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Research article

Introduction to spherical fuzzy metric spaces

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Abstract: In this paper, we introduced the notion of a spherical fuzzy metric space, which extends the classical fuzzy metric space and generalized fuzzy metric structures by integrating spherical fuzzy set theory. Then, we presented a detailed theoretical analysis to obtain the relationships between spherical fuzzy metric and fuzzy metric structures, including their extensions. Moreover, we compared these fuzzy metric structures through their applications to image processing tasks and demonstrated the effectiveness of spherical fuzzy metrics in handling uncertainty and preserving structural information. Finally, we established the Banach fixed point theorem in spherical fuzzy metric spaces.

Keywords: fuzzy metric; fixed point; image; t-norm
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1. Introduction

As a generalization of classical metric spaces, fuzzy metric spaces have made it possible to mathematically model situations involving ambiguity and vagueness. The notion of fuzzy metric spaces was defined differently by Kaleva and Seikkala [1] and Kramosil and Michálek [2]. Kaleva and Seikkala [1] defined this structure as the distance of ordinary two points as a fuzzy number, whereas Kramosil and Michálek [2] defined this notion as the distance of ordinary two points with respect to a parameter as a value in $[0, 1]$. The fuzzy metric approach introduced by Kramosil and Michálek [2], has garnered significant interest in the literature because it provides a more natural way to generalize the metric structure in fuzzy environments, enabling analyses performed on this structure to be conducted in a relatively more accessible manner. Following this fundamental study, various theoretical extensions of this structure have been proposed. Among them, George and Veeramani [3] modified the definition of Kramosil and Michálek in such a way that each fuzzy metric induces a

Hausdorff topology. In this way, the topological properties of fuzzy metric spaces could be investigated, and these structures could be analyzed within a topological framework. In addition, George and Veeramani [3] studied some well-known topological properties of fuzzy metric spaces. In this direction, Aygünoğlu et al. [4] constructed a fuzzy topology by means of a fuzzy metric and investigated its properties, establishing a direct link between fuzzy metric spaces and fuzzy topological spaces. Additional studies related to the topological view can be found in [5]. One of the most significant research areas has been the contributions to fixed point theory. Grabiec [6] and Gregori and Sapena [7] presented the classical Banach contraction theorem for fuzzy metric spaces by adopting different perspectives on contractive conditions. Moreover, another study extending the Banach contraction theorem has been given by Gregori and Miñana [8] using t-conorms. Mihet [9] defined ψ-contractive mappings, which form a wider class than the contractions considered by Gregori and Sapena [7], and proved fixed point theorems for such mappings in non-Archimedean fuzzy metric spaces. Since then, contractive conditions and the corresponding fixed point results in fuzzy metric spaces have remained an active research topic. As a contribution to this area, Martínez-Moreno et al. [10] introduced new classes of fuzzy contractions defined by means of fuzzy negations and established related fixed point theorems. In addition to these, applications of fixed point theory to image restoration have been given in [11, 12]. Along with theoretical developments, numerous applications have also been investigated in fuzzy metric spaces. For example, Ralević et al. [13] applied fuzzy metrics to image processing and demonstrated that these structures can be used to remove noise from images. Their work shows that fuzzy metric approaches can be used to address uncertainties in image data and improve image quality. The wide applicability and usefulness of fuzzy metrics have motivated researchers to extend this framework [14–16]. Motivated by these advantages, numerous generalizations have been developed in the literature to enlarge the theoretical scope of fuzzy metric spaces and to handle different types of uncertainty more effectively. These generalizations include intuitionistic fuzzy metric spaces [17], fuzzy partial metric spaces [18, 19], and t-conorm based fuzzy metric spaces [20]. In addition to these structural generalizations, numerous fixed point results have been obtained in these structures [21, 22].

Furthermore, spherical fuzzy set theory [23] has emerged as a powerful framework for handling higher-dimensional uncertainty. Spherical fuzzy sets, which generalize intuitionistic and Pythagorean fuzzy sets, enable independent control over the degrees of membership, hesitancy, and non-membership while satisfying a geometrically interpretable spherical constraint. This characteristic provides a more flexible and realistic modeling tool for problems involving complex uncertainty, hesitation, and partial truth-features that are frequently encountered in decision-making, pattern recognition, and image analysis. Initially, spherical fuzzy set theory was applied successfully to decision-making theory [24–26] and has solved practical real-life problems. Kutlu Gündoğdu and Kahraman [23] established the well-known TOPSIS method in the spherical fuzzy environment. Ashraf et al. [24] gave a decision-making technique based on aggregation operators. In addition, Mahmood et al. [25] presented a different decision-making technique and applied this method to the medical diagnosis problem. Additionally, some studies based on spherical fuzzy distance and similarity measures can be found in [27, 28]. Later, Ashraf et al. [29] transferred t-norms and t-conorms to spherical fuzzy numbers; the resulting operations are called spherical fuzzy t-norms and t-conorms. These operations enable truth degrees in fuzzy systems to be represented in a multidimensional way, and it was shown that the fundamental properties of classical t-norms remain

valid in the spherical case.

Motivated by these developments, we propose spherical fuzzy metric spaces, a framework that combines the structure of fuzzy metrics with the principles of spherical fuzzy sets. The proposed framework generalizes classical and generalized fuzzy metric structures by incorporating an additional layer of uncertainty representation. This integration not only enhances the descriptive capacity of metric-based fuzzy systems but also offers new insights into the nature of distance and proximity under spherical fuzzy uncertainty.

The main novelties and contributions of this work are outlined as follows:

(1) We define and develop the theory of spherical fuzzy metric spaces, establishing their basic properties and underlying axioms.

(2) We provide a detailed analysis of the relationships between spherical fuzzy metrics and existing fuzzy metric structures, including comparative discussions on their expressive capabilities.

(3) We show a comparative analysis of fuzzy metric structures using images with various noise levels to investigate the behavior of the proposed spherical metric structure on different data types.

(4) We introduce convergent and Cauchy sequences in spherical fuzzy metric spaces and prove a Banach-type fixed point theorem in this setting.

The paper is structured as follows: In Section 2, we recall the basic notions concerning spherical fuzzy sets and fuzzy metric structures. In Section 3, we introduce spherical fuzzy metric spaces, study their basic properties and relate them to the fuzzy metric structures recalled in Section 2. Furthermore, a comparison of spherical fuzzy metric spaces with existing fuzzy metric structures in terms of similarity values is presented using images with different noise levels. In Section 4, we present the Banach fixed point theorem with an illustrative example.

2. Preliminaries

In this section, we first recall fuzzy sets and some of their generalizations, namely intuitionistic, picture, and spherical fuzzy sets. Then, we give the fuzzy metric structures used in the sequel, including intuitionistic and picture fuzzy metric spaces.

2.1. Fuzzy set and its generalizations

In this subsection, fuzzy sets and some of their generalizations are reviewed, and the functions on spherical fuzzy sets needed in the rest of the study are given.

Definition 2.1. [30] A fuzzy set (FS) F on a nonempty set X is characterized by a membership function $\mu_F : X \to [0, 1]$ and is written as $F = \{\langle x, \mu_F(x)\rangle \mid x \in X\}$. Here, $\mu_F(x)$ denotes the membership degree of x in F.

Definition 2.2. [31] An intuitionistic fuzzy set (IFS) I on X is determined by two functions $\mu_I, \nu_I : X \to [0, 1]$ satisfying $\mu_I(x) + \nu_I(x) \leq 1$ for every $x \in X$ and is written as $I = \{\langle x, \mu_I(x), \nu_I(x)\rangle \mid x \in X\}$. For each $x \in X$, the values $\mu_I(x)$ and $\nu_I(x)$ are called the membership degree and the non-membership degree of x in I, respectively.

Definition 2.3. [32] A picture fuzzy set (PFS) P on X is determined by three functions $\mu_P, \iota_P, \nu_P : X \to [0, 1]$ satisfying $\mu_P(x) + \iota_P(x) + \nu_P(x) \leq 1$ for every $x \in X$ and is written as

$P = \{\langle x, \mu_P(x), \iota_P(x), \nu_P(x)\rangle \mid x \in X\}$. For each $x \in X$, the values $\mu_P(x)$, $\iota_P(x)$ and $\nu_P(x)$ are called the membership degree, the neutral membership degree and the non-membership degree of x in P, respectively.

Definition 2.4. [23] A spherical fuzzy set (SFS) S on X is determined by three functions $\mu_S, \iota_S, \nu_S : X \to [0,1]$ satisfying ${\mu_S}^2(x) + {\iota_S}^2(x) + {\nu_S}^2(x) \leq 1$ for every $x \in X$ and is written as $S = \{\langle x, \mu_S(x), \iota_S(x), \nu_S(x)\rangle \mid x \in X\}$. For each $x \in X$, the values $\mu_S(x)$, $\iota_S(x)$ and $\nu_S(x)$ are called the membership degree, the hesitancy degree and the non-membership degree of x in S, respectively, and we denote by $SFS(X)$ the family of all SFSs on X. For each $x \in X$, the value $\pi_S(x) = \sqrt{1 - {\mu_S}^2(x) - {\iota_S}^2(x) - {\nu_S}^2(x)}$ is called the refusal degree of x. Moreover, a triplet $S = (\mu_S, \iota_S, \nu_S)$ of numbers in $[0,1]$ with ${\mu_S}^2 + {\iota_S}^2 + {\nu_S}^2 \leq 1$ is called a spherical fuzzy number (SFN).

Score and accuracy functions that enable comparison among spherical fuzzy numbers have been defined in various ways by different authors in the literature.

Definition 2.5. Let $\mathscr{S}$ be the collection of the SFNs on X and $S, S_1, S_2 \in \mathscr{S}$. A score function $SF : \mathscr{S} \to [-1,1]$ is defined as

- $SF_1(S) = \frac{1}{3}(2 + \mu_S - \iota_S - \nu_S)$ [24],
- $SF_2(S) = (\mu_S - \iota_S)^2 - (\nu_S - \iota_S)^2$ [23],
- $SF_3(S) = \mu_S^2 - \nu_S^2$ [25],
- $SF_4(S) = \frac{1}{3}(2 + \mu_S^2 - \iota_S^2 - \nu_S^2)$ [33],
- $SF_5(S) = \mu_S^2 - \iota_S^2\nu_S^2$ [26].

Accuracy function $AF : \mathscr{S} \to [0,1]$ is defined as

- $AF_1(S) = \mu_S - \nu_S$ [24],
- $AF_2(S) = \mu_S^2 + \iota_S^2 + \nu_S^2$ [23,25],
- $AF_3(S) = \mu_S^2 + \iota_S^2\nu_S^2$ [26].

