0, \forall \ell \in \mathbb{Z}_+^n : \sup_{x\in\mathbb{R}^n} \left|x^\ell D^\kappa \varphi(x)\right| \leqslant C_\kappa B^{|\ell|} N_{|\ell|}.$$

By the Leibniz formula and the inequality (3.6), we obtain

$$\begin{aligned}
\left|x^\ell D^\kappa (f(x)\varphi(x))\right| &\leqslant \left|x^\ell\right| \sum_{\gamma\leq\kappa} \binom{\kappa}{\gamma} |D^\gamma f(x)| \left|D^{\kappa-\gamma}\varphi(x)\right| \\
&\leqslant \sum_{\gamma\leq\kappa} \binom{\kappa}{\gamma} C_\gamma |x^\ell| \left|D^{|\kappa-\gamma|}\varphi(x)\right| \\
&\leqslant \sum_{\gamma\leq\kappa} \binom{\kappa}{\gamma} C_\gamma C_{\kappa-\gamma} B^{|\ell|} N_{|\ell|} \\
&\leq C'_\kappa B^{|\ell|} N_{|\ell|}.
\end{aligned}$$

Thus, $f\varphi \in S_{\{N\},B}(\mathbb{R}^n)$.

For the proof of the boundedness of the operator of multiplication by the function f, we have from the above caculus for all $\kappa \in \mathbb{Z}_+^n$

$$\begin{aligned}
\|f\varphi\|_\kappa &= \sup_{x\in\mathbb{R}^n} \sup_{\ell\in\mathbb{Z}_+^n} \frac{\left|x^\ell D^\kappa (f\varphi)(x)\right|}{B^{|\ell|} N_{|\ell|}} \\
&\leq \sup_{x\in\mathbb{R}^n} \sup_{\ell\in\mathbb{Z}_+^n} \sum_{\gamma\leq\kappa} \binom{\kappa}{\gamma} C_\gamma \frac{\left|x^\ell D^{|\kappa-\gamma|}\varphi(x)\right|}{B^{|\ell|} N_{|\ell|}} \\
&\leq \sum_{\gamma\leq\kappa} \binom{\kappa}{\gamma} C_\gamma \|\varphi\|_{\kappa-\gamma},
\end{aligned}$$

which means that the operator of multiplication by any function f satisfying (3.6) is bounded from $S_{\{N\},B}(\mathbb{R}^n)$ to itself. □

Corollary 3.21. *The space $S_{\{N\}}(\mathbb{R}^n)$ is stable by multiplication by any function f satisfying* (3.6)*, and the operator of multiplication by the function f is bounded from $S_{\{N\}}(\mathbb{R}^n)$ to itself.*

3.3.3. Case of the space $S_{\{N\}}^{\{M\}}(\mathbb{R}^n)$

Theorem 3.22. *Let f be an infinitely differentiable function satisfying the condition* (3.5)*. Then, for all $A > 0$ and $B > 0$, the operator of multiplication by the function f is bounded from $S_{\{N\},B}^{\{M\},A}(\mathbb{R}^n)$ to $S_{\{N\},\max(B,BH)}^{\{M\},A+A_0}(\mathbb{R}^n)$, where H is the constant of the condition* (2.3).

Proof. Let $A > 0$ and $B > 0$, and let $\varphi \in S_{\{N\},B}^{\{M\},A}(\mathbb{R}^n)$ so that

$$\exists C > 0, \forall \kappa, \ell \in \mathbb{Z}_+^n : \sup_{x\in\mathbb{R}^n} \left| x^\ell D^\kappa \varphi(x) \right| \leqslant CA^{|\kappa|}B^{|\ell|}M_{|\kappa|}N_{|\ell|}.$$

By the Leibniz formula and the inequality (3.5), we obtain

$$\begin{aligned} \left| x^\ell D^\kappa (f(x)\varphi(x)) \right| &\leqslant \left| x^\ell \right| \sum_{\gamma\leq\kappa} \binom{\kappa}{\gamma} |D^\gamma f(x)| \left| D^{\kappa-\gamma}\varphi(x) \right| \\ &\leqslant \sum_{\gamma\leq\kappa} \binom{\kappa}{\gamma} C_0 A_0^{|\gamma|} M_{|\gamma|} \left(|x^\ell| + |x^{\ell+h}| \right) \left| D^{|\kappa-\gamma|}\varphi(x) \right| \\ &\leqslant \sum_{\gamma\leq\kappa} \binom{\kappa}{\gamma} C_0 A_0^{|\gamma|} M_{|\gamma|} C \left(B^{|\ell|}N_{|\ell|} + B^{|\ell+h|}N_{|\ell+h|} \right) A^{|\kappa-\gamma|} M_{|\kappa-\gamma|}, \end{aligned}$$

which gives, with the condition (2.2)-(2.3),

$$\begin{aligned} \left| x^\ell D^\kappa (f(x)\varphi(x)) \right| &\leqslant CC_0 \left(B^{|\ell|}N_{|\ell|} + (BH)^{|\ell+h|} N_{|\ell|}N_{|h|} \right) M_{|\kappa|} \sum_{\gamma\leq\kappa} A_0^{|\gamma|} A^{|\kappa-\gamma|} \\ &\leqslant CC_0 \left(1 + (BH)^{|h|} N_{|h|} \right) (\max(B, BH))^{|\ell|} N_{|\ell|} M_{|\kappa|} (A + A_0)^{|\kappa|} \\ &\leq C' M_{|\kappa|} (A + A_0)^{|\kappa|} (\max(B, BH))^{|\ell|} N_{|\ell|}, \end{aligned}$$

where C' depend on parameters B, H.

$$C' := CC_0 \left(1 + (BH)^{|h|} N_{|h|} \right).$$

Thus, $f\varphi \in S_{\{N\},\max(B,BH)}^{\{M\},A+A_0}(\mathbb{R}^n)$.

