Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(3): 5379–5408.
DOI: 10.3934/math.2026222
Received: 28 December 2025
Revised: 12 February 2026
Accepted: 26 February 2026
Published: 03 March 2026

Research article

Bivariate Chen distribution based on iterated FGM copula: Properties and applications

M. A. Alawady[1,2], Mohamed A. Abd Elgawad[3,*], Salem A. Alyami[3], H. M. Barakat[1], I. A. Husseiny[1], G. M. Mansour[1], T. S. Taher[1], and M. O. Mohamed[1,4]

[1] Department of Mathematics, Faculty of Science, Zagazig University, Zagazig 44519, Egypt
[2] Department of Statistics and Operations Research, College of Science, Qassim University, Buraydah 51482, Saudi Arabia
[3] Department of Mathematics and Statistics, Faculty of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11432, Saudi Arabia
[4] Faculty of Computers and Information Systems, Egyptian Chinese University, Cairo 11528, Egypt

* **Correspondence:** Email: moaasalem@imamu.edu.sa.

Abstract: This paper introduced a bivariate model constructed by coupling the iterated Farlie-Gumbel-Morgenstern (IFGM) copula with Chen marginals, yielding the bivariate IFGM-Chen distribution (IFGM-CD). The model was particularly tailored for reliability and survival analysis, as it accommodated bathtub-shaped hazard rates and captured a broader spectrum of dependence structures than traditional FGM-based models. Fundamental statistical properties were investigated, including product moments, conditional distributions, and reliability functions. A dedicated section on the stress-strength model within the IFGM-CD framework was provided, offering new insights into component reliability under dependent stress and strength. For parameter estimation, both maximum likelihood (ML) and Bayesian approaches were employed, supplemented by asymptotic and bootstrap confidence intervals. Extensive Monte Carlo simulations validated the performance of the estimators, and two real-data applications demonstrated the model's practicality and flexibility.

Keywords: iterated FGM copula; Chen distribution; bivariate lifetime model; stress-strength reliability; maximum likelihood estimation; Bayesian estimation; confidence intervals; bootstrap
Mathematics Subject Classification: 60B12, 62G30

1. Introduction

Copulas are functions that link multivariate distribution functions (DFs) to their one-dimensional marginal distributions. They provide a helpful framework for building bivariate or multivariate distributions by separating the dependence structure from the marginal behaviors (Nelsen [1]). This separation makes copulas especially useful for modeling complex relationships between random variables (RVs).

A copula $C(u, v)$ is a bivariate DF defined on the unit square $[0, 1]^2$ with uniform marginals. The dependence structure between two RVs X and Y is determined entirely by the choice of copula function, while their individual behaviors are described by their marginal distributions.

One main benefit of copulas in statistical modeling is that they allow for the separate estimation of marginal distributions and the dependence structure. This feature is particularly valuable in high-dimensional settings, where modeling the joint distribution directly can be tough.

For two continuous RVs X and Y with marginal DFs $F_X(x) = P(X \leq x)$ and $F_Y(y) = P(Y \leq y)$, Sklar's theorem (Sklar [2]) states that there exists a unique copula $C(., .)$ such that the joint DF (JDF) can be expressed as:

$$F_{X,Y}(x, y) = C(F_X(x), F_Y(y)). \tag{1.1}$$

If the marginal distributions are continuous, the copula C is unique. Also, if C is absolutely continuous with copula density $c(u, v) = \partial^2 C(u, v)/\partial u \partial v$, and if X and Y have marginal probability density functions (PDFs) $f_X(x)$ and $f_Y(y)$, then the joint PDF (JPDF) is given by

$$f_{X,Y}(x, y) = f_X(x) f_Y(y)\, c(F_X(x), F_Y(y)). \tag{1.2}$$

Equations (1.1) and (1.2) show how copulas break down the joint distribution into its marginal components and a pure dependence component represented by $c(u, v)$. This breakdown makes it easier to model complex dependence structures that traditional multivariate distributions may not capture.

The Farlie-Gumbel-Morgenstern (FGM) copula and its variants are among the most extensively studied families of copulas. A lot of research has been done on the FGM family of bivariate distributions, both for their theoretical features and their real-world uses. Gupta and Wong [3] extracted bivariate beta distributions with three and five parameters from the FGM family. Hassan and Chesneau [4] recently employed the FGM copula to develop a bivariate generalized half-logistic distribution.

Recent research has broadened the theoretical framework and applications of the FGM copula. Susam [5] proposed multiparameter generalized FGM copula family based on Bernstein polynomials. This approach increases model flexibility while keeping analytical clarity. Aliyu and Usman [6] introduced a bivariate Nadarajah-Haghighi distribution created through the FGM copula. They used Bayesian estimation with covariates and censoring and showed that their model fits better than other models. Blier-Wong et al. [7] looked at high-dimensional exchangeable FGM copulas through a multivariate Bernoulli representation. They provided geometric insights along with effective sampling and estimation methods. Dimitriou [8] studied overlap times in single-server queueing systems where service and inter-arrival times depend on the FGM copula. He derived clear steady-state time distributions and demonstrated the effects of dependence. Mansour et al. [9] explored the cumulative residual Tsallis entropy for concomitants of generalized order statistics based on the FGM copula. They also discussed applications in medical data.

The IFGM copula represents a significant extension of the classical FGM family, offering enhanced flexibility in modeling dependence structures. This copula is constructed through an iterative process that allows for more complex dependence patterns while maintaining mathematical tractability. The iterated Farlie-Gumbel-Morgenstern (IFGM) copula of order k is defined by Huang and Kotz [10], recursively,

$$C_{\boldsymbol{\theta}}^{(k)}(u,v) = uv + \sum_{j=1}^{k} \theta_j \phi_j(u)\phi_j(v),$$

where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$ are dependence parameters satisfying specific constraints, and $\{\phi_j\}_{j=1}^{k}$ is a set of orthonormal functions on $[0, 1]$ with $\int_0^1 \phi_j(t)dt = 0$.

The IFGM copula aims to improve the correlation between marginals of the FGM family. A single iteration could treble the covariance of some marginals, according to research by Huang and Kotz [10] using subsequent iterations. The IFGM copula (of second order) is defined by

$$C(u,v) = uv\left[1 + \gamma(1-u)(1-v) + \omega uv(1-u)(1-v)\right]. \tag{1.3}$$

The corresponding JPDF of the copula (1.3) is given by

$$c(u,v) = 1 + \gamma(1-2u)(1-2v) + \omega uv(2-3u)(2-3v). \tag{1.4}$$

By setting $\omega = 0$, the FGM copula can be obtained as a special instance of the IFGM copula (1.3) and (1.4). Huang and Kotz [10] proved that the natural parameter space Ω (which is the admissible set of the parameters γ and ω) is convex, where

$$\Omega = \left\{(\gamma, \omega) : -1 \le \gamma \le 1; -1 \le \gamma + \omega; \omega \le \frac{3 - \gamma + \sqrt{9 - 6\gamma - 3\gamma^2}}{2}\right\}.$$

The correlation coefficient associated with this copula is expressed as

$$\rho = \frac{\gamma}{3} + \frac{\omega}{12}.$$

Our research concentrates on this particular copula, which has been the subject of considerable investigation in recent literature [11–15].

As a general rule, a variety of censoring schemes is employed to draw inferences about unknown parameters in lifetime analyses, particularly those concerned with reliability data and corresponding hazard-rate characteristics. In many practical situations, the most commonly encountered hazard-rate shapes are constant, increasing, or decreasing. Classical lifetime models, such as the generalized exponential, gamma, Weibull, and lognormal distributions, have been widely used to describe these behaviors.

However, developing accurate inference becomes considerably more challenging when the underlying data exhibit a bathtub-shaped hazard-rate function, owing to the additional structural complexity of such models. For comprehensive discussions of distributions capable of modeling bathtub-shaped failure rates and their applications, see [16–18]. Historically, Davis [19], after examining the breakdown behavior of several motor buses, observed that the exponential distribution was inadequate for modeling the early-life failures, highlighting the need for more flexible lifetime models.

Bathtub-shaped models form an important class of distributions in reliability and life-testing studies. As Block and Savits [20] noted, such models frequently arise in the analysis of failure processes of electronic components, including switches, lighting systems, and circuits. Further applications in real-world reliability studies were discussed in Bebbington et al. [21, 22].

Chen [23] introduced a new two-parameter distribution known as the Chen distribution (CD) with positive parameters θ and β, capable of exhibiting either a bathtub-shaped or an increasing hazard-rate function. The DF and PDF of the CD are given, respectively, by

$$F_X(x;\theta,\beta) = 1 - e^{-\theta\left(e^{x^\beta}-1\right)}, \ \ x > 0, \ \theta,\beta > 0 \tag{1.5}$$

and

$$f_X(x;\theta,\beta) = \theta\beta x^{\beta-1} e^{x^\beta} e^{-\theta\left(e^{x^\beta}-1\right)}, \ \ x > 0, \ \theta,\beta > 0.$$

In the present work, we construct a bivariate distribution, referred to as the bivariate IFGM-CD, by combining the IFGM copula with Chen marginal distributions. A related construction based on the classical FGM copula was previously studied by El-Sherpieny et al. [24], who introduced what they termed the FGM-CD model.

Motivation for the IFGM-CD model

The bivariate IFGM-CD model is driven by significant issues encountered in reliability engineering, survival analysis, and multivariate statistical modeling. A lot of people employ classical lifetime distributions like the Weibull, exponential, and gamma models because they are easy to analyze. However, they don't always work well for characterizing bathtub-shaped hazard rates, which are ubiquitous in mechanical parts, electrical devices, and living systems. The univariate CD effectively addresses certain hazard-rate shapes; nevertheless, numerous practical issues include dependent failure mechanisms, making univariate models inadequate. Furthermore, several current bivariate lifetime models enforce stringent reliance structures or exhibit insufficient flexibility to accurately represent intricate real-world dependence patterns.

The CD is a good choice as a marginal model since it can show either increasing or bathtub-shaped hazard rates, depending on the values of the parameters. This adaptability makes it ideal for simulating various lifetime phenomena, such as electronic components that fail early in their life, then operate steadily before wearing out; mechanical systems that go through burn-in and aging phases; and biological processes that involve initial adaptation followed by deterioration.

The IFGM copula is a useful expansion of the standard FGM copula that may be used to model dependence between lifetimes. The IFGM copula has two dependence parameters, which makes it possible to have a wider range of connection patterns. The dependency is shown by the correlation coefficient $\rho = \frac{\gamma}{3} + \frac{\omega}{12}$, while still being mathematically manageable. This greater flexibility makes it possible to depict dependence more accurately than with the standard FGM copula, without having to use more complicated copula families that could slow down analytical and inferential progress.

From an applied perspective, the proposed IFGM-CD model addresses methodological gaps in several important areas. In material science, it facilitates the modeling of interdependent failure processes in multi-component systems with nonmonotone hazard rates. In medical and biological studies, it allows for the joint analysis of correlated survival endpoints exhibiting complex hazard behavior. In reliability engineering, it provides a flexible framework for assessing systems with competing or interacting failure modes. Consequently, the IFGM-CD model offers a coherent and versatile approach to bivariate lifetime modeling by combining flexible Chen marginals with an enriched dependence structure through the IFGM copula.

Structure of the paper: Section 2 develops the joint distribution and density functions of the IFGM-CD, deriving key structural properties, including product moments, the moment-generating function (MGF), conditional distributions, and the bivariate reliability function. Section 3 examines statistical measures of association. This includes Pearson's correlation and Kendall's tau. Section 4 introduces a dependent stress-strength model within the IFGM-CD framework and looks into its reliability properties. Section 5 presents the ML and Bayesian estimation methods for the model parameters. Section 6 discusses asymptotic and bootstrap confidence intervals, providing a simulation study to compare the performance of the estimators. Section 7 illustrates the model with two real data applications. Finally, Section 8 concludes the paper.

2. The proposed model and statistical characteristics of IFGM-CD

By using (1.3) and (1.5), the JDF of bivariate CD based on IFGM copula, denoted by IFGM-CD$(\theta_1,\beta_1;\theta_2,\beta_2)$, is given by

$$\begin{aligned} F_{X,Y}(x,y) &= \left(1-e^{-\theta_1\left(e^{x^{\beta_1}}-1\right)}\right)\left(1-e^{-\theta_2\left(e^{y^{\beta_2}}-1\right)}\right)\Bigg[1+\gamma e^{-\theta_1\left(e^{x^{\beta_1}}-1\right)}e^{-\theta_2\left(e^{y^{\beta_2}}-1\right)} \\ &+ \omega e^{-\theta_1\left(e^{x^{\beta_1}}-1\right)}e^{-\theta_2\left(e^{y^{\beta_2}}-1\right)}\left(1-e^{-\theta_1\left(e^{x^{\beta_1}}-1\right)}\right)\left(1-e^{-\theta_2\left(e^{y^{\beta_2}}-1\right)}\right)\Bigg], \end{aligned} \tag{2.1}$$

and the corresponding JPDF of (2.1) is given by

$$\begin{aligned} f_{X,Y}(x,y) &= \theta_1\beta_1\theta_2\beta_2 x^{\beta_1-1}y^{\beta_2-1}e^{x^{\beta_1}+y^{\beta_2}}e^{-\theta_1\left(e^{x^{\beta_1}}-1\right)}e^{-\theta_2\left(e^{y^{\beta_2}}-1\right)}\Bigg[1+\gamma\left(2e^{-\theta_1\left(e^{x^{\beta_1}}-1\right)}-1\right) \\ &\times \left(2e^{-\theta_2\left(e^{y^{\beta_2}}-1\right)}-1\right)+\omega\left(1-e^{-\theta_1\left(e^{x^{\beta_1}}-1\right)}\right)\left(1-e^{-\theta_2\left(e^{y^{\beta_2}}-1\right)}\right)\left(3e^{-\theta_1\left(e^{x^{\beta_1}}-1\right)}-1\right) \\ &\times \left(3e^{-\theta_2\left(e^{y^{\beta_2}}-1\right)}-1\right)\Bigg]. \end{aligned} \tag{2.2}$$

2.1. Product moments

It is easy to show that the (r_1,r_2)th, $r_1,r_2=1,2,...$, product moments of the IFGM-CD$(\theta_1,\beta_1;\theta_2,\beta_2)$, are given by

$$\begin{aligned} \mathrm{E}\left(X^{r_1}Y^{r_2}\right) &= \theta_1\theta_2\beta_1\beta_2\int_0^\infty\int_0^\infty x^{r_1}x^{\beta_1-1}y^{r_2}y^{\beta_2-1}e^{x^{\beta_1}}e^{y^{\beta_2}}e^{-\theta_1(e^{x^{\beta_1}}-1)}e^{-\theta_2(e^{y^{\beta_2}}-1)}\Big[1+\gamma\left(2e^{-\theta_1(e^{x^{\beta_1}}-1)}-1\right) \\ &\times\left(2e^{-\theta_2(e^{y^{\beta_2}}-1)}-1\right)+\omega\left(1-e^{-\theta_1(e^{x^{\beta_1}}-1)}\right)\left(1-e^{-\theta_2(e^{y^{\beta_2}}-1)}\right)\left(3e^{-\theta_1(e^{x^{\beta_1}}-1)}-1\right)\times\left(3e^{-\theta_2(e^{y^{\beta_2}}-1)}-1\right)\Big]dxdy \\ &= \prod_{n=1}^{2}\theta_n\alpha_n\Gamma(1+\frac{r_n}{\beta_n})\Big[1+\gamma\prod_{n=1}^{2}\Big(\frac{2\sum\limits_{j_n=0}^{\infty}\frac{(2\theta_n)^{j_n}}{j_n!}\sum\limits_{m_n=0}^{j_n}\binom{j_n}{m_n}(-1)^{m_n}(-1-m_n)^{-\frac{r_n+\beta_n}{\beta_n}}}{\alpha_n}-1\Big)+\omega\prod_{n=1}^{2} \end{aligned}$$

$$\times\Big(\frac{2\sum\limits_{j_n=0}^{\infty}\sum\limits_{m_n=0}^{j_n}\frac{(2\theta_n)^{j_n}}{j_n}\binom{j_n}{m_n}(-1)^{m_n}(-1-m_n)^{-\frac{r_n+\beta_n}{\beta_n}}-3\sum\limits_{l_n=0}^{\infty}\sum\limits_{s_n=0}^{l_n}\frac{(3\theta_n)^{l_n}}{l_n}\binom{l_n}{s_n}(-1)^{s_n}(-1-s_n)^{-\frac{r_n+\beta_n}{\beta_n}}}{\alpha_n}-1\Big)\Big],$$

where $\alpha_n=\sum\limits_{i_n=0}^{\infty}\sum\limits_{k_n=0}^{i_n}\frac{\theta_n^{i_n}}{i_n!}\binom{i_n}{k_n}(-1)^{k_n}(-1-k_n)^{-\frac{r_n+\beta_n}{\beta_n}}$. In particular, at $r_1=r_2=1$, the product moment is

$$\begin{aligned}\mathrm{E}\,(XY)&=\prod_{n=1}^{2}\theta_n\varphi_n\Gamma(1+\frac{1}{\beta_n})\Big[1+\gamma\prod_{n=1}^{2}\Big(\frac{2\sum\limits_{j_n=0}^{\infty}\frac{(2\theta_n)^{j_n}}{j_n}\sum\limits_{m_n=0}^{j_n}\binom{j_n}{m_n}(-1)^{m_n}(-1-m_n)^{-\frac{1+\beta_n}{\beta_n}}}{\varphi_n}-1\Big)+\omega\prod_{n=1}^{2}\\&\times\Big(\frac{2\sum\limits_{j_n=0}^{\infty}\sum\limits_{m_n=0}^{j_n}\frac{(2\theta_n)^{j_n}}{j_n}\binom{j_n}{m_n}(-1)^{m_n}(-1-m_n)^{-\frac{1+\beta_n}{\beta_n}}-3\sum\limits_{l_n=0}^{\infty}\sum\limits_{s_n=0}^{l_n}\frac{(3\theta_n)^{l_n}}{l_n}\binom{l_n}{s_n}(-1)^{s_n}(-1-s_n)^{-\frac{1+\beta_n}{\beta_n}}}{\varphi_n}-1\Big)\Big],\end{aligned}\tag{2.3}$$

where $\varphi_n=\sum_{i_n=0}^{\infty}\sum_{k_n=0}^{i_n}\frac{\theta_n^{i_n}}{i_n}\binom{i_n}{k_n}(-1)^{k_n}(-1-k_n)^{-\frac{1+\beta_n}{\beta_n}}$.