The comparison between SFNs through the score function and accuracy function is as follows:

(i) If $SF(S_1) < SF(S_2)$, then $S_1 \ll S_2$.
(ii) If $SF(S_1) > SF(S_2)$, then $S_1 \gg S_2$.
(iii) $SF(S_1) = SF(S_2)$, then
  (a) $AF(S_1) < AF(S_2)$, then $S_1 \ll S_2$,
  (b) $AF(S_1) > AF(S_2)$, then $S_1 \gg S_2$,
  (c) $AF(S_1) = AF(S_2)$, then $S_1 = S_2$.

2.2. Fuzzy metric structures

In this subsection, we recall the fuzzy metric structures that will be compared with spherical fuzzy metrics in Section 3. For this purpose, we first give the notions of t-norm and t-conorm. Then, fuzzy metric spaces and t-conorm based fuzzy metric spaces are recalled, and finally intuitionistic and picture fuzzy metric spaces are given.

Definition 2.6. [34] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (t-norm) if it is commutative and associative, it is monotone in the sense that $y \leq z$ implies $x * y \leq x * z$, and 1 is its neutral element, that is, $x * 1 = x$ for every $x \in [0,1]$.

Definition 2.7. [34] A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is called a triangular conorm (t-conorm) if it is commutative, associative and monotone in the same sense as above, and 0 is its neutral element, that is, $x \diamond 0 = x$ for every $x \in [0,1]$.

Next, we recall some fundamental notions related to t-norms and t-conorms.

Definition 2.8. [34] Let $*, \star$ be t-norms and $\diamond, \nabla$ be t-conorms.

(i) If $*$ (resp. $\diamond$) is continuous on $[0,1]\times[0,1]$, then it is called a continuous t-norm (resp. continuous t-conorm).
(ii) $*$ is called Archimedean if, for any $x, y \in (0,1)$, the inequality $x_*^n < y$ holds for some $n \in \mathbb{N}$, where $x_*^n = x * x * \cdots * x$ (n times). In this case, $x_*^n \to 0$ as $n \to \infty$. Similarly, $\diamond$ is called Archimedean if, for any $x, y \in (0,1)$, the inequality $x_\diamond^n > y$ holds for some $n \in \mathbb{N}$, and in this case $x_\diamond^n \to 1$ as $n \to \infty$.
(iii) If $x * y \le x \star y$ for all $x, y \in [0,1]$, then $*$ is called weaker than $\star$, and $\star$ is called stronger than $*$. If $x \nabla y \le x \diamond y$ for all $x, y \in [0,1]$, then $\diamond$ is called weaker than ∇, and ∇ is called stronger than $\diamond$.
(iv) The operation $\diamond_*$ defined by $x \diamond_* y = 1 - ((1-x) * (1-y))$ for all $x, y \in [0,1]$ is a t-conorm and is called the dual t-conorm associated with the t-norm $*$. Similarly, the operation $*_\diamond$ defined by $x *_\diamond y = 1 - ((1-x) \diamond (1-y))$ is a t-norm, and it is called the dual t-norm associated with the t-conorm $\diamond$.

We now recall the notion of fuzzy metric introduced by George and Veeramani [3].

Definition 2.9. [3] Let X be a nonempty set and $*$ be a continuous t-norm. A mapping $M : X \times X \times (0,\infty) \to [0,1]$ is called a fuzzy metric (FM) on X if the following conditions hold for all $x, y, z \in X$ and $t, s > 0$:
(FM1) $0 < M(x, y, t)$,
(FM2) $x = y \Leftrightarrow M(x, y, t) = 1$,
(FM3) $M(x, y, t) = M(y, x, t)$,
(FM4) $M(x, z, t + s) \ge M(x, y, t) * M(y, z, s)$,
(FM5) $M(x, y, \cdot) : (0,\infty) \to (0,1]$ is continuous.
In this case, the triple $(X, M, *)$ is called a fuzzy metric space (FMS).

For $\alpha \in (0,1]$, consider the mapping M_α defined by

$$M_\alpha(x, y, t) = \begin{cases} 1, & \text{if } x = y, \\ \alpha M(x, y, t), & \text{if } x \ne y. \end{cases}$$

It is easy to check that M_α is again a fuzzy metric on X; i.e., $(X, M_\alpha, *)$ is an FMS.

Remark 2.10. If $(X, M, \star)$ is an FMS and $*$ is a continuous t-norm weaker than $\star$, then $(X, M, *)$ is an FMS, too. Indeed, by (FM4), we have $M(x, z, t + s) \ge M(x, y, t) \star M(y, z, s) \ge M(x, y, t) * M(y, z, s)$.

By using t-conorms instead of t-norms, Grigorenko et al. [20] introduced the following notion.

Definition 2.11. [20] Let X be a nonempty set and $\diamond$ be a continuous t-conorm. A mapping $N : X \times X \times (0,\infty) \to [0,1]$ is called a t-conorm based fuzzy metric on X if the following conditions hold for all $x, y, z \in X$ and $t, s > 0$:

(CBFM1) $1 > N(x, y, t)$,
(CBFM2) $x = y \Leftrightarrow N(x, y, t) = 0$,
(CBFM3) $N(x, y, t) = N(y, x, t)$,
(CBFM4) $N(x, z, t + s) \leq N(x, y, t) \diamond N(y, z, s)$,
(CBFM5) $N(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

In this case, the triple $(X, N, \diamond)$ is called a t-conorm based fuzzy metric space (CB-FMS).

Similarly, for $\alpha \in (0, 1]$, the mapping αN defined by $(\alpha N)(x, y, t) = \alpha N(x, y, t)$ is again a t-conorm based fuzzy metric on X, so $(X, \alpha N, \diamond)$ is a CB-FMS.

Remark 2.12. If (X, N, ∇) is a CB-FMS and $\diamond$ is a continuous t-conorm weaker than ∇, then $(X, N, \diamond)$ is a CB-FMS, too. Indeed, by (CBFM4), we have $N(x, z, t + s) \leq N(x, y, t) \nabla N(y, z, s) \leq N(x, y, t) \diamond N(y, z, s)$.

The following remark shows that FMSs and CB-FMSs can be obtained from each other.

Remark 2.13. Let $\alpha \in (0, 1]$.

(i) If $(X, M, *)$ is an FMS, then $(X, \alpha(1 - M), \diamond_*)$ is a CB-FMS.
(ii) If $(X, N, \diamond)$ is a CB-FMS, then $(X, M_N, *_\diamond)$ is an FMS, where

$$M_N(x, y, t) = \begin{cases} 1, & \text{if } x = y, \\ \alpha(1 - N(x, y, t)), & \text{if } x \neq y. \end{cases}$$

Park [17] extended fuzzy metric spaces to the intuitionistic setting by using a t-norm and a t-conorm together.

Definition 2.14. [17] Let X be a nonempty set, $*$ be a continuous t-norm and $\diamond$ be a continuous t-conorm. A pair (M, N) of mappings $M, N : X \times X \times (0, \infty) \to [0, 1]$ is called an intuitionistic fuzzy metric (IFM) on X if the following conditions hold for all $x, y, z \in X$ and $t, s > 0$:
(IFM0) $M(x, y, t) + N(x, y, t) \leq 1$,
(IFM1) $M(x, y, t) > 0$ and $N(x, y, t) > 0$,
(IFM2) $M(x, y, t) = 1$ and $N(x, y, t) = 0 \Leftrightarrow x = y$,
(IFM3) $M(x, y, t) = M(y, x, t)$ and $N(x, y, t) = N(y, x, t)$,
(IFM4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(IFM5) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$,
(IFM6) $M(x, y, \cdot), N(x, y, \cdot) : (0, \infty) \to [0, 1]$ are continuous.

In this case, $(X, M, N, *, \diamond)$ is called an intuitionistic fuzzy metric space (IFMS).

The following remark, given in [17], connects these two structures.

Remark 2.15. [17] If $(X, M, *)$ is an FMS, then $(X, M, 1 - M, *, \diamond_*)$ is an IFMS.

The above remark can be expressed in a more general form using $\alpha, \beta \in (0, 1]$. This generalization ensures that the given remark describes a specific case and is valid under different conditions. Appropriately selecting the values of α and β increases flexibility in obtaining various generalizations. Thus, it enables deriving various IFMSs from an FMS.

Remark 2.16. Let $\alpha, \beta \in (0, 1]$.

(i) For an FMS $(X, M, *)$, the structure $(X, M_\alpha, \beta(1-M), *, \diamond_*)$ is an IFMS.
(ii) For a CB-FMS $(X, N, \diamond)$, the structure $(X, M_N, \beta N, *_\diamond, \diamond)$ is an IFMS.

Remark 2.17. Conversely, for an IFMS $(X, M, N, *, \diamond)$ and $\alpha \in (0, 1]$, the structures $(X, M_\alpha, *)$ and $(X, M_N, *_\diamond)$ are FMSs, and the structures $(X, \alpha N, \diamond)$ and $(X, \alpha(1-M), \diamond_*)$ are CB-FMSs.

The relations between IFMS with FMS and CB-FMS are shown in Figure 1.

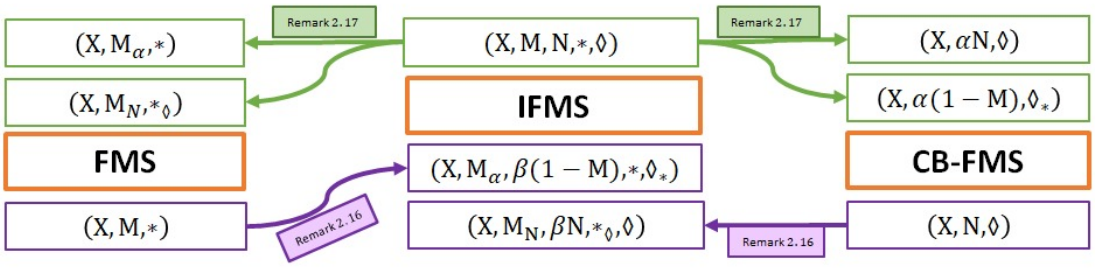

Figure 1. Relation between intuitionistic and fuzzy metric spaces.

A further generalization with three mappings is the picture fuzzy metric space given in [35].