For the boundedness of the operator of multiplication by any function f satisfying the condition (3.5), we have from the above inequality

$$\|f\varphi\| = \sup_{x\in\mathbb{R}^n} \sup_{\kappa,\ell\in\mathbb{Z}_+^n} \frac{\left| x^\ell D^\kappa (f(x)\varphi(x)) \right|}{(A + A_0)^{|\kappa|} (HB)^{|\ell|} M_{|\kappa|}N_{|\ell|}} \leq C'.$$

Then, for all $C > 0$ such that

$$\sup_{x\in\mathbb{R}^n} \sup_{\kappa,\ell\in\mathbb{Z}_+^n} \frac{\left| x^\ell D^\kappa \varphi(x) \right|}{A^{|\kappa|}B^{|\ell|}M_{|\kappa|}N_{|\ell|}} \leqslant C,$$

there is $C' > 0$ so that $\|f\varphi\| \leq C'$, which means that the range of a bounded set of $S_{\{N\},B}^{\{M\},A}(\mathbb{R}^n)$ by the operator of multiplication by any function f satisfying the condition (3.5) is a bounded set in the space $S_{\{N\},BH}^{\{M\},A+A_0}(\mathbb{R}^n)$. □

Remark 3.23. *This is the point where anisotropy plays a decisive role. Unlike in isotropic theory, one must control how the binomial distribution interacts with two independent growth regimes.*

Corollary 3.24. *The space $S_{\{N\}}^{\{M\}}(\mathbb{R}^n)$ is stable by multiplication by any function f satisfying the condition* (3.5)*, and the operator of multiplication by the function f is bounded from $S_{\{N\}}^{\{M\}}(\mathbb{R}^n)$ to itself.*

3.4. Translation operator

For all $h \in \mathbb{R}^n$, the spaces $S^{\{M\}}(\mathbb{R}^n)$, $S_{\{N\}}(\mathbb{R}^n)$, and $S^{\{M\}}_{\{N\}}(\mathbb{R}^n)$ are stable under translation by the vector h; furthermore, the operator of translation by h is bounded from each space to itself.

3.4.1. Case of the space $S^{\{M\}}(\mathbb{R}^n)$

Theorem 3.25. *The translation operator*

$$\varphi(x) \to \varphi(x-h) \quad h \in \mathbb{R}^n$$

is defined and bounded in any space $S^{\{M\},A}(\mathbb{R}^n)$ and transforms this space into itself.

Proof. Let $A > 0$, and let $\varphi \in S^{\{M\},A}(\mathbb{R}^n)$ so that

$$\forall \ell \in \mathbb{Z}_+^n, \exists C_\ell > 0, \forall \kappa \in \mathbb{Z}_+^n : \left|x^\ell D^\kappa \varphi(x)\right| \leqslant C_\ell A^{|\kappa|} M_{|\kappa|}.$$

Because

$$\sup_{x\in\mathbb{R}^n} \left|x^\ell D^\kappa \varphi(x-h)\right| = \sup_{x\in\mathbb{R}^n} \left|(x+h)^\ell D^\kappa \varphi(x)\right|,$$

we get

$$\begin{aligned}
\left|(x+h)^\ell D^\kappa \varphi(x)\right| &\leqslant \left|\sum_{\gamma\leq\ell} \binom{\ell}{\gamma} x^\gamma h^{\ell-\gamma} D^\kappa \varphi(x)\right| \\
&\leqslant \sum_{\gamma\leq\ell} \binom{\ell}{\gamma} \left|h^{\ell-\gamma}\right| \left|x^\gamma D^\kappa \varphi(x)\right| \\
&\leqslant \sum_{\gamma\leq\ell} \binom{\ell}{\gamma} \left|h^{\ell-\gamma}\right| C_\gamma A^{|\kappa|} M_{|\kappa|} \\
&\leq C'_\ell A^{|\kappa|} M_{|\kappa|}. \qquad (3.7)
\end{aligned}$$

Hence, $\varphi(\cdot - h) \in S^{\{M\},A}(\mathbb{R}^n)$.

Furthermore, from (3.7) and Lemma 2.22, we have for all $\ell \in \mathbb{Z}_+^n$

$$\begin{aligned}
\|\varphi(\cdot - h)\|_\ell &= \sup_{x\in\mathbb{R}^n} \sup_{\kappa\in\mathbb{Z}_+^n} \frac{\left|x^\ell D^\kappa \varphi(x-h)\right|}{A^{|\kappa|} M_{|\kappa|}} \\
&= \sup_{x\in\mathbb{R}^n} \sup_{\kappa\in\mathbb{Z}_+^n} \frac{\left|(x+h)^\ell D^\kappa \varphi(x)\right|}{A^{|\kappa|} M_{|\kappa|}} \\
&\leqslant \sup_{x\in\mathbb{R}^n} \sup_{\kappa\in\mathbb{Z}_+^n} \sum_{\gamma\leq\ell} \binom{\ell}{\gamma} \left|h^{\ell-\gamma}\right| \frac{\left|x^\gamma D^\kappa \varphi(x)\right|}{A^{|\kappa|} M_{|\kappa|}} \\
&\leq \sum_{\gamma\leq\ell} \binom{\ell}{\gamma} \left|h^{\ell-\gamma}\right| \|\varphi\|_\gamma,
\end{aligned}$$

which means that the translation operator is bounded in $S^{\{M\},A}(\mathbb{R}^n)$. □

Corollary 3.26. *The operator of translation is defined and bounded in the space $S^{\{M\}}(\mathbb{R}^n)$ and transforms this space into itself.*

3.4.2. Case of the space $S_{\{N\}}(\mathbb{R}^n)$

Theorem 3.27. *The translation operator*

$$\varphi(x) \to \varphi(x-h)$$

is defined and bounded in any space $S_{\{N\},B}(\mathbb{R}^n)$ *and transforms this space into the space* $S_{\{N\},B+|h|}(\mathbb{R}^n)$.