2.2. *Moment generating function*

Using (2.2), the MGF is given by

$$\begin{aligned}M_{X,Y}(t_1,t_2)&=\theta_1\theta_2\beta_1\beta_2\sum_{p_1=0}^{\infty}\frac{(t_1)^{p_1}}{p_1}\sum_{p_2=0}^{\infty}\frac{(t_2)^{p_2}}{p_2!}\int_0^{\infty}\int_0^{\infty}x^{p_1}x^{\beta_1-1}y^{p_2}y^{\beta_2-1}e^{x^{\beta_1}}e^{y^{\beta_2}}e^{-\theta_1(e^{x^{\beta_1}}-1)}e^{-\theta_2(e^{y^{\beta_2}}-1)}\\&\times\Big[1+\gamma\Big(2e^{-\theta_1(e^{x^{\beta_1}}-1)}-1\Big)\Big(2e^{-\theta_2(e^{y^{\beta_2}}-1)}-1\Big)+\omega\Big(1-e^{-\theta_1(e^{x^{\beta_1}}-1)}\Big)\Big(1-e^{-\theta_2(e^{y^{\beta_2}}-1)}\Big)\\&\times\Big(3e^{-\theta_1(e^{x^{\beta_1}}-1)}-1\Big)\Big(3e^{-\theta_2(e^{y^{\beta_2}}-1)}-1\Big)\Big]\,dxdy\\&=\sum_{p_1=0}^{\infty}\frac{(t_1)^{p_1}}{p_1}\sum_{p_2=0}^{\infty}\frac{(t_2)^{p_2}}{p_2}\prod_{n=1}^{2}\theta_n\Lambda_n\Gamma(1+\frac{p_n}{\beta_n})\Big[1+\gamma\prod_{n=1}^{2}\Big(\frac{2\sum\limits_{j_n=0}^{\infty}\frac{(2\theta_n)^{j_n}}{j_n}\sum\limits_{m_n=0}^{j_n}\binom{j_n}{m_n}(-1)^{m_n}(-1-m_n)^{-\frac{p_n+\beta_n}{\beta_n}}}{\Lambda_n}-1\Big)\\&+\omega\prod_{n=1}^{2}\Big(\frac{2\sum\limits_{j_n=0}^{\infty}\sum\limits_{m_n=0}^{j_n}\frac{(2\theta_n)^{j_n}}{j_n!}\binom{j_n}{m_n}(-1)^{m_n}(-1-m_n)^{-\frac{p_n+\beta_n}{\beta_n}}-3\sum\limits_{l_n=0}^{\infty}\sum\limits_{s_n=0}^{l_n}\frac{(3\theta_n)^{l_n}}{l_n!}\binom{l_n}{s_n}(-1)^{s_n}(-1-s_n)^{-\frac{p_n+\beta_n}{\beta_n}}}{\Lambda_n}-1\Big)\Big],\end{aligned}$$

where $\Lambda_n=\sum_{i_n=0}^{\infty}\sum_{k_n=0}^{i_n}\frac{\theta_n^{i_n}}{i_n}\binom{i_n}{k_n}(-1)^{k_n}(-1-k_n)^{-\frac{p_n+\beta_n}{\beta_n}}$.

2.3. *Conditional distribution*

The conditional DF of, Y given $X=x$, is given by

$$\begin{aligned}F_{Y|X}(y|x)&=\Big(1-e^{-\theta_2\left(e^{y^{\beta_2}}-1\right)}\Big)\Big[1+\gamma\Big(e^{-\theta_2(e^{y^{\beta_2}}-1)}\Big)\Big(2e^{-\theta_1(e^{x^{\beta_1}}-1)}-1\Big)+\omega\Big(1-e^{-\theta_1(e^{x^{\beta_1}}-1)}\Big)\\&\times\Big(1-e^{-\theta_2(e^{y^{\beta_2}}-1)}\Big)\Big(e^{-\theta_2(e^{y^{\beta_2}}-1)}\Big)\Big(3e^{-\theta_1(e^{x^{\beta_1}}-1)}-1\Big)\Big].\end{aligned}$$

2.4. Bivariate reliability function

Sreelakshmi [25] introduced the relationship between copula and reliability copula, which is defined as follows:

$$R(x, y) = 1 - F_X(x) - F_Y(y) + C\left(F_X(x), F_Y(y)\right).$$

The bivariate reliability function $R(x, y)$ for the IFGM-CD$(\theta_1, \beta_1; \theta_2, \beta_2)$, is

$$\begin{aligned} R(x, y) &= e^{-\theta_1\left(e^{x^{\beta_1}}-1\right)} e^{-\theta_2\left(e^{y^{\beta_2}}-1\right)} \left[1 + \left(1 - e^{-\theta_1\left(e^{x^{\beta_1}}-1\right)}\right)\left(1 - e^{-\theta_2\left(e^{y^{\beta_2}}-1\right)}\right)\left[\gamma + \omega\left(1 - e^{-\theta_1\left(e^{x^{\beta_1}}-1\right)}\right)\right.\right. \\ &\times \left.\left.\left(1 - e^{-\theta_2\left(e^{y^{\beta_2}}-1\right)}\right)\right]\right]. \end{aligned}$$

We provide surface plots in Figure 1 in which the marginal parameters are held fixed while only the copula parameters vary. Specifically, we fix $(\theta_1, \beta_1, \theta_2, \beta_2)$ at representative values and construct reliability surfaces for different combinations of (γ, ω) selected to correspond approximately to prescribed levels of dependence as measured by Kendall's τ. This controlled setup allows changes in the joint reliability surface to be attributed solely to variations in the dependence parameters, thereby providing clearer insight into the role of γ and ω.

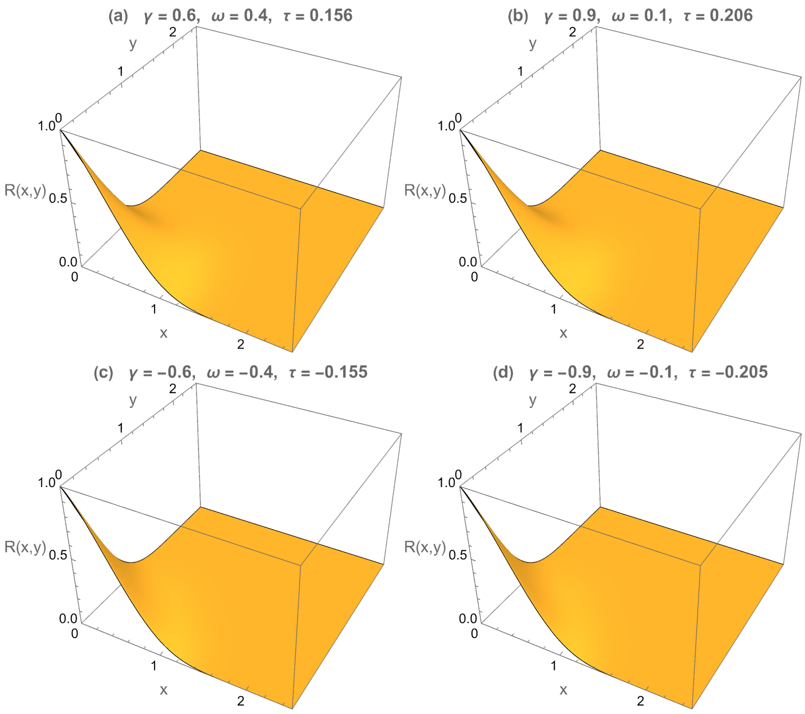


Figure 1. Reliability surfaces of the IFGM–CD model with fixed marginal parameters $(\theta_1, \beta_1, \theta_2, \beta_2) = (1.2, 1.3, 1.5, 1.1)$ and varying dependence parameters (γ, ω). Panels (a) and (b) correspond to positive dependence, whereas panels (c) and (d) illustrate negative dependence. The observed differences across panels are induced solely by changes in the dependence parameters of the IFGM-CD.

Figure 1 displays the $R(x, y)$ for fixed marginal parameters and varying dependence structures. Panels (a) and (b) correspond to positive dependence with $\tau(0.6, 0.4) \approx 0.1561$ and

$\tau(0.9, 0.1) \approx 0.2058$, respectively, while panels (c) and (d) illustrate negative dependence with $\tau(-0.6, -0.4) \approx -0.1550$ and $\tau(-0.9, -0.1) \approx -0.20544$, respectively. Consistent with the model structure, stronger positive dependence increases the joint reliability for larger values of (x, y), whereas negative dependence leads to a more rapid decay of the reliability surface. These plots confirm that the parameters γ and ω govern the dependence structure, while the marginal parameters control the baseline reliability behavior.

3. Measures of association

As established, the IFGM copula is central to capturing the dependence structure between RVs. Beyond its role in modeling, the copula framework also provides a foundation for quantifying the strength of this dependence. Within this framework, we assess dependence using three key measures: Pearson's correlation coefficient, Kendall's tau, and a regression-based approach.

3.1. Pearson's correlation coefficient

The Pearson correlation coefficient of the bivariate IFGM–CD is defined as

$$\rho_c = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)}\,\sqrt{\mathrm{Var}(Y)}} = \frac{\mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y)}{\sqrt{\mathrm{Var}(X)}\,\sqrt{\mathrm{Var}(Y)}}. \tag{3.1}$$

To evaluate ρ_c, the marginal moments of X and Y are required. For the Chen marginal distribution with parameters (θ_1, β_1), the r_1th raw moment of X is given by

$$\begin{aligned} \mathrm{E}(X^{r_1}) &= \theta_1\beta_1 \int_0^\infty x^{r_1+\beta_1-1} \exp\left(x^{\beta_1}\right) \exp\left[-\theta_1(e^{x^{\beta_1}} - 1)\right] dx \\ &= \theta_1 \sum_{i_1=0}^{\infty} \sum_{k_1=0}^{i_1} \frac{\theta_1^{i_1}}{i_1} \binom{i_1}{k_1} (-1)^{k_1} (-1-k_1)^{-\frac{r_1+\beta_1}{\beta_1}} \Gamma\left(1 + \frac{r_1}{\beta_1}\right), \quad r_1 > 0. \end{aligned} \tag{3.2}$$

The corresponding moments $\mathrm{E}(Y^{r_2})$ are obtained analogously by replacing (θ_1, β_1) with (θ_2, β_2).

Using (3.2), the variance of X is computed as $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$, where $\mathrm{E}(X)$ and $\mathrm{E}(X^2)$ follow directly from (3.2) with $r_1 = 1$ and $r_1 = 2$, respectively. The variance of Y is obtained in the same manner.

Substituting the marginal moments and variances into (3.1), together with the joint moment $\mathrm{E}(XY)$ derived from the IFGM-CD density, yields the Pearson correlation coefficient ρ_c.

We evaluate the maximum attainable Pearson correlation coefficient for the bivariate IFGM–CD with identical Chen marginals (θ, β) at the boundary point $(\gamma, \omega) = (1, 1)$. For $\theta = 1$ and $\beta = 0.5, 1, 1.5, 2$, the computed maximum values of ρ_c are approximately 0.3510, 0.4179, 0.4161, and 0.4058, respectively. For $\theta = 1.5$, the corresponding maxima are 0.3292, 0.4134, 0.4179, and 0.4101. For $\theta = 2$ they are 0.3130, 0.4092, 0.4181, and 0.4123, while for $\theta = 2.5$ they are 0.2986, 0.4048, 0.4169, and 0.4107.

For each fixed value of θ, the maximum correlation, $\rho_c^{\max}$, increases as β moves from 0.5 to approximately 1–1.5, and then decreases slightly at $\beta = 2$. This indicates that moderate hazard–shape parameters allow stronger linear dependence than very flat or strongly increasing hazard rates.

For a fixed β, the effect of θ on $\rho_c^{\max}$ is non–monotone: the maximum correlation increases from $\theta = 1$ to $\theta = 1.5$ and then gradually decreases as θ becomes larger. Among all considered parameter combinations, the largest attainable correlation is $\rho_c^{\max} \approx 0.4181$, which occurs at $(\theta, \beta, \gamma, \omega) = (2,\ 1.5,\ 1,\ 1)$. This shows that, although the IFGM copula substantially enlarges the dependence range compared to the classical FGM copula, the achievable linear dependence remains moderate.

Restricting the IFGM–CD model to $\omega = 0$ reduces it to the classical FGM–CD. In this case, the Pearson correlation coefficient depends only on the parameter γ, and its maximum over the admissible space is attained at $\gamma = 1$.

Repeating the numerical evaluation of the marginal and joint moments for identical Chen marginals, the maximum attainable Pearson correlation coefficients are substantially smaller than those obtained under the full IFGM structure. In particular, for $(\theta, \beta, \gamma, \omega) = (2,\ 1.5,\ 1,\ 0)$, the maximum Pearson correlation coefficient is found to be $\rho_c^{\max} \approx 0.183$.

This value is markedly lower than the corresponding IFGM–CD maximum $\rho_c^{\max} \approx 0.4181$ at $(\theta, \beta, \gamma, \omega) = (2, 1.5, 1, 1)$, demonstrating that the iterated term ω plays a crucial role in enlarging the attainable dependence range while preserving the same Chen marginal distributions.

3.2. Kendall's tau correlation

Kendall's τ measures the difference between the probabilities of concordance and discordance between two RVs and provides a copula–based measure of the strength and direction of dependence. For an absolutely continuous copula C, Kendall's τ is given by

$$\tau = 1 - 4 \int_0^1 \int_0^1 \frac{\partial C(u,v)}{\partial u} \frac{\partial C(u,v)}{\partial v} \, du \, dv.$$

For the IFGM copula considered in this work, the partial derivatives are

$$\frac{\partial C(u,v)}{\partial u} = v + \gamma v(1-v)(1-2u) + \omega u v^2 (1-v)(2-3u)$$

and

$$\frac{\partial C(u,v)}{\partial v} = u + \gamma u(1-u)(1-2v) + \omega v u^2 (1-u)(2-3v).$$

Substituting these expressions into the definition of Kendall's τ and evaluating the integrals yields the closed–form expression

$$\tau(\gamma, \omega) = \frac{25\omega + \gamma(100 + \omega)}{450}.$$

To determine the maximum attainable value of τ over the admissible parameter space Ω, we note that $\tau(\gamma, \omega)$ is increasing in ω. Hence, the maximum occurs on the upper boundary $\omega = U(\gamma)$, where

$$U(\gamma) = \frac{3 - \gamma + \sqrt{9 - 6\gamma - 3\gamma^2}}{2}, \qquad 9 - 6\gamma - 3\gamma^2 \geq 0,$$

which implies $\gamma \in [-3, 1]$.