Definition 2.18. [35] Let X be a nonempty set, $*$ be a continuous t-norm, and $\diamond$ be a continuous t-conorm. A triple (M, N, W) of mappings $M, N, W : X \times X \times (0, \infty) \to [0, 1]$ is called a picture fuzzy metric (PFM) on X if the following conditions hold for all $x, y, z \in X$ and $t, s > 0$:
(PFM0) $M(x, y, t) + N(x, y, t) + W(x, y, t) \leq 1$,
(PFM1) $M(x, y, t) > 0$, $N(x, y, t) > 0$ and $W(x, y, t) > 0$,
(PFM2) $M(x, y, t) = 1$, $N(x, y, t) = 0$ and $W(x, y, t) = 0 \Leftrightarrow x = y$,
(PFM3) $M(x, y, t) = M(y, x, t)$, $N(x, y, t) = N(y, x, t)$ and $W(x, y, t) = W(y, x, t)$,
(PFM4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(PFM5) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$,
(PFM6) $W(x, y, t) \diamond W(y, z, s) \geq W(x, z, t + s)$,
(PFM7) $M(x, y, \cdot), N(x, y, \cdot), W(x, y, \cdot) : (0, \infty) \to (0, 1]$ are continuous.
In this case, $(X, M, N, W, *, \diamond)$ is called a picture fuzzy metric space (PFMS).

Next, we show that we can induce PFMSs from FMSs, CB-FMSs, and IFMSs by using the appropriate t-norms and t-conorms.

Remark 2.19. Let $\alpha, \beta, \gamma \in (0, 1]$ be such that $\alpha + \beta + \gamma \leq 1$.

(i) For an FMS $(X, M, *)$, the structure $(X, M_\alpha, \beta(1-M), \gamma(1-M), *, \diamond_*)$ is a PFMS.
(ii) For a CB-FMS $(X, N, \diamond)$, the structure $(X, M_N, \beta N, \gamma N, *_\diamond, \diamond)$ is a PFMS.
(iii) For an IFMS $(X, M, N, *, \diamond)$, the structures $(X, M_\alpha, \beta N, \gamma N, *, \diamond)$, $(X, M_\alpha, \beta(1-M), \gamma(1-M), *, \diamond_*)$ and $(X, M_N, \beta N, \gamma N, *_\diamond, \diamond)$ are PFMSs.

In the following, we provide the relationships between the components of a PFM and the corresponding metric structures.

Remark 2.20. Assume that $(X, M, N, W, *, \diamond)$ is a PFMS and $\alpha, \beta \in (0, 1]$.

(i) The structures $(X, M_\alpha, *)$, $(X, M_N, *_\diamond)$ and $(X, M_W, *_\diamond)$ are FMSs, and the structures $(X, \alpha N, \diamond)$, $(X, \alpha W, \diamond)$ and $(X, \alpha(1-M), \diamond_*)$ are CB-FMSs.

(ii) If, in addition, $\alpha+\beta \leq 1$, then the structures $(X, M_\alpha, \beta N, *, \diamond)$, $(X, M_\alpha, \beta W, *, \diamond)$, $(X, M_N, \beta N, *_\diamond, \diamond)$, $(X, M_W, \beta W, *_\diamond, \diamond)$ and $(X, M_\alpha, \beta(1-M), *, \diamond_*)$ are IFMSs.

The relations between PFMS with FMS, CB-FMS and IFMS are shown in Figure 2.

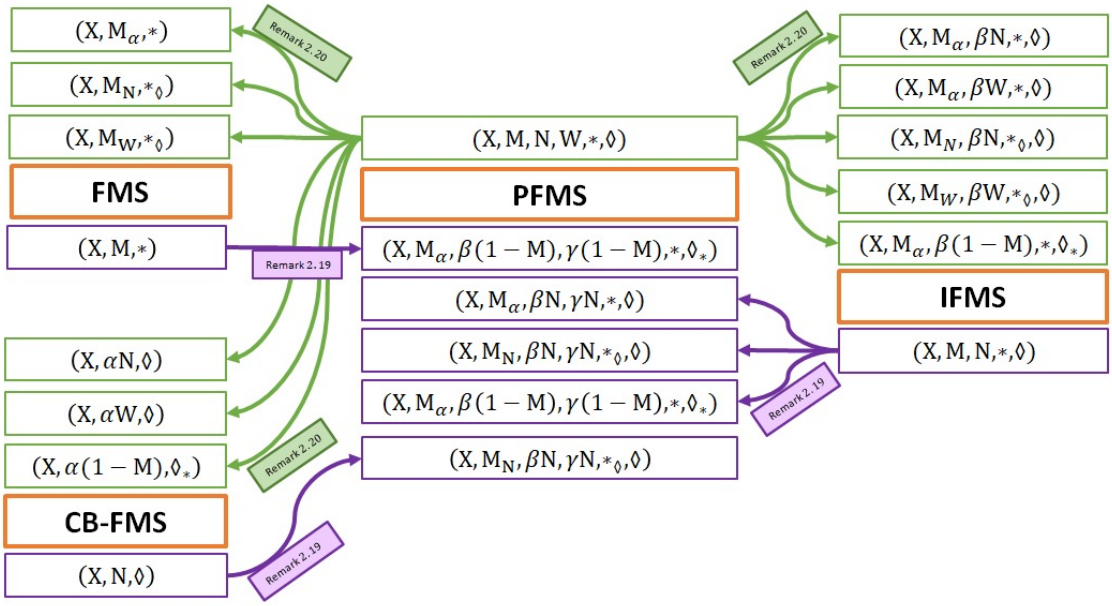

Figure 2. Relation among picture and other fuzzy metric spaces.

3. Spherical fuzzy metric spaces

This section consists of two subsections. The first subsection includes the representation of the spherical fuzzy t-norm and the concept of spherical fuzzy metrics, providing their basic definitions and properties. Subsequently, the relationships between spherical fuzzy metric spaces and other fuzzy metric structures are examined in detail. In the second subsection, different fuzzy metric structures are compared on gray-level images with various noise levels.

3.1. Notion of spherical fuzzy metric space and its relationships with other fuzzy metric structures

Throughout this section, $\mathscr{S}$ denotes the set of all spherical fuzzy numbers, that is $\mathscr{S} = \{a = (a_1, a_2, a_3) : a_1^2 + a_2^2 + a_3^2 \leq 1, a_i \in [0,1] \text{ for all } i = 1,2,3\}$. The unit elements of $\mathscr{S}$ are given by $1_\mathscr{S} = (1,0,0)$ and $0_\mathscr{S} = (0,0,1)$. For each $a = (a_1, a_2, a_3) \in \mathscr{S}$, we also consider the subset $I(a)$ of $\mathscr{S}$ given by

$$I(a) = \{b \in \mathscr{S} : b = (a_1, b_2, a_3) \text{ with } 0 \leq b_2^2 \leq 1 - a_1^2 - a_3^2\}.$$

The order relation $\preceq$ on $\mathscr{S}$ was defined by Ashraf et al. [29] as follows:

$$\begin{aligned} a \preceq b &\Leftrightarrow (a_1, a_2, a_3) \preceq (b_1, b_2, b_3) \\ &\Leftrightarrow ((a_1 < b_1) \wedge (a_3 \geq b_3)) \vee ((a_1 = b_1) \wedge (a_3 > b_3)) \vee ((a_1 = b_1) \wedge (a_3 = b_3) \wedge (a_2 \leq b_2)) \end{aligned} \tag{3.1}$$

for all $a, b \in \mathscr{S}$. Since $\preceq$ is not a total order, some elements of $\mathscr{S}$ cannot be compared: If $a \npreceq b$ and $b \npreceq a$, we write $a \|_{\preceq} b$.

We also use the following strict version of $\preceq$:

$$\begin{aligned} a \prec b &\Leftrightarrow (a_1, a_2, a_3) \prec (b_1, b_2, b_3) \\ &\Leftrightarrow ((a_1 < b_1) \wedge (a_3 \geq b_3)) \vee ((a_1 = b_1) \wedge (a_3 > b_3)) \vee ((a_1 = b_1) \wedge (a_3 = b_3) \wedge (a_2 < b_2)). \end{aligned} \tag{3.2}$$

Definition 3.1. [29] An operation $\tilde{*} : \mathscr{S} \times \mathscr{S} \to \mathscr{S}$ is called a spherical fuzzy t-norm (SF t-norm, for short) if it is commutative and associative, it satisfies $a\tilde{*}1_{\mathscr{S}} \in I(a)$, and it is monotone in the sense that $b \preceq c$ implies $a\tilde{*}b \preceq a\tilde{*}c$, for any $a, b, c \in \mathscr{S}$.

In this definition, $\tilde{*}$ acts componentwise, i.e., $a\tilde{*}b = (a_1 *_1 b_1, a_2 *_2 b_2, a_3 *_3 b_3)$ for $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3) \in \mathscr{S}$, where $*_1, *_2, *_3$ are binary operations on $[0, 1]$. If these three operations are continuous, then $\tilde{*}$ is called a continuous SF t-norm.

The notion of representability was first introduced for intuitionistic fuzzy t-norms by Deschrijver et al. [36]. Since intuitionistic fuzzy t-norms involve two components, an intuitionistic fuzzy t-norm is called representable when the first component is defined by a t-norm and the second by a t-conorm. Ashraf et al. [29] extended this idea to spherical fuzzy t-norms by considering two t-norms and one t-conorm for their three components. In this paper, we further extend this notion by also allowing a t-conorm in the second component.

Definition 3.2. Let $\tilde{*}$ be an SF t-norm. We say that $\tilde{*}$ is representable if, for all $a, b \in \mathscr{S}$, it can be written in one of the forms

$$a\tilde{\bigstar}b = (a_1 * b_1, a_2 \star b_2, a_3 \diamond b_3) \quad \text{or} \quad a\tilde{\blacksquare}b = (a_1 * b_1, a_2 \square b_2, a_3 \diamond b_3),$$

where $*, \star$ are t-norms and $\square, \diamond$ are t-conorms on $[0, 1]$. When it is not necessary to specify whether the operation in the second component is a t-norm or a t-conorm, we simply write

$$a\tilde{\bullet}b = (a_1 * b_1, a_2 \circ b_2, a_3 \diamond b_3),$$

where $\circ$ stands for either of them.

From now on, all representable SF t-norms are assumed to be continuous. Accordingly, $\tilde{\bigstar}$, $\tilde{\blacksquare}$, and $\tilde{\bullet}$ always denote continuous representable SF t-norms, and in the case of $\tilde{\bullet}$, the operation $\circ$ is a continuous t-norm or a continuous t-conorm.

