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Research article

The super integrable hierarchies associated with Lie superalgebra $\mathcal{D}(2,1)$

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Abstract: Based on the Lie superalgebra $\mathcal{D}(2,1)$, we constructed super integrable hierarchies via the zero curvature equation. Isospectral super integrable hierarchies of the super Ablowitz–Kaup–Newell–Segur type were derived, and the corresponding super Hamiltonian structures were established by employing the supertrace identity. The nonisospectral zero curvature equation was further considered, and the associated nonisospectral super integrable hierarchies were obtained. In addition, both isospectral and nonisospectral $(2+1)$-dimensional problems were investigated, and their super Hamiltonian structures were presented.

Keywords: Lie superalgebra; super Hamiltonian structure; super integrable hierarchy; isospectral problem; nonisospectral problem
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1. Introduction

Integrable systems provide an important framework for studying nonlinear evolution equations and soliton phenomena [1]. The Lax pair formulation expresses a nonlinear equation as the compatibility condition of two linear spectral problems [2], leading to the zero curvature equation. A time-independent spectral parameter gives rise to isospectral hierarchies, whereas a time-dependent parameter leads to nonisospectral hierarchies [3] and [4]. This approach underlies the construction of many classical integrable hierarchies, including the Ablowitz–Kaup–Newell–Segur (AKNS), Kaup–Newell, and Dirac hierarchies.

Lie-algebraic methods underpin integrable hierarchies and their Hamiltonian structures [5–7]. The trace identity offers an effective way to derive Hamiltonian structures. The Hamiltonian theory of soliton equations was developed extensively by Faddeev and Takhtajan [8]. Variational identities have also been extended to semidirect sums of Lie algebras [9], broadening the range of admissible algebraic constructions. Related spectral methods have been applied to the construction of $(2+1)$-dimensional hierarchies [10].

The extension of these methods to Lie superalgebras makes it possible to construct integrable systems with both bosonic and fermionic variables. [11] introduced the supertrace identity and used the Lie superalgebra $B(0,1)$ to derive the super-AKNS and super-Dirac hierarchies together with their super Hamiltonian structures. [12] subsequently developed an enlarged spectral matrix method for nonlinear super integrable Hamiltonian couplings. Complementary approaches include the Riemann–Hilbert method for constructing exact solutions of nonlinear integrable equations [13] and symbolic methods for computing integrability structures and recursion operators [14, 15].

Nonisospectral extensions constitute another important direction in this theory. [16, 17] developed spectral approaches for generating isospectral and nonisospectral hierarchies. Further nonisospectral constructions have been obtained for generalized Lie algebras [18]. Meanwhile, investigations of extended Lie superalgebras, generalized super-AKNS hierarchies, and multi-component super Hamiltonian hierarchies have enriched the available algebraic constructions [19–21]. These developments motivate a combined investigation of isospectral and nonisospectral super integrable hierarchies, their higher-dimensional extensions, and the corresponding super Hamiltonian structures.

In this paper, we consider the Lie superalgebra $D(2,1)$. It is isomorphic to $\mathfrak{osp}(4|2)$ and can be identified with a special member of the one-parameter family $D(2,1;\alpha)$, namely, $D(2,1;-\frac{1}{2})$ [22–24]. Our construction concerns this fixed algebra. The existing methods for super integrable hierarchies provide a basis for investigating how different spectral representations of $D(2,1)$ generate distinct hierarchies and how these constructions extend to nonisospectral and higher-dimensional settings.

We construct two types of super integrable hierarchies associated with $D(2,1)$, using spectral problems with eight and twelve dependent variables, respectively. Starting from the zero curvature equations, we derive the isospectral (1 + 1)-dimensional hierarchies and establish their super Hamiltonian structures by the supertrace identity. We then study the corresponding nonisospectral hierarchies and develop both isospectral and nonisospectral (2 + 1)-dimensional extensions. The resulting flow equations, recursive relations, and super Hamiltonian formulations place these constructions within a common algebraic framework.

The paper is organized as follows: Section 2 presents the algebraic preliminaries and the two isospectral (1 + 1)-dimensional hierarchies. Section 3 treats their nonisospectral counterparts. Sections 4 and 5 develop the isospectral and nonisospectral (2 + 1)-dimensional hierarchies, respectively. Sections 6 and 7 give the conclusions and discuss possible applications and future directions.

2. The isospectral (1 + 1)-dimensional super integrable hierarchy of $\mathcal{D}(2,1)$

In this section, we will construct two types of isospectral (1 + 1)-dimensional super integrable hierarchies associated with the Lie superalgebra $\mathcal{D}(2,1)$. The distinction between these two types does not come from the underlying Lie superalgebra, but from different choices of spectral representations and corresponding dependent variables. The first hierarchy type is constructed from an 8-component spectral problem, where the dependent variables are $(u_1,\ldots,u_8)$. The second hierarchy type is generated from another spectral representation with a 12-component variable set $(u_1,\ldots,u_{12})$.

Starting from the corresponding spectral problem, we derive the zero curvature equation and obtain the evolution equations of the hierarchy. Furthermore, the supertrace identity is employed to establish the associated super Hamiltonian structures. Consequently, the two hierarchies possess different recursion operators and Hamiltonian structures. The results in this section provide the foundation

for the subsequent construction of nonisospectral and higher-dimensional super integrable hierarchies.

2.1. Preliminaries

For Lie superalgebra $\mathcal{D}(2,1)$, its basis is

$$E_1 = e_{55} - e_{66},\ E_2 = e_{56},\ E_3 = e_{65},\ E_4 = e_{11} - e_{33},\ E_5 = e_{12} - e_{43},$$
$$E_6 = e_{21} - e_{34},\ E_7 = e_{22} - e_{44},\ E_8 = e_{14} - e_{23},\ E_9 = e_{32} - e_{41},$$
$$E_{10} = e_{15} + e_{63},\ E_{11} = e_{16} - e_{53},\ E_{12} = e_{25} + e_{64},\ E_{13} = e_{26} - e_{54},$$
$$E_{14} = e_{35} + e_{61},\ E_{15} = e_{36} - e_{51},\ E_{16} = e_{45} + e_{62},\ E_{17} = e_{46} - e_{52},$$

where e_{ij} stands for the matrix whose (i, j)-entry is 1 and all other entries are zero and satisfy the following commutative relations:

