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Research article

Identities with inverses on matrix rings over a division ring of characteristic two

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Abstract: Let $\mathcal{D}$ be a division ring with $\mathrm{char}(\mathcal{D}) = 2$. Let $\mathcal{R} = M_n(\mathcal{D})$ be the ring of all $n \times n$ matrices over $\mathcal{D}$, where $n \geq 2$. Let $f, g : \mathcal{R} \to \mathcal{R}$ be two additive maps such that

$$f(A) + Ag(A^{-1})A = 0$$

for all invertible $A \in \mathcal{R}$. If $|\mathcal{D}| \neq 2, 4, 8$, then

$$f(A) = AQ + \delta(A) \quad \text{and} \quad g(A) = QA + \delta(A)$$

for all invertible $A \in \mathcal{R}$, where $A \in \mathcal{R}$ and δ is a derivation of $\mathcal{R}$. This result affirmatively answers a question posed by Argac et al. under a mild condition.

Keywords: identities with inverses on rings; derivation; matrix ring; division ring
Mathematics Subject Classification: 16R60, 16K40

1. Introduction

Identities involving all elements on rings and algebras have been investigated systematically, that is, the theory of function identities on rings (see [3]). In the last decades, many results on identities involving inverses on rings and algebras have been obtained (see [5, 6, 12]).

Let $\mathcal{D}$ be a division ring and let $\mathcal{R} = M_n(\mathcal{D})$ be the set of all $n \times n$ matrices over $\mathcal{D}$. By $\mathcal{R}^\times$ we denote the set of all invertible matrices of $\mathcal{R}$.

In 2018, Catalano [4] characterized two additive maps f, g of $\mathcal{R}$ with $\mathrm{char}(\mathcal{D}) \neq 2, 3$ such that

$$f(A)A^{-1} + Ag(A^{-1}) = 0 \tag{1.1}$$

for all $A \in \mathcal{R}^\times$. In 2020, Argac et al. [1] removed the assumption $\mathrm{char}(\mathcal{D}) \neq 3$ as follows:

Theorem 1.1. *[1, Theorem 1.5] Suppose that char($\mathcal{D}$) $\neq 2$. Let $f, g : \mathcal{R} \to \mathcal{R}$ be two additive maps such that*

$$f(A)A^{-1} + Ag(A^{-1}) = 0$$

for all $A \in \mathcal{R}^{\times}$. Then,

$$f(A) = AQ + \delta(A) \quad \textit{and} \quad g(y) = QA + \delta(A)$$

for all $A \in \mathcal{R}$, where $Q \in \mathcal{R}$ and δ is a derivation of $\mathcal{R}$.

They proposed the following question:

Question 1.1. *[1, Question 1.9] Is Theorem 1.1 still true if char($\mathcal{D}$) = 2?*

Our goal of the paper is to affirmatively answer Question 1.1 under a mild condition as follows:

Theorem 1.2. *Suppose that char($\mathcal{D}$) = 2. Let $f, g : \mathcal{R} \to \mathcal{R}$ be two additive maps such that*

$$f(A)A^{-1} + Ag(A^{-1}) = 0$$

for all $A \in \mathcal{R}^{\times}$. If $|\mathcal{D}| \neq 2, 4, 8$, then

$$f(A) = AQ + \delta(A) \quad \textit{and} \quad g(A) = QA + \delta(A)$$

for all $A \in \mathcal{R}$, where $Q \in \mathcal{R}$ and δ is a derivation of $\mathcal{R}$.

In Section 2, we give some technical results. In Section 3, we give the proof of Theorem 1.2. In the last section, we introduce some relevant works on functional equations on alternative division algebras

We remark that the proof of Theorem 1.2 does not hold for $|D| = 2, 4, 8$. It is an interesting topic to discuss these special cases.

2. Some technical results

Throughout this paper, we always assume that char($\mathcal{D}$) = 2. Let $\mathcal{F} = \{0, 1\}$ be the prime field $\mathbb{F}_2$ of $\mathcal{D}$ and $\mathcal{R} = M_n(\mathcal{D})$. For the sake of convenience, we use $\alpha \in \mathcal{D}$ to represent $\alpha \cdot 1$, where 1 is the identity matrix of $\mathcal{R}$.

It is well-known that every finite division ring is a field (Wedderburn's theorem, see [2, Theorem 1.38]). By the theory of finite fields, we have that "$|\mathcal{D}| \neq 2, 4, 8$" in Theorem 1.2 is equivalent to "$\dim_{\mathcal{F}}(\mathcal{D}) \geq 4$".

Lemma 2.1. *Suppose that $dim_{\mathcal{F}}(\mathcal{D}) \geq 4$. For any $a, b \in \mathcal{D}$ and $1 \leq i, j, k, l \leq n$, we set $A = ae_{ij} + be_{kl} \in \mathcal{R}$. Then, there exist $A_1, A_2 \in \mathcal{R}^{\times}$ such that*

$$A_i + 1, A_i + A, A_i + A + 1, A_1 + A_2, A_1 + A_2 + 1, A_1 + A_2 + A, A_1 + A_2 + A + 1 \in \mathcal{R}^{\times}$$

for $i = 1, 2$.

Proof. We discuss two cases as follows:

Case 1. $a, b, 1$ are $\mathcal{F}$-independent. Since $dim_{\mathcal{F}}(\mathcal{D}) \geq 4$, we have that there exists $d \in \mathcal{D}$ such that $d, a, b, 1$ are $\mathcal{F}$-independent.