Example 3.3. For $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ in $\mathscr{S}$, each of the operations below is a continuous representable SF t-norm:

(a) $a\tilde{\bigstar}_1 b = (a_1 \wedge b_1,\ a_2 b_2,\ a_3 \vee b_3)$,

(b) $a\tilde{\bigstar}_2 b = (a_1 b_1,\ a_2 b_2,\ a_3 \vee b_3)$,

(c) $a\tilde{\bigstar}_3 b = \left(a_1 b_1,\ a_2 b_2,\ \sqrt{a_3^2 + b_3^2 - a_3^2 b_3^2}\right)$,

(d) $a\tilde{\bigstar}_4 b = (a_1 \wedge b_1, a_2 \wedge b_2, a_3 \vee b_3)$,

(e) $a\tilde{\blacksquare}_1 b = (a_1 \wedge b_1, a_2 \vee b_2, a_3 \vee b_3)$,

(f) $a\tilde{\blacksquare}_2 b = (a_1 b_1, a_2 \vee b_2, a_3 \vee b_3)$,

(g) $a\tilde{\blacksquare}_3 b = \left(a_1 b_1,\ \sqrt{a_2^2 + b_2^2 - a_2^2 b_2^2},\ \sqrt{a_3^2 + b_3^2 - a_3^2 b_3^2}\right)$,

(h) $a\tilde{\blacksquare}_4 b = \left(a_1 \wedge b_1,\ \sqrt{a_2^2 + b_2^2 - a_2^2 b_2^2},\ \sqrt{a_3^2 + b_3^2 - a_3^2 b_3^2}\right)$.

Definition 3.4. An SF t-norm $\tilde{*}$ is said to be Archimedean if $a\tilde{*}a \prec a$ holds for all $a \in \mathscr{S}(a \neq 0_{\mathscr{S}}, 1_{\mathscr{S}})$.

Remark 3.5. Let $a = (a_1, a_2, a_3) \in \mathscr{S}$. Then we have the following properties for representable SF t-norms:

- $a\tilde{\bigstar}a \preceq a$,
- $a\tilde{\bigstar}1_{\mathscr{S}} \in I(a)$,
- $a\tilde{\bigstar}0_{\mathscr{S}} = 0_{\mathscr{S}}$,
- $a\tilde{\blacksquare}a \preceq a$ if $a_2 \square a_2 = a_2$,
- $a\tilde{\blacksquare}1_{\mathscr{S}} = a$,
- $a\tilde{\blacksquare}0_{\mathscr{S}} = 0_{\mathscr{S}}$ if and only if $a_2 = 0$.

In the following, we propose the concept of spherical fuzzy metric spaces by using SF t-norms and giving fewer axioms than those in IFSM and PFSM given in Definitions 2.14 and 2.18. In the definitions of IFMS and PFMS, the mappings M, N and M, N, W, respectively, are given different mappings; instead of this, we consider these mappings as the components of the metric structure and we do not use different t-norms or t-conorms in the triangle inequality; we use an SF t-norm. On the other hand, in PFMSs, the mapping N behaves similarly to W and in the examples provided for these spaces, N is always taken to be equal to W. We also remove this restriction. Hence, with the proposed approach, we do not need to require many axioms, and this is appropriate to the nature of fuzzy metric spaces.

Definition 3.6. Let X be a nonempty set and $\tilde{*}$ be a continuous SF t-norm. A mapping $\mathbf{S} : X \times X \times (0, \infty) \to \mathscr{S}$ is said to be a spherical fuzzy metric (SFM) on X if the following conditions hold for all $x, y, z \in X$ and $t, s > 0$:
(SFM1) $0_{\mathscr{S}} \prec \mathbf{S}(x, y, t)$,
(SFM2) $x = y \Leftrightarrow \mathbf{S}(x, y, t) = 1_{\mathscr{S}}$,
(SFM3) $\mathbf{S}(x, y, t) = \mathbf{S}(y, x, t)$,
(SFM4) $\mathbf{S}(x, z, t + s) \succeq \mathbf{S}(x, y, t)\tilde{*}\mathbf{S}(y, z, s)$,
(SFM5) $\mathbf{S}(x, y, \cdot) : (0, \infty) \to \mathscr{S}$ is continuous.

In this case, the triple $(X, \mathbf{S}, \tilde{*})$ is called a spherical fuzzy metric space (SFMS).

The value $\mathbf{S}(x, y, t)$ can be regarded as the spherical closeness degree of x and y with respect to t. If we denote $\mathbf{S}(x, y, t) = (M(x, y, t), N(x, y, t), W(x, y, t))$, then we can consider the components $M(x, y, t)$, $N(x, y, t)$ and $W(x, y, t)$ give the degrees of closeness, uncertainty and non-closeness of x and y, respectively. Throughout the rest of the paper, when we write $\mathbf{S} = (M, N, W)$, we mean that $\mathbf{S}(x, y, t) = (M(x, y, t), N(x, y, t), W(x, y, t))$ for all $x, y \in X$ and $t > 0$.

In the following, we present an example based on a representable SF t-norm and the Euclidean distance.

Example 3.7. Let $X = \mathbb{R}$ and let $\tilde{\bigstar}$ be the representable SF t-norm obtained from the product t-norm in the first two components and the maximum t-conorm in the third one, that is, $a\tilde{\bigstar}b = (a_1 b_1, a_2 b_2, a_3 \vee b_3)$

for all $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3) \in \mathscr{S}$. Define $\mathbf{S} : X \times X \times (0, \infty) \to \mathscr{S}$ by

$$\mathbf{S}(x, y, t) = \begin{cases} 1_{\mathscr{S}}, & \text{if } x = y, \\ \left(\frac{1}{3}e^{-\frac{|x-y|}{t}}, \frac{1}{3}e^{-\frac{|x-y|}{t}}, 1 - \frac{2}{3}e^{-\frac{|x-y|}{t}}\right), & \text{if } x \neq y. \end{cases}$$

One can check that $\mathbf{S}$ fulfils (SFM1)–(SFM5); hence $(X, \mathbf{S}, \tilde{\bigstar})$ is an SFMS.

In contrast to Example 3.7, the following example provides an SFM that is independent of the Euclidean distance.

Example 3.8. Let $X = \mathbb{N}$ and let $\tilde{\bigstar}$ be the representable SF t-norm given in Example 3.7. Define $\mathbf{S} : X \times X \times (0, \infty) \to \mathscr{S}$ by

$$\mathbf{S}(x, y, t) = \begin{cases} (\frac{x}{y}, \frac{y-x}{t+y-x}, \frac{y-x}{t}), & \text{if } x \leq y, \\ (\frac{y}{x}, \frac{x-y}{t+x-y}, \frac{x-y}{t}), & \text{if } x \geq y. \end{cases}$$

One can check that $\mathbf{S}$ fulfils (SFM1)–(SFM5); hence $(X, \mathbf{S}, \tilde{\bigstar})$ is an SFMS.

In the following, we investigate the relationships between the components of an SFM and the corresponding metric structures defined in Section 2.

Remark 3.9. Assume that $(X, \mathbf{S}, \tilde{\bigstar})$ is an SFMS with $\mathbf{S} = (M, N, W)$ and $a\tilde{\bigstar}b = (a_1 * b_1, a_2 \star b_2, a_3 \diamond b_3)$. Then, for all $x, y, z \in X$ and $t, s > 0$, the components of $\mathbf{S}$ are related to the structures in Section 2 as follows:

(i) $(X, M, *)$ is an FMS, provided that $M(x, y, t) > 0$.
(ii) $(X, N_1, \star)$ is an FMS, provided that $N(x, y, t) > 0$ and $N(x, y, t) \star N(y, z, s) \leq N(x, z, t + s)$, where

$$N_1(x, y, t) = \begin{cases} 1, & \text{if } x = y, \\ N(x, y, t), & \text{if } x \neq y. \end{cases}$$

(iii) $(X, W, \diamond)$ is a CB-FMS.
(iv) $(X, M, W, *, \diamond)$ is an IFMS, provided that $M(x, y, t) > 0$ and $M(x, y, t) + W(x, y, t) \leq 1$.
(v) $(X, N_1, W, \star, \diamond)$ is an IFMS, provided that $N(x, y, t) > 0$, $N_1(x, y, t) + W(x, y, t) \leq 1$ and $N(x, y, t) \star N(y, z, s) \leq N(x, z, t + s)$.

Next, we remark on the conditions under which an SFMS can be considered a PFMS.

Remark 3.10. Let $(X, \mathbf{S}, \tilde{\bullet})$ be an SFMS with $\mathbf{S} = (M, N, W)$ and $a\tilde{\bullet}b = (a_1 * b_1, a_2 \circ b_2, a_3 \diamond b_3)$. If M and N take only positive values, $M + N + W \leq 1$, and $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$ for all $x, y, z \in X$ and $t, s > 0$, then $(X, M, N, W, *, \diamond)$ is a PFMS.

We show the monotonicity of $S(x, y, t)$ according to the parameter t depends on the properties of the underlying SF t-norm.

Theorem 3.11. *Let $(X, \mathbf{S}, \tilde{*})$ be an SFMS with $\mathbf{S} = (M, N, W)$.*
(i) If $\tilde{*} = \tilde{\bigstar}$ and $N(x, y, \cdot)$ is non-decreasing on $(0, \infty)$, then so is $\mathbf{S}(x, y, \cdot)$.
(ii) If $\tilde{*} = \tilde{\blacksquare}$, then $\mathbf{S}(x, y, \cdot)$ is non-decreasing on $(0, \infty)$.

Proof. Let $x, y \in X$ and $0 < s < t$. Since $\mathbf{S}(y, y, t - s) = 1_{\mathscr{S}}$, (SFM4) gives

$$\mathbf{S}(x, y, s) \,\tilde{*}\, 1_{\mathscr{S}} \preceq \mathbf{S}(x, y, t).$$

(i) Let $a\tilde{\bigstar}b = (a_1 * b_1, a_2 \star b_2, a_3 \diamond b_3)$. Since 1 is neutral for $*$, 0 is absorbing for $\star$ and 0 is neutral for $\diamond$, the above inequality becomes $(M(x, y, s), 0, W(x, y, s)) \preceq \mathbf{S}(x, y, t)$. If this holds by the first or the second clause of (3.1), the same clause yields $\mathbf{S}(x, y, s) \preceq \mathbf{S}(x, y, t)$, since these clauses do not involve the second components. Otherwise, $M(x, y, s) = M(x, y, t)$ and $W(x, y, s) = W(x, y, t)$, and since $N(x, y, \cdot)$ is non-decreasing, $N(x, y, s) \leq N(x, y, t)$. Hence, the third clause of (3.1) gives $\mathbf{S}(x, y, s) \preceq \mathbf{S}(x, y, t)$.
(ii) Let $a\tilde{\blacksquare}b = (a_1 * b_1, a_2 \square b_2, a_3 \diamond b_3)$. Since 0 is neutral for $\square$ and $\diamond$, we have $\mathbf{S}(x, y, s) \,\tilde{\blacksquare}\, 1_{\mathscr{S}} = \mathbf{S}(x, y, s)$, and the above inequality yields $\mathbf{S}(x, y, s) \preceq \mathbf{S}(x, y, t)$. □

The following theorem demonstrates that every classical metric space induces an SFMS.