Proof. Let $B>0$, and let $\varphi \in S_{\{N\},B}(\mathbb{R}^n)$ so that

$$\forall \kappa \in \mathbb{Z}_+^n, \exists C_\kappa > 0, \forall \ell \in \mathbb{Z}_+^n : \left|x^\ell D^\kappa \varphi(x)\right| \leqslant C_\kappa B^{|\ell|} N_{|\ell|}.$$

Because

$$\sup_{x\in\mathbb{R}^n} \left|x^\ell D^\kappa \varphi(x-h)\right| = \sup_{x\in\mathbb{R}^n} \left|(x+h)^\ell D^\kappa \varphi(x)\right|,$$

we get

$$\begin{aligned}
\left|(x+h)^\ell D^\kappa \varphi(x)\right| &\leqslant \left|\sum_{\gamma\le\ell}\binom{\ell}{\gamma} x^\gamma h^{\ell-\gamma} D^\kappa\varphi(x)\right| \\
&\leqslant \sum_{\gamma\le\ell}\binom{\ell}{\gamma}\left|h^{\ell-\gamma}\right| |x^\gamma D^\kappa \varphi(x)| \\
&\leqslant \sum_{\gamma\le\ell}\binom{\ell}{\gamma}|h|^{|\ell-\gamma|} C_\kappa B^{|\gamma|} N_{|\gamma|} \\
&\leq C_\kappa (B+|h|)^{|\ell|} N_{|\ell|}.
\end{aligned}$$

Hence, $\varphi(\cdot - h) \in S_{\{N\},B+|h|}(\mathbb{R}^n)$.

Furthermore, from the above estimates, we have for all $\kappa \in \mathbb{Z}_+^n$

$$\begin{aligned}
\|\varphi(\cdot-h)\|_\kappa &= \sup_{x\in\mathbb{R}^n}\sup_{\ell\in\mathbb{Z}_+^n} \frac{\left|x^\ell D^\kappa\varphi(x-h)\right|}{(B+|h|)^{|\ell|}N_{|\ell|}} \\
&= \sup_{x\in\mathbb{R}^n}\sup_{\ell\in\mathbb{Z}_+^n} \frac{\left|(x+h)^\ell D^\kappa\varphi(x)\right|}{(B+|h|)^{|\ell|}N_{|\ell|}} \\
&\leqslant \sup_{x\in\mathbb{R}^n}\sup_{\ell\in\mathbb{Z}_+^n} \sum_{\gamma\le\ell}\binom{\ell}{\gamma}\left|h^{\ell-\gamma}\right| \frac{|x^\gamma D^\kappa\varphi(x)|}{(B+|h|)^{|\ell|}N_{|\ell|}} \\
&\leqslant \sup_{x\in\mathbb{R}^n}\sup_{\ell\in\mathbb{Z}_+^n} \sum_{\gamma\le\ell}\binom{\ell}{\gamma}\frac{|h|^{|\ell-\gamma|}B^{|\gamma|}}{(B+|h|)^{|\ell|}} \frac{|x^\gamma D^\kappa\varphi(x)|}{B^{|\gamma|}N_{|\gamma|}} \\
&\leqslant \sup_{x\in\mathbb{R}^n}\sup_{\ell\in\mathbb{Z}_+^n} \sum_{\gamma\le\ell}\binom{\ell}{\gamma}\frac{|h|^{|\ell-\gamma|}B^{|\gamma|}}{(B+|h|)^{|\ell|}} \sup_{\gamma\le\ell}\frac{|x^\gamma D^\kappa\varphi(x)|}{B^{|\gamma|}N_{|\gamma|}} \\
&\leqslant \sup_{x\in\mathbb{R}^n}\sup_{\ell\in\mathbb{Z}_+^n}\sup_{\gamma\le\ell} \frac{|x^\gamma D^\kappa\varphi(x)|}{B^{|\gamma|}N_{|\gamma|}} \\
&\leq \|\varphi\|_\kappa,
\end{aligned}$$

which means that the translation operator is bounded from $S_{\{N\},B}(\mathbb{R}^n)$ to $S_{\{N\},B+|h|}(\mathbb{R}^n)$. □

Corollary 3.28. *The operator of translation is defined and bounded in the space* $S_{\{N\}}(\mathbb{R}^n)$ *and transforms this space into itself.*

3.4.3. Case of the space $S^{\{M\}}_{\{N\}}(\mathbb{R}^n)$

Theorem 3.29. *The translation operator*

$$\varphi(x) \to \varphi(x-h)$$

is defined and bounded in any space $S^{\{M\},A}_{\{N\},B}(\mathbb{R}^n)$ *and transforms this space into* $S^{\{M\},A}_{\{N\},B+|h|}(\mathbb{R}^n)$.

Proof. Let $A>0$ and $B>0$, and let $\varphi \in S^{\{M\},A}_{\{N\},B}(\mathbb{R}^n)$ so that

$$\exists C>0, \forall \kappa,\ell \in \mathbb{Z}^n_+ : \left|x^\ell D^\kappa \varphi(x)\right| \leqslant CA^{|\kappa|}B^{|\ell|}M_{|\kappa|}N_{|\ell|}.$$

Because

$$\sup_{x\in\mathbb{R}^n}\left|x^\ell D^\kappa\varphi(x-h)\right| = \sup_{x\in\mathbb{R}^n}\left|(x+h)^\ell D^\kappa\varphi(x)\right|,$$

we get

$$\begin{aligned}
\left|(x+h)^\ell D^\kappa\varphi(x)\right| &\leqslant \left|\sum_{\gamma\le\ell}\binom{\ell}{\gamma}x^\gamma h^{\ell-\gamma}D^\kappa\varphi(x)\right| \\
&\leqslant \sum_{\gamma\le\ell}\binom{\ell}{\gamma}\left|h^{\ell-\gamma}x^\gamma D^\kappa\varphi(x)\right| \\
&\leqslant \sum_{\gamma\le\ell}\binom{\ell}{\gamma}|h|^{|\ell-\gamma|}\,CA^{|\kappa|}B^{|\gamma|}M_{|\kappa|}N_{|\gamma|} \\
&\le CA^{|\kappa|}(B+|h|)^{|\ell|}M_{|\kappa|}N_{|\ell|}.
\end{aligned}$$

Hence, $\varphi(\cdot - h) \in S^{\{M\},A}_{\{N\},B+|h|}(\mathbb{R}^n)$.