Defining $\widetilde{\tau}(\gamma) = \tau(\gamma, U(\gamma))$, the optimization problem reduces to maximizing $\widetilde{\tau}(\gamma)$ over $\gamma \in [-3, 1]$. Differentiating with respect to γ and simplifying leads to the first–order condition

$$(89 - \gamma) \sqrt{9 - 6\gamma - 3\gamma^2} = \sqrt{3} \, (\gamma^2 + 14\gamma + 11).$$

Squaring both sides and simplifying yields the quartic equation

$$\gamma^4 - 74\gamma^3 + 3890\gamma^2 - 31082\gamma - 11821 = 0.$$

Solving this equation numerically produces two complex roots and two real roots, of which only one lies within the admissible interval $[-3, 1]$, namely, $\gamma^* \approx 0.93851$. Evaluating $U(\gamma)$ at this value gives $\omega^* = U(\gamma^*) \approx 1.45692$. Therefore, the maximizer of Kendall's τ over the admissible region Ω is

$$(\gamma^*, \omega^*) \approx (0.93851,\ 1.45692),$$

with corresponding maximum value

$$\tau_{\max} = \tau(\gamma^*, \omega^*) \approx 0.29254.$$

For comparison, setting $\omega = 0$ reduces the model to the classical FGM–CD copula studied by El–Sherpieny et al. [24]. In this case,

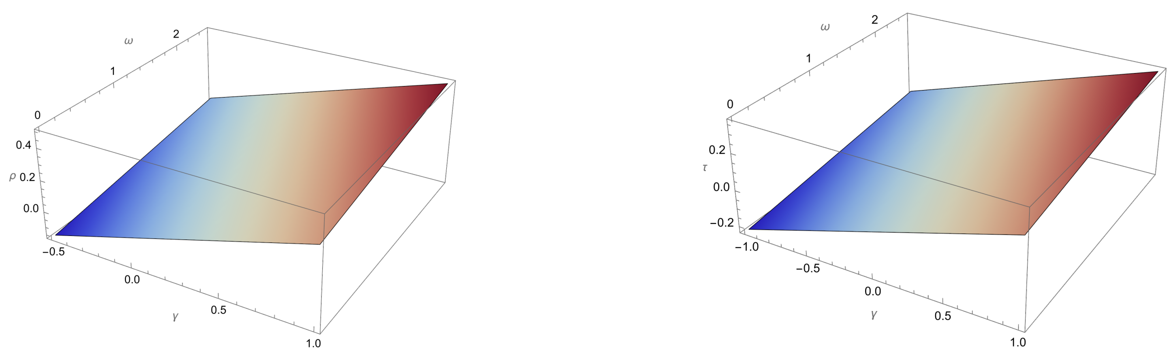
$$\tau(\gamma, 0) = \frac{100\gamma}{450} = \frac{2\gamma}{9},$$

which is maximized over $-1 \le \gamma \le 1$ at $\gamma = 1$, yielding

$$\gamma^* = 1, \qquad \omega^* = 0, \qquad \tau_{\max} = \frac{2}{9} \approx 0.22222.$$

This comparison clearly demonstrates that the IFGM structure substantially enlarges the attainable range of dependence relative to the classical FGM copula, while preserving the same Chen marginal distributions.

Figure 2 investigates the effect of the dependence parameters on the Pearson's correlation coefficient ρ and Kendall's tau τ. The plots confirm that both γ and ω jointly influence the strength and direction of dependence, with higher values generally indicating a stronger positive association. However, even at maximum admissible values, the attainable correlation remains moderate consistent with the IFGM family's lack of tail dependence. This visualization helps underscore the flexibility gained by extending the classical FGM copula through the iterated parameter ω, while also highlighting the model's limitations in capturing extreme co-movements.



(a) 3D surface plot of Pearson's correlation coefficient ρ for the IFGM-CD with fixed parameters $(\theta_1, \beta_1) = (0.0013, 0.7072)$ and $(\theta_2, \beta_2) = (0.0094, 0.4324)$.

(b) 3D surface plot of Kendall's tau τ for the IFGM copula as a function of the dependence parameters γ and ω.

Figure 2. 3D surface plots of ρ and τ for the IFGM copula.

3.3. Conditional moments

A conditional distribution represents the probability distribution of an RV, given that another RV is known to have a specific value. It shows how the probabilities are adjusted when additional information is available.

$$\begin{aligned}
\mathrm{E}(X|y) &= \int_0^\infty x f_{X|Y}(x|y)dx = \theta_1\beta_1 \int_0^\infty x^{\beta_1} e^{x^{\beta_1}} e^{-\theta_1(e^{x^{\beta_1}}-1)} \Big[1+\gamma\Big(2e^{-\theta_1(e^{x^{\beta_1}}-1)}-1\Big)\Big(2e^{-\theta_2(e^{y^{\beta_2}}-1)}-1\Big) \\
&+ \omega\Big(1-e^{-\theta_1(e^{x^{\beta_1}}-1)}\Big)\Big(1-e^{-\theta_2(e^{y^{\beta_2}}-1)}\Big)\Big(3e^{-\theta_1(e^{x^{\beta_1}}-1)}-1\Big)\Big(3e^{-\theta_2(e^{y^{\beta_2}}-1)}-1\Big)\Big] dx \\
&= \frac{\theta_1}{\beta_1}\Psi\Big[1+\gamma\Big(2e^{-\theta_2(e^{y^{\beta_2}}-1)}-1\Big)\Big(\frac{2\sum_{j_1=0}^{\infty}\frac{(2\theta_1)^{j_1}}{j_1}\sum_{m_1=0}^{j_1}\binom{j_1}{m_1}(-1)^{m_1}(-1-m_1)^{-\frac{1+\beta_1}{\beta_1}}\Gamma(\frac{1}{\beta_1})}{\Psi}-1\Big)+\omega \\
&\times \Big(\frac{2\sum_{j_1=0}^{\infty}\sum_{m_1=0}^{j_1}\frac{(2\theta_1)^{j_1}}{j_1}\binom{j_1}{m_1}(-1)^{m_1}(-1-m_1)^{-\frac{1+\beta_1}{\beta_1}}-3\sum_{l_1=0}^{\infty}\sum_{s_1=0}^{l_1}\frac{(3\theta_1)^{l_1}}{l_1}\binom{l_1}{s_1}(-1)^{s_1}(-1-s_1)^{-\frac{1+\beta_1}{\beta_1}}\Gamma(\frac{1}{\beta_1})}{\Psi} \\
&-1\Big)\Big(1-e^{-\theta_2(e^{y^{\beta_2}}-1)}\Big)\Big(3e^{-\theta_2(e^{y^{\beta_2}}-1)}-1\Big)\Big],
\end{aligned}$$

where

$$\Psi = \sum_{i_1=0}^{\infty}\sum_{k_1=0}^{i_1}\frac{\theta_1^{i_1}}{i_1}\binom{i_1}{k_1}(-1)^{k_1}(-1-k_1)^{-\frac{1+\beta_1}{\beta_1}}\Gamma(\frac{1}{\beta_1}).$$

3.4. Tail dependence properties

Tail dependence measures the probability of extreme co-movements between RVs and is particularly relevant in reliability and survival analysis, where simultaneous extreme events (e.g., catastrophic failures) are of interest. For a bivariate copula $C(u, v)$, the coefficients of lower and upper tail dependence are defined, cf. Nelsen ([1], Section 5.4), respectively, as:

$$\lambda_L = \lim_{u\to 0^+}\frac{C(u,u)}{u}, \tag{3.3}$$

$$\lambda_U = \lim_{u\to 1^-}\frac{1-2u+C(u,u)}{1-u}. \tag{3.4}$$

For the IFGM copula specified in (1.3), we calculate $C(u,u) = u^2[1+\gamma(1-u)^2+\omega u^2(1-u)^2]$. By substituting into Eqs (3.3) and (3.4), we get

$$\lambda_L = \lim_{u\to 0^+}\frac{u^2[1+\gamma(1-u)^2+\omega u^2(1-u)^2]}{u} = 0,$$

$$\lambda_U = \lim_{u\to 1^-}\frac{1-2u+u^2[1+\gamma(1-u)^2+\omega u^2(1-u)^2]}{1-u} = 0.$$

Thus, the IFGM copula–like the classical FGM copula–exhibits neither lower nor upper tail dependence. This characteristic distinguishes it from popular Archimedean copulas, such as:

(i) Gumbel copula: defined as

$$C_\theta^{\text{Gumbel}}(u,v) = \exp\left[-\Big((-\ln u)^\theta + (-\ln v)^\theta\Big)^{1/\theta}\right], \quad \theta \geq 1,$$

which exhibits upper tail dependence with coefficient $\lambda_U = 2 - 2^{1/\theta}$.

(ii) Clayton copula: defined as

$$C_\theta^{\text{Clayton}}(u, v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}, \quad \theta > 0,$$

which exhibits lower tail dependence with coefficient $\lambda_L = 2^{-1/\theta}$.

Because the IFGM-CD model lacks tail dependence, it may not be the best choice for situations where extreme simultaneous failures (or survivals) are the most critical. The model is still excellent at describing mild dependence structures with bathtub-shaped hazard rates. This scenario is a common situation in reliability engineering, where parts often have associated but not necessarily extreme failure patterns. Researchers dealing with systems susceptible to extreme co-failures may explore alternate copula families exhibiting tail dependence, potentially in conjunction with Chen marginals, as a prospective avenue for further investigation.

4. Reliability in dependent stress-strength model with IFGM-CD

In reliability engineering, the stress-strength model provides a fundamental framework for assessing component reliability. Consider a component with random strength Z subjected to random stress T. The component performs satisfactorily when $Z > T$, making the reliability measure $R = P(T \leq Z)$ a crucial indicator of component performance. Traditional models often assume independence between stress and strength, which may not hold in practical scenarios where both variables are influenced by common environmental factors or manufacturing processes.

Assuming that the strength Z and stress T follow a bivariate IFGM-CD, the reliability measure R can be expressed as:

$$\begin{aligned} R = P(T < Z) &= \int_0^\infty \int_0^z f_{Z,T}(z, t) dt dz \\ &= \int_0^\infty \int_0^z f_Z(z) f_T(t) c(F_Z(z), F_T(t)) dt dz, \end{aligned}$$

where $c(u, v)$ is the IFGM copula density defined by (1.4). Substituting and separating terms yields the following linear decomposition:

$$R = R_0 + \gamma R_1 + \omega R_2, \tag{4.1}$$

where

$$\begin{aligned} R_0 &= \int_0^\infty f_Z(z) \left[\int_0^z f_T(t) dt\right] dz = \int_0^\infty f_Z(z) F_T(z) dz, \\ R_1 &= \int_0^\infty f_Z(z)(1 - 2F_Z(z)) \left[\int_0^z f_T(t)(1 - 2F_T(t)) dt\right] dz, \\ R_2 &= \int_0^\infty f_Z(z) F_Z(z)(2 - 3F_Z(z)) \left[\int_0^z f_T(t) F_T(t)(2 - 3F_T(t)) dt\right] dz. \end{aligned}$$

R_0 represents the reliability under the independence assumption between stress and strength, and R_1 and R_2 capture the effects of dependence structure through parameters γ and ω. The parameters γ

and ω modulate the strength and nature of dependence, allowing the model to accommodate various dependence patterns encountered in practical applications.

Using the CD marginals with parameters (θ_Z, β_Z) for strength Z and (θ_T, β_T) for stress T, the integrals R_0, R_1, and R_2 can be computed numerically using appropriate quadrature methods or Monte Carlo simulation. For specific parameter values, these computations provide valuable insights into how dependence between stress and strength affects overall component reliability. In the case of independent components, i.e., $\gamma = \omega = 0$, we get

$$R = R_0 = \int_0^\infty f_Z(z) F_T(z) dz.$$

In this case, $R \geq \frac{1}{2}$, if and only if, the strength variable Z is stochastically larger than the stress variable T (denoted by $T \geq_{st} Z$), i.e., $F_Z(x) \leq F_T(x)$ for all $x \geq 0$.

Lemma 4.1 (Bounds on reliability components)**.** *The reliability components satisfy:*

$$0 < R_0 < 1, \qquad |R_1| \leq \frac{1}{4}, \text{ and } |R_2| \leq \frac{1}{9}.$$

Proof. For R_1, let $I(z) = \int_0^z f_T(t)(1 - 2F_T(t))dt$. Using the substitution $u = F_T(t)$, we get

$$I(z) = \int_0^{F_T(z)} (1 - 2u) du = F_T(z) - F_T^2(z).$$

Since $F_T(z) \in [0, 1]$, the maximum of $|F_T(z) - F_T^2(z)|$ is $\frac{1}{4}$ at $F_T(z) = \frac{1}{2}$. Therefore, $|I(z)| \leq \frac{1}{4}$ and, consequently, $|R_1| \leq \frac{1}{4}$. For R_2, similarly define $J(z) = \int_0^z f_T(t)F_T(t)(2 - 3F_T(t))dt$. With $u = F_T(t)$, we get

$$J(z) = \int_0^{F_T(z)} u(2 - 3u) du = F_T^2(z) - F_T^3(z).$$

The function $g(u) = u^2 - u^3$ achieves its maximum absolute value of $\frac{4}{27} \approx 0.148$ on $[0, 1]$, giving $|R_2| \leq \frac{1}{9}$ as a conservative bound. □

Theorem 4.1 (Dependence-adjusted reliability bounds)**.** *For the IFGM-structure stress-strength model:*

$$R_0 - \frac{|\gamma|}{4} - \frac{|\omega|}{9} \leq R \leq R_0 + \frac{|\gamma|}{4} + \frac{|\omega|}{9}.$$

Moreover, if $R_0 > \frac{1}{2} + \frac{|\gamma|}{4} + \frac{|\omega|}{9}$*, then* $R > \frac{1}{2}$*, and if* $R_0 < \frac{1}{2} - \frac{|\gamma|}{4} - \frac{|\omega|}{9}$*, then* $R < \frac{1}{2}$.

Proof. From (4.1) and Lemma 4.1:

$$|R - R_0| = |\gamma R_1 + \omega R_2| \leq |\gamma||R_1| + |\omega||R_2| \leq \frac{|\gamma|}{4} + \frac{|\omega|}{9}.$$

The threshold conditions follow directly from these bounds. □

We need the following elementary lemma:

Lemma 4.2 (Positive-negative quadrant dependent)**.** *The IFGM copula defined by* (1.3) *is positive quadrant dependent (PQD), i.e.,* $C(u, v) \geq uv,\ \forall\gamma, \omega \in \Omega$, *if and only if,*

$$\gamma \geq 0 \quad and \quad \gamma + \omega \geq 0, \tag{4.2}$$

provided that the parameters (γ, ω) *also belong to the admissible set* Ω *(which ensures that* $C(u, v)$ *is a valid copula). Moreover, the IFGM copula defined by* (1.3) *is negative quadrant dependent (NQD), i.e.,* $C(u, v) \leq uv,\ \forall\gamma, \omega \in \Omega$, *if and only if,*

$$\gamma \leq 0 \quad and \quad \gamma + \omega \leq 0, \tag{4.3}$$

provided that the parameters (γ, ω) *also belong to the admissible set* Ω *(which ensures that* $C(u, v)$ *is a valid copula).*

Proof. The condition (4.2) follows from the requirement that $C(u, v) \geq uv$ for all $u, v \in [0, 1]$, which simplifies to $\gamma + \omega uv \geq 0$ for all $uv \in [0, 1]$. The linear function $f(t) = \gamma + \omega t$ attains its minimum over $t \in [0, 1]$ either at $t = 0$ (if $\omega \geq 0$) or at $t = 1$ (if $\omega < 0$), leading to the two inequalities in (4.2). Thus, within the valid parameter space, the IFGM copula captures positive dependence between the marginals precisely when both γ and $\gamma + \omega$ are nonnegative. On the other hand, the condition (4.3) follows from the requirement that $C(u, v) \leq uv$ for all $u, v \in [0, 1]$, which simplifies to $\gamma + \omega uv \leq 0$ for all $uv \in [0, 1]$. The linear function $f(t) = \gamma + \omega t$ attains its maximum over $t \in [0, 1]$ either at $t = 0$ (if $\omega \leq 0$) or at $t = 1$ (if $\omega \geq 0$), leading to the two inequalities (4.3). Thus, within the valid parameter space, the IFGM copula captures negative dependence between the marginals precisely when both γ and $\gamma + \omega$ are nonpositive. □

Theorem 4.2 (Stochastic ordering and reliability)**.** *Let* Z *and* T *have CD with parameters* (θ_Z,β_Z) *and* (θ_T,β_T), *respectively.*

(1) If $\theta_Z \leq \theta_T$ *and* $\beta_Z = \beta_T$, *then* $Z \geq_{st} T$.
(2) If $Z \geq_{st} T$ *and the IFGM copula is PQD, then* $R \geq \frac{1}{2}$.
(3) If $Z \leq_{st} T$ *and the IFGM copula is NQD, then* $R \leq \frac{1}{2}$.