Theorem 3.12. *Let (X, d) be a metric space, and for all $x, y \in X$ and $t > 0$ put*

$$\begin{aligned} \mathbf{S}_d(x, y, t) &= (M_d(x, y, t), N_d(x, y, t), W_d(x, y, t)) \\ &= \Big(\frac{t}{t + 2d(x, y)}, \frac{d(x, y)}{t + 2d(x, y)}, \frac{d(x, y)}{t + 2d(x, y)}\Big). \end{aligned} \tag{3.3}$$

Then $\mathbf{S}_d$ is an SFM on X with respect to $a\tilde{\bullet}b = (a_1 b_1, a_2 \circ b_2, a_3 \vee b_3)$, that is, $(X, \mathbf{S}_d, \tilde{\bullet})$ is an SFMS.

Proof. Since $M_d(x, y, t) + N_d(x, y, t) + W_d(x, y, t) = 1$ and all components lie in $[0, 1]$, we have $M_d^2(x, y, t) + N_d^2(x, y, t) + W_d^2(x, y, t) \leq 1$, so $\mathbf{S}_d$ takes its values in $\mathscr{S}$. We check the conditions of Definition 3.6.
(SFM1) Since $M_d(x, y, t) > 0$, we have $0_{\mathscr{S}} \prec \mathbf{S}_d(x, y, t)$.
(SFM2) If $x = y$, then $d(x, y) = 0$, and hence $\mathbf{S}_d(x, y, t) = (1, 0, 0) = 1_{\mathscr{S}}$. Conversely, $\mathbf{S}_d(x, y, t) = 1_{\mathscr{S}}$ gives $M_d(x, y, t) = 1$, which forces $d(x, y) = 0$, i.e., $x = y$.
(SFM3) This follows from the symmetry of d.
(SFM4) Let $a = d(x, y)$, $b = d(y, z)$ and $c = d(x, z)$. If $a = b = 0$, then $x = y = z$ and the inequality is trivial. Otherwise, the identity

$$(t + s)(t + 2a)(s + 2b) - ts(t + s + 2a + 2b) = 2bt^2 + 2as^2 + 4ab(t + s) > 0$$

together with $c \leq a + b$ gives

$$\begin{aligned} M_d(x, y, t)\, M_d(y, z, s) = \frac{ts}{(t + 2a)(s + 2b)} &< \frac{t + s}{t + s + 2a + 2b} \\ &\leq \frac{t + s}{t + s + 2c} = M_d(x, z, t + s). \end{aligned}$$

Moreover, since $u \mapsto \frac{u}{r+2u}$ is increasing on $[0, \infty)$ for each $r > 0$, we get

$$\begin{aligned} W_d(x, z, t + s) = \frac{c}{t + s + 2c} &\leq \frac{a + b}{(t + 2a) + (s + 2b)} \\ &\leq \max\Big\{\frac{a}{t + 2a}, \frac{b}{s + 2b}\Big\} = W_d(x, y, t) \vee W_d(y, z, s), \end{aligned}$$

where the last step is the elementary mediant inequality. Therefore, the first clause of (3.1) yields $\mathbf{S}_d(x, y, t) \,\tilde{\bullet}\, \mathbf{S}_d(y, z, s) \preceq \mathbf{S}_d(x, z, t + s)$.
(SFM5) Since $d(x, y)$ does not depend on t, each component of $\mathbf{S}_d(x, y, \cdot)$ is continuous on $(0, \infty)$. □

Definition 3.13. We call $(\mathbf{S}_d, \tilde{\bullet})$ in Theorem 3.12 the standard spherical fuzzy metric induced by d.

For instance, if we take $d(x, y) = \rho(x, y)$ for all $x, y \in X$ where ρ is the discrete metric on X in Eq (3.3), we have the following example:

Example 3.14. Let (X, ρ) be the discrete metric space and take $a\tilde{\bullet}b = (a_1b_1, a_2 \circ b_2, a_3 \vee b_3)$. Then $(\mathbf{S}_\rho, \tilde{\bullet})$ is the standard SFM on X where the mapping $\mathbf{S}_\rho$ is given as

$$\mathbf{S}_\rho(x, y, t) = \begin{cases} 1_{\mathscr{S}}, & \text{if } x = y \\ \left(\frac{t}{t+2}, \frac{1}{t+2}, \frac{1}{t+2}\right), & \text{if } x \neq y \end{cases}$$

for all $x, y \in X$ and $t > 0$.

We also note that, for all $x, y \in X$ and $t > 0$ with $k, m, n \in \mathbb{R}^+$,

$$\begin{aligned} \mathbf{S}_d(x, y, t) &= (M_d(x, y, t), N_d(x, y, t), W_d(x, y, t)) \\ &= \Big(\frac{kt^n}{kt^n + 2md(x, y)}, \frac{md(x, y)}{kt^n + 2md(x, y)}, \frac{md(x, y)}{kt^n + 2md(x, y)}\Big) \end{aligned} \tag{3.4}$$

is an SFM on X. Thus, we can obtain different SFMs from each classical metric.

We investigate that we can induce different SFMs by considering the order relation together with the above equality, since the order relation enables us to relax the second component, as shown in the following way.

Remark 3.15. Let (X, d) be a classical metric space. Take $M_d(x, y, t) = \frac{kt^n}{kt^n+2md(x,y)}$ and $W_d(x, y, t) = \frac{md(x,y)}{kt^n+2md(x,y)}$ for all $x, y \in X$, $t > 0$ and $k, m, n \in \mathbb{R}^+$. Assume that f is a function on $X \times X \times (0, \infty)$, which satisfies the following conditions:

(i) $M_d^2(x, y, t) + f^2(x, y, t) + W_d^2(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$,
(ii) $f(x, x, t) = 0$ for all $t > 0$,
(iii) $f(x, y, t) = f(y, x, t)$ for all $x, y \in X$ and $t > 0$,
(iv) $f(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

Define the mapping $\mathbf{S}_{d,f}$ as follows:

$$\mathbf{S}_{d,f}(x, y, t) = (M_d(x, y, t), f(x, y, t), W_d(x, y, t)) = \Big(\frac{kt^n}{kt^n + 2md(x, y)}, f(x, y, t), \frac{md(x, y)}{kt^n + 2md(x, y)}\Big).$$

Then $(X, \mathbf{S}_{d,f}, \tilde{\bullet})$ is an SFMS.

Example 3.16. Let (X, ρ) be the discrete metric space and take $a\tilde{\bullet}b = (a_1b_1, a_2 \circ b_2, a_3 \vee b_3)$. Then $(\mathbf{S}_{\rho,0}, \tilde{\bullet})$ is an SFM on X where the mapping $\mathbf{S}_{\rho,0}$ is given as

$$\mathbf{S}_{\rho,0}(x, y, t) = \begin{cases} 1_{\mathscr{S}}, & \text{if } x = y \\ \left(\frac{t}{t+2}, 0, \frac{1}{t+2}\right), & \text{if } x \neq y \end{cases}.$$

To show that $(\mathbf{S}_{\rho,0}, \tilde{\bullet})$ is an SFM on X, let us take $d(x, y) = \rho(x, y)$ and $f(x, y, t) = 0$ in Remark 3.15. Here, the function $f(x, y, t) = 0$ satisfies conditions (ii)–(iv) in Remark 3.15. We also have $lim_{t\to\infty}\left(\left(\frac{t}{t+2}\right)^2 + \left(\frac{1}{t+2}\right)^2\right) = lim_{t\to\infty}\frac{t^2+1}{(t+2)^2} = 1$, and this equality guarantees that condition (i) of Remark 3.15 is satisfied. Hence, $(X.\mathbf{S}_{\rho,0}, \tilde{\bullet})$ is an SFMS.

The following theorem gives a method of obtaining an SFMS from any FMS.

Theorem 3.17. *Let $(X, M, *)$ be an FMS.*

(i) Let $\alpha, \beta, \gamma \in (0, 1]$ with $\alpha^2 + \beta^2 + \gamma^2 \leq 1$, and let $a\tilde{\bigstar}b = (a_1 * b_1, a_2 * b_2, a_3 \diamond_* b_3)$. If $\mathbf{S}_M$ is defined by

$$\mathbf{S}_M(x, y, t) = \begin{cases} 1_{\mathscr{S}}, & x = y, \\ (\alpha M(x, y, t), \beta M(x, y, t), \gamma(1 - M(x, y, t))), & x \neq y, \end{cases}$$

then $(X, \mathbf{S}_M, \tilde{\bigstar})$ is an SFMS.

(ii) Let $\alpha, \beta \in (0, 1]$ with $\alpha^2 + 2\beta^2 \leq 1$, and let $a\tilde{\blacksquare}b = (a_1 * b_1, a_2 \diamond_* b_2, a_3 \diamond_* b_3)$. If $\mathbf{S}_M$ is defined by

$$\mathbf{S}_M(x, y, t) = \begin{cases} 1_{\mathscr{S}}, & x = y, \\ (\alpha M(x, y, t), \beta(1 - M(x, y, t)), \beta(1 - M(x, y, t))), & x \neq y, \end{cases}$$

then $(X, \mathbf{S}_M, \tilde{\blacksquare})$ is an SFMS.