Furthermore, from the above estimates, we have

$$\begin{aligned}
\|\varphi(\cdot-h)\| &= \sup_{x\in\mathbb{R}^n}\sup_{\kappa,\ell\in\mathbb{Z}^n_+}\frac{\left|x^\ell D^\kappa\varphi(x-h)\right|}{A^{|\kappa|}(B+|h|)^{|\ell|}M_{|\kappa|}N_{|\ell|}} \\
&= \sup_{x\in\mathbb{R}^n}\sup_{\kappa,\ell\in\mathbb{Z}^n_+}\frac{\left|(x+h)^\ell D^\kappa\varphi(x)\right|}{A^{|\kappa|}(B+|h|)^{|\ell|}M_{|\kappa|}N_{|\ell|}} \\
&\leqslant \sup_{x\in\mathbb{R}^n}\sup_{\kappa,\ell\in\mathbb{Z}^n_+}\sum_{\gamma\le\ell}\binom{\ell}{\gamma}\left|h^{\ell-\gamma}\right|\frac{|x^\gamma D^\kappa\varphi(x)|}{A^{|\kappa|}(B+|h|)^{|\ell|}M_{|\kappa|}N_{|\ell|}} \\
&\leqslant \sup_{x\in\mathbb{R}^n}\sup_{\kappa,\ell\in\mathbb{Z}^n_+}\sum_{\gamma\le\ell}\binom{\ell}{\gamma}\frac{|h|^{|\ell-\gamma|}B^{|\gamma|}}{(B+|h|)^{|\ell|}}\sup_{\gamma\le\ell}\frac{|x^\gamma D^\kappa\varphi(x)|}{A^{|\kappa|}B^{|\gamma|}M_{|\kappa|}N_{|\gamma|}} \\
&\leqslant \sup_{x\in\mathbb{R}^n}\sup_{\kappa,\ell\in\mathbb{Z}^n_+}\sup_{\gamma\le\ell}\frac{|x^\gamma D^\kappa\varphi(x)|}{A^{|\kappa|}B^{|\gamma|}M_{|\kappa|}N_{|\gamma|}} \\
&\le \|\varphi\|,
\end{aligned}$$

which means that the translation operator maps every bounded set of $S^{\{M\},A}_{\{N\},B}(\mathbb{R}^n)$ to a bounded set of $S^{\{M\},A}_{\{N\},B+|h|}(\mathbb{R}^n)$. □

Corollary 3.30. *The operator of translation is defined and bounded in the space* $S^{\{M\}}_{\{N\}}(\mathbb{R}^n)$ *and transforms this space into itself.*

3.5. *Dilatation operator*

For all $\lambda > 0$, the spaces $S^{\{M\}}(\mathbb{R}^n)$, $S_{\{N\}}(\mathbb{R}^n)$, and $S^{\{M\}}_{\{N\}}(\mathbb{R}^n)$ are stable under dilatation of ratio λ; furthermore, the operator of dilatation of ratio λ is bounded from each space to itself.

The result can be refined if we consider the subspaces $S_{\{N\},B}(\mathbb{R}^n)$, $S^{\{M\},A}(\mathbb{R}^n)$, and $S^{\{M\},A}_{\{N\},B}(\mathbb{R}^n)$ more precisely; we thus obtain the following results.

3.5.1. Case of the space $S^{\{M\}}(\mathbb{R}^n)$

Theorem 3.31. *Let $\lambda > 0$. The dilation operator defined by*

$$\varphi(x) \to \varphi(\lambda x)$$

is a well–defined and bounded linear operator from $S^{\{M\},A}(\mathbb{R}^n)$ into $S^{\{M\},\lambda A}(\mathbb{R}^n)$.

Proof. Let $A > 0$, and let $\varphi \in S^{\{M\},A}(\mathbb{R}^n)$ so that

$$\forall \ell \in \mathbb{Z}^n_+, \exists C_\ell > 0, \forall \kappa \in \mathbb{Z}^n_+ : \left|x^\ell D^\kappa \varphi(x)\right| \leqslant C_\ell A^{|\kappa|} M_{|\kappa|}.$$

Because

$$\left|x^\ell D^\kappa \left[\varphi(\lambda x)\right]\right| = \left|x^\ell \lambda^{|\kappa|} (D^\kappa \varphi)(\lambda x)\right|,$$

we get

$$\begin{aligned}
\left|x^\ell D^\kappa \left[\varphi(\lambda x)\right]\right| &= \left|\frac{(\lambda x)^\ell}{\lambda^{|\ell|}} \lambda^{|\kappa|} (D^\kappa \varphi)(\lambda x)\right| \\
&= \left|\frac{\lambda^{|\kappa|}}{\lambda^{|\ell|}}\right| \left|(\lambda x)^\ell (D^\kappa \varphi)(\lambda x)\right| \\
&\leqslant \frac{\lambda^{|\kappa|}}{\lambda^{|\ell|}} C_\ell A^{|\kappa|} M_{|\kappa|} \\
&\leqslant C'_\ell (\lambda A)^{|\kappa|} M_{|\kappa|}.
\end{aligned}$$

Hence, $\varphi(\lambda \cdot) \in S^{\{M\},\lambda A}(\mathbb{R}^n)$.