Subcase 1.1. $i = j = k = l$ and $A = (a+b)e_{ii}$. We set

$$A_1 = d \quad \text{and} \quad A_2 = d + a.$$

Note that

$$A_i, A_i + 1, A_i + A, A_i + A + 1 \in \mathcal{R}^\times$$

for $i = 1, 2$. We get

$$\begin{aligned} A_1 + A_2 &= a \in \mathcal{R}^\times; \\ A_1 + A_2 + 1 &= a + 1 \in \mathcal{R}^\times; \\ A_1 + A_2 + A &= \sum_{\substack{1 \le j \le n \\ j \ne i}} ae_{jj} + be_{ii} \in \mathcal{R}^\times; \\ A_1 + A_2 + A + 1 &= \sum_{\substack{1 \le j \le n \\ j \ne i}} (a+1)e_{jj} + (b+1)e_{ii} \in \mathcal{R}^\times. \end{aligned}$$

Subcase 1.2. $i = j$, $k = l$, $i \neq k$, and $A = ae_{ii} + be_{kk}$. We set

$$A_1 = d \quad \text{and} \quad A_2 = d + a + b.$$

It is clear that

$$A_i, A_i + 1, A_i + A, A_i + A + 1 \in \mathcal{R}^\times$$

for $i = 1, 2$. Note that $A_1 + A_2 = a + b$. It is clear that

$$A_1 + A_2, A_1 + A_2 + 1, A_1 + A_2 + A, A_1 + A_2 + A + 1 \in \mathcal{R}^\times.$$

Subcase 1.3. $i = j$, $k \neq l$, and $A = ae_{ii} + be_{kl}$. It is clear that

$$A_1 = d \quad \text{and} \quad A_2 = d + b.$$

It is clear that

$$A_i, A_i + 1, A_i + A, A_i + A + 1 \in \mathcal{R}^\times$$

for $i = 1, 2$. Note that $A_1 + A_2 = b$. We get

$$A_1 + A_2, A_1 + A_2 + 1, A_1 + A_2 + A, A_1 + A_2 + A + 1 \in \mathcal{R}^\times.$$

Subcase 1.4. $i \neq j$, $k = l$, and $A = ae_{ij} + be_{kk}$. Set

$$A_1 = d \quad \text{and} \quad A_2 = d + a.$$

It is clear that

$$A_i, A_i + 1, A_i + A, A_i + A + 1 \in \mathcal{R}^\times$$

for $i = 1, 2$. Note that $A_1 + A_2 = a$. It is clear that

$$A_1 + A_2, A_1 + A_2 + 1, A_1 + A_2 + A, A_1 + A_2 + A + 1 \in R^\times.$$

Subcase 1.5. $i \neq j, k \neq l, (i, j) \neq (l, k)$, and $A = ae_{ij} + be_{kl}$. We set

$$A_1 = d \quad \text{and} \quad A_2 = d + a.$$

It is easy to check that

$$A_i, A_i + 1, A_i + A, A_i + A + 1, A_1 + A_2, A_1 + A_2 + 1, A_1 + A_2 + A, A_1 + A_2 + A + 1 \in \mathcal{R}^{\times}.$$

Subcase 1.6. $i \neq j, k \neq l, (i, j) = (l, k)$, and $A = ae_{ij} + be_{ji}$. We consider the case of $i < j$. The other case can be discussed similarly. Set

$$\begin{aligned} A_1 &= d + ae_{ij} \\ A_2 &= \sum_{\substack{1 \leq s \leq n \\ s \neq j}} (d + a)e_{ss} + (d + b + 1)e_{jj} + ae_{ij}. \end{aligned}$$

It is easy to check that

$$A_1, A_1 + 1, A_2, A_2 + 1 \in \mathcal{R}^{\times}.$$

Note that

$$\begin{aligned} A_1 + A &= d + be_{ji}; \\ A_1 + A + 1 &= d + 1 + be_{ji}; \\ A_2 + A &= \sum_{\substack{1 \leq s \leq n \\ s \neq j}} (d + a)e_{ss} + (d + b + 1)e_{jj} + be_{ji}; \\ A_2 + A + 1 &= \sum_{\substack{1 \leq s \leq n \\ s \neq j}} (d + a + 1)e_{ss} + (d + b)e_{jj} + be_{ji}. \end{aligned}$$

It is easy to check that

$$A_i + A, A_i + A + 1 \in \mathcal{R}^{\times}$$

for $i = 1, 2$. Note that

$$\begin{aligned} A_1 + A_2 &= \sum_{\substack{1 \leq s \leq n \\ s \neq j}} ae_{ss} + (b + 1)e_{jj}; \\ A_1 + A_2 + 1 &= \sum_{\substack{1 \leq s \leq n \\ s \neq j}} (a + 1)e_{ss} + be_{jj}. \end{aligned}$$

It is easy to check that

$$A_1 + A_2, A_1 + A_2 + 1 \in \mathcal{R}^{\times}.$$

Note that

$$\begin{aligned} A_1 + A_2 + A &= \sum_{\substack{1 \leq s \leq n \\ s \neq j}} ae_{ss} + (b + 1)e_{jj} + ae_{ij} + be_{ji}; \\ A_1 + A_2 + A + 1 &= \sum_{\substack{1 \leq s \leq n \\ s \neq j}} (a + 1)e_{ss} + be_{jj} + ae_{ij} + be_{ji}. \end{aligned}$$

It is easy to check that

$$(1 + ba^{-1}e_{ji})(A_1 + A_2 + A) = \sum_{\substack{1\le s\le n\\ s\ne j}} ae_{ss} + e_{jj} + ae_{ij} \in \mathcal{R}^{\times}.$$