$$[E_1, E_2] = 2E_2,\ [E_1, E_3] = -2E_3,\ [E_1, E_{10}] = -E_{10},\ [E_1, E_{11}] = E_{11},$$
$$[E_1, E_{12}] = -E_{12},\ [E_1, E_{13}] = E_{13},\ [E_1, E_{14}] = -E_{14},\ [E_1, E_{15}] = E_{15},$$
$$[E_1, E_{16}] = -E_{16},\ [E_1, E_{17}] = E_{17},\ [E_2, E_3] = E_1,\ [E_2, E_{10}] = -E_{11},$$
$$[E_2, E_{12}] = -E_{13},\ [E_2, E_{14}] = -E_{15},\ [E_2, E_{16}] = -E_{17},\ [E_3, E_{11}] = -E_{10},$$
$$[E_3, E_{13}] = -E_{12},\ [E_3, E_{15}] = -E_{14},\ [E_3, E_{17}] = -E_{16},\ [E_4, E_5] = E_5,$$
$$[E_4, E_6] = -E_6,\ [E_4, E_8] = E_8,\ [E_4, E_9] = -E_9,\ [E_4, E_{10}] = E_{10},$$
$$[E_4, E_{11}] = E_{11},\ [E_4, E_{14}] = -E_{14},\ [E_4, E_{15}] = -E_{15},\ [E_5, E_6] = E_4 - E_7,$$
$$[E_5, E_7] = E_5,\ [E_5, E_{12}] = E_{10},\ [E_5, E_{13}] = E_{11},\ [E_5, E_{14}] = -E_{16},$$
$$[E_5, E_{15}] = -E_{17},\ [E_6, E_7] = -E_6,\ [E_6, E_{10}] = E_{12},\ [E_6, E_{11}] = E_{13},$$
$$[E_6, E_{16}] = -E_{14},\ [E_6, E_{17}] = -E_{15},\ [E_7, E_8] = E_8,\ [E_7, E_9] = -E_9,$$
$$[E_7, E_{12}] = E_{12},\ [E_7, E_{13}] = E_{13},\ [E_7, E_{16}] = -E_{16},\ [E_7, E_{17}] = -E_{17},$$
$$[E_8, E_9] = -E_4 - E_7,\ [E_8, E_{14}] = -E_{12},\ [E_8, E_{15}] = -E_{13},\ [E_8, E_{16}] = E_{10},$$
$$[E_8, E_{17}] = E_{11},\ [E_9, E_{10}] = -E_{16},\ [E_9, E_{11}] = -E_{17},\ [E_9, E_{12}] = E_{14},$$
$$[E_9, E_{13}] = E_{15},\ [E_{10}, E_{13}]_+ = -E_8,\ [E_{10}, E_{14}]_+ = 2E_3,\ [E_{10}, E_{15}]_+ = -E_4 - E_1,$$
$$[E_{10}, E_{17}]_+ = -E_5,\ [E_{11}, E_{12}]_+ = E_8,\ [E_{11}, E_{14}]_+ = E_4 - E_1,\ [E_{11}, E_{15}]_+ = -2E_2,$$
$$[E_{11}, E_{16}]_+ = E_5,\ [E_{12}, E_{15}]_+ = -E_6,\ [E_{12}, E_{16}]_+ = 2E_3,\ [E_{12}, E_{17}]_+ = -E_7 - E_1,$$
$$[E_{13}, E_{14}]_+ = E_6,\ [E_{13}, E_{16}]_+ = E_7 - E_1,\ [E_{13}, E_{17}]_+ = -2E_2,\ [E_{14}, E_{17}]_+ = -E_9,$$
$$[E_{15}, E_{16}]_+ = E_9,$$

where all remaining Lie superbracket relations vanish. The parity assignments of the basis elements are summarized as follows:

Basis elements	Parity	Graded subspace
$E_1, E_2, \ldots, E_9$	0 (even)	$\mathfrak{G}_{\bar{0}}$
$E_{10}, E_{11}, \ldots, E_{17}$	1 (odd)	$\mathfrak{G}_{\bar{1}}$

.

Accordingly,

$$\mathfrak{G} = \mathfrak{G}_{\bar{0}} \oplus \mathfrak{G}_{\bar{1}},$$

where

$$\mathfrak{G}_{\bar{0}} = \text{span}\{E_1, \ldots, E_9\}, \qquad \mathfrak{G}_{\bar{1}} = \text{span}\{E_{10}, \ldots, E_{17}\}.$$

For homogeneous elements $X, Y \in \mathfrak{G}$, the Lie superbracket is defined by

$$[X, Y] = XY - (-1)^{|X||Y|}YX,$$

where $|X|, |Y| \in \{0, 1\}$ denote their parities. Thus, the superbracket reduces to the ordinary commutator when at least one element is even and to the anticommutator when both elements are odd.

Let $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ be a differential supercommutative algebra over $\mathbb{C}$, equipped with the even derivation $\partial = \partial_x$. For homogeneous elements $a, b \in \mathcal{A}$, multiplication satisfies

$$ab = (-1)^{|a||b|}ba.$$

Differentiation preserves parity.

The dependent variables $u_1, \ldots, u_4$ in the first construction and $u_1, \ldots, u_6$ in the second construction are even, while the remaining dependent variables are odd. Accordingly, the coefficient functions A_i, B_i, C_i, D_i, F_i, G_i, H_i, I_i, K_i are even, whereas M_i, N_i, P_i, Q_i, S_i, T_i, W_i, Z_i are odd.

For the first spectral representation, we introduce an auxiliary matrix V_1 with the formal Laurent expansion

$$V_1 = \sum_{i=0}^{\infty} V_{1,i}\lambda^{-i},$$

where $V_{1,i}$ denotes the coefficient matrix of λ^{-i}, whose entries are the coefficient functions specified above, arranged according to the matrix structure given below.

Throughout this paper, the auxiliary matrices V_j $(j = 1, \ldots, 8)$ have the same general matrix structure and formal Laurent expansion form. However, their coefficient matrices are determined independently according to the corresponding spectral problems and zero-curvature equations. Therefore, different subscripts are used to distinguish the auxiliary matrices associated with different spectral representations. The matrix structure is specified separately whenever the corresponding spectral representation is introduced.

The formal integration operator ∂^{-1} is used with integration constants fixed by the prescribed normalization. In particular,

$$(a\partial^{-1}b)(f) = a\partial^{-1}(bf).$$

All products are taken in the indicated order.

2.2. *The first type of super integrable hierarchy*

We consider an isospectral problem

$$\begin{cases} \varphi_x = U_1\varphi, \\ \varphi_t = V_1\varphi, \\ \lambda_t = 0, \end{cases}$$

where

$$U_1 = \begin{pmatrix} \lambda & 0 & 0 & u_3 & 0 & u_5 \\ 0 & \lambda & -u_3 & 0 & 0 & u_6 \\ 0 & u_4 & -\lambda & 0 & u_7 & 0 \\ -u_4 & 0 & 0 & -\lambda & u_8 & 0 \\ 0 & 0 & -u_5 & -u_6 & \lambda & u_1 \\ u_7 & u_8 & 0 & 0 & u_2 & -\lambda \end{pmatrix},$$

$$V_1 = \begin{pmatrix} D & F & 0 & I & M & N \\ G & \mathcal{H} & -I & 0 & P & Q \\ 0 & K & -D & -G & S & T \\ -K & 0 & -F & -\mathcal{H} & W & Z \\ -T & -Z & -N & -Q & A & B \\ S & W & M & P & C & -A \end{pmatrix} = \sum_{i=0}^{n} \begin{pmatrix} D_i & F_i & 0 & I_i & M_i & N_i \\ G_i & \mathcal{H}_i & -I_i & 0 & P_i & Q_i \\ 0 & K_i & -D_i & -G_i & S_i & T_i \\ -K_i & 0 & -F_i & -\mathcal{H}_i & W_i & Z_i \\ -T_i & -Z_i & -N_i & -Q_i & A_i & B_i \\ S_i & W_i & M_i & P_i & C_i & -A_i \end{pmatrix} \lambda^{-i}, \tag{2.1}$$

here u_5, u_6, u_7, and u_8 are Grassmann variables satisfying the following anticommutation and nilpotency relations:

$$\begin{aligned} &u_5^2 = u_6^2 = u_7^2 = u_8^2 = 0, \\ &u_i u_j + u_j u_i = 0, \quad i, j = 5, 6, 7, 8, \ \textit{and}\ i \neq j. \end{aligned}$$

Substituting the formal Laurent expansion of V_1 into the stationary zero curvature equation $V_{1,x} = [U_1, V_1]$ and comparing the coefficients of λ^{-i}, we obtain the following recurrence relations for $i \geq 0$.