For the boundedness of the operator of dilatation, we have from the above inequality

$$\|\varphi(\lambda \cdot)\|_\ell = \sup_{x \in \mathbb{R}^n} \sup_{\kappa \in \mathbb{Z}^n_+} \frac{\left|x^\ell D^\kappa \left[\varphi(\lambda x)\right]\right|}{(\lambda A)^{|\kappa|} M_{|\kappa|}} \leq \frac{C_\ell}{\lambda^{|\ell|}},$$

where C_ℓ is such that

$$\sup_{x \in \mathbb{R}^n} \sup_{\kappa \in \mathbb{Z}^n_+} \frac{\left|x^\ell D^\kappa \varphi(x)\right|}{A^{|\kappa|} M_{|\kappa|}} \leqslant C_\ell,$$

which means that the dilatation operator maps every bounded set of $S^{\{M\},A}(\mathbb{R}^n)$ to a bounded set of $S^{\{M\},\lambda A}(\mathbb{R}^n)$. □

Corollary 3.32. *The operator of dilatation is defined and bounded in the space $S^{\{M\}}(\mathbb{R}^n)$ and transforms this space into itself.*

3.5.2. Case of the space $S_{\{N\}}(\mathbb{R}^n)$

Theorem 3.33. *The operator of dilatation of ratio λ*

$$\varphi(x) \to \varphi(\lambda x)$$

is defined and bounded in any space $S_{\{N\},B}(\mathbb{R}^n)$ and transforms this space into $S_{\{N\},B/\lambda}(\mathbb{R}^n)$.

Proof. Let $B > 0$, and let $\varphi \in S_{\{N\},B}(\mathbb{R}^n)$ so that

$$\forall \kappa \in \mathbb{Z}_+^n, \exists C_\kappa > 0, \forall \ell \in \mathbb{Z}_+^n : \left| x^\ell D^\kappa \varphi(x) \right| \leqslant C_\kappa B^{|\ell|} N_{|\ell|}.$$

Because

$$\left| x^\ell D^\kappa \left[\varphi(\lambda x) \right] \right| = \left| x^\ell \lambda^{|\kappa|} (D^\kappa \varphi)(\lambda x) \right|,$$

we get

$$\begin{aligned}
\left| x^\ell D^\kappa \left[\varphi(\lambda x) \right] \right| &= \left| \frac{(\lambda x)^\ell}{\lambda^{|\ell|}} \lambda^{|\kappa|} (D^\kappa \varphi)(\lambda x) \right| \\
&= \left| \frac{\lambda^{|\kappa|}}{\lambda^{|\ell|}} \right| \left| (\lambda x)^\ell (D^\kappa \varphi)(\lambda x) \right| \\
&\leqslant \frac{\lambda^{|\kappa|}}{\lambda^{|\ell|}} C_\kappa B^{|\ell|} N_{|\ell|} \\
&\leqslant C'_\kappa \left(\frac{B}{\lambda} \right)^{|\ell|} N_{|\ell|}. \qquad (3.8)
\end{aligned}$$

Hence, $\varphi(\lambda \cdot) \in S_{\{N\},B/\lambda}(\mathbb{R}^n)$.

Furthermore, from (3.8) and Lemma 2.22, we have for all $\kappa \in \mathbb{Z}_+^n$

$$\|\varphi(\lambda \cdot)\|_\kappa = \sup_{x \in \mathbb{R}^n} \sup_{\ell \in \mathbb{Z}_+^n} \frac{\left| x^\ell D^\kappa \left[\varphi(\lambda x) \right] \right|}{(B/\lambda)^{|\ell|} N_{|\ell|}} \leq \lambda^{|\kappa|} C_\kappa,$$

where C_κ is such that

$$\|\varphi\|_\kappa = \sup_{x \in \mathbb{R}^n} \sup_{\ell \in \mathbb{Z}_+^n} \frac{\left| x^\ell D^\kappa \varphi(x) \right|}{B^{|\ell|} N_{|\ell|}} \leq C_\kappa,$$

which means that the dilatation operator maps every bounded set of $S_{\{N\},B}(\mathbb{R}^n)$ to a bounded set of $S_{\{N\},B/\lambda}(\mathbb{R}^n)$. □

Corollary 3.34. *The operator of dilatation is defined and bounded in the space $S_{\{N\}}(\mathbb{R}^n)$ and transforms this space into itself.*

3.5.3. Case of the space $S_{\{N\}}^{\{M\}}(\mathbb{R}^n)$

Theorem 3.35. *The operator of dilatation of ratio λ*

$$\varphi(x) \to \varphi(\lambda x)$$

is defined and bounded in any space $S_{\{N\},B}^{\{M\},A}(\mathbb{R}^n)$ and transforms this space into $S_{\{N\},B/\lambda}^{\{M\},\lambda A}(\mathbb{R}^n)$.

Proof. Let $A > 0$ and $B > 0$, and let $\varphi \in S_{\{N\},B}^{\{M\},A}(\mathbb{R}^n)$ so that

$$\exists C > 0, \forall \kappa, \ell \in \mathbb{Z}_+^n : \left|x^\ell D^\kappa \varphi(x)\right| \leqslant CA^{|\kappa|}B^{|\ell|}M_{|\kappa|}N_{|\ell|}.$$

Because

$$\left|x^\ell D^\kappa \left[\varphi(\lambda x)\right]\right| = \left|x^\ell \lambda^{|\kappa|}(D^\kappa \varphi)(\lambda x)\right|,$$

we get

$$\begin{aligned}\left|x^\ell D^\kappa \left[\varphi(\lambda x)\right]\right| &= \left|\frac{(\lambda x)^\ell}{\lambda^{|\ell|}}\lambda^{|\kappa|}(D^\kappa\varphi)(\lambda x)\right| \\ &= \left|\frac{\lambda^{|\kappa|}}{\lambda^{|\ell|}}\right|\left|(\lambda x)^\ell(D^\kappa\varphi)(\lambda x)\right| \\ &\leqslant \frac{\lambda^{|\kappa|}}{\lambda^{|\ell|}}CA^{|\kappa|}B^{|\ell|}M_{|\kappa|}N_{|\ell|} \\ &\leqslant C\,(\lambda A)^{|\kappa|}\left(\frac{B}{\lambda}\right)^{|\ell|}M_{|\kappa|}N_{|\ell|}.\end{aligned}$$

Hence, $\varphi(\lambda\cdot) \in S_{\{N\},B/\lambda}^{\{M\},\lambda A}(\mathbb{R}^n)$.