Proof. **Part 1:** The partial derivative

$$\frac{\partial F_Z(z)}{\partial\theta_Z} = (e^{z^{\beta_Z}} - 1)e^{-\theta_Z(e^{z^{\beta_Z}}-1)} > 0 \quad \text{for } z > 0,$$

yields that $F_Z(z)$ is increasing in θ_Z, meaning Z becomes stochastically smaller. Consequently, the stress-strength probability $R = P(Z < T)$ decreases as θ_Z increases.

Part 2: Similarly,

$$\frac{\partial F_T(t)}{\partial\theta_T} = (e^{t^{\beta_T}} - 1)e^{-\theta_T(e^{t^{\beta_T}}-1)} > 0 \quad \text{for } t > 0,$$

hence, $F_T(t)$ is increasing in θ_T, making T stochastically smaller and therefore increasing R.

Part 3: The shape parameters β_Z and β_T affect both the scale and the shape of the hazard rate, which can lead to non-monotonic behavior of R. To demonstrate this, we construct an explicit counterexample. Consider the IFGM-CD model with the independence copula ($\gamma = \omega = 0$), and set $\theta_Z = \theta_T = 1, \beta_T = 1.5$. The stress-strength probability then simplifies to

$$R = \int_0^\infty [1 - F_T(z)]f_Z(z)\, dz.$$

When $\beta_Z = \beta_T = 1.5$, the two marginals are identical, so $R(1.5) := R = 1/2$. Numerical integration (using, e.g., the `copula` package in R program) yields the approximations:

$$R(0.5) \approx 0.0002 \ \text{ and } \ R(2.5) \approx 0.179.$$

Since $R(1.5) = 0.5$ is larger than both $R(0.5)$ and $R(2.5)$, the function $\beta_Z \mapsto R$ is not monotonic. A parallel construction can be made for β_T. Hence, in general, R is non-monotonic in β_Z and β_T. $\square$

5. Methods of estimation

This section presents two approaches for estimating the unknown parameters of the IFGM-CD, namely, the maximum likelihood (ML) method and Bayesian framework. In addition, asymptotic confidence intervals (CIs) for the model parameters are obtained through the Fisher information matrix (FIM).

5.1. The ML estimation

The ML method is a powerful tool for parameter estimation in statistics. It works by selecting the parameter values that maximize the likelihood of the observed data. Estimates obtained through ML are known to exhibit appealing large sample properties, including consistency, asymptotic efficiency, and asymptotic normality. The log likelihood function $\ln L$ is obtained as

$$\begin{aligned}\ln L = {} & n\ln\theta_1 + n\ln\beta_1 + (\beta_1 - 1)\sum_{i=1}^{n}\ln(x_i) + \sum_{i=1}^{n} x_i^{\beta_1} - \theta_1\sum_{i=1}^{n}(e^{x_i^{\beta_1}} - 1) + n\ln\theta_2 + n\ln\beta_2 + (\beta_2 - 1)\\ & \times\sum_{i=1}^{n}\ln(y_i) + \sum_{i=1}^{n} y_i^{\beta_2} - \theta_2\sum_{i=1}^{n}(e^{y_i^{\beta_2}} - 1) + \sum_{i=1}^{n}\ln\Big[1 + \gamma(1 - 2e^{-\theta_1(e^{x_i^{\beta_1}}-1)})(1 - 2e^{-\theta_2(e^{y_i^{\beta_2}}-1)}) + \omega\\ & \times(1 - e^{-\theta_1(e^{x_i^{\beta_1}}-1)})(1 - e^{-\theta_2(e^{y_i^{\beta_2}}-1)})(3e^{-\theta_1(e^{x_i^{\beta_1}}-1)} - 1)(3e^{-\theta_2(e^{y_i^{\beta_2}}-1)} - 1)\Big].\end{aligned}$$

The normal equations are obtained by setting the partial derivatives of $\ln L$ with respect to $\Theta = (\theta_1, \beta_1, \theta_2, \beta_2, \gamma, \omega)$ to zero. These derivatives are given as follows:

$$\begin{aligned}\frac{\partial\ln L(\Theta)}{\partial\theta_l} = {} & \sum_{i=1}^{n}\frac{2A_{l,i}(1 - e^{z_{l,i}^{\beta_l}})\left[-\gamma(1 - 2A_{j,i}) + \omega(1 - A_{j,i})(3A_{j,i} - 1)(2 - 3A_{l,i})\right]}{1 + \gamma(1 - 2A_{l,i})(1 - 2A_{j,i}) + \omega(1 - A_{l,i})(1 - A_{j,i})(3A_{l,i} - 1)(3A_{j,i} - 1)}\\ & + \frac{n}{\theta_l} + \sum_{i=1}^{n}(1 - e^{z_{l,i}^{\beta_1}}),\end{aligned}$$

$$\begin{aligned}\frac{\partial\ln L(\Theta)}{\partial\beta_l} = {} & \sum_{i=1}^{n}\frac{2z_{l,i}^{\beta_l}\ln(z_{l,i})\theta_l e^{z_{l,i}^{\beta_l}}A_{l,i}\left[\gamma(1 - 2A_{j,i}) - \omega(1 - A_{j,i})(3A_{j,i} - 1)(2 - 3A_{l,i})\right]}{1 + \gamma(1 - 2A_{l,i})(1 - 2A_{j,i}) + \omega(1 - A_{l,i})(1 - A_{j,i})(3A_{l,i} - 1)(3A_{j,i} - 1)}\\ & + \frac{n}{\beta_l} + \sum_{i=1}^{n} z_{l,i}^{\beta_l}\ln(z_{l,i}) - \theta_l\sum_{i=1}^{n} z_{l,i}^{\beta_l}\ln(z_{l,i})(e^{z_{l,i}^{\beta_l}}) + \sum_{i=1}^{n}\ln(z_{l,i}),\end{aligned}$$

$$\frac{\partial \ln L(\Theta)}{\partial \gamma} = \sum_{i=1}^{n} \frac{(1-2A_{l,i})(1-2A_{j,i})}{1+\gamma(1-2A_{l,i})(1-2A_{j,i})+\omega(1-A_{l,i})(1-A_{j,i})(3A_{l,i}-1)(3A_{j,i}-1)},$$

and

$$\frac{\partial \ln L(\Theta)}{\partial \omega} = \sum_{i=1}^{n} \frac{(1-A_{l,i})(1-A_{j,i})(3A_{l,i}-1)(3A_{j,i}-1)}{1+\gamma(1-2A_{l,i})(1-2A_{j,i})+\omega(1-A_{l,i})(1-A_{j,i})(3A_{l,i}-1)(3A_{j,i}-1)},$$

where $A_{k,i} = e^{-\theta_k(e^{z_{k,i}^{\beta_k}}-1)}$, $k = l, j$; $l, j = 1, 2$, and $l \neq j$. The normal equations, which are produced by the preceding derivatives, are a system of nonlinear equations that has no explicit, easy solution. Therefore, we use the Mathcad 2004 software; see Subsection 6.2.

5.2. Bayesian estimation

The Bayesian estimation technique is a powerful tool for estimating unknown parameters based on observable data. Using the Bayes theorem, a concept in probability theory, new information can be gathered to update the probability of a hypothesis. Because this approach takes prior knowledge into account when estimating, it provides some advantages over the traditional ML approach. Moreover, it is capable of assessing the degree of uncertainty surrounding each parameter. We must select an acceptable prior PDF and hyper-parameter values that reflect our belief regarding the data. For the parameters θ_l and β_l, $l = 1, 2$, we select gamma-independent priors, specifically,

$$\Pi_1(\theta_l) \propto \theta_l^{b_l-1} e^{-a_l\theta_l}, \;\; \theta_l > 0, a_l, b_l > 0, \;\; l = 1, 2,$$

and

$$\Pi_2(\beta_l) \propto \beta_l^{d_l-1} e^{-c_l\beta_l}, \;\; \beta_l > 0, c_l, d_l > 0, \;\; l = 1, 2.$$

While the copula parameters γ and ω have uniform prior distribution, the prior JPDF is given by

$$\Pi(\Theta) \propto \theta_1^{b_1-1} e^{-a_1\theta_1} \beta_1^{d_1-1} e^{-c_1\beta_1} \theta_2^{b_2-1} e^{-a_2\theta_2} \beta_2^{d_2-1} e^{-c_2\beta_2}.$$

The likelihood method's estimate and variance-covariance matrix can be used to determine how to elicit the independent joint prior's hyper-parameters. Gamma prior's mean and variance can be used to represent the derived hyper-parameters. For information see, Abd Elgawad et al. [26], Dey et al. [27], and Mansour et al. [28]. The parameters θ_l and β_l where $l = 1, 2$, of bivariate IFGM-CD, should be well-known and positive. The likelihood function is given by

$$L(\Theta) = \theta_1^n \beta_1^n \prod_{i=1}^{n} x_i^{\beta_1-1} e^{x_i^{\beta_1}} e^{-\theta_1(e^{x_i^{\beta_1}}-1)} \theta_2^n \beta_2^n \prod_{i=1}^{n} y_i^{\beta_2-1} e^{y_i^{\beta_2}} e^{-\theta_2(e^{y_i^{\beta_2}}-1)} \prod_{i=1}^{n} \Bigg[1+\gamma(1-2e^{-\theta_1(e^{x_i^{\beta_1}}-1)})$$
$$\times (1-2e^{-\theta_2(e^{y_i^{\beta_2}}-1)}) + \omega(1-e^{-\theta_1(e^{x_i^{\beta_1}}-1)})(1-e^{-\theta_2(e^{y_i^{\beta_2}}-1)})(3e^{-\theta_1(e^{x_i^{\beta_1}}-1)}-1)(3e^{-\theta_2(e^{y_i^{\beta_2}}-1)}-1) \Bigg].$$

The corresponding posterior density is given as follows:

$$\Pi(\Theta|x,y) = \theta_1^{n+b_1-1} \beta_1^{n+d_1-1} \theta_2^{n+b_2-1} \beta_2^{n+d_2-1} e^{-a_1\theta_1-a_2\theta_2-c_1\beta_1-c_2\beta_2} e^{\sum_{i=1}^n x_i^{\beta_1}+\sum_{i=1}^n y_i^{\beta_2}} e^{-\theta_1\sum_{i=1}^n(e^{x_i^{\beta_1}}-1)}$$
$$\times e^{-\theta_2\sum_{i=1}^n(e^{y_i^{\beta_2}}-1)} \prod_{i=1}^{n} x_i^{\beta_1-1} y_i^{\beta_2-1} \Bigg[1+\gamma(1-2e^{-\theta_1(e^{x_i^{\beta_1}}-1)})(1-2e^{-\theta_2(e^{y_i^{\beta_2}}-1)})+\omega(1-e^{-\theta_1(e^{x_i^{\beta_1}}-1)})$$

$$\times\,(1-e^{-\theta_2(e^{y_i^{\beta_2}}-1)})(3e^{-\theta_1(e^{x_i^{\beta_1}}-1)}-1)(3e^{-\theta_2(e^{y_i^{\beta_2}}-1)}-1)\Bigg].$$

The marginal posterior distributions $\Pi(\theta_1|\beta_1,\theta_2,\beta_2,\gamma,\omega;x,y)$, $\Pi(\theta_2|\theta_1,\beta_1,\beta_2,\gamma,\omega;x,y)$, $\Pi(\beta_1|\theta_1,\beta_2,\theta_2,\gamma,\omega;x,y)$, and $\Pi(\beta_2|\theta_1,\beta_1,\theta_2,\gamma,\omega;x,y)$ of the parameters θ_l and β_l, $l=1,2$, may be found by integrating out the nuisance parameters from the posterior distribution $\Pi(\Theta|x,y)$ as follows:

$$\Pi(\theta_1|x,y)\propto\theta_1^{n+b_1-1}e^{-a_1\theta_1}e^{-\theta_1\sum_{i=1}^{n}(e^{x_i^{\beta_1}}-1)}\prod_{i=1}^{n}\Bigg[1+\gamma(1-2e^{-\theta_1(e^{x_i^{\beta_1}}-1)})(1-2e^{-\theta_2(e^{y_i^{\beta_2}}-1)})$$
$$+\,\omega(1-e^{-\theta_1(e^{x_i^{\beta_1}}-1)})(1-e^{-\theta_2(e^{y_i^{\beta_2}}-1)})(3e^{-\theta_1(e^{x_i^{\beta_1}}-1)}-1)(3e^{-\theta_2(e^{y_i^{\beta_2}}-1)}-1)\Bigg],$$
$$\Pi(\theta_2|x,y)\propto\theta_2^{n+b_2-1}e^{-a_2\theta_2}e^{-\theta_2\sum_{i=1}^{n}(e^{y_i^{\beta_2}}-1)}\prod_{i=1}^{n}\Bigg[1+\gamma(1-2e^{-\theta_1(e^{x_i^{\beta_1}}-1)})(1-2e^{-\theta_2(e^{y_i^{\beta_2}}-1)})$$
$$+\,\omega(1-e^{-\theta_1(e^{x_i^{\beta_1}}-1)})(1-e^{-\theta_2(e^{y_i^{\beta_2}}-1)})(3e^{-\theta_1(e^{x_i^{\beta_1}}-1)}-1)(3e^{-\theta_2(e^{y_i^{\beta_2}}-1)}-1)\Bigg],$$
$$\Pi(\beta_1|x,y)\propto\beta_1^{n+d_1-1}e^{-c_1\beta_1}e^{\sum_{i=1}^{n}x_i^{\beta_1}}e^{-\theta_1\sum_{i=1}^{n}e^{x_i^{\beta_1}}}\prod_{i=1}^{n}x_i^{\beta_1-1}\Bigg[1+\gamma(1-2e^{-\theta_1(e^{x_i^{\beta_1}}-1)})$$
$$\times\,(1-2e^{-\theta_2(e^{y_i^{\beta_2}}-1)})+\omega(1-e^{-\theta_1(e^{x_i^{\beta_1}}-1)})(1-e^{-\theta_2(e^{y_i^{\beta_2}}-1)})(3e^{-\theta_1(e^{x_i^{\beta_1}}-1)}-1)(3e^{-\theta_2(e^{y_i^{\beta_2}}-1)}-1)\Bigg],$$
$$\Pi(\beta_2|x,y)\propto\beta_2^{n+d_2-1}e^{-c_2\beta_2}e^{\sum_{i=1}^{n}y_i^{\beta_2}}e^{-\theta_2\sum_{i=1}^{n}e^{y_i^{\beta_2}}}\prod_{i=1}^{n}y_i^{\beta_2-1}\Bigg[1+\gamma(1-2e^{-\theta_1(e^{x_i^{\beta_1}}-1)})$$
$$\times\,(1-2e^{-\theta_2(e^{y_i^{\beta_2}}-1)})+\omega(1-e^{-\theta_1(e^{x_i^{\beta_1}}-1)})(1-e^{-\theta_2(e^{y_i^{\beta_2}}-1)})(3e^{-\theta_1(e^{x_i^{\beta_1}}-1)}-1)(3e^{-\theta_2(e^{y_i^{\beta_2}}-1)}-1)\Bigg],$$
$$\Pi(\gamma|x,y)\propto\prod_{i=1}^{n}\Bigg[1+\gamma(1-2e^{-\theta_1(e^{x_i^{\beta_1}}-1)})(1-2e^{-\theta_2(e^{y_i^{\beta_2}}-1)})+\omega(1-e^{-\theta_1(e^{x_i^{\beta_1}}-1)})(1-e^{-\theta_2(e^{y_i^{\beta_2}}-1)})$$
$$\times\,(3e^{-\theta_1(e^{x_i^{\beta_1}}-1)}-1)(3e^{-\theta_2(e^{y_i^{\beta_2}}-1)}-1)\Bigg],$$

and

$$\Pi(\omega|x,y)\propto\prod_{i=1}^{n}\Bigg[1+\gamma(1-2e^{-\theta_1(e^{x_i^{\beta_1}}-1)})(1-2e^{-\theta_2(e^{y_i^{\beta_2}}-1)})+\omega(1-e^{-\theta_1(e^{x_i^{\beta_1}}-1)})(1-e^{-\theta_2(e^{y_i^{\beta_2}}-1)})$$
$$\times\,(3e^{-\theta_1(e^{x_i^{\beta_1}}-1)}-1)(3e^{-\theta_2(e^{y_i^{\beta_2}}-1)}-1)\Bigg].$$

We use the well-known squared error loss function that yields the posterior means as the Bayes estimates of Θ, say $(\hat{\theta}_l,\hat{\beta}_l,\hat{\gamma},\hat{\omega})$, which are given by

$$\hat{\theta}_l=\int_0^\infty\theta_l\Pi(\theta_l|x,y)d\theta_l,\quad\hat{\beta}_l=\int_0^\infty\beta_l\Pi(\beta_l|x,y)d\beta_l,$$
$$\hat{\gamma}=\int_{-1}^{1}\gamma\Pi(\gamma|x,y)d\gamma,\quad\hat{\omega}=\int_{-\omega_{max}}^{\omega_{max}}\omega\Pi(\omega|x,y)d\omega,$$

where ω_{max} depends on the validity of $c(u, v) \geq 0$. In Subsection 6.2, these estimates will be further studied with comparison to ML estimates via a simulation study.