Since $1 + ba^{-1}e_{ji} \in \mathcal{R}^{\times}$, we get that $A_1 + A_2 + A \in \mathcal{R}^{\times}$. Similarly, we get

$$\begin{aligned}&(1 + b(a + 1)^{-1}e_{ji})(A_1 + A_2 + A + 1)\\ &\qquad = \sum_{\substack{1\le s\le n\\ s\ne j}} (a + 1)e_{ss} + (b - b(a + 1)^{-1}a)e_{jj} + ae_{ij} \in \mathcal{R}^{\times}.\end{aligned}$$

Since $1 + b(a + 1)^{-1}e_{ji} \in \mathcal{R}^{\times}$, we get that $A_1 + A_2 + A + 1 \in \mathcal{R}^{\times}$. We get that $A_1 + A_2 + A + 1 \in \mathcal{R}^{\times}$.

Case 2. $a, b, 1$ are $\mathcal{F}$-dependent over F. Since $dim_{\mathcal{F}}(\mathcal{D}) \ge 4$, we have that there exists $d_1 \in \mathcal{D}$ such that $d_1 \notin L(1, a, b)$, where $L(1, a, b)$ is a subspace of $\mathcal{D}$ generated by $1, a, b$. It is clear that

$$dim_{\mathcal{F}}(L(1, a, b, d_1)) \le 3,$$

where $L(1, a, b, d_1)$ is a subspace of $\mathcal{D}$ generated by $1, a, b, d_1$. We get that there exists $d_2 \in \mathcal{D}$ such that $d_2 \notin L(1, a, b, d_1)$. We consider the following subcases:

Subcase 2.1. $i = j = k = l$ and $A = (a + b)e_{ii}$. Set

$$A_1 = d_1 \quad \text{and} \quad A_2 = d_2.$$

It is easy to check that

$$A_i, A_i + 1, A_i + A, A_i + A + 1, A_1 + A_2, A_1 + A_2 + 1, A_1 + A_2 + A, A_1 + A_2 + A + 1 \in \mathcal{R}^{\times}$$

for $i = 1, 2$.

Subcase 2.2. $i = j$, $k = l$, $i \ne k$, and $A = ae_{ii} + be_{kk}$. Set

$$A_1 = d_1 \quad \text{and} \quad A_2 = d_2.$$

It is easy to check that

$$A_i, A_i + 1, A_i + A, A_i + A + 1, A_1 + A_2, A_1 + A_2 + 1, A_1 + A_2 + A, A_1 + A_2 + A + 1 \in \mathcal{R}^{\times}$$

for $i = 1, 2$.

Subcase 2.3. $i = j$, $k \ne l$, and $A = ae_{ii} + be_{kl}$. Set

$$A_1 = d_1 \quad \text{and} \quad A_2 = d_2.$$

It is easy to check that

$$A_i, A_i + 1, A_i + A, A_i + A + 1, A_1 + A_2, A_1 + A_2 + 1, A_1 + A_2 + A, A_1 + A_2 + A + 1 \in \mathcal{R}^{\times}$$

for $i = 1, 2$.

Subcase 2.4. $i \neq j$, $k = l$, and $T = ae_{ij} + be_{kk}$. Set

$$A_1 = d_1 \quad \text{and} \quad A_2 = d_2.$$

It is easy to check that

$$A_i, A_i + 1, A_i + A, A_i + A + 1, A_1 + A_2, A_1 + A_2 + 1, A_1 + A_2 + A, A_1 + A_2 + A + 1 \in \mathcal{R}^{\times}$$

for $i = 1, 2$.

Subcase 2.5. $i \neq j$, $k \neq l$, and $(i, j) \neq (l, k)$. Note that $T = ae_{ij} + be_{kl}$. Set

$$A_1 = d_1 \quad \text{and} \quad A_2 = d_2.$$

It is easy to check that

$$A_i, A_i + 1, A_i + A, A_i + A + 1, A_1 + A_2, A_1 + A_2 + 1, A_1 + A_2 + A, A_1 + A_2 + A + 1 \in \mathcal{R}^{\times}$$

for $i = 1, 2$.

Subcase 2.6. $i \neq j$, $k \neq l$, and $(i, j) = (l, k)$. Note that $A = ae_{ij} + be_{ji}$. We may assume that $i < j$. The case of $i > j$ can be discussed analogously. Suppose that either $a = 0$ or $b = 0$. Set

$$A_1 = d_1 \quad \text{and} \quad A_2 = d_2.$$

We easily check that

$$A_i, A_i + 1, A_i + A, A_i + A + 1, A_1 + A_2, A_1 + A_2 + 1, A_1 + A_2 + A, A_1 + A_2 + A + 1 \in \mathcal{R}^{\times}.$$

We now consider the case of $a, b \neq 0$. Suppose first that $a \neq 1$ and $b \neq 1$. We set

$$\begin{aligned} A_1 &= d_1 + ae_{ij} \\ A_2 &= \sum_{\substack{1 \leq s \leq n \\ s \neq j}} (d_1 + a)e_{ss} + (d_1 + b + 1)e_{jj} + ae_{ij}. \end{aligned}$$