Furthermore, from the above inequality, we have

$$\|\varphi(\lambda\cdot)\| = \sup_{x\in\mathbb{R}^n}\sup_{\kappa,\ell\in\mathbb{Z}_+^n}\frac{\left|x^\ell D^\kappa\left[\varphi(\lambda x)\right]\right|}{(\lambda A)^{|\kappa|}\left(B/\lambda\right)^{|\ell|}M_{|\kappa|}N_{|\ell|}} \leq C,$$

where C is such that

$$\sup_{x\in\mathbb{R}^n}\sup_{\kappa,\ell\in\mathbb{Z}_+^n}\frac{\left|x^\ell D^\kappa\varphi(x)\right|}{A^{|\kappa|}B^{|\ell|}M_{|\kappa|}N_{|\ell|}} \leq C,$$

which means that the dilatation operator maps every bounded set of $S_{\{N\},B}^{\{M\},A}(\mathbb{R}^n)$ to a bounded set of $S_{\{N\},B/\lambda}^{\{M\},\lambda A}(\mathbb{R}^n)$. □

Corollary 3.36. *The operator of dilatation is defined and bounded in the space* $S_{\{N\}}^{\{M\}}(\mathbb{R}^n)$ *and transforms this space into itself.*

3.6. Differential operators in Gelfand–Shilov spaces of Roumieu type

Let $\gamma \in \mathbb{Z}_+^n$, then the spaces $S^{\{M\}}(\mathbb{R}^n)$, $S_{\{N\}}(\mathbb{R}^n)$, and $S_{\{N\}}^{\{M\}}(\mathbb{R}^n)$ are stable under the operator of derivation D^γ; furthermore, the operator of derivation D^γ is bounded from each space to itself.

More precisely, the operator of derivation D^γ is bounded in each of the subspaces $S_{\{N\},B}(\mathbb{R}^n)$, $S^{\{M\},A}(\mathbb{R}^n)$, and $S_{\{N\},B}^{\{M\},A}(\mathbb{R}^n)$.

3.6.1. Case of the space $S^{\{M\}}(\mathbb{R}^n)$

Theorem 3.37. *The differentiation operator* D^γ *is defined and bounded in the spaces* $S^{\{M\},A}(\mathbb{R}^n)$ *and transforms this space into* $S^{\{M\},AH}(\mathbb{R}^n)$.

Proof. Let $A > 0$, and let $\varphi \in S^{\{M\},A}(\mathbb{R}^n)$. This means that the inequality

$$\forall \ell \in \mathbb{Z}_+^n, \exists C_\ell > 0, \forall \kappa \in \mathbb{Z}_+^n : \left| x^\ell D^\kappa \varphi(x) \right| \leqslant C_\ell A^{|\kappa|} M_{|\kappa|}$$

is satisfied. Then, we have

$$\left| x^\ell D^\kappa D^\gamma \varphi(x) \right| = \left| x^\ell D^{\kappa+\gamma} \varphi(x) \right| \leqslant C_\ell A^{|\kappa+\gamma|} M_{|\kappa+\gamma|},$$

which gives, with the condition (2.3),

$$\left| x^\ell D^\kappa D^\gamma \varphi(x) \right| \leqslant C_\ell H^{|\kappa+\gamma|} A^{|\gamma|} M_{|\gamma|} A^{|\kappa|} M_{|\kappa|} = C_\ell^{'} (AH)^{|\kappa|} M_{|\kappa|}.$$

Furthermore, from above, we have for all $\ell \in \mathbb{Z}_+^n$

$$\|D^\gamma \varphi\|_\ell = \sup_{x \in \mathbb{R}^n} \sup_{\kappa \in \mathbb{Z}_+^n} \frac{\left| x^\ell D^\kappa D^\gamma \varphi(x) \right|}{(AH)^{|\kappa|} M_{|\kappa|}} \leq C_\ell H^{|\gamma|} A^{|\gamma|} M_{|\gamma|},$$

where C_ℓ is such that

$$\sup_{x \in \mathbb{R}^n} \sup_{\kappa \in \mathbb{Z}_+^n} \frac{\left| x^\ell D^\kappa \varphi(x) \right|}{A^{|\kappa|} M_{|\kappa|}} \leqslant C_\ell.$$

Therefore, the range of a bounded set in the space $S^{\{M\},A}(\mathbb{R}^n)$ is a bounded set in $S^{\{M\},AH}(\mathbb{R}^n)$. □

Corollary 3.38. *The differentiation operator is defined and bounded in the space $S^{\{M\}}(\mathbb{R}^n)$ and transforms this space into itself.*

3.6.2. Case of the space $S_{\{N\}}(\mathbb{R}^n)$

Theorem 3.39. *The differentiation operator is defined and bounded in the spaces $S_{\{N\},B}(\mathbb{R}^n)$ and transforms this space into itself.*

Proof. Let $B > 0$, and let $\varphi \in S_{\{N\},B}(\mathbb{R}^n)$. This means that the inequality

$$\forall \kappa \in \mathbb{Z}_+^n, \exists C_\kappa > 0, \forall \ell \in \mathbb{Z}_+^n : \left| x^\ell D^\kappa \varphi(x) \right| \leqslant C_\kappa B^{|\ell|} N_{|\ell|}$$

is satisfied. Then, we have

$$\left| x^\ell D^\kappa D^\gamma \varphi(x) \right| = \left| x^\ell D^{\kappa+\gamma} \varphi(x) \right| \leqslant C_{\kappa+\gamma} B^{|\ell|} N_{|\ell|} = C_\kappa^{'} B^{|\ell|} N_{|\ell|}.$$