$$\begin{cases} A_{ix} = u_1 C_i - u_2 B_i - u_5 S_i - u_6 W_i - u_7 N_i - u_8 Q_i, \\ B_{ix} = 2B_{i+1} - 2u_1 A_i - 2u_5 T_i - 2u_6 Z_i, \\ C_{ix} = -2C_{i+1} + 2u_2 A_i + 2u_7 M_i + 2u_8 P_i, \\ D_{ix} = -u_3 K_i + u_4 I_i + u_5 S_i + u_7 N_i, \\ F_{ix} = u_5 W_i + u_8 N_i, \\ G_{ix} = u_6 S_i + u_7 Q_i, \\ \mathcal{H}_{ix} = -u_3 K_i + u_4 I_i + u_6 W_i + u_8 Q_i, \\ I_{ix} = 2I_{i+1} - u_3 D_i - u_3 \mathcal{H}_i - u_6 M_i + u_5 P_i, \\ K_{ix} = -2K_{i+1} + u_4 D_i + u_4 \mathcal{H}_i - u_7 Z_i + u_8 T_i, \end{cases} \quad \begin{cases} M_{ix} = -u_2 N_i + u_3 W_i + u_5 C_i - u_8 I_i, \\ N_{ix} = 2N_{i+1} - u_1 M_i + u_3 Z_i - u_5 A_i - u_5 D_i - u_6 F_i, \\ P_{ix} = -u_2 Q_i - u_3 S_i + u_6 C_i + u_7 I_i, \\ Q_{ix} = 2Q_{i+1} - u_1 P_i - u_3 T_i - u_5 G_i - u_6 A_i - u_6 \mathcal{H}_i, \\ S_{ix} = -2S_{i+1} - u_2 T_i + u_4 P_i + u_7 A_i + u_7 D_i + u_8 G_i, \\ T_{ix} = -u_1 S_i + u_4 Q_i - u_6 K_i + u_7 B_i, \\ W_{ix} = -2W_{i+1} - u_2 Z_i - u_4 M_i + u_7 F_i + u_8 A_i + u_8 \mathcal{H}_i, \\ Z_{ix} = -u_1 W_i - u_4 N_i + u_5 K_i + u_8 B_i. \end{cases} \tag{2.2}$$

We take the initial values

$$\begin{aligned} &A_0 = \alpha, D_0 = \beta, \mathcal{H}_0 = \eta, \quad \alpha, \beta, \eta \in \mathbb{C}, \\ &B_0 = C_0 = F_0 = G_0 = I_0 = K_0 = M_0 = N_0 = P_0 = Q_0 = S_0 = T_0 = W_0 = Z_0 = 0. \end{aligned}$$

From (2.2), we obtain

$$A_1 = D_1 = \mathcal{H}_1 = 0,\ B_1 = \alpha u_1,\ C_1 = \alpha u_2,\ F_1 = \frac{\eta-\beta}{2}\partial^{-1}(u_5u_8),\ G_1 = \frac{\beta-\eta}{2}\partial^{-1}(u_6u_7),$$
$$I_1 = \frac{\beta+\eta}{2}u_3,\ K_1 = \frac{\beta+\eta}{2}u_4,\ M_1 = \frac{\alpha-\beta}{2}\partial^{-1}(u_2u_5+u_3u_8),\ N_1 = \frac{\alpha+\beta}{2}u_5,$$
$$P_1 = \frac{\alpha-\eta}{2}\partial^{-1}(u_2u_6-u_3u_7),\ Q_1 = \frac{\alpha+\eta}{2}u_6,\ S_1 = \frac{\alpha+\beta}{2}u_7,\ W_1 = \frac{\alpha+\eta}{2}u_8,$$
$$T_1 = \frac{\alpha-\beta}{2}\partial^{-1}(u_1u_7+u_4u_6),\ Z_1 = \frac{\alpha-\eta}{2}\partial^{-1}(u_1u_8-u_4u_5),$$
$$B_2 = \frac{1}{2}B_{1x}+u_5T_1+u_6Z_1,\ C_2 = -\frac{1}{2}C_{1x}+u_7M_1+u_8P_1,\ I_2 = \frac{1}{2}I_{1x}+\frac{u_6}{2}M_1-\frac{u_5}{2}P_1,$$
$$K_2 = -\frac{1}{2}K_{1x}-\frac{u_7}{2}Z_1+\frac{u_8}{2}T_1,\ N_2 = \frac{1}{2}N_{1x}+\frac{u_1}{2}M_1-\frac{u_3}{2}Z_1+\frac{u_6}{2}F_1,$$
$$Q_2 = \frac{1}{2}Q_{1x}+\frac{u_1}{2}P_1+\frac{u_3}{2}T_1+\frac{u_5}{2}G_1,\ S_2 = -\frac{1}{2}S_{1x}-\frac{u_2}{2}T_1+\frac{u_4}{2}P_1+\frac{u_8}{2}G_1,$$
$$W_2 = -\frac{1}{2}W_{1x}-\frac{u_2}{2}Z_1-\frac{u_4}{2}M_1+\frac{u_7}{2}F_1,$$
$$\ldots.$$

Let

$$g_i = \begin{pmatrix} -C_i & -B_i & -2K_i & -2I_i & -2S_i & -2W_i & 2N_i & 2Q_i \end{pmatrix}^{\mathrm{T}}.$$

The first nonzero coefficient vector is

$$g_1 = \begin{pmatrix} -\alpha u_2 & -\alpha u_1 & -(\beta+\eta)u_4 & -(\beta+\eta)u_3 & -(\alpha+\beta)u_7 & -(\alpha+\eta)u_8 & (\alpha+\beta)u_5 & (\alpha+\eta)u_6 \end{pmatrix}^{\mathrm{T}}.$$

For an arbitrary column vector $g = (g_1, \ldots, g_8)^{\mathrm{T}}$ with the corresponding parity assignments, define

$$A = \partial^{-1}\left(-u_1g_1+u_2g_2+\frac{1}{2}u_5g_5+\frac{1}{2}u_6g_6-\frac{1}{2}u_7g_7-\frac{1}{2}u_8g_8\right),$$
$$D = \frac{1}{2}\partial^{-1}(u_3g_3-u_4g_4-u_5g_5+u_7g_7),$$
$$H = \frac{1}{2}\partial^{-1}(u_3g_3-u_4g_4-u_6g_6+u_8g_8),$$
$$F = \frac{1}{2}\partial^{-1}(-u_5g_6+u_8g_7),$$
$$G = \frac{1}{2}\partial^{-1}(-u_6g_5+u_7g_8),$$
$$M = \partial^{-1}\left(-\frac{1}{2}u_2g_7-\frac{1}{2}u_3g_6-u_5g_1+\frac{1}{2}u_8g_4\right),$$
$$P = \partial^{-1}\left(-\frac{1}{2}u_2g_8+\frac{1}{2}u_3g_5-u_6g_1-\frac{1}{2}u_7g_4\right),$$
$$T = \partial^{-1}\left(\frac{1}{2}u_1g_5+\frac{1}{2}u_4g_8+\frac{1}{2}u_6g_3-u_7g_2\right),$$
$$Z = \partial^{-1}\left(\frac{1}{2}u_1g_6-\frac{1}{2}u_4g_7-\frac{1}{2}u_5g_3-u_8g_2\right).$$

The operator L_1 is defined by

$$L_1 g = \begin{pmatrix} -\frac{1}{2}g_{1,x} - u_2 A - u_7 M - u_8 P \\ \frac{1}{2}g_{2,x} - u_1 A - u_5 T - u_6 Z \\ -\frac{1}{2}g_{3,x} - u_4(D+H) + u_7 Z - u_8 T \\ \frac{1}{2}g_{4,x} - u_3(D+H) - u_6 M + u_5 P \\ -\frac{1}{2}g_{5,x} + u_2 T - u_4 P - u_7(A+D) - u_8 G \\ -\frac{1}{2}g_{6,x} + u_2 Z + u_4 M - u_7 F - u_8(A+H) \\ \frac{1}{2}g_{7,x} + u_1 M - u_3 Z + u_5(A+D) + u_6 F \\ \frac{1}{2}g_{8,x} + u_1 P + u_3 T + u_5 G + u_6(A+H) \end{pmatrix}.$$

The coefficient recursion is therefore

$$g_{i+1} = L_1 g_i, \qquad i \geq 1.$$

Here and below, ∂^{-1} and operator products are interpreted according to the conventions specified in Section 2.1.