6. Asymptotic CIs

Based on the asymptotic normality of the ML estimates, the FIM is frequently used to create asymptotic CIs for the unknown parameters in Θ. Under specific assumptions of regularity, the ML estimates denoted as $\hat{\Theta}$ are distributed according to the normal distribution. As the sample size n tends to infinity, the distribution of the estimator $\hat{\Theta} = (\hat{\theta}_1, \hat{\beta}_1, \hat{\theta}_2, \hat{\beta}_2, \hat{\gamma}, \hat{\omega})$ approaches a normal distribution with mean Θ and covariance matrix equal to the inverse of the FIM, $I^{-1}(\Theta)$. The FIM consists of the negative expected values of the second-order derivatives of $\ln L$, as expressed by

$$I(\hat{\Theta}) = -E\begin{bmatrix} I_{\hat{\theta}_1\hat{\theta}_1} & & & & & \\ I_{\hat{\beta}_1\hat{\theta}_1} & I_{\hat{\beta}_1\hat{\beta}_1} & & & & \\ I_{\hat{\theta}_2\hat{\theta}_1} & I_{\hat{\theta}_2\hat{\beta}_1} & I_{\hat{\theta}_2\hat{\theta}_2} & & & \\ I_{\hat{\beta}_2\hat{\theta}_1} & I_{\hat{\beta}_2\hat{\beta}_1} & I_{\hat{\beta}_2\hat{\theta}_2} & I_{\hat{\beta}_2\hat{\beta}_2} & & \\ I_{\hat{\gamma}\hat{\theta}_1} & I_{\hat{\gamma}\hat{\beta}_1} & I_{\hat{\gamma}\hat{\theta}_2} & I_{\hat{\gamma}\hat{\beta}_2} & I_{\hat{\gamma}\hat{\gamma}} & \\ I_{\hat{\omega}\hat{\theta}_1} & I_{\hat{\omega}\hat{\beta}_1} & I_{\hat{\omega}\hat{\theta}_2} & I_{\hat{\omega}\hat{\beta}_2} & I_{\hat{\omega}\hat{\gamma}} & I_{\hat{\omega}\hat{\omega}} \end{bmatrix},$$

where

$$\begin{aligned} I_{\theta_l\theta_l} = \frac{\partial^2 L(\Theta)}{\partial\theta_l^2} =& -\sum_{i=1}^{n} \frac{\left[2A_{l,i}(1-e^{z_{l,i}^{\beta_l}})\left[-\gamma(1-2A_{j,i})+\omega(1-A_{j,i})(3A_{j,i}-1)(2-3A_{l,i})\right]\right]^2}{[1+\gamma(1-2A_{l,i})(1-2A_{j,i})+\omega(1-A_{l,i})(1-A_{j,i})(3A_{l,i}-1)(3A_{j,i}-1)]^2} \\ &+\sum_{i=1}^{n} \frac{-2A_{l,i}(1-e^{z_{l,i}^{\beta_l}})^2\left[\gamma(1-2A_{j,i})+2\omega(1-A_{j,i})(3A_{j,i}-1)(3A_{l,i}-1)\right]}{[1+\gamma(1-2A_{l,i})(1-2A_{j,i})+\omega(1-A_{l,i})(1-A_{j,i})(3A_{l,i}-1)(3A_{j,i}-1)]} - \frac{n}{\theta_l^2}, \end{aligned}$$

$$\begin{aligned} I_{\theta_l\theta_j} =& -\sum_{i=1}^{n} \frac{\left(2A_{j,i}(1-e^{z_{j,i}^{\beta_j}})\left[-\gamma(1-2A_{l,i})+\omega(1-A_{l,i})(3A_{l,i}-1)(2-3A_{j,i})\right]\right)}{\left[1+\gamma(1-2A_{l,i})(1-2A_{j,i})+\omega(1-A_{l,i})(1-A_{j,i})(3A_{l,i}-1)(3A_{j,i}-1)\right]^2} \\ &\times\left(2A_{l,i}(1-e^{z_{l,i}^{\beta_l}})\left[-\gamma(1-2A_{j,i})+\omega(1-A_{j,i})(3A_{j,i}-1)(2-3A_{l,i})\right]\right) \\ &+\sum_{i=1}^{n} \frac{4A_{l,i}A_{j,i}(1-e^{z_{l,i}^{\beta_l}})(1-e^{z_{j,i}^{\beta_j}})\left[\gamma+\omega\left(9A_{l,i}A_{j,i}-6A_{l,i}-6A_{j,i}+4\right)\right]}{[1+\gamma(1-2A_{l,i})(1-2A_{j,i})+\omega(1-A_{l,i})(1-A_{j,i})(3A_{l,i}-1)(3A_{j,i}-1)]}, \end{aligned}$$

$$\begin{aligned} I_{\beta_l\beta_l} =& -\sum_{i=1}^{n} \frac{\left(2z_{l,i}^{\beta_l}\ln(z_{l,i})\theta_l A_{l,i}e^{z_{l,i}^{\beta_l}}\left[\gamma(1-2A_{j,i})-\omega(1-A_{j,i})(3A_{j,i}-1)(2-3A_{l,i})\right]\right)^2}{\left[1+\gamma(1-2A_{l,i})(1-2A_{j,i})+\omega(1-A_{l,i})(1-A_{j,i})(3A_{l,i}-1)(3A_{j,i}-1)\right]^2} - \frac{n}{\beta_l^2} \\ &+\sum_{i=1}^{n} z_{l,i}^{\beta_l}\ln(z_{l,i})^2 + \sum_{i=1}^{n} \frac{\left(\gamma(1-2A_{j,i})-\omega(1-A_{j,i})(3A_{j,i}-1)(2-3A_{l,i}[1-z_{l,i}^{\beta_l}\theta_l e^{z_{l,i}^{\beta_l}}])\right)}{[1+\gamma(1-2A_{l,i})(1-2A_{j,i})+\omega(1-A_{l,i})(1-A_{j,i})(3A_{l,i}-1)(3A_{j,i}-1)]} \\ &\times\left(2z_{l,i}^{\beta_l}\ln(z_{l,i})^2\theta_l A_{l,i}e^{z_{l,i}^{\beta_l}}[1+z_{l,i}^{\beta_l}(1-\theta_l e^{z_{l,i}^{\beta_l}})]\right) - \theta_l\sum_{i=1}^{n} z_{l,i}^{\beta_l}\ln(z_{l,i})^2 e^{z_{l,i}^{\beta_l}}(1+z_{l,i}^{\beta_l}), \end{aligned}$$

$$I_{\beta_l\beta_j}=-\sum_{i=1}^{n}\frac{\left(2z_{l,i}^{\beta_l}\ln(z_{l,i})\theta_l e^{z_{l,i}^{\beta_l}}A_{l,i}\left[\gamma(1-2A_{j,i})-\omega(1-A_{j,i})(3A_{j,i}-1)(2-3A_{l,i})\right]\right)}{\left[1+\gamma(1-2A_{l,i})(1-2A_{j,i})+\omega(1-A_{l,i})(1-A_{j,i})(3A_{l,i}-1)(3A_{j,i}-1)\right]^2}$$
$$\times\left(2z_{j,i}^{\beta_j}\ln(z_{j,i})\theta_j e^{z_{j,i}^{\beta_j}}A_{j,i}\left[\gamma(1-2A_{l,i})-\omega(1-A_{l,i})(3A_{l,i}-1)(2-3A_{j,i})\right]\right)$$
$$+\sum_{i=1}^{n}\frac{2z_{l,i}^{\beta_l}z_{j,i}^{\beta_j}\ln(z_{l,i})\ln(z_{j,i})\theta_l\theta_j e^{z_{l,i}^{\beta_l}}e^{z_{j,i}^{\beta_j}}A_{l,i}A_{j,i}\left(2\gamma+\omega\left[(1-A_{j,i})(2-5A_{l,i})-(3A_{j,i}-1)(2-3A_{l,i})\right]\right)}{[1+\gamma(1-2A_{l,i})(1-2A_{j,i})+\omega(1-A_{l,i})(1-A_{j,i})(3A_{l,i}-1)(3A_{j,i}-1)]},$$

$$I_{\beta_l\theta_l}=-\sum_{i=1}^{n}\frac{\left(2(1-e^{z_{l,i}^{\beta_l}})A_{l,i}\left[-\gamma(1-2A_{j,i})+\omega(1-A_{j,i})(3A_{j,i}-1)(2-3A_{l,i})\right]\right)}{\left[1+\gamma(1-2A_{l,i})(1-2A_{j,i})+\omega(1-A_{l,i})(1-A_{j,i})(3A_{l,i}-1)(3A_{j,i}-1)\right]^2}$$
$$\times\left(2z_{l,i}^{\beta_l}\ln(z_{l,i})\theta_l e^{z_{l,i}^{\beta_l}}A_{l,i}\left[\gamma(1-2A_{j,i})-\omega(1-A_{j,i})(3A_{j,i}-1)(2-3A_{l,i})\right]\right)$$
$$+\sum_{i=1}^{n}\frac{\left(\gamma(1-2A_{j,i})(1+\theta_l(1-e^{z_{l,i}^{\beta_l}}))-\omega(1-A_{j,i})(3A_{j,i}-1)\left((2-3A_{l,i})-2\theta_l(1-e^{z_{l,i}^{\beta_l}})(3A_{l,i}-1)\right)\right)}{[1+\gamma(1-2A_{l,i})(1-2A_{j,i})+\omega(1-A_{l,i})(1-A_{j,i})(3A_{l,i}-1)(3A_{j,i}-1)]}$$
$$\times\left(2A_{l,i}z_{l,i}^{\beta_l}\ln(z_{l,i})e^{z_{l,i}^{\beta_l}}\right)-\sum_{i=1}^{n}z_{l,i}^{\beta_l}\ln(z_{l,i})e^{z_{l,i}^{\beta_l}},$$

$$I_{\beta_l\theta_j}=-\sum_{i=1}^{n}\frac{\left(2(1-e^{z_{j,i}^{\beta_j}})A_{j,i}\left[-\gamma(1-2A_{l,i})+\omega(1-A_{l,i})(3A_{l,i}-1)(2-3A_{j,i})\right]\right)}{\left[1+\gamma(1-2A_{l,i})(1-2A_{j,i})+\omega(1-A_{l,i})(1-A_{j,i})(3A_{l,i}-1)(3A_{j,i}-1)\right]^2}$$
$$\times\left(2z_{l,i}^{\beta_l}\ln(z_{l,i})\theta_l e^{z_{l,i}^{\beta_l}}A_{l,i}\left[\gamma(1-2A_{j,i})-\omega(1-A_{j,i})(3A_{j,i}-1)(2-3A_{l,i})\right]\right)$$
$$-\sum_{i=1}^{n}\frac{4\theta_l A_{l,i}A_{j,i}z_{l,i}^{\beta_l}\ln(z_{l,i})e^{z_{l,i}^{\beta_l}}(1-e^{z_{j,i}^{\beta_j}})\left(\gamma+\omega(2-3A_{j,i})(2-3A_{l,i})\right)}{[1+\gamma(1-2A_{l,i})(1-2A_{j,i})+\omega(1-A_{l,i})(1-A_{j,i})(3A_{l,i}-1)(3A_{j,i}-1)]},$$

$$I_{\theta_l\gamma}\;=\;-\sum_{i=1}^{n}\frac{2(1-e^{z_{l,i}^{\beta_l}})A_{l,i}(1-2A_{j,i})}{1+\gamma(1-2A_{l,i})(1-2A_{j,i})+\omega(1-A_{l,i})(1-A_{j,i})(3A_{l,i}-1)(3A_{j,i}-1)}$$
$$-\sum_{i=1}^{n}\frac{(1-2A_{l,i})(1-2A_{j,i})\left(2(1-e^{z_{l,i}^{\beta_l}})A_{l,i}\left[-\gamma(1-2A_{j,i})+\omega(1-A_{j,i})(3A_{j,i}-1)(2-3A_{l,i})\right]\right)}{\left(1+\gamma(1-2A_{l,i})(1-2A_{j,i})+\omega(1-A_{l,i})(1-A_{j,i})(3A_{l,i}-1)(3A_{j,i}-1)\right)^2},$$

$$I_{\beta_l\gamma}=\sum_{i=1}^{n}\frac{2z_{l,i}^{\beta_l}\ln(z_{l,i})\theta_l e^{z_{l,i}^{\beta_l}}A_{l,i}(1-2A_{j,i})}{1+\gamma(1-2A_{l,i})(1-2A_{j,i})+\omega(1-A_{l,i})(1-A_{j,i})(3A_{l,i}-1)(3A_{j,i}-1)}$$
$$-\sum_{i=1}^{n}\frac{(1-2A_{l,i})(1-2A_{j,i})\left(2z_{l,i}^{\beta_l}\ln(z_{l,i})\theta_l e^{z_{l,i}^{\beta_l}}A_{l,i}\left[\gamma(1-2A_{j,i})-\omega(1-A_{j,i})(3A_{j,i}-1)(2-3A_{l,i})\right]\right)}{\left(1+\gamma(1-2A_{l,i})(1-2A_{j,i})+\omega(1-A_{l,i})(1-A_{j,i})(3A_{l,i}-1)(3A_{j,i}-1)\right)^2},$$

$$I_{\gamma\gamma} = \sum_{i=1}^{n} \frac{(1-2A_{l,i})^2(1-2A_{j,i})^2}{\left(1+\gamma(1-2A_{l,i})(1-2A_{j,i})+\omega(1-A_{l,i})(1-A_{j,i})(3A_{l,i}-1)(3A_{j,i}-1)\right)^2},$$

$$\begin{aligned} I_{\theta_l\omega} =& \sum_{i=1}^{n} \frac{2A_{l,i}(1-A_{j,i})(3A_{j,i}-1)(1-e^{z_{l,i}^{\beta_l}})(2-3A_{l,i})}{1+\gamma(1-2A_{l,i})(1-2A_{j,i})+\omega(1-A_{l,i})(1-A_{j,i})(3A_{l,i}-1)(3A_{j,i}-1)} \\ -& \sum_{i=1}^{n} \frac{\left[-\gamma(1-2A_{j,i})+\omega(1-A_{j,i})(3A_{j,i}-1)(2-3A_{l,i})\right]}{\left(1+\gamma(1-2A_{l,i})(1-2A_{j,i})+\omega(1-A_{l,i})(1-A_{j,i})(3A_{l,i}-1)(3A_{j,i}-1)\right)^2} \\ \times& \left(2A_{l,i}(1-A_{l,i})(3A_{l,i}-1)(1-A_{j,i})(3A_{j,i}-1)(1-e^{z_{l,i}^{\beta_l}})\right), \end{aligned}$$