Furthermore, from above, we have for all $\kappa \in \mathbb{Z}_+^n$

$$\|D^\gamma \varphi\|_\kappa = \sup_{x \in \mathbb{R}^n} \sup_{\ell \in \mathbb{Z}_+^n} \frac{\left| x^\ell D^\kappa D^\gamma \varphi(x) \right|}{B^{|\ell|} N_{|\ell|}} \leq C_{\kappa+\gamma},$$

where C_δ is such that

$$\sup_{x \in \mathbb{R}^n} \sup_{\ell \in \mathbb{Z}_+^n} \frac{\left| x^\ell D^\delta \varphi(x) \right|}{B^{|\ell|} N_{|\ell|}} \leq C_\delta.$$

Therefore, the range of a bounded set in the space $S_{\{N\},B}(\mathbb{R}^n)$ is a bounded set in $S_{\{N\},B}(\mathbb{R}^n)$. □

Corollary 3.40. *The differentiation operator is defined and bounded in the space $S_{\{N\}}(\mathbb{R}^n)$ and transforms this space into itself.*

3.6.3. Case of the space $S_{\{N\}}^{\{M\}}(\mathbb{R}^n)$

Theorem 3.41. *The differentiation operator is defined and bounded in the spaces* $S_{\{N\},B}^{\{M\},A}(\mathbb{R}^n)$ *and transforms this space into* $S_{\{N\},B}^{\{M\},AH}(\mathbb{R}^n)$.

Proof. Let $A>0$ and $B>0$, and let $\varphi \in S_{\{N\},B}^{\{M\},A}(\mathbb{R}^n)$. This means that the inequality

$$\exists C>0, \forall \kappa, \ell \in \mathbb{Z}_+^n : \left|x^{\ell} D^{\kappa} \varphi(x)\right| \leqslant C A^{|\kappa|} B^{|\ell|} M_{|\kappa|} N_{|\ell|}$$

is satisfied. Then, we have

$$\left|x^{\ell} D^{\kappa} D^{\gamma} \varphi(x)\right| = \left|x^{\ell} D^{\kappa+\gamma} \varphi(x)\right| \leqslant C A^{|\kappa+\gamma|} B^{|\ell|} M_{|\kappa+\gamma|} N_{|\ell|},$$

which gives, with the condition (2.3),

$$\left|x^{\ell} D^{\kappa} D^{\gamma} \varphi(x)\right| \leqslant C H^{|\kappa+\gamma|} A^{|\gamma|} M_{|\gamma|} A^{|\kappa|} B^{|\ell|} M_{|\kappa|} N_{|\ell|} = C'(AH)^{|\kappa|} B^{|\ell|} M_{|\kappa|} N_{|\ell|},$$

where C' depend on parameters A, H.

$C' = C(AH)^{|\gamma|} M_{|\gamma|}$. Therefore, $D^{\gamma}\varphi \in S_{\{N\},B}^{\{M\},AH}(\mathbb{R}^n)$.

Furthermore, from above, we have

$$\|D^{\gamma}\varphi\| = \sup_{x\in\mathbb{R}^n} \sup_{\kappa,\ell\in\mathbb{Z}_+^n} \frac{\left|x^{\ell} D^{\kappa} D^{\gamma} \varphi(x)\right|}{(AH)^{|\kappa|} B^{|\ell|} M_{|\kappa|} N_{|\ell|}} \leq C(AH)^{|\gamma|} M_{|\gamma|},$$

where C is such that

$$\sup_{x\in\mathbb{R}^n} \sup_{\kappa,\ell\in\mathbb{Z}_+^n} \frac{\left|x^{\ell} D^{\kappa} \varphi(x)\right|}{A^{|\kappa|} B^{|\ell|} M_{|\kappa|} N_{|\ell|}} \leq C.$$

Therefore, the range of a bounded set in the space $S_{\{N\},B}^{\{M\},A}(\mathbb{R}^n)$ is a bounded set in $S_{\{N\},B}^{\{M\},AH}(\mathbb{R}^n)$. □

Corollary 3.42. *The differentiation operator is defined and bounded in the space* $S_{\{N\}}^{\{M\}}(\mathbb{R}^n)$ *and transforms this space into itself.*

Remark 3.43. *By virtue of the results of Section 3.2 concerning multiplication in Gelfand–Shilov spaces of Roumieu type and the results of this subsection, we conclude that if the sequences* (M_p) *and* (N_p) *satisfy the condition* (2.2)-(2.3)*, then the spaces* $S^{\{M\}}(\mathbb{R}^n)$, $S_{\{N\}}(\mathbb{R}^n)$*, and* $S_{\{N\}}^{\{M\}}(\mathbb{R}^n)$ *are differential subalgebras of* $C^{\infty}(\mathbb{R}^n)$.

4. Conclusions

In this article, we have introduced and thoroughly analyzed generalized anisotropic Gelfand–Shilov spaces of Roumieu type $S_{\{N\}}^{\{M\}}(\mathbb{R}^n)$ defined via sequences of positive real numbers satisfying natural growth and convexity conditions. Our main results establish that these spaces possess rich algebraic and topological structures, being stable under standard operations including multiplication, translation, dilatation, and differentiation. We proved boundedness for the associated operators and showed that the spaces $S^{\{M\}}(\mathbb{R}^n)$, $S_{\{N\}}(\mathbb{R}^n)$, and $S_{\{N\}}^{\{M\}}(\mathbb{R}^n)$ form differential subalgebras of the space of continuous functions. These findings significantly deepen the understanding of function spaces of type S and provide a robust framework for further developments in microlocal analysis and the theory of partial differential equations with ultradifferentiable functions and ultradistributions. The results also open pathways for practical applications involving pseudodifferential operators and evolution equations in these highly regular function spaces. Future work may explore the interplay of these spaces with more general classes of operators and their applications in mathematical physics and signal analysis.