For each $n \geq 0$, define the temporal matrix by

$$V_1^{(n)} = -(\lambda^n V_1)_+ = -\sum_{i=0}^{n} V_{1,i}\lambda^{n-i},$$

where the subscript (+) denotes the projection onto the nonnegative powers of λ. The compatibility condition of $\phi_x = U_1\phi, \phi_{t_{1,n}} = V_1^{(n)}\phi$ is $U_{1,t_{1,n}} - (V_1^{(n)})_x + [U_1, V_1^{(n)}] = 0$. We can obtain the following supersoliton hierarchy:

$$u_{t_{1,n}} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix}_{t_{1,n}} = \begin{pmatrix} -2B_{n+1} \\ 2C_{n+1} \\ -2I_{n+1} \\ 2K_{n+1} \\ -2N_{n+1} \\ -2Q_{n+1} \\ 2S_{n+1} \\ 2W_{n+1} \end{pmatrix} = J_1 \begin{pmatrix} -C_{n+1} \\ -B_{n+1} \\ -2K_{n+1} \\ -2I_{n+1} \\ -2S_{n+1} \\ -2W_{n+1} \\ 2N_{n+1} \\ 2Q_{n+1} \end{pmatrix} = J_1 L_1 \begin{pmatrix} -C_n \\ -B_n \\ -2K_n \\ -2I_n \\ -2S_n \\ -2W_n \\ 2N_n \\ 2Q_n \end{pmatrix},$$

where the Hamiltonian operator is given by

$$J_1 = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

Next, we construct the super Hamiltonian structure. By direct calculation, we obtain

$$\left\langle V_1, \frac{\partial U_1}{\partial \lambda} \right\rangle = 2D + 2\mathcal{H} - 2A, \left\langle V_1, \frac{\partial U_1}{\partial u_1} \right\rangle = -C, \left\langle V_1, \frac{\partial U_1}{\partial u_2} \right\rangle = -B, \left\langle V_1, \frac{\partial U_1}{\partial u_3} \right\rangle = -2K,$$
$$\left\langle V_1, \frac{\partial U_1}{\partial u_4} \right\rangle = -2I, \left\langle V_1, \frac{\partial U_1}{\partial u_5} \right\rangle = -2S, \left\langle V_1, \frac{\partial U_1}{\partial u_6} \right\rangle = -2W, \left\langle V_1, \frac{\partial U_1}{\partial u_7} \right\rangle = 2N, \left\langle V_1, \frac{\partial U_1}{\partial u_8} \right\rangle = 2Q.$$

Substituting the above equations into the supertrace identity and comparing the coefficients of λ^{-n-1} on both sides, we obtain

$$\frac{\delta}{\delta u} \int (2D_{n+1} + 2\mathcal{H}_{n+1} - 2A_{n+1})dx = (\gamma - n) \begin{pmatrix} -C_n \\ -B_n \\ -2K_n \\ -2I_n \\ -2S_n \\ -2W_n \\ 2N_n \\ 2Q_n \end{pmatrix}.$$

Setting $n = 1$ yields $\gamma = 0$. Thus, we have

$$u_{t_{1,n}} = J_1 \frac{\delta H_{1,n+1}}{\delta u}, \quad H_{1,n+1} = -2 \int \frac{(D_{n+2} + \mathcal{H}_{n+2} - A_{n+2})}{n+1} dx, \quad n \geq 0.$$

2.3. The second type of super integrable hierarchy

We consider an isospectral problem

$$\begin{cases} \varphi_x = U_2 \varphi, \\ \varphi_t = V_2 \varphi, \\ \lambda_t = 0. \end{cases}$$

The auxiliary matrix V_2 has the same matrix structure as V_1 in (2.1), but its coefficient matrices are determined independently by

$$V_{2,x} = [U_2, V_2].$$

The formal Laurent expansion is given by

$$V_2 = \sum_{i=0}^{\infty} V_{2,i} \lambda^{-i},$$

and U_2 is defined as follows:

$$U_2 = \begin{pmatrix} \lambda & u_3 & 0 & u_5 & 0 & u_7 \\ u_4 & 0 & -u_5 & 0 & u_8 & u_9 \\ 0 & u_6 & -\lambda & -u_4 & u_{10} & 0 \\ -u_6 & 0 & -u_3 & 0 & u_{11} & u_{12} \\ 0 & -u_{12} & -u_7 & -u_9 & \lambda & u_1 \\ u_{10} & u_{11} & 0 & u_8 & u_2 & -\lambda \end{pmatrix},$$

where $u_7, u_8, u_9, u_{10}, u_{11}$, and u_{12} are Grassmann variables satisfying the following anticommutation and nilpotency relations:

$$u_7^2 = u_8^2 = u_9^2 = u_{10}^2 = u_{11}^2 = u_{12}^2 = 0,$$
$$u_i u_j + u_j u_i = 0, \quad i, j = 7, 8, 9, 10, 11, 12, \text{ and } i \neq j.$$