$$\begin{aligned} I_{\beta_l\omega} =& -\sum_{i=1}^{n} \frac{2z_{l,i}^{\beta_l}\ln(z_{l,i})\theta_l A_{l,i}(1-A_{j,i})(3A_{j,i}-1)e^{z_{l,i}^{\beta_l}}(2-3A_{l,i})}{1+\gamma(1-2A_{l,i})(1-2A_{j,i})+\omega(1-A_{l,i})(1-A_{j,i})(3A_{l,i}-1)(3A_{j,i}-1)} \\ -& \sum_{i=1}^{n} \frac{\left[\gamma(1-2A_{j,i})-\omega(1-A_{j,i})(3A_{j,i}-1)(2-3A_{l,i})\right]}{\left(1+\gamma(1-2A_{l,i})(1-2A_{j,i})+\omega(1-A_{l,i})(1-A_{j,i})(3A_{l,i}-1)(3A_{j,i}-1)\right)^2} \\ \times& \left(2\theta_l z_{l,i}^{\beta_l}\ln(z_{l,i})A_{l,i}(1-A_{l,i})(3A_{l,i}-1)(1-A_{j,i})(3A_{j,i}-1)e^{z_{l,i}^{\beta_l}}\right), \end{aligned}$$

and

$$I_{\omega\omega} = \sum_{i=1}^{n} \frac{(1-A_{l,i})^2(3A_{l,i}-1)^2(1-A_{j,i})^2(3A_{j,i}-1)^2}{\left(1+\gamma(1-2A_{l,i})(1-2A_{j,i})+\omega(1-A_{l,i})(1-A_{j,i})(3A_{l,i}-1)(3A_{j,i}-1)\right)^2},$$

where $l, j = 1,2,\ l \neq j$. When the expected FIM is difficult to compute directly, a common solution is to approximate it using the observed FIM. Following the approach of Jia et al. [29], this is achieved by calculating the negative Hessian (the matrix of second-order partial derivatives) of the log-likelihood function, evaluated at the ML estimates $\hat{\Theta}$. Thus, the estimated FIM is given by $I(\hat{\Theta})$. Using this approximation, an asymptotic $100(1-\alpha)\%$ CI for the parameter Θ can be constructed as:

$$\hat{\theta}_l \pm Z_{\frac{\alpha}{2}}\sqrt{J_{\hat{\theta}_l\hat{\theta}_l}}, \quad \hat{\beta}_l \pm Z_{\frac{\alpha}{2}}\sqrt{J_{\hat{\beta}_l\hat{\beta}_l}},$$

$$\hat{\gamma} \pm Z_{\frac{\alpha}{2}}\sqrt{J_{\hat{\gamma}\hat{\gamma}}}, \quad \hat{\omega} \pm Z_{\frac{\alpha}{2}}\sqrt{J_{\hat{\omega}\hat{\omega}}},\ l = 1,2,$$

where $J_{x_lx_l},\ x_l = \hat{\theta}_l, \hat{\beta}_l, \hat{\gamma}, \hat{\omega}$ is the element in $I^{-1}(\hat{\Theta})$, such that corresponding to the element Ix_lx_l in the estimated matrix $I(\hat{\Theta})$ and $Z_{\frac{\alpha}{2}}$ is the percentile of the standard normal distribution with right tail probability $\frac{\alpha}{2}$.

Remark 6.1. *The second-order partial derivatives required for the construction of the Hessian matrix and the FIM were obtained using symbolic and numerical differentiation routines implemented in standard statistical software. Due to their length, these expressions are presented only insofar as they are needed for asymptotic inference and simulation-based evaluation of the estimators.*

6.1. Bootstrap CI

The bootstrap is a resampling-based method for statistical inference, originally introduced by Efron [30]. It treats the observed data as a proxy for the population and generates multiple re-samples to approximate the sampling distribution of estimators. Unlike asymptotic approaches, the bootstrap often provides more reliable results in small or complex samples. Kotz et al. [31] recommended its use for constructing CIs. To implement the bootstrap for the parameters of the bivariate IFGM-CD, the following procedure is applied:

- **Parameter estimation:** Obtain the ML estimates of the parameters based on the observed bivariate sample $\{(\underline{t}_1, \underline{t}_2)\} = \{(t_{11}, t_{12}), (t_{21}, t_{22}), ..., (t_{n1}, t_{n2})\}$. The ML estimators for the parameters in the IFGM-CD, using the results of Subsection 5.1, $\hat{\beta}_1 = s_1(\underline{t}_1, \underline{t}_2)$, $\hat{\beta}_2 = s_2(\underline{t}_1, \underline{t}_2)$, $\hat{\theta}_1 = s_3(\underline{t}_1, \underline{t}_2)$, $\hat{\theta}_2 = s_4(\underline{t}_1, \underline{t}_2)$, $\hat{\gamma} = s_5(\underline{t}_1, \underline{t}_2)$, and $\hat{\omega} = s_6(\underline{t}_1, \underline{t}_2)$.
- **Bootstrap re-sampling:** Draw B independent bootstrap bivariate samples (with replacement) from the original data. Each re-sample is denoted by $\{(\underline{t}_1^b, \underline{t}_2^b)\} = \{(t_{11}^b, t_{12}^b), (t_{21}^b, t_{22}^b), ..., (t_{n1}^b, t_{n2}^b)\}$, $b = 1, 2, ..., B$, where in practice $B = 2000$ is used to ensure accuracy.
- **Bootstrap estimators:** For each bootstrap sample, compute the parameter estimates $\beta_1^*(b) = s_1(\underline{t}_1^b, \underline{t}_2^b), \beta_2^*(b) = s_2(\underline{t}_1^b, \underline{t}_2^b)$, $\theta_1^*(b) = s_3(\underline{t}_1^b, \underline{t}_2^b), \theta_2^*(b) = s_4(\underline{t}_1^b, \underline{t}_2^b)$, $\gamma^*(b) = s_5(\underline{t}_1^b, \underline{t}_2^b)$, and $\omega^*(b) = s_6(\underline{t}_1^b, \underline{t}_2^b)$, $b = 1, 2, ..., B$.
- **Mean squared errors:** The mean squared error (MSE) of each parameter is estimated by

$$\widehat{MSE}_{B\Theta} = \frac{1}{B}\sum_{b=1}^{B}(\Theta^*(b) - \Theta^*(.))^2, \quad \Theta^*(.) = \frac{\sum_{b=1}^{B}\Theta^*(b)}{B},$$

  where $\Theta = (\theta_1, \beta_1, \theta_2, \beta_2, \gamma, \omega)$.
- **Bootstrap CIs:** Finally, approximate CIs for the unknown parameters are obtained as

$$\left(\Theta^*(.) \pm Z_{\frac{\alpha}{2}} \sqrt{\widehat{MSE}_{B\Theta}}\right),$$

  for each parameter $\Theta \in (\theta_1, \beta_1, \theta_2, \beta_2, \gamma, \omega)$.

6.2. Simulation

This subsection provides a numerical assessment of the ML, Bayesian, and bootstrap estimators. The comparison is based on analytically derived results together with the observed performance of each estimation method. All simulation results reported in this subsection are based on 5000 Monte Carlo replications. All computations were carried out using the Mathcad package.

The combinations of parameter values considered in the simulation study were selected to represent a range of practically relevant scenarios. In particular, the chosen values of the Chen marginal parameters capture different hazard-rate shapes, while the IFGM copula parameters were varied to reflect weak, moderate, and relatively strong dependence structures within the admissible parameter space. This choice allows for a transparent assessment of the finite-sample performance of the proposed estimators across diverse dependence settings, while maintaining numerical stability and comparability with related studies. The parameter settings for the experiments are summarized as follows:

- In Table 1: $\beta_1 = 1, \beta_2 = 2$, $\theta_1 = 1$, $\theta_2 = 2$, $\gamma = 0.9$, and $\omega = 1.4$,
- In Table 2: $\beta_1 = 2, \beta_2 = 0.3$, $\theta_1 = 2$, $\theta_2 = 0.3$, $\gamma = 0.9$, and $\omega = -1.8$,
- In Table 3: $\beta_1 = 1.5, \beta_2 = 0.9$, $\theta_1 = 1.5$, $\theta_2 = 0.9$, $\gamma = -0.5$, and $\omega = 2.5$, and
- In Table 4: $\beta_1 = 0.2, \beta_2 = 2$, $\theta_1 = 0.2$, $\theta_2 = 2$, $\gamma = -0.5$, and $\omega = -0.4$.

The study considers different sample sizes with $n = 20, 50, 100$, and 150. Simulation outcomes for bias and MSE, obtained from 5000 Monte Carlo replications, are reported in Tables 1–4. In addition, the asymptotic and bootstrap CIs are displayed in Tables 3 and 4. A set of conclusions is then drawn from the numerical findings presented in Tables 1–4.

- For almost all parameters $(\beta_1, \beta_2, \theta_1, \theta_2, \gamma, \omega)$, the bias reduces steadily as n increases from 20 to 150. This indicates consistency of both ML and Bayesian estimators.
- In most cases, Bayesian estimates have smaller bias and lower MSE compared to ML estimates, particularly for the parameters β_2, γ, and θ_2. This suggests that Bayesian methods provide more efficient estimation for the IFGM-CD model.
- The MSE of the estimators decreases with larger sample sizes, confirming the consistency of the estimates as they approach the true parameter values.
- Empirical results indicate that the bootstrap CIs tend to be more precise, as measured by their average length, compared to the intervals derived from asymptotic theory.

Table 1. ML and Bayesian estimation methods for the parameters of bivariate IFGM-CD model.

		$\beta_1 = 1, \beta_2 = 2, \theta_1 = 1$ $\theta_2 = 2, \gamma = 0.9, \omega = 1.4$				$\beta_1 = 2, \beta_2 = 0.3, \theta_1 = 2$ $\theta_2 = 0.3, \gamma = 0.9, \omega = -1.8$			
		ML estimate		Bayesian estimate		ML estimate		Bayesian estimate	
n		*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*
20	β_1	0.088	0.257	0.055	0.255	0.082	0.552	0.064	0.442
	β_2	0.112	0.784	0.088	0.599	0.222	0.282	0.096	0.221
	θ_1	0.085	0.345	0.078	0.303	0.074	0.165	0.058	0.113
	θ_2	0.076	0.321	0.065	0.323	0.073	0.189	0.068	0.176
	γ	0.084	0.457	0.064	0.316	0.063	0.284	0.039	0.142
	ω	0.056	0.231	0.046	0.222	0.034	0.213	0.029	0.189
50	β_1	0.079	0.241	0.028	0.333	0.076	0.483	0.056	0.432
	β_2	0.097	0.698	0.072	0.487	0.105	0.251	0.084	0.186
	θ_1	0.071	0.299	0.069	0.289	0.068	0.145	0.052	0.165
	θ_2	0.067	0.343	0.057	0.321	0.066	0.178	0.057	0.165
	γ	0.069	0.421	0.051	0.287	0.034	0.241	0.020	0.130
	ω	0.043	0.212	0.041	0.208	0.031	0.211	0.025	0.181
100	β_1	0.063	0.239	0.019	0.298	0.061	0.442	0.047	0.408
	β_2	0.083	0.631	0.065	0.414	0.084	0.198	0.071	0.140
	θ_1	0.067	0.289	0.056	0.256	0.059	0.134	0.044	0.154
	θ_2	0.057	0.321	0.046	0.312	0.058	0.166	0.048	0.155
	γ	0.057	0.401	0.047	0.245	0.028	0.208	0.012	0.099
	ω	0.037	0.201	0.032	0.198	0.029	0.205	0.024	0.175
150	β_1	0.059	0.214	0.010	0.167	0.050	0.383	0.030	0.460
	β_2	0.076	0.543	0.054	0.389	0.066	0.152	0.063	0.210
	θ_1	0.058	0.277	0.048	0.249	0.046	0.123	0.038	0.244
	θ_2	0.049	0.312	0.041	0.308	0.051	0.154	0.033	0.243
	γ	0.053	0.289	0.038	0.212	0.023	0.162	0.010	0.182
	ω	0.025	0.187	0.022	0.168	0.027	0.199	0.021	0.170

Table 2. ML and Bayesian estimation methods for the parameters of bivariate IFGM-CD model.

		$\beta_1 = 1.5, \beta_2 = 0.9, \theta_1 = 1.5$ $\theta_2 = 0.9, \gamma = -0.5, \omega = 2.5$				$\beta_1 = 0.2, \beta_2 = 2, \theta_1 = 0.2$ $\theta_2 = 2, \gamma = -0.5, \omega = -0.4$			
		ML estimate		Bayesian estimate		ML estimate		Bayesian estimate	
n		*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*
20	β_1	0.058	0.179	0.037	0.148	0.072	0.401	0.066	0.347
	β_2	0.069	0.204	0.049	0.188	0.079	0.544	0.078	0.468
	θ_1	0.059	0.277	0.055	0.242	0.069	0.177	0.064	0.187
	θ_2	0.077	0.287	0.068	0.255	0.087	0.254	0.071	0.221
	γ	0.011	0.432	0.010	0.325	0.069	0.237	0.058	0.164
	ω	0.036	0.211	0.031	0.198	0.042	0.376	0.041	0.361
50	β_1	0.049	0.148	0.028	0.136	0.065	0.372	0.055	0.313
	β_2	0.045	0.185	0.035	0.168	0.067	0.479	0.065	0.452
	θ_1	0.044	0.264	0.048	0.221	0.062	0.165	0.061	0.165
	θ_2	0.066	0.276	0.059	0.244	0.082	0.244	0.065	0.212
	γ	0.092	0.377	0.076	0.311	0.061	0.231	0.049	0.132
	ω	0.029	0.198	0.028	0.197	0.041	0.366	0.038	0.355
100	β_1	0.032	0.131	0.015	0.122	0.055	0.311	0.045	0.277
	β_2	0.033	0.164	0.022	0.155	0.052	0.402	0.064	0.413
	θ_1	0.039	0.254	0.035	0.211	0.053	0.154	0.055	0.145
	θ_2	0.055	0.265	0.055	0.221	0.074	0.221	0.061	0.201
	γ	0.069	0.368	0.065	0.301	0.058	0.177	0.042	0.104
	ω	0.027	0.188	0.024	0.188	0.035	0.333	0.031	0.321
150	β_1	0.023	0.124	0.008	0.101	0.048	0.278	0.038	0.235
	β_2	0.015	0.141	0.011	0.135	0.046	0.373	0.042	0.377
	θ_1	0.029	0.245	0.028	0.187	0.047	0.145	0.048	0.132
	θ_2	0.047	0.255	0.043	0.212	0.066	0.211	0.055	0.189
	γ	0.061	0.305	0.059	0.288	0.049	0.141	0.034	0.098
	ω	0.021	0.178	0.019	0.176	0.029	0.311	0.029	0.301

Table 3. Lower and upper bootstrap CI limits for the bivariate IFGM-CD model.