It is clear that

$$A_1, A_1 + 1, A_2, A_2 + 1 \in \mathcal{R}^{\times}.$$

Hence

$$\begin{aligned} A_1 + A &= d_1 + be_{ji} \in \mathcal{R}^{\times}; \\ A_1 + A + 1 &= d_1 + 1 + be_{ji} \in \mathcal{R}^{\times}; \\ A_2 + A &= \sum_{\substack{1 \leq s \leq n \\ s \neq j}} (d_1 + a)e_{ss} + (d_1 + b + 1)e_{jj} + be_{ji} \in \mathcal{R}^{\times}; \\ A_2 + A + 1 &= \sum_{\substack{1 \leq s \leq n \\ s \neq j}} (d_1 + a + 1)e_{ss} + (d_1 + b)e_{jj} + be_{ji} \in \mathcal{R}^{\times}. \end{aligned}$$

It is easy to check that

$$A_1 + A_2 = \sum_{\substack{1\le s\le n\\ s\neq j}} ae_{ss} + (b+1)e_{jj} \in \mathcal{R}^{\times}$$
$$A_1 + A_2 + 1 = \sum_{\substack{1\le s\le n\\ s\neq j}} (a+1)e_{ss} + be_{jj} \in \mathcal{R}^{\times}.$$

Note that

$$A_1 + A_2 + A = \sum_{\substack{1\le s\le n\\ s\neq j}} ae_{ss} + (b+1)e_{jj} + ae_{ij} + be_{ji};$$
$$A_1 + A_2 + A + 1 = \sum_{\substack{1\le s\le n\\ s\neq j}} (a+1)e_{ss} + be_{jj} + ae_{ij} + be_{ji}.$$

It is clear that

$$(1 + ba^{-1}e_{ji})(A_1 + A_2 + A) = \sum_{\substack{1\le s\le n\\ s\neq j}} ae_{ss} + e_{jj} + ae_{ij} \in \mathcal{R}^{\times}.$$

Since $1 + ba^{-1}e_{ji} \in \mathcal{R}^{\times}$, we get that $A_1 + A_2 + A \in \mathcal{R}^{\times}$. It is easy to check

$$\begin{aligned}&(1 + b(a+1)^{-1}e_{ji})(A_1 + A_2 + A + 1)\\ &\qquad = \sum_{\substack{1\le s\le n\\ s\neq j}} (a+1)e_{ss} + (b - b(a+1)^{-1}a)e_{jj} + ae_{ij} \in \mathcal{R}^{\times}.\end{aligned}$$

Since $1 + b(a+1)^{-1}e_{ji} \in \mathcal{R}^{\times}$, we get that $A_1 + A_2 + A + 1 \in \mathcal{R}^{\times}$.

Suppose next that either $a = 1$ or $b = 1$. We only consider the case of $a = 1$. The case of $b = 1$ can be discussed analogously. We set

$$A_1 = d_1 + e_{ij}$$
$$A_2 = \sum_{\substack{1\le s\le n\\ s\neq j}} (d_2+1)e_{ss} + (d_2+b)e_{jj} + e_{ij}.$$

It is clear that

$$A_1, A_1 + 1, A_2, A_2 + 1 \in \mathcal{R}^{\times}.$$

Furthermore, we get

$$A_1 + A = d_1 + be_{ji} \in \mathcal{R}^{\times};$$
$$A_1 + A + 1 = d_1 + 1 + be_{ji} \in \mathcal{R}^{\times};$$
$$A_2 + A = \sum_{\substack{1\le s\le n\\ s\neq j}} (d_2+1)e_{ss} + (d_2+b)e_{jj} + be_{ji} \in \mathcal{R}^{\times};$$
$$A_2 + A + 1 = \sum_{\substack{1\le s\le n\\ s\neq j}} d_2e_{ss} + (d_2+b+1)e_{jj} + be_{ji} \in \mathcal{R}^{\times};$$

$$A_1 + A_2 = \sum_{\substack{1 \le s \le n \\ s \ne j}} (d_1 + d_2 + 1)e_{ss} + (d_1 + d_2 + b)e_{jj} \in \mathcal{R}^{\times};$$
$$A_1 + A_2 + 1 = \sum_{\substack{1 \le s \le n \\ s \ne j}} (d_1 + d_2)e_{ss} + (d_1 + d_2 + b + 1)e_{jj} \in \mathcal{R}^{\times}.$$

Note that

$$A_1 + A_2 + A = \sum_{\substack{1 \le s \le n \\ s \ne j}} (d_1 + d_2 + 1)e_{ss} + (d_1 + d_2 + b)e_{jj} + e_{ij} + be_{ji};$$
$$A_1 + A_2 + A + 1 = \sum_{\substack{1 \le s \le n \\ s \ne j}} (d_1 + d_2)e_{ss} + (d_1 + d_2 + b + 1)e_{jj} + e_{ij} + be_{ji}.$$

It is easy to check that

$$\begin{aligned}&(1 + b(d_1 + d_2 + 1)^{-1}e_{ji})(A_1 + A_2 + A)\\ &= \sum_{\substack{1 \le s \le n \\ s \ne j}} (d_1 + d_2 + 1)e_{ss} + (d_1 + d_2 + b - b(d_1 + d_2 + 1)^{-1})e_{jj} + e_{ij}.\end{aligned}$$

Suppose that

$$d_1 + d_2 + b - b(d_1 + d_2 + 1)^{-1} = 0.$$

We get that $d_1 + d_2 + 1 + b = 0$, a contradiction. We obtain

$$d_1 + d_2 + b - b(d_1 + d_2 + 1)^{-1} \ne 0.$$

This implies that

$$(1 + b(d_1 + d_2 + 1)^{-1}e_{ji})(A_1 + A_2 + A) \in \mathcal{R}^{\times}.$$

Since $1 + b(d_1 + d_2 + 1)^{-1}e_{ji} \in \mathcal{R}^{\times}$, we obtain that $A_1 + A_2 + A \in \mathcal{R}^{\times}$. Similarly, we note that

$$\begin{aligned}&(1 + b(d_1 + d_2)^{-1}e_{ji})(A_1 + A_2 + A + 1)\\ &= \sum_{\substack{1 \le s \le n \\ s \ne j}} (d_1 + d_2)e_{ss} + (d_1 + d_2 + b + 1 - b(d_1 + d_2)^{-1})e_{jj} + e_{ij}.\end{aligned}$$