The stationary zero curvature representation $V_{2,x} = [U_2, V_2]$ gives

$$\begin{cases} A_{ix} = u_1 C_i - u_2 B_i - u_7 S_i - u_8 Z_i - u_9 W_i - u_{10} N_i - u_{11} Q_i - u_{12} P_i, \\ B_{ix} = 2B_{i+1} - 2u_1 A_i - 2u_7 T_i - 2u_9 Z_i - 2u_{12} Q_i, \\ C_{ix} = -2C_{i+1} + 2u_2 A_i + 2u_8 W_i + 2u_{10} M_i + 2u_{11} P_i, \\ D_{ix} = u_3 G_i - u_4 F_i - u_5 K_i + u_6 I_i + u_7 S_i + u_{10} N_i, \\ F_{ix} = F_{i+1} - u_3 D_i + u_3 \mathcal{H}_i + u_7 W_i + u_{11} N_i - u_{12} M_i \\ G_{ix} = -G_{i+1} + u_4 D_i - u_4 \mathcal{H}_i - u_8 T_i + u_9 S_i + u_{10} Q_i, \\ \mathcal{H}_{ix} = -u_3 G_i + u_4 F_i - u_5 K_i + u_6 I_i - u_8 Z_i + u_9 W_i + u_{11} Q_i - u_{12} P_i, \\ I_{ix} = I_{i+1} - u_5 D_i - u_5 \mathcal{H}_i + u_7 P_i + u_8 N_i - u_9 M_i, \\ K_{ix} = -K_{i+1} + u_6 D_i + u_6 \mathcal{H}_i - u_{10} Z_i + u_{11} T_i - u_{12} S_i, \\ M_{ix} = -u_2 N_i + u_3 P_i + u_5 W_i + u_7 C_i - u_8 F_i - u_{11} I_i, \\ N_{ix} = 2N_{i+1} - u_1 M_i + u_3 Q_i + u_5 Z_i - u_7 A_i - u_7 D_i - u_9 F_i - u_{12} I_i, \\ P_{ix} = -P_{i+1} - u_2 Q_i + u_4 M_i - u_5 S_i + u_8 A_i - u_8 \mathcal{H}_i + u_9 C_i + u_{10} I_i, \\ Q_{ix} = Q_{i+1} - u_1 P_i + u_4 N_i - u_5 T_i - u_7 G_i + u_8 B_i - u_9 A_i - u_9 \mathcal{H}_i, \\ S_{ix} = -2S_{i+1} - u_2 T_i - u_4 W_i + u_6 P_i - u_8 K_i + u_{10} A_i + u_{10} D_i + u_{11} G_i, \\ T_{ix} = -u_1 S_i - u_4 Z_i + u_6 Q_i - u_9 K_i + u_{10} B_i + u_{12} G_i, \\ W_{ix} = -W_{i+1} - u_2 Z_i - u_3 S_i - u_6 M_i + u_{10} F_i + u_{11} A_i + u_{11} \mathcal{H}_i + u_{12} C_i, \\ Z_{ix} = Z_{i+1} - u_1 W_i - u_3 T_i - u_6 N_i + u_7 K_i + u_{11} B_i - u_{12} A_i + u_{12} \mathcal{H}_i. \end{cases} \tag{2.3}$$

We take the initial values

$$A_0 = \alpha, D_0 = \beta, \mathcal{H}_0 = \eta,$$
$$B_0 = C_0 = F_0 = G_0 = I_0 = K_0 = M_0 = N_0 = P_0 = Q_0 = S_0 = T_0 = W_0 = Z_0 = 0.$$

From (2.3), we obtain

$$A_1 = D_1 = \mathcal{H}_1 = 0, B_1 = \alpha u_1, C_1 = \alpha u_2, F_1 = (\beta - \eta) u_3, G_1 = (\beta - \eta) u_4,$$
$$I_1 = (\beta + \eta) u_5, K_1 = (\beta + \eta) u_6, N_1 = \frac{\alpha + \beta}{2} u_7, P_1 = (\alpha - \eta) u_8, Q_1 = (\alpha + \eta) u_9,$$
$$S_1 = \frac{\alpha + \beta}{2} u_{10}, W_1 = (\alpha + \eta) u_{11}, Z_1 = (\alpha - \eta) u_{12},$$
$$M_1 = \frac{\alpha - \beta}{2} \partial^{-1}(u_2 u_7) + (\alpha - \beta) \partial^{-1}(u_3 u_8 + u_5 u_{11}),$$
$$T_1 = \frac{\alpha - \beta}{2} \partial^{-1}(u_1 u_{10}) + (\alpha - \beta) \partial^{-1}(u_6 u_9 - u_4 u_{12}),$$

$$B_2 = \frac{1}{2}B_{1x} + u_7T_1 - 2\eta u_9u_{12},\ C_2 = -\frac{1}{2}C_{1x} + u_{10}M_1 + 2\eta u_8u_{11},$$
$$F_2 = F_{1x} + u_{12}M_1 + \left(\frac{\beta-\alpha}{2} - \eta\right)u_7u_{11},\ G_2 = -G_{1x} - u_8T_1 + \left(\frac{\beta-\alpha}{2} - \eta\right)u_9u_{10},$$
$$I_2 = I_{1x} + u_9M_1 + \left(\frac{\beta-\alpha}{2} + \eta\right)u_7u_8,\ K_2 = -K_{1x} + u_{11}T_1 + \left(\frac{\beta-\alpha}{2} + \eta\right)u_{10}u_{12},$$
$$N_2 = \frac{1}{2}N_{1x} + \frac{1}{2}u_1M_1 + \left(\frac{\beta-\alpha}{2} - \eta\right)u_3u_9 + \left(\frac{\beta-\alpha}{2} + \eta\right)u_5u_{12},$$
$$P_2 = -P_{1x} + u_4M_1 - \eta u_2u_9 + \left(\frac{\beta-\alpha}{2} + \eta\right)u_5u_{10},$$
$$Q_2 = Q_{1x} + u_5T_1 - \eta u_1u_8 + \left(\frac{\beta-\alpha}{2} - \eta\right)u_4u_7,$$
$$S_2 = -\frac{1}{2}S_{1x} - \frac{1}{2}u_2T_1 + \left(\frac{\alpha-\beta}{2} - \eta\right)u_6u_8 + \left(\frac{\beta-\alpha}{2} - \eta\right)u_4u_{11},$$
$$W_2 = -W_{1x} - u_6M_1 + \eta u_2u_{12} + \left(\frac{\beta-\alpha}{2} - \eta\right)u_3u_{10},$$
$$Z_2 = Z_{1x} + u_3T_1 + \eta u_1u_{11} + \left(\frac{\alpha-\beta}{2} - \eta\right)u_6u_7,$$
$$\cdots.$$

According to (2.3), we obtain the following recursion relation:

$$\begin{pmatrix} -C_{n+1} \\ -B_{n+1} \\ 2G_{n+1} \\ 2F_{n+1} \\ -2K_{n+1} \\ -2I_{n+1} \\ -2S_{n+1} \\ 2Z_{n+1} \\ -2W_{n+1} \\ 2N_{n+1} \\ 2Q_{n+1} \\ -2P_{n+1} \end{pmatrix} = L_2 \begin{pmatrix} -C_n \\ -B_n \\ 2G_n \\ 2F_n \\ -2K_n \\ -2I_n \\ -2S_n \\ 2Z_n \\ -2W_n \\ 2N_n \\ 2Q_n \\ -2P_n \end{pmatrix}.$$

The recurrence relation is valid for $n \geq 1$; the initial coefficient vector g_1 is determined separately by the prescribed zeroth-order diagonal coefficients. Here, $L_2 = (l_{ij})_{12\times12}$ and l_{ij} is

$$l_{11} = -\frac{1}{2}\partial + u_2\partial^{-1}u_1 + u_{10}\partial^{-1}u_7,\ l_{12} = -u_2\partial^{-1}u_2,\ l_{14} = \frac{1}{2}u_{10}\partial^{-1}u_8,\ l_{16} = -\frac{1}{2}u_{10}\partial^{-1}u_{11},$$
$$l_{17} = -\frac{1}{2}u_2\partial^{-1}u_7,\ l_{18} = \frac{1}{2}u_2\partial^{-1}u_8,\ l_{19} = -l_{75} = \frac{1}{2}(-u_2\partial^{-1}u_9 + u_8 + u_{10}\partial^{-1}u_5),$$
$$l_{1,10} = -\frac{1}{2}l_{72} = \frac{1}{2}(u_2\partial^{-1}u_{10} + u_{10}\partial^{-1}u_2),\ l_{1,11} = \frac{1}{2}u_2\partial^{-1}u_{11},\ l_{21} = u_1\partial^{-1}u_1,$$
$$l_{1,12} = -l_{73} = \frac{1}{2}(-u_2\partial^{-1}u_{12} + u_{10}\partial^{-1}u_3 + u_{11}),\ l_{22} = \frac{1}{2}\partial - u_1\partial^{-1}u_2 + u_7\partial^{-1}u_{10},$$