		$\beta_1 = 1, \beta_2 = 2, \theta_1 = 1, \theta_2 = 2, \gamma = 0.9, \omega = 1.4$					$\beta_1 = 2, \beta_2 = 0.3, \theta_1 = 2, \theta_2 = 0.3, \gamma = 0.9, \omega = -1.8$			
n		*CI*	*Bootstrap CI*	*CI*	*Bootstrap CI*		*CI*	*Bootstrap CI*	*CI*	*Bootstrap CI*
20	$\beta_1 = 1$	(0.245,1.443)	(0.343,1.476)	(0.443,1.554)	(0.523,1.578)	$\beta_1 = 2$	(1.243,3.443)	(1.342,3.476)	(1.441,3.554)	(1.521,3.578)
	$\beta_2 = 2$	(2.234,3.664)	(2.239,3.632)	(2.254,3.521)	(2.259,3.511)	$\beta_2 = 0.3$	(0.134,0.564)	(0.139,0.532)	(0.154,0.421)	(0.159,0.411)
	$\theta_1 = 1$	(0.356,1.622)	(0.365,1.611)	(0.376,1.640)	(0.379,1.605)	$\theta_1 = 2$	(1.356,3.622)	(1.365,3.611)	(1.376,3.640)	(1.379,3.605)
	$\theta_2 = 2$	(1.346,2.544)	(1.444,2.577)	(1.544,2.655)	(1.624,2.679)	$\theta_2 = 0.3$	(0.246,0.544)	(0.244,0.577)	(0.244,0.655)	(0.224,0.679)
	$\gamma = 0.9$	(0.366,1.632)	(0.366,1.622)	(0.378,1.639)	(0.381,1.611)	$\gamma = 0.9$	(0.866,1.232)	(0.866,1.222)	(0.878,1.239)	(0.881,1.211)
	$\omega = 1.4$	(1.311,1.499)	(1.314,1.489)	(1.315,1.497)	(1.319,1.485)	$\omega = -1.8$	(-1.732,-1.234)	(-1.722,-1.265)	(-1.729,-1.267)	(-1.721,-1.255)
50	$\beta_1 = 1$	(0.315,1.441)	(0.354,1.467)	(0.452,1.545)	(0.545,1.565)	$\beta_1 = 2$	(1.314,3.441)	(0.3544,3.467)	(0.452,3.441)	(0.544,3.461)
	$\beta_2 = 2$	(2.244,3.644)	(2.246,3.623)	(2.256,3.512)	(2.257,3.501)	$\beta_2 = 0.3$	(0.144,0.544)	(0.146,0.523)	(0.156,0.412)	(0.157,0.401)
	$\theta_1 = 1$	(0.378,1.629)	(0.367,1.620)	(0.379,1.631)	(0.381,1.607)	$\theta_1 = 2$	(1.378,3.629)	(1.367,3.620)	(1.379,3.631)	(1.381,3.607)
	$\theta_2 = 2$	(1.416,2.542)	(1.455,2.568)	(1.553,2.646)	(1.646,2.666)	$\theta_2 = 0.3$	(0.216,0.542)	(0.255,0.568)	(0.253,0.646)	(0.246,0.666)
	$\gamma = 0.9$	(0.379,1.629)	(0.369,1.620)	(0.381,1.631)	(0.383,1.607)	$\gamma = 0.9$	(0.879,1.229)	(0.869,1.220)	(0.881,1.231)	(0.883,1.207)
	$\omega = 1.4$	(1.315,1.487)	(1.322,1.481)	(1.319,1.491)	(1.325,1.481)	$\omega = -1.8$	(-1.754,-1.224)	(-1.744,-1.276)	(-1.737,-1.281)	(-1.731,-1.265)
100	$\beta_1 = 1$	(0.333,1.432)	(0.367,1.443)	(0.487,1.532)	(0.554,1.543)	$\beta_1 = 2$	(1.332,3.431)	(1.363,3.441)	(1.483,3.530)	(1.551,3.540)
	$\beta_2 = 2$	(2.246,3.621)	(2.249,3.601)	(2.259,3.502)	(2.260,3.499)	$\beta_2 = 0.3$	(0.146,0.522)	(0.149,0.501)	(0.159,0.302)	(0.160,0.399)
	$\theta_1 = 1$	(0.382,1.625)	(0.370,1.618)	(0.385,1.626)	(0.384,1.601)	$\theta_1 = 2$	(1.382,3.625)	(1.370,3.618)	(1.385,3.626)	(1.384,3.601)
	$\theta_2 = 2$	(1.434,2.533)	(1.468,1.544)	(1.588,2.633)	(1.655,2.644)	$\theta_2 = 0.3$	(0.234,0.533)	(0.268,0.244)	(0.588,0.633)	(0.255,0.644)
	$\gamma = 0.9$	(0.386,1.625)	(0.372,1.618)	(0.386,1.626)	(0.387,1.601)	$\gamma = 0.9$	(0.886,1.325)	(0.872,1.318)	(0.886,1.326)	(0.887,1.301)
	$\omega = 1.4$	(1.321,1.477)	(1.327,1.479)	(1.325,1.486)	(1.328,1.479)	$\omega = -1.8$	(-1.758,-1.229)	(-1.756,-1.278)	(-1.743,-1.289)	(-1.739,-1.271)
150	$\beta_1 = 1$	(0.338,1.467)	(0.376,1.438)	(0.517,1.543)	(0.565,1.521)	$\beta_1 = 2$	(1.335,3.462)	(1.371,3.434)	(1.512,3.541)	(1.562,3.520)
	$\beta_2 = 2$	(2.249,3.611)	(2.254,3.599)	(2.264,3.498)	(2.265,3.488)	$\beta_2 = 0.3$	(0.149,0.511)	(0.154,0.599)	(0.164,0.398)	(0.165,0.388)
	$\theta_1 = 1$	(0.388,1.614)	(0.373,1.612)	(0.390,1.612)	(0.389,1.589)	$\theta_1 = 2$	(1.388,3.614)	(1.373,3.612)	(1.390,3.612)	(1.389,3.589)
	$\theta_2 = 2$	(1.439,2.568)	(1.477,2.539)	(1.618,2.644)	(1.666,2.622)	$\theta_2 = 0.3$	(0.239,0.568)	(0.277,0.539)	(0.218,0.644)	(0.266,0.622)
	$\gamma = 0.9$	(0.389,1.614)	(0.374,1.612)	(0.393,1.612)	(0.389,1.589)	$\gamma = 0.9$	(0.889,1.314)	(0.874,1.312)	(0.893,1.312)	(0.889,1.389)
	$\omega = 1.4$	(1.331,1.471)	(1.333,1.468)	(1.336,1.468)	(1.335,1.461)	$\omega = -1.8$	(-1.761,-1.231)	(-1.760,-1.283)	(-1.750,-1.295)	(-1.745,-1.280)

Table 4. Lower and upper bootstrap CI limits for the bivariate IFGM-CD model.

		$\beta_1 = 1.5, \beta_2 = 0.9, \theta_1 = 1.5, \theta_2 = 0.9, \gamma = 0.5, \omega = 2.5$					$\beta_1 = 0.2, \beta_2 = 2, \theta_1 = 0.2, \theta_2 = 2, \gamma = -0.5, \omega = -0.4$			
n		CI	$Bootstrap\ CI$	CI	$Bootstrap\ CI$		CI	$Bootstrap\ CI$	CI	$Bootstrap\ CI$
20	$\beta_1 = 1.5$	(1.235,1.533)	(1.333,1.566)	(1.433,1.644)	(1.613,1.668)	$\beta_1 = 0.2$	(0.134,0.564)	(0.139,0.532)	(0.154,0.421)	(0.159,0.411)
	$\beta_2 = 0.9$	(0.801,0.989)	(0.811,0.995)	(0.805,0.999)	(0.815,0.996)	$\beta_2 = 2$	(1.335,2.533)	(1.433,2.566)	(1.533,2.644)	(1.613,2.668)
	$\theta_1 = 1.5$	(1.345,1.543)	(1.443,1.576)	(1.543,1.654)	(1.623,1.678)	$\theta_1 = 0.2$	(0.101,0.389)	(0.111,0.395)	(0.105,0.399)	(0.115,0.401)
	$\theta_2 = 0.9$	(0.311,0.988)	(0.314,0.987)	(0.315,0.987)	(0.319,0.985)	$\theta_2 = 2$	(1.346,2.544)	(1.444,2.577)	(1.544,2.655)	(1.624,2.679)
	$\gamma = 0.5$	(0.354,0.631)	(0.363,0.621)	(0.373,0.633)	(0.372,0.601)	$\gamma = -0.5$	(-0.532,-0.234)	(-0.522,-0.265)	(-0.529,-0.267)	(-0.521,-0.255)
	$\omega = 2.5$	(2.427,4.121)	(2.468,3.987)	(2.453,4.011)	(2.473,3977)	$\omega = -0.4$	(-0.422,-0.324)	(-0.432,-0.365)	(-0.428,-0.367)	(-0.429-0.355)
50	$\beta_1 = 1.5$	(1.305,1.531)	(1.344,1.557)	(1.442,1.635)	(1.535,1.655)	$\beta_1 = 0.2$	(0.144,0.544)	(0.146,0.523)	(0.156,0.412)	(0.157,0.401)
	$\beta_2 = 0.9$	(0.804,0.975)	(0.819,0.983)	(0.809,0.974)	(0.822,0.985)	$\beta_2 = 2$	(1.405,2.531)	(1.444,2.557)	(1.542,2.635)	(1.635,2.655)
	$\theta_1 = 1.5$	(1.415,1.541)	(1.454,1.567)	(1.452,1.645)	(1.466,1.665)	$\theta_1 = 0.2$	(0.104,0.375)	(0.119,0.383)	(0.109,0.374)	(0.122,0.400)
	$\theta_2 = 0.9$	(0.315,0.987)	(0.322,0.981)	(0.319,0.991)	(0.325,0.981)	$\theta_2 = 2$	(1.416,2.542)	(1.455,2.568)	(1.553,2.646)	(1.646,2.666)
	$\gamma = 0.5$	(0.374,0.630)	(0.364,0.629)	(0.374,0.636)	(0.376,0.617)	$\gamma = -0.5$	(-0.554,-0.224)	(-0.544,-0.276)	(-0.537,-0.281)	(-0.531,-0.265)
	$\omega = 2.5$	(2.441,4.011)	(2.488, 3.784)	(2.455,3.933)	(2.453,3.864)	$\omega = -0.4$	(-0.434,-0.327)	(-0.441,-0.376)	(-0.436,-0.381)	(-0.439,-0.365)
100	$\beta_1 = 1.5$	(1.323,1.522)	(1.357,1.533)	(1.477,1.622)	(1.564,1.633)	$\beta_1 = 0.2$	(0.146,0.521)	(0.149,0.501)	(0.159,0.402)	(0.160,0.399)
	$\beta_2 = 0.9$	(0.807,0.971)	(0.822,0.979)	(0.813,0.969)	(0.825,0.972)	$\beta_2 = 2$	(1.423,2.522)	(1.457,2.533)	(1.577,2.622)	(1.644,2.633)
	$\theta_1 = 1.5$	(1.433,1.532)	(1.467,1.543)	(1.477,1.632)	(1.474,1.643)	$\theta_1 = 0.2$	(0.107,0.371)	(0.122,0.379)	(0.113,0.369)	(0.125,0.399)
	$\theta_2 = 0.9$	(0.321,0.977)	(0.327,0.979)	(0.325,0.986)	(0.328,0.979)	$\theta_2 = 2$	(1.434,2.533)	(1.468,2.544)	(1.588,2.633)	(1.655,2.644)
	$\gamma = 0.5$	(0.381,0.620)	(0.369,0.617)	(0.384,0.625)	(0.381,0.599)	$\gamma = -0.5$	(-0.558,-0.229)	(-0.556,-0.278)	(-0.543,-0.289)	(-0.539,-0.271)
	$\omega = 2.5$	(2.442,3.912)	(2.432, 3.654)	(2.498,3.845)	(2.461,3.599)	$\omega = -0.4$	(-0.443,-0.329)	(-0.446,-0.378)	(-0.448-0.389)	(-0.441,-0.371)
150	$\beta_1 = 1.5$	(1.348,1.557)	(1.366,1.528)	(1.507,1.633)	(1.555,1.611)	$\beta_1 = 0.2$	(0.149,0.511)	(0.154,0.499)	(0.164,0.398)	(0.165,0.388)
	$\beta_2 = 0.9$	(0.814,0.969)	(0.825,0.975)	(0.819,0.961)	(0.829,0.971)	$\beta_2 = 2$	(1.438,2.557)	(1.476,2.528)	(1.617,2.633)	(1.665,2.611)
	$\theta_1 = 1.5$	(1.438,1.567)	(1.476,1.538)	(1.487,1.643)	(1.479,1.621)	$\theta_1 = 0.2$	(0.114,0.369)	(0.125,0.375)	(0.119,0.361)	(0.129,0.381)
	$\theta_2 = 0.9$	(0.331,0.971)	(0.333,0.968)	(0.336,0.968)	(0.335,0.961)	$\theta_2 = 2$	(1.439,2.568)	(1.477,2.539)	(1.618,2.644)	(1.666,2.622)
	$\gamma = 0.5$	(0.387,0.617)	(0.371,0.616)	(0.389,0.611)	(0.388,0.599)	$\gamma = -0.5$	(-0.561,-0.231)	(-0.560,-0.283)	(-0.550,-0.295)	(-0.545,-0.280)
	$\omega = 2.5$	(2.443,3.823)	(2.449, 3.594)	(2.465,3.745)	(2.483,3.589)	$\omega = -0.4$	(-0.451,-0.331)	(-0.455,-0.383)	(-0.454,-0.395)	(-0.448,-0.380)

7. Real data application

In this study, we focus on two distinct real-world datasets to assess and compare the performance of several copula models. The copulas considered in this research, appearing in the comparison tables, are IFGM, Cambanis (CAM) [32, 33], general setting Cambanis (GCAM) [34, 35], FGM [9], Huang-Kotz type 1 (HK1), Huang-Kotz type 2 (HK2) [36], Huang-Kotz type 3 (HK3), Kimeldorf and Sampson (KS), and Gumbel-Barnett (GumB) [37]. All the preceding abbreviations of copulas will be used in tables for comparison purposes.

The 1st dataset (Black cherry trees): The analysis begins with the classic black cherry trees dataset (Hand [38]), containing measurements from 31 trees. Following Saali et al. [39], we model the dependence between diameter and volume using the IFGM-CD framework.

The ML estimates for the fitted model are $\hat{\theta}_1 = 0.0013$, $\hat{\beta}_1 = 0.7072$, $\hat{\theta}_2 = 0.0094$, $\hat{\beta}_2 = 0.4324$, $\hat{\gamma} = 0.6990$, and $\hat{\omega} = 2.0643$. Moreover, the estimated Pearson correlation coefficient ρ and Kendall's tau correlation τ_c for the IFGM-CD model are approximately 0.398 and 0.23, respectively. On the other hand, those correlations for the FGM model are approximately 0.273 and 0.155, respectively.

As summarized in Tables 5 and 6, the IFGM model demonstrates superior performance based on log-likelihood and information criteria values when compared to alternative copulas. The nature of the dependence is further illustrated in Figure 3, where the joint density surface and contour plots reveal a distinct curvilinear relationship between the two variables.

The 2nd dataset (NFL scoring times): The second empirical analysis utilizes a bivariate dataset comprising initial and subsequent scoring times (X, Y) from 42 national football league (NFL) matches. This data, which has been examined in earlier statistical literature (Meintanis [40]; Khames and Mokhlis [41]; and El-Morshedy et al. [42]), is particularly valuable for testing models against complex real-world dependence patterns.

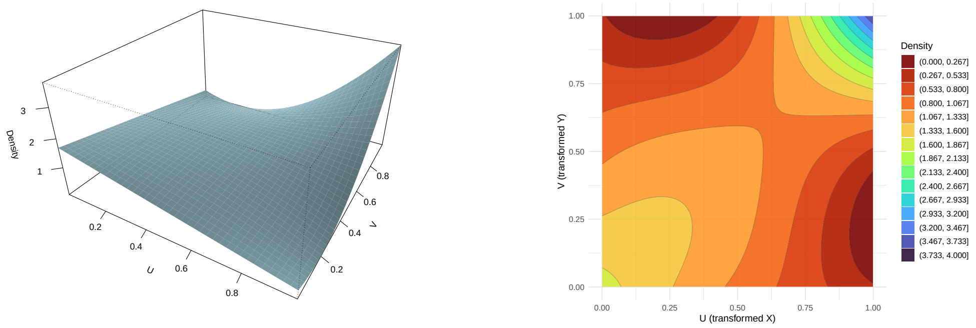
Application of the IFGM copula yielded the following ML estimates: $\hat{\theta}_1 = 0.0653$, $\hat{\beta}_1 = 0.4365$, $\hat{\theta}_2 = 0.0694$, $\hat{\beta}_2 = 0.3709$, $\hat{\gamma} = 0.9331$, and $\hat{\omega} = 1.4776$. Moreover, the estimated Pearson correlation coefficient ρ_c and Kendall's tau correlation τ for the IFGM-CD model are approximately 0.42 and 0.25, respectively. On the other hand, those correlations for the FGM model are approximately 0.29 and 0.207, respectively. The model demonstrates a strong fit, as evidenced by comparative measures in Table 7 and parameter estimates in Table 8. The resultant joint density surface and contour plots (shown in Figure 4) provide a clear visual representation of the asymmetric dependence structure captured by the model.

Table 5. Model comparison for the 1st dataset.

Model	ℓ	k	AIC	AICc	BIC	HQIC	CAIC
IFGM	-199.7549	6	411.5097	415.0097	420.1136	414.3144	426.1136
HK1	-200.9548	6	413.9096	417.4096	422.5135	416.7143	428.5135
HK3	-201.8551	6	415.7103	419.2103	424.3142	418.5150	430.3142
FGM	-204.2121	5	418.4241	420.8241	425.5941	420.7614	430.5941
CAM	-204.2120	6	420.4241	423.9241	429.0280	423.2287	435.0280
HK2	-204.2781	6	420.5563	424.0563	429.1602	423.3609	435.1602
KS	-205.1822	6	422.3643	425.8643	430.9683	425.1690	436.9683
GCAM	-207.8392	7	429.6783	434.5479	439.7162	432.9504	446.7162
GumB	-211.6363	5	433.2726	435.6726	440.4425	435.6098	445.4425

Table 6. Parameter estimates for selected copula models based on CD.