Author contributions

(S.Y.): Conceptualization, Methodology, Software, Data curation, Investigation, Validation, Formal analysis, Visualization, Resources, Writing - original draft, Writing - review & editing. **(M.S.S.)**: Conceptualization, Methodology, Validation, Supervision, Visualization, Project administration, Resources, Writing - original draft, Writing - review & editing. **(M.Aw.)**: Methodology, Software, Formal analysis, Writing - review & editing. **(M.Ar.)**: Conceptualization, Methodology, Software, Formal analysis, Writing - review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to thank the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia (Grant No. KFU260995) for their financial support, as well as the reviewers for their valuable comments.

Funding

This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [Grant No. KFU260995].

Conflict of interest

The authors declare no conflicts of interest.

References

1. V. Horodets'kyi, R. Petryshyn, O. Martynyuk, Evolutionary pseudodifferential equations with smooth symbols in S-type spaces, *Ukr. Math. J.*, **75** (2023), 861–888. http://doi.org/10.1007/s11253-023-02233-3
2. V. V. Horodets'kyi, R. S. Kolisnyk, N. M. Shevchuk, Multipoint (in time) problem for singular parabolic equations in spaces of type S, *J. Math. Sci.*, **277** (2023), 201–239. http://doi.org/10.1007/s10958-023-06828-w

3. K. Mahato, D. Pasawan, Estimates of pseudo-differential operators associated with fractional Hankel transform on the space of type S, *Integr. Transf. Spec. Funct.*, **32** (2020), 224–239. http://doi.org/10.1080/10652469.2020.1815197

4. V. Horodetskyi, O. Martynyuk, Evolution pseudodifferential equations with analytic symbols in spaces of S-type, *Carpathian Math. Publ.*, **13** (2021), 160–179. http://doi.org/10.15330/cmp.13.1.160-179

5. V. Horodetskyi, O. Martynyuk, R. Kolisnyk, Time-nonlocal problem for evolutionary equations with pseudodifferential operators in spaces of type S, *J. Math. Sci.*, **282** (2024), 912–934. http://doi.org/10.1007/s10958-024-07225-7

6. K. Mahato, D. Pasawan, Norm estimates for the pseudo-differential operator involving fractional Hankel-like transform on S-type spaces, *Adv. Oper. Theory*, **8** (2023), 2. http://doi.org/10.1007/s43036-022-00228-8

7. M. A. Soloviev, Characterization of the Moyal multiplier algebras for the Generalized spaces of type S, *Proc. Steklov Inst. Math.*, **309** (2020), 271–283. http://doi.org/10.1134/S0081543820030207

8. V. V. Gorodetskiy, R. S. Kolisnyk, N. M. Shevchuk, On one evolution equation of parabolic type with fractional differentiation operator in S spaces, *Int. J. Differ. Equ.*, **2020** (2020), 1673741. http://doi.org/10.1155/2020/1673741

9. M. Bensaid, R. Chaili, Iterates of differential operators of Shubin type in anisotropic Roumieu Gelfand–Shilov spaces, *Bull. Sci. Math.*, **192** (2024), 103404. http://doi.org/10.1016/j.bulsci.2024.103404

10. M. Bensaid, R. Chaili, On the non-triviality of anisotropic Roumieu Gelfand–Shilov spaces and inclusion between them, *Georgian Math. J.*, **31** (2024), 181–186. http://doi.org/10.1515/gmj-2023-2087

11. R. Chaili, Systems of differential operators in anisotropic Roumieu classes, *Rend. Circ. Mat. Palermo*, **62** (2013), 189–198. https://doi.org/10.1007/s12215-012-0101-7

12. P. Wahlberg, Propagation of anisotropic Gelfand–Shilov wave front sets, *J. Pseudo-Differ. Oper. Appl.*, **14** (2023), 7. https://doi.org/10.1007/s11868-022-00502-6

13. K. Mahato, P. Singh, Continuity of the fractional Hankel wavelet transform on the spaces of type S, *Math. Methods Appl. Sci.*, **42** (2019), 1941–1954. http://doi.org/10.1002/mma.5486

14. F. Colombo, D. C. Struppa, A. Yger, Superoscillating sequences towards approximation in S or S-type spaces and extrapolation, *J. Fourier Anal. Appl.*, **25** (2019), 242–266. http://doi.org/10.1007/s00041-018-9592-8

15. M. A. Soloviev, Spaces of type S and deformation quantization, *Theor. Math. Phys.*, **201** (2019), 1682–1700. http://doi.org/10.1134/S004057791912002X

16. M. A. Soloviev, Spaces of type S as topological algebras under twisted convolution and star product, *Proc. Steklov Inst. Math.*, **306** (2019), 220–241. http://doi.org/10.1134/S0081543819050195

17. J. Chung, S. Y. Chung, D. Kim, Characterizations of the Gelfand–Shilov spaces via Fourier transforms, *Proc. Amer. Math. Soc.*, **124** (1996), 2101–2108. http://doi.org/10.1090/S0002-9939-96-03291-1

18. M. Pascu, On the characterization of Gelfand–Shilov–Roumieu spaces, *J. Operator Theory*, **72** (2014), 475–485. http://doi.org/10.7900/jot.2013jun04.2009

19. I. M. Gelfand, G. E. Shilov, *Generalized functions*, Spaces of Fundamental and Generalized Functions, Vol. 2, Academic Press, 1968.

20. H. Komatsu, Ultradistributions. I. Structure theorems and a characterization, *J. Fac. Sci. Univ. Tokyo Sec. IA*, **20** (1973), 25–105.