$$l_{23} = -\frac{1}{2}u_7\partial^{-1}u_{12},\ l_{25} = -\frac{1}{2}u_7\partial^{-1}u_9,\ l_{27} = \frac{1}{2}l_{10,1} = -\frac{1}{2}(u_1\partial^{-1}u_7 + u_7\partial^{-1}u_1),$$
$$l_{28} = -l_{10,4} = \frac{1}{2}(u_1\partial^{-1}u_8 + u_7\partial^{-1}u_4 - u_9),\ l_{29} = -\frac{1}{2}u_1\partial^{-1}u_9,\ l_{2,10} = \frac{1}{2}u_1\partial^{-1}u_{10},$$
$$l_{2,11} = l_{10,6} = \frac{1}{2}(u_1\partial^{-1}u_{11} - u_7\partial^{-1}u_6 - u_{12}),\ l_{2,12} = -\frac{1}{2}u_1\partial^{-1}u_{12},\ l_{32} = 2u_8\partial^{-1}u_{10},$$
$$l_{33} = -\partial + 2u_4\partial^{-1}u_3 - u_8\partial^{-1}u_{12},\ l_{34} = -2u_4\partial^{-1}u_4,\ l_{35} = -u_8\partial^{-1}u_9,$$
$$l_{37} = -\frac{1}{2}l_{12,1} = -u_4\partial^{-1}u_7 - u_8\partial^{-1}u_1 - u_9,\ l_{38} = l_{12,4} = u_4\partial^{-1}u_8 + u_8\partial^{-1}u_4,\ l_{39} = u_4\partial^{-1}u_9,$$
$$l_{3,10} = u_4\partial^{-1}u_{10},\ l_{3,11} = l_{12,6} = -u_4\partial^{-1}u_{11} - u_8\partial^{-1}u_6 + u_{10},\ l_{3,12} = -u_4\partial^{-1}u_{12},$$
$$l_{41} = -2u_{12}\partial^{-1}u_7,\ l_{43} = 2u_3\partial^{-1}u_3,\ l_{44} = \partial - 2u_3\partial^{-1}u_4 - u_{12}\partial^{-1}u_8,\ l_{46} = u_{12}\partial^{-1}u_{11},$$
$$l_{47} = -u_3\partial^{-1}u_7,\ l_{48} = u_3\partial^{-1}u_8,\ l_{49} = l_{85} = u_3\partial^{-1}u_9 + u_7 - u_{12}\partial^{-1}u_5,$$
$$l_{4,10} = -\frac{1}{2}l_{82} = u_3\partial^{-1}u_{10} - u_{11} - u_{12}\partial^{-1}u_2,\ l_{4,11} = -u_3\partial^{-1}u_{11},$$
$$l_{4,12} = -l_{83} = -u_3\partial^{-1}u_{12} - u_{12}\partial^{-1}u_3,\ l_{52} = 2u_{11}\partial^{-1}u_{10},\ l_{53} = -u_{11}\partial^{-1}u_{12},$$
$$l_{55} = -\partial - 2u_6\partial^{-1}u_5 - u_{11}\partial^{-1}u_9,\ l_{56} = 2u_6\partial^{-1}u_6,\ l_{57} = -\frac{1}{2}l_{91} = u_6\partial^{-1}u_7 - u_{11}\partial^{-1}u_1 - u_{12},$$
$$l_{58} = -l_{94} = u_6\partial^{-1}u_8 + u_{10} + u_{11}\partial^{-1}u_4,\ l_{59} = u_6\partial^{-1}u_9,\ l_{5,10} = -u_6\partial^{-1}u_{10},$$
$$l_{5,11} = -l_{96} = -u_6\partial^{-1}u_{11} - u_{11}\partial^{-1}u_6,\ l_{5,12} = -u_6\partial^{-1}u_{12},\ l_{61} = 2u_9\partial^{-1}u_7,\ l_{64} = u_9\partial^{-1}u_8,$$
$$l_{65} = -2u_5\partial^{-1}u_5,\ l_{66} = \partial + 2u_5\partial^{-1}u_6 - u_9\partial^{-1}u_{11},\ l_{67} = u_5\partial^{-1}u_7,\ l_{68} = u_5\partial^{-1}u_8,$$
$$l_{69} = l_{11,5} = u_5\partial^{-1}u_9 + u_9\partial^{-1}u_5,\ l_{6,10} = \frac{1}{2}l_{11,2} = -u_5\partial^{-1}u_{10} + u_8 + u_9\partial^{-1}u_2,$$
$$l_{6,11} = -u_5\partial^{-1}u_{11},\ l_{6,12} = -l_{11,3} = -u_5\partial^{-1}u_{12} - u_7 + u_9\partial^{-1}u_3,\ l_{71} = u_{10}\partial^{-1}u_1,$$
$$l_{74} = \frac{1}{2}u_{10}\partial^{-1}u_4,\ l_{76} = \frac{1}{2}u_{10}\partial^{-1}u_6,\ l_{77} = -\frac{1}{2}\partial + \frac{1}{2}u_2\partial^{-1}u_1,\ l_{78} = -\frac{1}{2}u_2\partial^{-1}u_4 + \frac{1}{2}u_{10}\partial^{-1}u_8,$$
$$l_{79} = -\frac{1}{2}u_4 - \frac{1}{2}u_{10}\partial^{-1}u_9,\ l_{7,11} = \frac{1}{2}u_2\partial^{-1}u_6 + \frac{1}{2}u_{10}\partial^{-1}u_{11},\ l_{7,12} = \frac{1}{2}u_6 - \frac{1}{2}u_{10}\partial^{-1}u_{12},$$
$$l_{81} = -2u_{12}\partial^{-1}u_1,\ l_{84} = -u_{12}\partial^{-1}u_4,\ l_{86} = u_{12}\partial^{-1}u_6,\ l_{87} = u_3\partial^{-1}u_1 + u_{12}\partial^{-1}u_7,$$
$$l_{88} = \partial - u_3\partial^{-1}u_4,\ l_{89} = -u_1 + 2u_{12}\partial^{-1}u_9,\ l_{8,10} = u_6 - u_{12}\partial^{-1}u_{10},\ l_{8,11} = u_3\partial^{-1}u_6 - 2u_{12}\partial^{-1}u_{11},$$
$$l_{92} = -2u_{11}\partial^{-1}u_2,\ l_{93} = u_{11}\partial^{-1}u_3,\ l_{95} = -u_{11}\partial^{-1}u_5,\ l_{97} = -u_3 - u_{11}\partial^{-1}u_7,$$
$$l_{98} = u_2 + 2u_{11}\partial^{-1}u_8,\ l_{99} = -\partial - u_6\partial^{-1}u_5,\ l_{9,10} = -u_6\partial^{-1}u_2 + u_{11}\partial^{-1}u_{10},$$
$$l_{9,12} = -u_6\partial^{-1}u_3 - 2u_{11}\partial^{-1}u_{12},\ l_{10,2} = u_7\partial^{-1}u_2,\ l_{10,3} = \frac{1}{2}u_7\partial^{-1}u_3,\ l_{10,5} = \frac{1}{2}u_7\partial^{-1}u_5,$$
$$l_{10,8} = -\frac{1}{2}u_5 - \frac{1}{2}u_7\partial^{-1}u_8,\ l_{10,9} = -\frac{1}{2}u_1\partial^{-1}u_5 + \frac{1}{2}u_7\partial^{-1}u_9,\ l_{10,10} = \frac{1}{2}\partial - \frac{1}{2}u_1\partial^{-1}u_2,$$
$$l_{10,11} = -\frac{1}{2}u_3 - \frac{1}{2}u_7\partial^{-1}u_{11},\ l_{10,12} = -\frac{1}{2}u_1\partial^{-1}u_3 + \frac{1}{2}u_7\partial^{-1}u_{12},\ l_{11,1} = -2u_9\partial^{-1}u_1,\ l_{11,4} = u_9\partial^{-1}u_4,$$
$$l_{11,6} = -u_9\partial^{-1}u_6,\ l_{11,7} = u_5\partial^{-1}u_1 + u_9\partial^{-1}u_7,\ l_{11,8} = -u_5\partial^{-1}u_4 - 2u_9\partial^{-1}u_8,$$
$$l_{11,10} = -u_4 - u_9\partial^{-1}u_{10},\ l_{11,11} = \partial + u_5\partial^{-1}u_6,\ l_{11,12} = -u_1 + 2u_9\partial^{-1}u_{12},\ l_{12,2} = -2u_8\partial^{-1}u_2,$$
$$l_{12,3} = -u_8\partial^{-1}u_3,\ l_{12,5} = u_8\partial^{-1}u_5,\ l_{12,7} = -u_5 - u_8\partial^{-1}u_7,\ l_{12,9} = u_4\partial^{-1}u_5 - 2u_8\partial^{-1}u_9,$$
$$l_{12,10} = u_4\partial^{-1}u_2 + u_8\partial^{-1}u_{10},\ l_{12,11} = u_2 + 2u_8\partial^{-1}u_{11},\ l_{12,12} = -\partial + u_4\partial^{-1}u_3.$$