If $d_1 + d_2 + b + 1 - b(d_1 + d_2)^{-1} = 0$, we get

$$(d_1 + d_2 + 1)(d_1 + d_2 + b) = 0,$$

this is a contradiction. It follows that

$$(1 + b(d_1 + d_2)^{-1}e_{ji})(A_1 + A_2 + A + 1) \in \mathcal{R}^{\times}.$$

Since $1 + b(d_1 + d_2)^{-1}e_{ji} \in \mathcal{R}^{\times}$, we get that $A_1 + A_2 + A + 1 \in \mathcal{R}^{\times}$. The proof of the result is now complete. □

We call a ring $\mathcal{B}$ *prime* if for any $a, b \in \mathcal{B}$, $a\mathcal{B}b = 0$ implies $a = 0$ or $b = 0$ (see [11] for details). Note that $M_n(\mathcal{D})$ is a prime ring.

Let $\mathcal{B}$ be a prime ring. By $Q_{ml}(\mathcal{B})$ we denote the maximal left ring of quotients of $\mathcal{B}$, and by C we denote the center of $Q_{ml}(\mathcal{B})$, which is called the *extended centroid* of $\mathcal{B}$. Note that $Q_{ml}(\mathcal{B})$ is also a prime ring and C is a field (see [2, Chapter 7] for details).

An additive map $\theta : \mathcal{B} \to Q_{ml}(\mathcal{B})$ is called a *Jordan derivation* if

$$\theta(a^2) = \theta(a)a + a\theta(a)$$

for all $a \in \mathcal{B}$ (see [10] for details).

3. The proof of Theorem 1.2

We assume that

$$f(A) = Ag(A^{-1})A \tag{3.1}$$

for all $A \in \mathcal{R}^\times$. Set $Q = f(1) = g(1)$ and

$$f_1(A) = f(A) + QA \quad \text{and} \quad g_1(A) = g(A) + AQ$$

for all $A \in \mathcal{R}$. Using the invertibility of the identity matrix and (3.1), we note that $f_1(1) = 0 = g_1(1)$. We get from (3.1) that

$$f_1(A) = Ag_1(A^{-1})A \tag{3.2}$$

for all $A \in \mathcal{R}^\times$. We get from (3.2) that

$$g_1(A^{-1}) = A^{-1}f_1(A)A^{-1}$$

for all $A \in \mathcal{R}^\times$. Substituting A with A^{-1} implies

$$g_1(A) = Af_1(A^{-1})A \tag{3.3}$$

for all $A \in \mathcal{R}^\times$. From both (3.2) and (3.3), we note that f_1 and g_1 have the same properties.

Lemma 3.1. *We claim that*

$$f_1(A^2) = f_1(A)A + Af_1(A) \quad \textit{and} \quad g_1(A^2) = g_1(A)A + Ag_1(A)$$

for all $A \in \mathcal{R}^\times$ with $A + 1 \in \mathcal{R}^\times$.

Proof. Take $A \in \mathcal{R}^\times$ with $A + 1 \in \mathcal{R}^\times$. We get

$$(A^{-1} + (A + 1)^{-1})^{-1} = A(A + 1).$$

It follows from (3.3) that

$$\begin{aligned}
0 &= g_1(A^{-1} + (A + 1)^{-1}) + (A^{-1} + (A + 1)^{-1})f_1(A(A + 1))(A^{-1} + (A + 1)^{-1}) \\
&= g_1(A^{-1} + (A + 1)^{-1}) + (A(A + 1))^{-1}(f_1(A^2) + f_1(A))(A(A + 1))^{-1} \\
&= g_1(A^{-1}) + g_1((A + 1)^{-1}) + (A(A + 1))^{-1}f_1(A^2)(A(A + 1))^{-1} \\
&\quad + (A(A + 1))^{-1}f_1(A)(A(A + 1))^{-1} \\
&= A^{-1}f_1(A)A^{-1} + (A + 1)^{-1}f_1(A)(A + 1)^{-1} + (A(A + 1))^{-1}f_1(A^2)(A(A + 1))^{-1} \\
&\quad + (A(A + 1))^{-1}f_1(A)(A(A + 1))^{-1}.
\end{aligned}$$

Multiplying the last relation by $A(A+1)$ from both left and right sides, we obtain

$$\begin{aligned} f_1(A^2) &= (A+1)f_1(A)(A+1) + Af_1(A)A + f_1(A) \\ &= f_1(A)A + Af_1(A) + 2Af_1(A)A + 2f_1(A) \end{aligned}$$

for all $A \in \mathcal{R}^\times$ with $A+1 \in \mathcal{R}^\times$. Since $\mathrm{Char}(\mathcal{D}) = 2$, we get that $2Af_1(A)A + 2f_1(A) = 0$. Hence

$$f_1(A^2) = f_1(A)A + Af_1(A)$$

for all $A \in \mathcal{R}^\times$ with $A+1 \in \mathcal{R}^\times$. Similarly, we can obtain that

$$g_1(A^2) = g_1(A)A + Ag_1(A)$$

for all $A \in \mathcal{R}^\times$ with $A+1 \in \mathcal{R}^\times$. □

For $A, B \in \mathcal{R}$, we define the Jordan product $A \circ B = AB + BA$.