Model	θ_1	β_1	θ_2	β_2	γ	ω	λ
IFGM	0.0013	0.7072	0.0094	0.4324	0.6990	2.0643	—
HK1	0.0014	0.7043	0.0107	0.4291	0.2273	4.3994	—
HK3	0.0012	0.7093	0.0089	0.4335	0.5000	1.0000	—
FGM	0.0010	0.7133	0.0079	0.4365	1.0000	—	—
CAM	0.0010	0.7133	0.0079	0.4365	0.0000	1.0000	—
HK2	0.0010	0.7131	0.0079	0.4364	0.9900	1.0001	—
KS	0.0012	0.7082	0.0088	0.4328	0.5000	0.1250	—
GCAM	0.0084	0.6198	0.0383	0.3728	-0.9999	-0.9999	0.9999
GumB	0.0016	0.6940	0.0108	0.4229	0.0000	—	—



(a) 3D surface of the IFGM-CD **(b)** Contour plot of the IFGM-CD

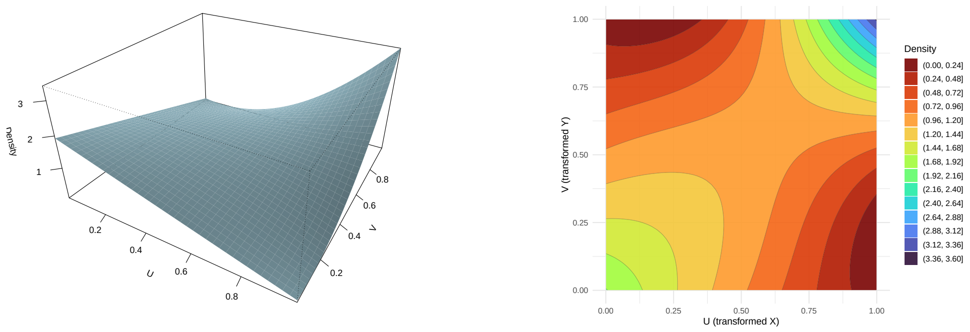
Figure 3. Plots of the IFGM-CD for the 1st dataset.

Table 7. Model comparison for the 2nd dataset.

Model	ℓ	k	AIC	AICc	BIC	HQIC	CAIC
IFGM	-274.4441	6	560.8882	563.3588	571.1696	564.6321	577.1696
FGM	-276.4464	5	562.8928	564.6070	571.4606	566.0127	576.4606
HK1	-275.9850	6	563.9700	566.4406	574.2515	567.7140	580.2515
HK3	-276.0544	6	564.1089	566.5794	574.3903	567.8528	580.3903
CAM	-276.4464	6	564.8927	567.3633	575.1742	568.6367	581.1742
HK2	-276.5005	6	565.0009	567.4715	575.2823	568.7448	581.2823
GCAM	-276.3094	7	566.6187	570.0127	578.6137	570.9867	585.6137
KS	-277.6436	6	567.2872	569.7578	577.5687	571.0312	583.5687
GumB	-282.8530	5	575.7060	577.4203	584.2739	578.8259	589.2739

Table 8. Parameter estimates for selected copula models based on CD.

Model	θ_1	β_1	θ_2	β_2	γ	ω	λ
IFGM	0.0653	0.4365	0.0694	0.3709	0.9331	1.4776	—
FGM	0.0623	0.4398	0.0661	0.3743	1.0000	—	—
HK1	0.0645	0.4377	0.0685	0.3722	0.6056	1.6511	—
HK3	0.0653	0.4368	0.0693	0.3713	0.5000	1.0000	—
CAM	0.0623	0.4398	0.0661	0.3743	-0.0000	1.0000	—
HK2	0.0623	0.4398	0.0661	0.3743	0.9900	1.0001	—
GCAM	0.1331	0.3915	0.1415	0.3309	-0.7310	-0.7288	0.9973
KS	0.0628	0.4386	0.0665	0.3733	0.5000	0.1250	—
GumBA	0.0635	0.4353	0.0666	0.3708	0.0000	—	—



(a) 3D surface of the IFGM-CD **(b)** Contour plot of the IFGM-CD

Figure 4. Plot of the IFGM-CD for the 2nd dataset.

8. Conclusions

This paper introduced the bivariate IFGM-CD, a flexible lifetime model created by combining the IFGM copula with Chen marginals. The model is particularly suited for reliability and survival applications, especially when hazard rates show bathtub-shaped behavior. It addresses the shortcomings of classical FGM-based models in capturing moderate dependence. Key properties of the IFGM-CD were derived, including joint and conditional distributions, moments, and the bivariate reliability function.

The IFGM copula significantly extends the range of dependence that can be achieved within the standard FGM framework. This is evident in the higher values of Pearson's correlation and Kendall's tau. Parameter estimation using ML and Bayesian methods was studied through Monte Carlo simulations. The results showed that Bayesian estimators usually perform better with small samples. Both methods are consistent for larger samples. Bootstrap confidence intervals proved to be more accurate than asymptotic intervals.

Applications to black cherry trees and NFL score times datasets showed the practical benefits of the IFGM-CD. In both scenarios, the proposed model offered better fits than competing copula models based on standard goodness-of-fit criteria and successfully captured complex dependence structures. These findings demonstrate that the IFGM-CD is a valuable tool for modeling dependent lifetime data in reliability and survival analysis.

The proposed IFGM-CD model can be naturally extended to accommodate censored data, which is common in reliability engineering and medical survival analysis. In reliability studies, components may be removed from testing before failure (right-censoring), while medical studies often involve patients lost to follow-up or surviving beyond the study period. The likelihood framework presented in Section 5 can be adapted to handle such censoring mechanisms by modifying the likelihood function to account for incomplete observations. For instance, under right-censoring, the contribution of a censored observation (x_i, y_i) would involve the survival function $S(x_i, y_i)$ rather than the density $f(x_i, y_i)$. Bayesian inference via the Markov chain Monte Carlo would similarly incorporate censoring through the appropriate likelihood terms. Such extensions would broaden the applicability of the IFGM-CD model to real-world scenarios where complete data is unavailable.

Author contributions

The authors contributed equally to the paper. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors are grateful to the editor and anonymous referees for their insightful comments and suggestions, which helped to improve the paper's presentation.

Conflict of interest

The authors have declared that no competing interests exist.

References

1. R. B. Nelsen, *An Introduction to Copulas*, New York: Springer-Verlag, 2006. https://doi.org/10.1007/0-387-28678-0
2. A. Sklar, Random variables, joint distributions functions, and copulas, *Kybernetica*, **9** (1973), 449–460.
3. A. K. Gupta, C. F. Wong, On three and five parameter bivariate beta distributions, *Metrika*, **32** (1985), 85–91. https://doi.org/10.1007/BF01897803
4. M. K. H. Hassan, C. Chesneau, Bivariate generalized half-logistic distribution: Properties and its application in household financial affordability in KSA, *Math. Comput. Appl.*, **27** (2022), 72. https://doi.org/10.3390/mca27040072
5. S. O. Susam, A multi-parameter generalized Farlie-Gumbel-Morgenstern bivariate copula family via Bernstein polynomial, *Hacettepe J. Math. Stat.*, **51** (2022), 618–631. https://doi.org/10.15672/hujms.993698
6. Y. Aliyu, U. Usman, On bivariate Nadarajah-Haghighi distribution derived from FGM copula in the presence of covariates, *J. Nigerian Soc. Phys. Sci.*, **5** (2023), 871. https://doi.org/10.46481/jnsps.2023.871
7. C. Blier-Wong, H. Cossette, E. Marceau, Exchangeable FGM copulas, *Adv. Appl. Prob.*, **56** (2024), 205–234. https://doi.org/10.1017/apr.2023.19
8. I. Dimitriou, On the overlap times in queues with dependence under a FGM copula, preprint paper, 2025. https://doi.org/10.48550/arXiv.2509.14100
9. G. M. Mansour, H. M. Barakat, I. A. Husseiny, M. Nagy, A. H. Mansi, M. A. Alawady, Measures of cumulative residual Tsallis entropy for concomitants of generalized order statistics based on the Morgenstern family with application to medical data, *Math. Bio. Eng.*, **22** (2025a), 1572–1597. https://doi.org/10.3934/mbe.2025058
10. J. S. Huang, S. Kotz, Correlation structure in iterated Farlie-Gumbel-Morgenstern distributions, *Biometrika*, **71** (1984), 633–636. https://doi.org/10.1093/biomet/71.3.633
11. M. A. Abd Elgawad, H. M. Barakat, A. F. Hashem, G. M. Mansour, I. A. Husseiny, M. A. Alawady, et al., The bivariate Weibull distribution based on the GFGM copula, *AIMS Math.*, **10** (2025), 27862–27897. https://doi.org/10.3934/math.20251224
12. H. M. Barakat, E. M. Nigm, I. A. Husseiny, Measures of information in order statistics and their concomitants for the single iterated Farlie-Gumbel-Morgenstern bivariate distribution, *Math. Popul. Stud.*, **28** (2021), 154–175. https://doi.org/10.1080/08898480.2020.1767926
13. H. M. Barakat, E. M. Nigm, M. A. Alawady, I. A. Husseiny, Concomitants of order statistics and record values from iterated FGM type bivariate-generalized exponential distribution, *Revstat. Stat. J.*, **19** (2021), 291–307. https://doi.org/10.2991/jsta.d.190822.001
14. N. Chandra, A. James, F. Domma, H. Rehman, Bivariate iterated Farlie-Gumbel-Morgenstern stress-strength reliability model for Rayleigh margins: Properties and estimation, *Stat. Theory Rel. Fields*, **8** (2024), 315–334. https://doi.org/10.1080/24754269.2024.2398987
15. I. A. Husseiny, M. A. Alawady, S. A. Alyami, M. A. Abd Elgawad, Measures of extropy based on concomitants of generalized order statistics under a general framework from iterated Morgenstern family, *Mathematics*, **11** (2023), 1377. https://doi.org/10.3390/math11061377

16. D. Kundu, Bayesian inference and life testing plan for the Weibull distribution in presence of progressive censoring, *Technometrics*, **50** (2008), 144–154. https://doi.org/10.1198/004017008000000217

17. D. Kundu, B. Pradhan, Bayesian inference and life testing plans for generalized exponential distribution, *Sci. China Ser. A: Math.*, **52** (2009), 1373–1388. https://doi.org/10.1007/s11425-009-0085-8

18. S. Singh, Y. M. Tripathi, S. J. Wu, On estimating parameters of a progressively censored lognormal distribution, *J. Stat. Comput. Simul.*, **85** (2015), 1071–1089. https://doi.org/10.1080/00949655.2013.861838

19. D. J. F. Davis, An analysis of some failure data, *J. Amer. Stat. Ass.*, **47** (1952), 113–150. https://doi.org/10.1080/01621459.1952.10501160

20. H. W. Block, T. H. Savits, Burn-in, *Statist. Sci.*, **12** (1997), 1–19. https://doi.org/10.1214/ss/1029963258

21. M. Bebbington, C. D. Lai, R. Zitikis, Useful periods for lifetime distributions with bathtub shaped hazard rate functions, *IEEE Trans. Reliab.*, **5** (2006), 245–251. https://doi.org/10.1109/TR.2001.874943

22. M. Bebbington, C. D. Lai, R. Zitikis, Modeling human mortality using mixtures of bathtub shaped failure distributions, *J. Theor. Biol.*, **245** (2007), 528–538. https://doi.org/10.1016/j.jtbi.2006.11.011

23. Z. Chen, A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function, *Stat. Prob. Lett.*, **49** (2000), 155–161. https://doi.org/10.1016/S0167-7152(00)00044-4

24. E. S. A. El-Sherpieny, H. Z. Muhammed, E. M. Almetwally, Bivariate Chen distribution based on copula function: Properties and application of diabetic Nephropathy, *J. Stat. Theory Pract.*, **16** (2022), 54. https://doi.org/10.1007/s42519-022-00275-7

25. N. Sreelakshmi, An introduction to copula-based bivariate reliability concepts, *Commun. Stat. Theory Meth.*, **47** (2018), 996–1012. https://doi.org/10.1080/03610926.2017.1316396

26. M. A. Abd Elgawad, M. A. Alawady, H. M. Barakat, G. M. Mansour, I. A. Husseiny, A. F. Hashem, et al., Bivariate power Lomax Sarmanov distribution: Statistical properties, Reliability measures, and Parameter estimation, *Alex. Eng. J.*, **113** (2025), 593–610. https://doi.org/10.1016/j.aej.2024.10.074

27. S. Dey, S. Singh, Y. M. Tripathi, A. Asgharzadeh, Estimation and prediction for a progressively censored generalized inverted exponential distribution, *Stat. Methodol.*, **32** (2016), 185–202. https://doi.org/10.1016/j.stamet.2016.05.007

28. G. M. Mansour, M. A. Abd Elgawad, A. S. Al-Moisheer, H. M. Barakat, M. A. Alawady, I. A. Husseiny, et al., Bivariate Epanechnikov-Weibull distribution based on sarmanov copula: Properties, simulation, and uncertainty measures with applications, *AIMS Math.*, **10** (2025), 12689–12725. https://doi.org/10.3934/math.2025572

29. X. Jia, D. Wang, P. Jiang, B. Guo, Inference on the reliability of weibull distribution with multiply type-I censored data, *Reliab. Eng. Syst. Saf.*, **150** (2016), 171–181. https://doi.org/10.1016/j.ress.2016.01.025

30. B. Efron, Bootstrap methods: Another look at the Jackknife, In: *Kotz, S., Johnson, N. L. (eds) Breakthroughs in Statistics*, New York: Springer, 1992. https://doi.org/10.1007/978-1-4612-4380-9-41
31. S. Kotz, Y. Lumelskii, M. Pensky, *The Stress-Strength Model and Its Generalizations: Theory and Applications*, New York: World Scientific, 2003. https://doi.org/10.1142/9789812564511-0001
32. M. A. Alawady, H. M. Barakat, T. S. Taher, I. A. Husseiny, Measures of extropy for k-record values and their concomitants based on Cambanis family, *J. Stat. Theory Pract.*, **19** (2025), 11. https://doi.org/10.1007/s42519-024-00423-1
33. M. A. Alawady, H. M. Barakat, G. M. Mansour, I. A. Husseiny, Information measures and concomitants of $k-$record values based on Sarmanov family of bivariate distributions, *Bull. Malays. Math. Sci. Soc.*, **46** (2023), 9. https://doi.org/10.1007/s40840-022-01396-9
34. H. M. Barakat, M. A. Alawady, T. S. Taher, I. A. Husseiny, Second-order concomitants of order statistics from bivariate Cambanis family: Some information measures–estimation, *Commun. Stat. Simul. Comput.*, **54** (2025), 3195–3221. https://doi.org/10.1080/03610918.2024.2344023
35. I. A. Husseiny, H. M. Barakat, T. S. Taher, M. A. Alawady, Fisher information in order statistics and their concomitants for Cambanis bivariate distribution, *Math. Slovaca*, **74** (2024), 501–520. https://doi.org/10.1515/ms-2024-0038
36. G. M. Mansour, H. M. Barakat, M. A. Alawady, M. A. Abd Elgawad, H. Alqifari, T. S. Taher, et al., Uncertainty measures for concomitants of upper k-record values based on the Huang-Kotz-Morgenstern type II family, *AIMS Math.*, **10** (2025), 29071–29106. https://doi.org/10.3934/math.20251279
37. V. Barnett, Some bivariate uniform distributions, *Commun. Stat. Theory Meth.*, **9** (1980), 453–461. https://doi.org/10.1080/03610928008827893
38. D. J. Hand, *A Handbook of Small Data Sets*, Boca Raton: Chapman & Hall/CRC, 1993.
39. T. Saali, M. Mesfioui, A. Shabri, Multivariate extension of Raftery copula, *Mathematics*, **11** (2023), 414. https://doi.org/10.3390/math11020414
40. S. G. Meintanis, Test of fit for Marshall-Olkin distributions with applications, *J. Stat. Plan. Inf.*, **137** (2007), 3954–3963. https://doi.org/10.1016/j.jspi.2007.04.013
41. S. K. Khames, N. A. Mokhlis, Bivariate general exponential models with stress-strength reliability application, *J. Egyp. Math. Soc.*, **28** (2020), 9. https://doi.org/10.1186/s42787-020-0069-y
42. M. El-Morshedy, M. S. Eliwa, M. H. Tahir, M. Alizadeh, R. El-Desokey, A. Al-Bossly, et al., A bivariate extension to exponentiated inverse flexible Weibull distribution: Shock model, features, and inference to model asymmetric data, *Symmetry*, **15** (2023), 411. https://doi.org/10.3390/sym15020411

AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)