All other elements $l_{ij} = 0$ are identically equal to zero.

Taking $V_2^{(n)} = V_{2+}^{(n)}$, from the zero curvature equation $U_{2,t} - V_{2,+x}^{(n)} + [U_2, V_{2,+}^{(n)}] = 0$, we can obtain the following supersoliton hierarchy:

$$u_{t_{2,n}} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{pmatrix}_{t_{2,n}} = \begin{pmatrix} -2B_{n+1} \\ 2C_{n+1} \\ -F_{n+1} \\ G_{n+1} \\ -I_{n+1} \\ K_{n+1} \\ -2N_{n+1} \\ P_{n+1} \\ -Q_{n+1} \\ 2S_{n+1} \\ W_{n+1} \\ -Z_{n+1} \end{pmatrix} = J_2 \begin{pmatrix} -C_{n+1} \\ -B_{n+1} \\ 2G_{n+1} \\ 2F_{n+1} \\ -2K_{n+1} \\ -2I_{n+1} \\ -2S_{n+1} \\ 2Z_{n+1} \\ -2W_{n+1} \\ 2N_{n+1} \\ 2Q_{n+1} \\ -2P_{n+1} \end{pmatrix} = J_2 L_2 \begin{pmatrix} -C_n \\ -B_n \\ 2G_n \\ 2F_n \\ -2K_n \\ -2I_n \\ -2S_n \\ 2Z_n \\ -2W_n \\ 2N_n \\ 2Q_n \\ -2P_n \end{pmatrix},$$

where the Hamiltonian operator is given by

$$J_2 = \begin{pmatrix} 2J_1' & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2}J_1' & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}J_1' & 0 & 0 \\ 0 & 0 & 0 & 0 & J_2' \\ 0 & 0 & 0 & J_2' & 0 \end{pmatrix},$$

where

$$J_1' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J_2' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}.$$

Proposition 2.1. *Proposition. Under the parity assignments specified in Sections 2.2 and 2.3, the constant-coefficient matrices J_1 and J_2 define graded Poisson structures.*

Proof. For an even-graded Poisson structure, the matrix entries must satisfy

$$(J_k)_{ij} = -(-1)^{p_i p_j}(J_k)_{ji}, \qquad k = 1, 2,$$

where $p_i = |u_i| \in \{0, 1\}$ denotes the parity of u_i. For J_1, the even-variable block is skew-symmetric, the odd-variable block is symmetric, and the mixed-parity blocks vanish. For J_2, the even-variable blocks are skew-symmetric, while the odd-variable block is symmetric because $(J_2')^T = J_2'$. Thus, both matrices satisfy the graded skew-symmetry condition.

Moreover, the entries of J_1 and J_2 are independent of the dependent variables. Hence, the corresponding constant-coefficient graded Poisson brackets satisfy the graded Jacobi identity by

the standard constant-coefficient super-Poisson criterion, under the adopted variational-derivative convention. Therefore, J_1 and J_2 define graded Poisson structures. □

Next, we construct the super Hamiltonian structure. By direct calculation, we obtain

$$\left\langle V_2, \frac{\partial U_2}{\partial \lambda} \right\rangle = 2D - 2A, \left\langle V_2, \frac{\partial U_2}{\partial u_1} \right\rangle = -C, \left\langle V_2, \frac{\partial U_2}{\partial u_2} \right\rangle = -B, \left\langle V_2, \frac{\partial U_2}{\partial u_3} \right\rangle = 2G, \left\langle V_2, \frac{\partial U_2}{\partial u_4} \right\rangle = 2F,$$
$$\left\langle V_2, \frac{\partial U_2}{\partial u_5} \right\rangle = -2K, \left\langle V_2, \frac{\partial U_2}{\partial u_6} \right\rangle = -2I, \left\langle V_2, \frac{\partial U_2}{\partial u_7} \right\rangle = -2S, \left\langle V_2, \frac{\partial U_2}{\partial u_8} \right\rangle = 2Z, \left\langle V_2, \frac{\partial U_2}{\partial u_9} \right\rangle = -2W,$$
$$\left\langle V_2, \frac{\partial U_2}{\partial u_{10}} \right\rangle = 2N, \left\langle V_2, \frac{\partial U_2}{\partial u_{11}} \right\rangle = 2Q, \left\langle V_2, \frac{\partial U_2}{\partial u_{12}} \right\rangle = -2P.$$

Substituting the above equations into the supertrace identity and comparing the coefficients of λ^{-n-1} on both sides, we obtain

$$\frac{\delta}{\delta u} \int (2D_{n+1} - 2A_{n+1}) dx = (\gamma - n) \begin{pmatrix} -C_n \\ -B_n \\ 2G_n \\ 2F_n \\ -2K_n \\ -2I_n \\ -2S_n \\ 2Z_n \\ -2W_n \\ 2N_n \\ 2Q_n \\ -2P_n \end{pmatrix}.$$

Setting $n = 1$ yields $\gamma = 0$. Thus, we have

$$u_{t_{2,n}} = J_2 \frac{\delta H_{2,n+1}}{\delta u}, \quad H_{2,n+1} = -2 \int \frac{(D_{n+2} - A_{n+2})}{n+1} dx, \quad n \geq 0.$$

3. The nonisospectral (1 + 1)-dimensional integrable hierarchy based on Lie superalgebra $\mathcal{D}(2, 1)$

For consistency, the spectral matrices in this section are denoted by U_3 and U_4. They correspond, respectively, to the first and second types of constructions introduced in Section 2, with different spectral parameters and evolution conditions.