Lemma 3.2. *Take $a, b \in \mathcal{D}$ and $1 \le i, j, k, l \le n$, and we claim that*

$$f_1((ae_{ij} + be_{kl})^2) = f_1(ae_{ij} + be_{kl}) \circ (ae_{ij} + be_{kl}). \tag{3.4}$$

In particular, we have that

$$f_1((ae_{ij})^2) = f_1(ae_{ij}) \circ ae_{ij}. \tag{3.5}$$

Proof. We set

$$A = ae_{ij} + be_{kl}.$$

By Lemma 2.1, we get that there exist $A_1, A_2 \in \mathcal{R}^\times$ such that

$$A_i + 1, A_i + A, A_i + A + 1, A_1 + A_2 + A, A_1 + A_2 + A + 1 \in \mathcal{R}^\times$$

for $i = 1, 2$. Using Lemma 3.1, we get

$$f_1((A_1 + A)^2) = f_1(A_1 + A) \circ (A_1 + A).$$

Expanding the last relation, we get

$$f_1(A_1^2) + f_1(A_1 \circ A) + f_1(A^2) = f_1(A_1) \circ A_1 + f_1(A_1) \circ A + f_1(A) \circ A_1 + f_1(A) \circ A. \tag{3.6}$$

In view of Lemma 3.1, we have that

$$f_1(A_1^2) = f_1(A_1) \circ A_1. \tag{3.7}$$

It follows from both (3.6) and (3.7) that

$$f_1(A_1 \circ A) + f_1(A^2) = f_1(A_1) \circ A + f_1(A) \circ A_1 + f_1(A) \circ A. \tag{3.8}$$

Similarly, we can obtain

$$f_1(A_2 \circ A) + f_1(A^2) = f_1(A_2) \circ A + f_1(A) \circ A_2 + f_1(A) \circ A \tag{3.9}$$

and

$$f_1((A_1 + A_2) \circ A) + f_1(A^2) = f_1(A_1 + A_2) \circ A + f_1(A) \circ (A_1 + A_2) + f_1(A) \circ A. \tag{3.10}$$

Adding (3.8) into (3.9), we get

$$f_1((A_1 + A_2) \circ A) = f_1(A_1 + A_2) \circ A + f_1(A) \circ (A_1 + A_2). \tag{3.11}$$

It follows from both (3.10) and (3.11) that $f_1(T^2) = f_1(T) \circ T$. □

Lemma 3.3. *We claim that both f_1 and g_1 are Jordan derivations of $\mathcal{R}$.*

Proof. Take $a, b \in \mathcal{D}$ and $1 \le i, j, k, l \le n$. Since f_1 is linear, we get

$$f_1((ae_{ij} + be_{kl})^2) = f_1((ae_{ij})^2) + f_1(ae_{ij} \circ be_{kl}) + f_1((be_{kl})^2). \tag{3.12}$$

Using (3.4), we get

$$\begin{aligned} f_1((ae_{ij} + be_{kl})^2) &= f_1(ae_{ij} + be_{kl}) \circ (ae_{ij} + be_{kl}) \\ &= f_1(ae_{ij} \circ (ae_{ij}) + f_1(ae_{ij} \circ (be_{kl}) \\ &\quad + f_1(be_{kl}) \circ (ae_{ij}) + f_1(be_{kl}) \circ (be_{kl}) \\ &= f_1((ae_{ij})^2) + f_1(ae_{ij} \circ (be_{kl}) \\ &\quad + f_1(be_{kl}) \circ (ae_{ij}) + f_1((be_{kl})^2). \end{aligned} \tag{3.13}$$

Comparing (3.12) and (3.13), we get

$$f_1(ae_{ij} \circ be_{kl}) = f_1(ae_{ij}) \circ be_{kl} + ae_{ij} \circ f_1(be_{kl}). \tag{3.14}$$

Since f_1 is linear and $\{e_{ij} \mid 1 \le i, j \le n\}$ is the standard base of $\mathcal{R}$, we easily get from (3.14) that

$$f_1(A^2) = f_1(A) \circ A$$

for all $A \in \mathcal{R}$. That is, f_1 is a Jordan derivation of $\mathcal{R}$. Similarly, we can obtain that g_1 is a Jordan derivation of $\mathcal{R}$. □

We are given:

Proof of Theorem 1.2. By Lemma 3.3 we have that both f_1 and g_1 are Jordan derivations of $\mathcal{R}$. Note that $\mathcal{R}$ is a prime ring. It follows from [10, Theorem 2.2] that

$$f_1(A) = \delta(A) + \mu(A) \quad \text{and} \quad g_1(A) = \delta_1(A) + \mu_1(A) \tag{3.15}$$

for all $A \in \mathcal{R}$, where $\delta, \delta_1 : \mathcal{R} \to Q_{ml}(\mathcal{R})$ are derivations and $\mu, \mu_1 : \mathcal{R} \to C$.