Proof. (i) Let $x, y, z \in X$ and $t, s > 0$. Since $M(x, y, t) \in (0, 1)$ for $x \neq y$, we have $\alpha^2 M^2(x, y, t) + \beta^2 M^2(x, y, t) + \gamma^2(1 - M(x, y, t))^2 < \alpha^2 + \beta^2 + \gamma^2 \leq 1$, so $\mathbf{S}_M$ takes its values in $\mathscr{S}$. We now verify the conditions of Definition 3.6.
(SFM1) Since $M(x, y, t) > 0$, we get $\alpha M(x, y, t) > 0$, and hence $0_{\mathscr{S}} \prec \mathbf{S}_M(x, y, t)$.
(SFM2) and (SFM3) follow directly from the definition of $\mathbf{S}_M$ and the symmetry of M.
(SFM4) As noted after the definition of fuzzy metric, $(X, M_\alpha, *)$ and $(X, M_\beta, *)$ are FMSs, and by Remark 2.13(i), $(X, \gamma(1 - M), \diamond_*)$ is a CB-FMS. Therefore,

$$\begin{aligned} &\alpha M(x, y, t) * \alpha M(y, z, s) \leq \alpha M(x, z, t + s), \\ &\quad \gamma(1 - M(x, z, t + s)) \leq \gamma(1 - M(x, y, t)) \diamond_* \gamma(1 - M(y, z, s)). \end{aligned}$$

If the first inequality is strict, the first clause of (3.1) gives $\mathbf{S}_M(x, y, t)\tilde{\bigstar}\mathbf{S}_M(y, z, s) \leq \mathbf{S}_M(x, z, t + s)$. Moreover, if it is an equality and the second inequality is strict, the same conclusion follows from the second clause of (3.1). Finally, if both are equalities, we use $\beta M(x, y, t) * \beta M(y, z, s) \leq \beta M(x, z, t + s)$, which holds since $(X, M_\beta, *)$ is an FMS, and the third clause of (3.1) completes the argument.
(SFM5) The continuity of $\mathbf{S}_M(x, y, \cdot)$ follows from that the continuity of $M(x, y, \cdot)$.
(ii) It can be proved in the same way as (i). □

In a similar way, an SFMS can be obtained from a CB-FMS by using the appropriate representable SF t-norms.

Remark 3.18. Let $(X, N, \diamond)$ be a CB-FMS.

(i) If $\alpha, \beta, \gamma \in (0, 1]$ with $\alpha^2 + \beta^2 + \gamma^2 \leq 1$, $a\tilde{\bigstar}b = (a_1 *_\diamond b_1, a_2 *_\diamond b_2, a_3 \diamond b_3)$ and

$$\mathbf{S}_N(x, y, t) = \begin{cases} 1_{\mathscr{S}}, & x = y, \\ (\alpha(1 - N(x, y, t)), \beta(1 - N(x, y, t)), \gamma N(x, y, t)), & x \neq y, \end{cases}$$

then $(X, \mathbf{S}_N, \tilde{\bigstar})$ is an SFMS.

(ii) If $\alpha, \beta \in (0, 1]$ with $\alpha^2 + 2\beta^2 \leq 1$, $a\tilde{\blacksquare}b = (a_1 *_\diamond b_1, a_2 \diamond b_2, a_3 \diamond b_3)$ and

$$\mathbf{S}_N(x, y, t) = \begin{cases} 1_{\mathscr{S}}, & x = y, \\ (\alpha(1 - N(x, y, t)), \beta N(x, y, t), \beta N(x, y, t)), & x \neq y, \end{cases}$$

then $(X, \mathbf{S}_N, \tilde{\blacksquare})$ is an SFMS.

Example 3.19. Consider the FMS $(X, M, *)$ given in [3], where X is a nonempty set, $a * b = ab$ and

$$M(x, y, t) = \begin{cases} 1, & \text{if } x = y, \\ e^{-\frac{1}{t}}, & \text{if } x \neq y. \end{cases}$$

Taking $\alpha = \frac{1}{3}, \beta = \frac{1}{5}$ and $\gamma = \frac{1}{4}$ in Theorem 3.17(i), we get

$$\mathbf{S}_M(x, y, t) = \begin{cases} 1_{\mathscr{S}}, & \text{if } x = y, \\ \left(\frac{1}{3}e^{-\frac{1}{t}}, \frac{1}{5}e^{-\frac{1}{t}}, \frac{1}{4} - \frac{1}{4}e^{-\frac{1}{t}}\right), & \text{if } x \neq y. \end{cases}$$

Hence, $(X, \mathbf{S}_M, \tilde{\bigstar})$ is an SFMS, where $a\tilde{\bigstar}b = (a_1b_1, a_2b_2, a_3 + b_3 - a_3b_3)$.

Remark 3.20. In Theorem 3.17(i), SF t-norm can be chosen such as $a\tilde{\bigstar}b = (a_1 * b_1, a_2 \star b_2, a_3 \diamond_* b_3)$ or $a\tilde{\bigstar}b = (a_1 \star b_1, a_2 \star b_2, a_3 \diamond_* b_3)$ or $a\tilde{\bigstar}b = (a_1 \star b_1, a_2 \star b_2, a_3 \diamond_\star b_3)$ where $\star$ is any continuous t-norm which is weaker than $*$. Similarly, in Theorem 3.17(ii), it can be chosen such as $a\tilde{\blacksquare}b = (a_1 \star b_1, a_2 \diamond_* b_2, a_3 \diamond_* b_3)$ or $a\tilde{\blacksquare}b = (a_1 \star b_1, a_2 \diamond_\star b_2, a_3 \diamond_* b_3)$ or $a\tilde{\blacksquare}b = (a_1 \star b_1, a_2 \diamond_\star b_2, a_3 \diamond_\star b_3)$ under same condition.

SFMs can also be constructed from IFMs as follows:

Remark 3.21. Let $(X, M, W, *, \diamond)$ be an IFMS.

(i) Let

$$N(x, y, t) = \begin{cases} 0, & \text{if } x = y, \\ M(x, y, t), & \text{if } x \neq y, \end{cases}$$

and suppose that $M^2(x, y, t) + N^2(x, y, t) + W^2(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$. Then $\mathbf{S} = (M, N, W)$ is an SFM on X with respect to $a\tilde{\bigstar}b = (a_1 * b_1, a_2 * b_2, a_3 \diamond b_3)$, that is, $(X, \mathbf{S}, \tilde{\bigstar})$ is an SFMS.

(ii) Suppose that $M^2(x, y, t) + 2W^2(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$. Then $\mathbf{S} = (M, W, W)$ is an SFM on X with respect to $a\tilde{\blacksquare}b = (a_1 * b_1, a_2 \diamond b_2, a_3 \diamond b_3)$, that is, $(X, \mathbf{S}, \tilde{\blacksquare})$ is an SFMS.

Since SFMS is a generalization of PFMS, every PFMS can be taken as an SFMS, as stated in the following remark.

Remark 3.22. Let $(X, M, N, W, *, \diamond)$ be a PFMS. If we take $\mathbf{S}(x, y, t) = (M(x, y, t), N(x, y, t), W(x, y, t))$ for all $x, y \in X$ and $t > 0$, $a\tilde{\blacksquare}b = (a_1 * b_1, a_2 \diamond b_2, a_3 \diamond b_3)$, then $(X, \mathbf{S}, \tilde{\blacksquare})$ is an SFMS.

The relations between SFMS with FMS, CB-FMS, IFSM and PFMS are shown in Figure 3.

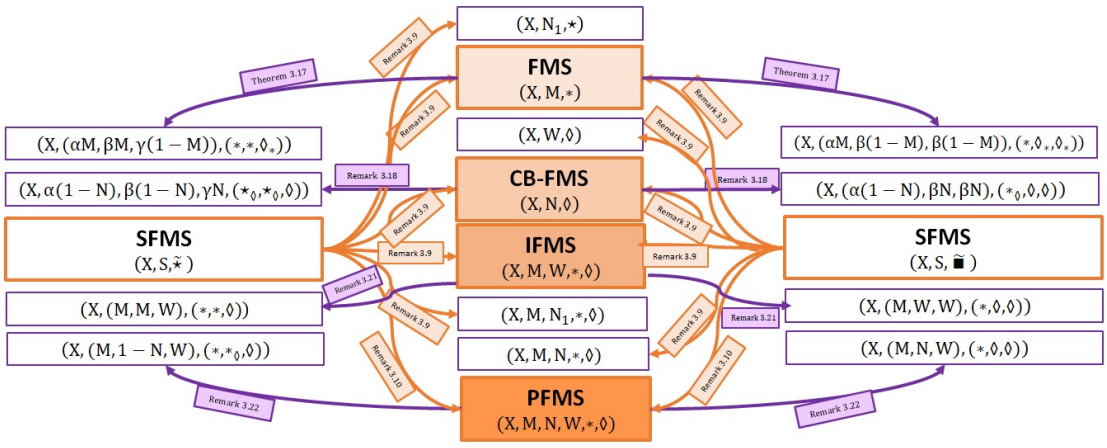

Figure 3. Relation among spherical and other fuzzy metric spaces.

3.2. Analysis of the sensitivity of fuzzy metric structures on noisy images

In image processing, noise is defined as unwanted disturbances that affect the quality of images, signals, or data in applications, including communication systems and signal analysis. Salt-and-pepper noise is among the most frequently encountered types of image noise and belongs to the category of impulse noise. It can arise from faulty camera sensors, memory cell defects, or errors during image capture and data transmission. This noise is characterized by randomly appearing black and white pixels, whose intensity values are typically set to 0 or 1, respectively. Such random changes may result in the loss of valuable image information, a decrease in visual quality, and difficulties in maintaining important structural features.

In this subsection, the similarity between the noisy images obtained by applying salt-and-pepper noise at different rates to a given color image and the original image is calculated using FM, IFM, and SFM. We prefer salt-and-pepper noise because it abruptly distorts individual pixels of an image, which makes it possible to analyze, by means of similarity matrices, how the fuzzy structures react to the damage. To investigate these similarity results, we first take the color image (given in Figure 4). The dimensions of this image are 135×198×3. We then convert this image to a gray-level image, as shown in Figure 4.



Figure 4. Original image and gray-level of the original image.

Then, salt-and-pepper noise is applied at rates of 0.01, 0.1, and 0.5 to obtain noisy images at various levels in Figure 5.



Figure 5. Noise-added versions of the gray-level image at densities of 0.01, 0.1, 0.5.

Here, we first calculate the similarities between the pixels of the original image and the noisy images using the FM, IFM, and SFM. We use the mappings $M(x, y, t) = e^{-\frac{|x-y|}{t}}$ (which is an FM), $S_M(x, y, t) = \begin{cases} (1, 0, 0), & x = y, \\ \left(\frac{1}{3}e^{-\frac{|x-y|}{t}}, \frac{1}{3}e^{-\frac{|x-y|}{t}}, 1 - \frac{2}{3}e^{-\frac{|x-y|}{t}}\right), & x \neq y, \end{cases}$ (which is SFM and given in Example 3.7), and $(M, N) = \left(e^{-\frac{|x-y|}{t}}, 1 - e^{-\frac{|x-y|}{t}}\right)$ (which is IFM and induced by the same FM). Furthermore, we consider the five different score functions defined in Definition 2.5 to calculate involving the SFM to compute the SFM1–SFM5 values, thereby comparing the score functions with one another. Then, we obtain the overall similarity values for each fuzzy metric by averaging the elements of these similarity matrices, and we present the results in Table 1.