We first claim that $\delta = \delta_1$. Since δ is a derivation, we get from both (3.3) and (3.15) that

$$\begin{aligned} g_1(A) &= Af_1(A^{-1})A \\ &= A\left(\delta(A^{-1}) + \mu(A^{-1})\right)A \\ &= A\delta(A^{-1})A + \mu(A^{-1})A^2 \\ &= \delta(A) + \mu(A^{-1})A^2 \end{aligned} \tag{3.16}$$

for all $A \in \mathcal{R}^\times$. We get from both the second relation in (3.15) and (3.16) that

$$(\delta + \delta_1)(A) = \mu(A^{-1})A^2 + \mu_1(A) \tag{3.17}$$

for all $A \in \mathcal{R}^\times$. Take $a \in \mathcal{D}$ and $1 \le i, j \le n$ with $i \ne j$. It is clear that $(1 + ae_{ij})^2 = 1$. We get from (3.17) that

$$(\delta + \delta_1)(1 + ae_{ij}) = \mu(1 + ae_{ij}) + \mu_1(1 + ae_{ij}). \tag{3.18}$$

Since $(\delta+\delta_1)(1)=0$, we get from (3.18) that

$$(\delta+\delta_1)(ae_{ij})\in C. \tag{3.19}$$

Since $\dim_{\mathcal{F}}(D)\geq 4$, we get that there exists $b\in\mathcal{D}^{\times}$ such that $b\neq a, a+1, 1$. We set

$$A=\sum_{\substack{1\leq j\leq n\\ j\neq i}} e_{jj}+be_{ii}.$$

It is clear that $A, A+ae_{ii}\in\mathcal{R}^{\times}$ are diagonal matrices. On one hand, we get from (3.17) that

$$\begin{aligned}(\delta+\delta_1)(ae_{ii}) &= (\delta+\delta_1)(A+ae_{ii}+A)\\ &= (\delta+\delta_1)(A+ae_{ii})+(\delta+\delta_1)(A)\\ &= \mu((A+ae_{ii})^{-1})(A+ae_{ii})^2+\mu_1(A+ae_{ii})\\ &\quad +\mu(A^{-1})A^2+\mu_1(A).\end{aligned} \tag{3.20}$$

Since both A^2 and $(A+ae_{ii})^2$ are diagonal matrices, we get from (3.20) that

$$\begin{aligned}e_{ii}(\delta+\delta_1)(ae_{ii})e_{ii} &= \mu((A+ae_{ii})^{-1})e_{ii}(A+ae_{ii})^2e_{ii}+\mu_1(A+ae_{ii})e_{ii}\\ &\quad +\mu(A^{-1})e_{ii}A^2e_{ii}+\mu_1(A)e_{ii}\\ &= \mu((A+ae_{ii})^{-1})(A+ae_{ii})^2+\mu_1(A+ae_{ii})e_{ii}\\ &\quad +\mu(A^{-1})A^2+\mu_1(A)e_{ii}.\end{aligned} \tag{3.21}$$

On the other hand, since $\delta+\delta_1$ is a derivation, we get from (3.19) that

$$\begin{aligned}(\delta+\delta_1)(ae_{ii}) &= (\delta+\delta_1)(ae_{ij}e_{ji})\\ &= (\delta+\delta_1)(ae_{ij})e_{ji}+ae_{ij}(\delta+\delta_1)(e_{ji})\in Ce_{ij}+Cae_{ji}\end{aligned} \tag{3.22}$$

where $1\leq j\leq n$ with $j\neq i$. It follows from (3.22) that

$$e_{ii}(\delta+\delta_1)(ae_{ii})e_{ii}=0. \tag{3.23}$$

We get from both (3.21) and (3.23) that

$$\mu((A+ae_{ii})^{-1})(A+ae_{ii})^2+\mu_1(A+ae_{ii})e_{ii}+\mu(A^{-1})a^2+\mu_1(A)e_{ii}=0. \tag{3.24}$$

Multiplying (3.24) with e_{jj} from the left side, where $j\neq i$, we get

$$\begin{aligned}0 &= \mu((A+ae_{ii})^{-1})e_{jj}(A+ae_{ii})^2+\mu(A^{-1})e_{jj}A^2\\ &= \mu((A+ae_{ii})^{-1})(A+ae_{ii})^2+\mu(A^{-1})A^2.\end{aligned} \tag{3.25}$$

Taking (3.25) into (3.20), we obtain

$$(\delta+\delta_1)(ae_{ii})=\mu_1(A+ae_{ii})+\mu_1(A)\in C. \tag{3.26}$$

For any $A=\sum_{1\leq i,j\leq n} a_{ij}e_{ij}\in\mathcal{R}$, we get from both (3.19) and (3.26) that

$$(\delta+\delta_1)(A)=\sum_{1\leq i,j\leq n}(\delta+\delta_1)(a_{ij}e_{ij})\in C.$$

This implies that

$$[(\delta + \delta_1)(A), A] = 0$$

for all $A \in \mathcal{R}$. Since $\delta + \delta_1$ is a derivation, we get from the famous Posner's theorem [11, Theorem 2] that $\delta + \delta_1 = 0$, and so $\delta = \delta_1$, as desired.

We next claim that $\mu = \mu_1 = 0$. We get from (3.17) that

$$\mu(A^{-1})A^2 + \mu_1(A) = 0$$

for all $A \in \mathcal{R}^\times$. This implies that

$$\mu(A^{-1})A^2 \in C \tag{3.27}$$

for all $A \in \mathcal{R}^\times$. Replacing A by A^{-1} in (3.27), we get that

$$\mu(A)A^{-2} \in C \tag{3.28}$$

for all $A \in \mathcal{R}^\times$. Since C is a field, we get from (3.28) that either $\mu(A) = 0$ or $A^{-2} \in C$. This implies that either $\mu(A) = 0$ or $A^2 \in C$. We get

$$\mu(A)A^2 \in C \tag{3.29}$$

for all $A \in \mathcal{R}^\times$. Take $a \in \mathcal{D}$ and $1 \leq i \leq n$, and we first claim that $\mu(ae_{ii}) = 0$.