Table 1. Similarity results of FM, IFM, and SFM (with different score values) under different noise densities.

Metric structure	Noise density		
	0.01	0.1	0.5
FM	0.9936	0.9406	0.7031
IFM	0.9885	0.8804	0.4058
SFMI (S_M and SF_1)	0.9940	0.9422	0.7107
SFM2 (S_M and SF_2)	0.9934	0.9396	0.6911
SFM3 (S_M and SF_3)	0.9862	0.8487	0.2305
SFM4 (S_M and SF_4)	0.9953	0.9491	0.7437
SFM5 (S_M and SF_5)	0.9897	0.9004	0.5091

The results (given in Table 1) show that similarity values generally decrease as noise density increases, although the magnitude of this decrease varies across the considered fuzzy metric structures and score functions. At a noise density of 0.01, all methods yield high similarity values, ranging from 0.9862 to 0.9953. At a noise density of 0.1, the similarity values decrease slightly compared with those obtained at a noise density of 0.01, with values ranging from 0.8487 to 0.9491. SFM3 shows a more substantial reduction, yielding a similarity value of 0.8487. At the highest noise density (0.5), the differences become more pronounced. SFM3 again gives the lowest value (0.2305), indicating a substantial reduction in similarity results. This large decrease suggests that SFM3 is more sensitive to increasing noise density and better distinguishes low-noise and high-noise conditions. Thus, these findings suggest that the chosen fuzzy metric structure affects the similarity values obtained under noisy conditions, while the choice of score function has a considerable effect on the

behavior of SFM. These results are shown in Figure 6.

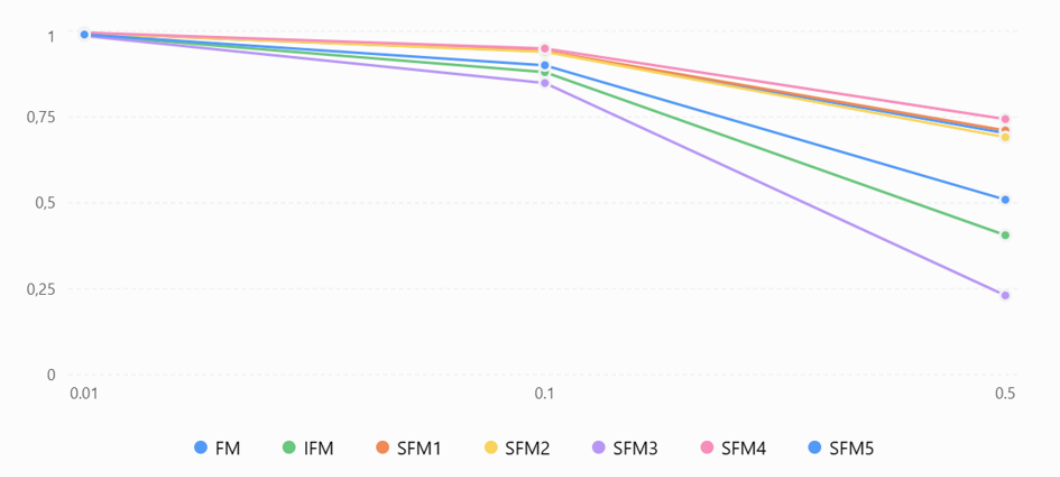

Figure 6. Effects of noise density on similarity values of FM, IFM, and SFM.

As a consequence, we can interpret that SFM with the score function SF_3 is more sensitive to increasing noise density than the FM and IFM approaches, making changes in image similarity more pronounced under noisy conditions.

4. Banach fixed point theorem in spherical fuzzy metric spaces

This section is devoted to sequences in SFMSs. After introducing convergence, Cauchy sequences, and completeness, and describing the limits they impose on SFNs, we prove a Banach-type fixed point theorem and illustrate it with an example.

Definition 4.1. Let $(X, \mathbf{S}, \tilde{*})$ be an SFMS. A sequence (x_n) in X is said to converge to $x \in X$, and we write $x_n \xrightarrow{\mathbf{S}} x$, if for every $\varepsilon \succ 0_{\mathscr{S}}$ there exists $n_0 \in \mathbb{N}$ such that $\mathbf{S}(x_n, x, t) \succ \mathscr{N}_0(\varepsilon)$ for all $n \geq n_0$ and all $t > 0$. Here, $\mathscr{N}_0$ is the negation on SFNs introduced in [29], given by $\mathscr{N}_0(\varepsilon) = (\varepsilon_3, 0, \varepsilon_1)$ for $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \mathscr{S}$.

Throughout, limits of $\mathbf{S}$ are understood componentwise.

Proposition 4.2. *Let $(X, \mathbf{S}, \tilde{*})$ be an SFMS, $x \in X$ and (x_n) be a sequence in X. Then the following holds:*

$$x_n \xrightarrow{\mathbf{S}} x \Rightarrow \lim_{n\to\infty} \mathbf{S}(x_n, x, t) = (l_1, l_2, 0)$$

where $l_1 \leq 1$ and $l_2 \geq 0$.

Proof. Assume that $x_n \xrightarrow{\mathbf{S}} x$ and take $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \succ 0_{\mathscr{S}}$. Then, there is an $n_0 \in \mathbb{N}$ such that $\mathbf{S}(x_n, x, t) \succ \mathscr{N}_0(\varepsilon)$ for all $t > 0$ whenever $n \geq n_0$. If $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, then we have three cases: $(\varepsilon_1 > 0 \wedge \varepsilon_3 \leq 1)$ or $(\varepsilon_1 = 0 \wedge \varepsilon_3 < 1)$ or $(\varepsilon_1 = 0 \wedge \varepsilon_3 = 0 \wedge \varepsilon_2 > 0)$. However, the case $(\varepsilon_1 = 0 \wedge \varepsilon_3 = 1 \wedge \varepsilon_2 > 0)$ is not possible since $\varepsilon \in \mathscr{S}$. So, we need to consider two cases mentioned above. Similarly, if we have $\mathbf{S}(x_n, x, t) \succ \mathscr{N}_0(\varepsilon) = (\varepsilon_3, 0, \varepsilon_1)$ for all $t > 0$ whenever $n \geq n_0$, then we need to handle again three cases: $(M(x_n, x, t) > \varepsilon_3 \wedge W(x_n, x, t) \leq \varepsilon_1)$ or $(M(x_n, x, t) = \varepsilon_3 \wedge W(x_n, x, t) < \varepsilon_1)$

or $(M(x_n, x, t) = \varepsilon_3 \wedge W(x_n, x, t) = \varepsilon_1 \wedge N(x_n, x, t) > 0)$. When we consider these cases, we have the followings:

(I) If $(M(x_n, x, t) > \varepsilon_3 \wedge W(x_n, x, t) \leq \varepsilon_1)$ when $(\varepsilon_1 > 0 \wedge \varepsilon_3 \leq 1)$ or $(\varepsilon_1 = 0 \wedge \varepsilon_3 < 1)$, we obtain $lim_{n\to\infty} M(x_n, x, t) = 1$ and $lim_{n\to\infty} W(x_n, x, t) = 0$. So, $lim_{n\to\infty} N(x_n, x, t)$ should be zero and these imply that $\lim_{n\to\infty} \mathbf{S}(x_n, x, t) = (1, 0, 0) = 1_{\mathscr{S}}$.

(II) If $(M(x_n, x, t) = \varepsilon_3 \wedge W(x_n, x, t) < \varepsilon_1)$ when $(\varepsilon_1 > 0 \wedge \varepsilon_3 \leq 1)$ or $(\varepsilon_1 = 0 \wedge \varepsilon_3 < 1)$, we obtain $lim_{n\to\infty} M(x_n, x, t) \leq 1$ and $lim_{n\to\infty} W(x_n, x, t) = 0$. Now, $lim_{n\to\infty} N(x_n, x, t)$ may be non-zero and these imply that $\lim_{n\to\infty} \mathbf{S}(x_n, x, t) = (l_1, l_2, 0)$ where $l_1 \leq 1$ and $l_2 \geq 0$.

(III) If $(M(x_n, x, t) = \varepsilon_3 \wedge W(x_n, x, t) = \varepsilon_1 \wedge N(x_n, x, t) > 0)$ when $(\varepsilon_1 > 0 \wedge \varepsilon_3 \leq 1)$ or $(\varepsilon_1 = 0 \wedge \varepsilon_3 < 1)$, again we $\lim_{n\to\infty} \mathbf{S}(x_n, x, t) = (l_1, l_2, 0)$ where $l_1 \leq 1$ and $l_2 \geq 0$ similar to (II). □

If, in addition, the SF t-norm is representable and the first component of the SFM is positive, then the limit in Proposition 4.2 is necessarily $1_{\mathscr{S}}$.

Corollary 4.3. *Let $(X, \mathbf{S}, \tilde{\bullet})$ be an SFMS with $\mathbf{S} = (M, N, W)$ such that $M(x, y, t) > 0$ for all $x, y \in X$ and $t > 0$. If a sequence (x_n) in X converges to $x \in X$, then $\lim_{n\to\infty} \mathbf{S}(x_n, x, t) = 1_{\mathscr{S}}$.*

Definition 4.4. A sequence (x_n) in an SFMS $(X, \mathbf{S}, \tilde{*})$ is called a Cauchy sequence if for every $\varepsilon \succ 0_{\mathscr{S}}$ there exists $n_0 \in \mathbb{N}$ such that $\mathbf{S}(x_n, x_m, t) \succ \mathscr{N}_0(\varepsilon)$ for all $n, m \geq n_0$ and all $t > 0$.

Analogous to Proposition 4.2, we have the following property for Cauchy sequences in SFMSs.

Proposition 4.5. *Let $(X, \mathbf{S}, \tilde{*})$ be an SFMS and (x_n) be a sequence in X. Then the following holds:*

$$x_n \text{ is a Cauchy sequence in } (X, \mathbf{S}, \tilde{*}) \Rightarrow \lim_{n,m\to\infty} \mathbf{S}(x_n, x_m, t) = (l_1, l_2, 0)$$

where $l_1 \leq 1$ and $l_2 \geq 0$.

Proof. The proof can be provided on a case-by-case basis, as in the proof of Proposition 4.2. □

Corollary 4.6. *Let $(X, \mathbf{S}, \tilde{\bullet})$ be an SFMS with $\mathbf{S} = (M, N, W)$ such that $M(x, y, t) > 0$ for all $x, y \in X$ and $t > 0$. If (x_n) is a Cauchy sequence in X, then $\lim_{n,m\to\infty} \mathbf{S}(x_n, x_m, t) = 1_{\mathscr{S}}$.*

Proposition 4.7. *Every convergent sequence is a Cauchy sequence in $(X, \mathbf{S}, \tilde{\bullet})$.*

Proof. Let (x_n) be a convergent sequence to a point $x \in Z$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \succ 0_{\mathscr{S}}$. Then, there is an $n_0 \in \mathbb{N}$ such that $\mathbf{S}(x_n, x, t) \succ \mathscr{N}_0(\varepsilon)$ for all $t > 0$ whenever $n \geq n_0$. Additionally, there exists an $\varepsilon^* = (\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*) \succ 0_{\mathscr{S}}$ such that $(\varepsilon_3^*, 0, \varepsilon_1^*)\tilde{\bullet}(\varepsilon_3^*, 0, \varepsilon_1^*) = (\varepsilon_3^* * \varepsilon_3^*, 0, \varepsilon_1^* \diamond \varepsilon_1^*) \succ (\varepsilon_3, 0, \varepsilon_1)$ since $*$ and $\diamond$ are continuous. This implies that $\mathscr{M}(\varepsilon^*)\tilde{\bullet}\mathscr{M}(\varepsilon^*) \succ \mathscr{M}(\varepsilon)$. Hence, we obtain

$$\mathbf{S}(x_n, x_m, t) \geq \mathbf{S}(x_n, x, \frac{t}{2})\tilde{\bullet}\mathbf{S}(x, x_m, \frac{t}{2}) \succ \mathscr{M}(\varepsilon^*)\tilde{\bullet}\mathscr{M}(\varepsilon^*) \succ \mathscr{M}(\varepsilon)$$

for all $t > 0$ whenever $n, m \geq n_0$. This shows that (x_n) is a Cauchy sequence. □

Definition 4.8. An SFMS $(X, \mathbf{S}, \tilde{*})$ is called complete if each of its Cauchy sequences is convergent in X.

We are now in a position to state and prove the Banach fixed point theorem for SFMSs.

Theorem 4.9. *Let $(X, \mathbf{S}, \tilde{\blacksquare})$ be a complete SFMS with $\mathbf{S} = (M, N, W)$ such that $M(x, y, t) > 0$ and $\lim_{t\to\infty} \mathbf{S}(x, y, t) = 1_{\mathscr{S}}$ for all $x, y \in X$ and $t > 0$. If $T : X \to X$ is a mapping for which there exists $k \in (0, 1)$ satisfying*

$$\mathbf{S}(x, y, t) \preceq \mathbf{S}(T(x), T(y), kt)$$

for all $x, y \in X$ and $t > 0$, then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ and $x_n = T^n(x_0)$ be an iteration sequence for all $n \in \mathbb{N}$. First, we need to show that the sequence (x_n) is a Cauchy sequence. Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \succ 0_{\mathscr{S}}$. There exists $\varepsilon_1^* > 0$ and $\varepsilon_3^* > 0$ such that $(1 - \varepsilon_1^*) * (1 - \varepsilon_1^*) * ... * (1 - \varepsilon_1^*) > \varepsilon_3$ and $\varepsilon_3 \diamond \varepsilon_3 \diamond ... \diamond \varepsilon_3 < \varepsilon_1$ since $*$ and $\diamond$ are continuous. From the contractive condition, we have

$$\begin{aligned}
\mathbf{S}(x_n, x_{n+1}, t) &= \mathbf{S}(T(x_{n-1}), T(x_n), t) \succeq \mathbf{S}(x_{n-1}, x_n, \frac{t}{k}) = \mathbf{S}(T(x_{n-2}), T(x_{n-1}), \frac{t}{k}) \\
&\succeq \mathbf{S}(x_{n-2}, x_{n-1}, \frac{t}{k^2}) \succeq ... \succeq \mathbf{S}(x_0, x_1, \frac{t}{k^n}).
\end{aligned}$$

Suppose that $n < m$ for $n, m \in \mathbb{N}$. Then, we have

$$\begin{aligned}
\mathbf{S}(x_n, x_m, t) &\succeq \mathbf{S}(x_n, x_{n+1}, \frac{t}{m-n}) \tilde{*} \mathbf{S}(x_{n+1}, x_{n+2}, \frac{t}{m-n}) \tilde{*} ... \tilde{*} \mathbf{S}(x_{m-1}, x_m, \frac{t}{m-n}) \\
&\succeq \mathbf{S}(x_0, x_1, \frac{t}{(m-n)k^n}) \tilde{*} \mathbf{S}(x_0, x_1, \frac{t}{(m-n)k^{n+1}}) \tilde{*} ... \tilde{*} \mathbf{S}(x_0, x_1, \frac{t}{(m-n)k^{m-1}}) \\
&= \Big(M(x_0, x_1, \frac{t}{(m-n)k^n}) * M(x_0, x_1, \frac{t}{(m-n)k^{n+1}}) * ... * M(x_0, x_1, \frac{t}{(m-n)k^{m-1}}), \\
&\quad N(x_0, x_1, \frac{t}{(m-n)k^n}) \square N(x_0, x_1, \frac{t}{(m-n)k^{n+1}}) \square ... \square N(x_0, x_1, \frac{t}{(m-n)k^{m-1}}), \\
&\quad W(x_0, x_1, \frac{t}{(m-n)k^n}) \diamond W(x_0, x_1, \frac{t}{(m-n)k^{n+1}}) \diamond ... \diamond W(x_0, x_1, \frac{t}{(m-n)k^{m-1}}) \Big).
\end{aligned}$$

Since $\lim_{t\to\infty} \mathbf{S}(x, y, t) = 1_{\mathscr{S}}$ for all $x, y \in X$, we obtain $\lim_{t\to\infty} M(x, y, t) = 1$, $\lim_{t\to\infty} N(x, y, t) = 0$ and $\lim_{t\to\infty} W(x, y, t) = 0$. Since $\lim_{t\to\infty} M(x, y, t) = 1$ and $\varepsilon_1^* > 0$, there exists $n_1 > 0$ such that $M(x, y, t) > 1 - \varepsilon_1^*$ for all $t > n_1$. Similarly, since $\lim_{t\to\infty} N(x, y, t) = 0$ and $\varepsilon_3^* > 0$, there exists $n_2 > 0$ such that $N(x, y, t) < \varepsilon_3^*$ for all $t > n_2$. If we chose $n^* = max\{n_1, n_2\}$, then we obtain $\mathbf{S}(x_n, x_m, t) \succ (\varepsilon_3, 0, \varepsilon_1) = \mathscr{N}(\varepsilon)$ for all $t > n^*$ and $n, m \geq n_0$. Hence, (x_n) is a Cauchy sequence in X. There exists a point $x \in X$ such that $x_n \xrightarrow{\mathbf{S}} x$ since $(X, \mathbf{S}, \tilde{\blacksquare})$ is a complete SFMS. We now claim that x is a fixed point of T. To prove this claim we need to show that $\mathbf{S}(Tx, x, t) = 1_{\mathscr{S}}$ for all $t > 0$. Since $M(x, y, t) > 0$, then we have $M(Tx, x, t) \geq M(Tx, x_n, \frac{t}{2}) * M(x_n, x, \frac{t}{2})$ which implies that $M(Tx, x, t) = 1$ from Corollary 4.3. Hence, we obtain $\mathbf{S}(Tx, x, t) = (1, 0, 0) = 1_{\mathscr{S}}$ for all $t > 0$. This shows that x is a fixed point of T. To show the uniqeness, suppose that $y \in X$ is another fixed point of T. Then, we obtain

$$\mathbf{S}(x, y, t) = \mathbf{S}(Tx, Ty, t) \succeq \mathbf{S}(x, y, \frac{t}{k}) = \mathbf{S}(Tx, Ty, \frac{t}{k}) \succeq \mathbf{S}(x, y, \frac{t}{k^2})$$

which gives a contradiction since $k \in (0, 1)$ and $\mathbf{S}(x, y, \cdot) : (0, \infty) \to [0, 1]^3$ is non-decreasing. As a result, T has a unique fixed point in X. □

In the following, we provide an illustrative example for Banach fixed point theorem:

Example 4.10. Let $X = \mathbb{R}$ and consider $\mathbf{S} = (M, N, W)$ given by

$$M(x, y, t) = \frac{t}{t + |x - y|}, \quad N(x, y, t) = 0, \quad W(x, y, t) = \frac{|x - y|}{t + |x - y|}$$

for all $x, y \in \mathbb{R}$ and $t > 0$. Then $\mathbf{S}$ is an SFM on $\mathbb{R}$ with respect to $a\tilde{\blacksquare}b = (a_1b_1, a_2 \vee b_2, a_3 \vee b_3)$, and $(X, \mathbf{S}, \tilde{\blacksquare})$ is a complete SFMS. Thus, M takes only positive values and $\lim_{t\to\infty} \mathbf{S}(x, y, t) = 1_{\mathscr{S}}$. Let $T : \mathbb{R} \to \mathbb{R}$ be given by $Tx = \frac{x}{2}$ and fix $k \in (\frac{1}{2}, 1)$. Since $|Tx - Ty| = \frac{1}{2}|x - y|$, we get

$$M(Tx, Ty, kt) = \frac{2kt}{2kt + |x - y|} \geq M(x, y, t),$$
$$W(Tx, Ty, kt) = \frac{|x - y|}{2kt + |x - y|} \leq W(x, y, t),$$

so $\mathbf{S}(x, y, t) \preceq \mathbf{S}(Tx, Ty, kt)$ for all $x, y \in \mathbb{R}$ and $t > 0$. Hence, T satisfies all assumptions of Theorem 4.9, and indeed 0 is its unique fixed point.

5. Conclusions

In this study, we introduce the concept of spherical fuzzy metric spaces, extending classical fuzzy metric spaces by incorporating the spherical fuzzy set framework. We establish its fundamental properties, examine its relations with existing fuzzy metric structures, and demonstrate its practical effectiveness in image processing applications. In addition, the well-known Banach fixed point theorem is given in the proposed spaces. For future research, spherical fuzzy metric theory can be further explored through the study of continuity, convergence, and completeness properties. Additionally, some potential applications such as optimization, machine learning, and medical image analysis can be investigated using this new concept.

Author contributions

Başak Aldemir: Conceptualization, Methodology, Investigation, Writing original draft, Visualization; Ebru Aydoğdu: Conceptualization, Methodology, Investigation, Writing–review and editing; Elif Güner: Conceptualization, Methodology, Validation, Formal analysis, Writing–review and editing; Halis Aygün: Conceptualization, Methodology, Funding acquisition, Supervision, Validation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The author declares that no Artificial Intelligence (AI) tools were used in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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