Since $\dim_{\mathcal{F}}(\mathcal{D}) \geq 4$ (implying $|\mathcal{D}| \geq 16$), we get that there exists $b \in \mathcal{D}^\times$ such that $b \neq a, 1 + a, 1$. We set

$$A = \sum_{\substack{1 \leq j \leq n \\ j \neq i}} e_{jj} + be_{ii}.$$

It is clear that $A, A + ae_{ii} \in \mathcal{R}^\times$. We claim that $A^2 \notin C$. Indeed, we have

$$A^2 e_{ij} = b^2 e_{ij} \neq e_{ij} = e_{ij}A^2$$

for $1 \leq j \leq n$ with $j \neq i$. This implies that $A^2 \notin C$, as desired. Similarly, we can obtain that $(A + ae_{ii})^2 \notin C$. It follows from (3.29) that

$$\mu(A)A^2 \in C.$$

Since $A^2 \notin C$, we get that $\mu(A) = 0$. It follows from (3.29) that

$$\mu(A + ae_{ii})(A + ae_{ii})^2 \in C.$$

Since $(A + ae_{ii})^2 \notin C$, we get that

$$\mu(A + ae_{ii}) = 0$$

and so $\mu(ae_{ii}) = 0$, as desired. We next claim that $\mu(ae_{ij}) = 0$ for all $a \in \mathcal{D}$ and $1 \leq i, j \leq n$ with $i \neq j$.

Since $\dim_{\mathcal{F}}(\mathcal{D}) \geq 4$ (implying $|\mathcal{D}| \geq 16$), we get that there exists $b \in \mathcal{D}^\times$ such that $b \neq 1$. We set

$$A = \sum_{\substack{1 \leq j \leq n \\ j \neq i}} e_{jj} + be_{ii}.$$

It is easy to check that $A, A + ae_{ij} \in \mathcal{R}^\times$ and $A^2 \notin C$. It follows from (3.29) that

$$\mu(A)A^2 \in C.$$

Since $A^2 \notin C$, we get that $\mu(A) = 0$. We claim that $(A + ae_{ij})^2 \notin C$. Indeed, we have

$$(A + ae_{ij})^2 = A^2 + A \circ ae_{ij}.$$

Note that

$$e_{ij}(A^2 + A \circ ae_{ij}) = e_{ij} \neq b^2 e_{ij} = (A^2 + a \circ ae_{ij})e_{ij}.$$

This implies that $(A + ae_{ij})^2 \notin C$. It follows from (3.29) that

$$\mu(A + ae_{ij})(A + ae_{ij})^2 \in C.$$

Since $(A + ae_{ij})^2 \notin C$, we get

$$\mu(A + ae_{ij}) = 0$$

and so $\mu(ae_{ij}) = 0$. For any $A = \sum_{1 \leq i,j \leq n} a_{ij}e_{ij} \in \mathcal{R}$, we obtain that

$$\mu(A) = \sum_{1 \leq i,j \leq n} \mu(ae_{ij}) = 0.$$

Hence, $\mu = 0$. Similarly, we can obtain that $\mu_1 = 0$. We have that

$$f_1 = \delta = \delta_1 = g_1.$$

This implies that $\delta = \delta_1 : \mathcal{R} \to \mathcal{R}$ and

$$f(A) = Aq + \delta(A) \quad \text{and} \quad g(A) = qA + \delta(A)$$

for all $A \in \mathcal{R}$. The proof of the result is now complete. □

4. Relevant works on functional equations in alternative division algebras

We remark that the functional identity (1.1) is closely related to a broader class of functional equations of the form

$$F(x) + M(x)G(1/x) = 0,$$

which have been extensively studied in different algebraic settings. In particular, the above equation has been completely classified over fields of characteristic two in the recent paper by Kawai and Ferreira [9]. In addition, functional equations on alternative division rings have been discussed in [7, 8].

In view of Theorem 1.2 and the recent developments in the theory of alternative division algebras, we give the following:

Conjecture 4.1. *Suppose that D is an alternative division algebra with char$(D) = 2$. Let $f, g : D \to D$ be additive maps satisfying*

$$f(A)A^{-1} + Ag(A^{-1}) = 0$$

for all invertible $A \in D$. If $|D| \neq 2, 4, 8$, then

$$f(A) = AQ + \delta(A) \quad \text{and} \quad g(A) = QA + \delta(A)$$

for all $A \in D$, where $Q \in D$ and δ is a derivation of D.

5. Conclusions

In this paper, we investigate the following functional identity with inverses on matrix rings over a division ring of characteristic two:

$$f(A)A^{-1} + Ag(A^{-1}) = 0$$

for all invertible $A \in M_n(\mathcal{D})$, which has been investigated for the case of characteristic not two in 2020. We remark that our main result (Theorem 1.2) affirmatively answers a question posed by Argac et al. under a mild condition that $|\mathcal{D}| \neq 2, 4, 8$. The proof method of the main theorem (Theorem 1.2) has several innovations: First, it uses the prime field of division ring to determine the independence of elements and then employs complex calculations to identify some invertible matrices. Second, we prove that additive mappings that satisfy the conditions of the theorem are Jordan derivations. Finally, we use a result on Jordan derivations of prime rings to complete the proof of the main theorem.

It should be noted that the main result of this paper is somewhat related to the problem of identities with inverses on alternative division algebras.

Author contributions

Yingyu Luo: Writing original draft; Qian Chen: Validation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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