3.1. The first type of super integrable hierarchy

We consider a nonisospectral problem

$$\begin{cases} \varphi_x = U_3 \varphi, \\ \varphi_t = V_3 \varphi, \\ \lambda_t = \sum\limits_{i \geq 0} k_i(t) \lambda^{-i}, \end{cases}$$

where

$$U_3 = \begin{pmatrix} \lambda & 0 & 0 & u_3 & 0 & u_5 \\ 0 & \lambda & -u_3 & 0 & 0 & u_6 \\ 0 & u_4 & -\lambda & 0 & u_7 & 0 \\ -u_4 & 0 & 0 & -\lambda & u_8 & 0 \\ 0 & 0 & -u_5 & -u_6 & \lambda & u_1 \\ u_7 & u_8 & 0 & 0 & u_2 & -\lambda \end{pmatrix},$$

$$V_3 = \begin{pmatrix} D & F & 0 & I & M & N \\ G & \mathcal{H} & -I & 0 & P & Q \\ 0 & K & -D & -G & S & T \\ -K & 0 & -F & -\mathcal{H} & W & Z \\ -T & -Z & -N & -Q & A & B \\ S & W & M & P & C & -A \end{pmatrix} = \sum_{i\geq 0}^{n} \begin{pmatrix} D_i & F_i & 0 & I_i & M_i & N_i \\ G_i & \mathcal{H}_i & -I_i & 0 & P_i & Q_i \\ 0 & K_i & -D_i & -G_i & S_i & T_i \\ -K_i & 0 & -F_i & -\mathcal{H}_i & W_i & Z_i \\ -T_i & -Z_i & -N_i & -Q_i & A_i & B_i \\ S_i & W_i & M_i & P_i & C_i & -A_i \end{pmatrix} \lambda^{-i}, \tag{3.1}$$

where u_5, u_6, u_7, and u_8 are Grassmann variables satisfying the following anticommutation and nilpotency relations:

$$u_5^2 = u_6^2 = u_7^2 = u_8^2 = 0,$$
$$u_i u_j + u_j u_i = 0, \quad i, j = 5, 6, 7, 8, \text{ and } i \neq j.$$

The stationary zero curvature representation $V_{3,x} = \frac{\partial U_3}{\partial \lambda}\lambda_t + [U_3, V_3]$ gives

$$\begin{cases} A_{ix} = u_1 C_i - u_2 B_i - u_5 S_i - u_6 W_i - u_7 N_i - u_8 Q_i + k_i(t), \\ B_{ix} = 2B_{i+1} - 2u_1 A_i - 2u_5 T_i - 2u_6 Z_i, \\ C_{ix} = -2C_{i+1} + 2u_2 A_i + 2u_7 M_i + 2u_8 P_i, \\ D_{ix} = -u_3 K_i + u_4 I_i + u_5 S_i + u_7 N_i + k_i(t), \\ F_{ix} = u_5 W_i + u_8 N_i, \\ G_{ix} = u_6 S_i + u_7 Q_i, \\ \mathcal{H}_{ix} = -u_3 K_i + u_4 I_i + u_6 W_i + u_8 Q_i + k_i(t), \\ I_{ix} = 2I_{i+1} - u_3 D_i - u_3 \mathcal{H}_i - u_6 M_i + u_5 P_i, \\ K_{ix} = -2K_{i+1} + u_4 D_i + u_4 \mathcal{H}_i - u_7 Z_i + u_8 T_i, \\ M_{ix} = -u_2 N_i + u_3 W_i + u_5 C_i - u_8 I_i, \\ N_{ix} = 2N_{i+1} - u_1 M_i + u_3 Z_i - u_5 A_i - u_5 D_i - u_6 F_i, \\ P_{ix} = -u_2 Q_i! - u_3 S_i + u_6 C_i + u_7 I_i, \\ Q_{ix} = 2Q_{i+1} - u_1 P_i - u_3 T_i - u_5 G_i - u_6 A_i - u_6 \mathcal{H}_i, \\ S_{ix} = -2S_{i+1} - u_2 T_i + u_4 P_i + u_7 A_i + u_7 D_i + u_8 G_i, \\ T_{ix} = -u_1 S_i + u_4 Q_i - u_6 K_i + u_7 B_i, \\ W_{ix} = -2W_{i+1} - u_2 Z_i - u_4 M_i + u_7 F_i + u_8 A_i! + u_8 \mathcal{H}_i, \\ Z_{ix} = -u_1 W_i - u_4 N_i + u_5 K_i + u_8 B_i. \end{cases} \tag{3.2}$$

We take the initial values

$$A_0 = \alpha(t), D_0 = \beta(t), \mathcal{H}_0 = \eta(t),$$
$$B_0 = C_0 = F_0 = G_0 = I_0 = K_0 = M_0 = N_0 = P_0 = Q_0 = S_0 = T_0 = W_0 = Z_0 = k_0(t) = 0.$$

From (3.2), we obtain

$$A_1 = D_1 = \mathcal{H}_1 = k_1(t)x,\ B_1 = \alpha u_1,\ C_1 = \alpha u_2,\ F_1 = \frac{\eta-\beta}{2}\partial^{-1}(u_5u_8),\ G_1 = \frac{\beta-\eta}{2}\partial^{-1}(u_6u_7),$$
$$I_1 = \frac{\beta+\eta}{2}u_3,\ K_1 = \frac{\beta+\eta}{2}u_4,\ M_1 = \frac{\alpha-\beta}{2}\partial^{-1}(u_2u_5+u_3u_8),\ N_1 = \frac{\alpha+\beta}{2}u_5,$$
$$P_1 = \frac{\alpha-\eta}{2}\partial^{-1}(u_2u_6-u_3u_7),\ Q_1 = \frac{\alpha+\eta}{2}u_6,\ S_1 = \frac{\alpha+\beta}{2}u_7,\ W_1 = \frac{\alpha+\eta}{2}u_8,$$
$$T_1 = \frac{\alpha-\beta}{2}\partial^{-1}(u_1u_7+u_4u_6),\ Z_1 = \frac{\alpha-\eta}{2}\partial^{-1}(u_1u_8-u_4u_5),$$
$$B_2 = \frac{1}{2}B_{1x}+u_5T_1+u_6Z_1+u_1k_1(t)x,\ C_2 = -\frac{1}{2}C_{1x}+u_7M_1+u_8P_1+u_2k_1(t)x,$$
$$I_2 = \frac{1}{2}I_{1x}+\frac{u_6}{2}M_1-\frac{u_5}{2}P_1+u_3k_1(t)x,\ K_2 = -\frac{1}{2}K_{1x}-\frac{u_7}{2}Z_1+\frac{u_8}{2}T_1+u_4k_1(t)x,$$
$$N_2 = \frac{1}{2}N_{1x}+\frac{u_1}{2}M_1-\frac{u_3}{2}Z_1+\frac{u_6}{2}F_1+u_5k_1(t)x,\ Q_2 = \frac{1}{2}Q_{1x}+\frac{u_1}{2}P_1+\frac{u_3}{2}T_1+\frac{u_5}{2}G_1+u_6k_1(t)x,$$
$$S_2 = -\frac{1}{2}S_{1x}-\frac{u_2}{2}T_1+\frac{u_4}{2}P_1+\frac{u_8}{2}G_1+u_7k_1(t)x,$$
$$W_2 = -\frac{1}{2}W_{1x}-\frac{u_2}{2}Z_1-\frac{u_4}{2}M_1+\frac{u_7}{2}F_1+u_8k_1(t)x,$$
$$\ldots.$$

According to (3.2), we obtain the following recursion relation:

$$\begin{pmatrix} -C_{n+1}\\ -B_{n+1}\\ -2K_{n+1}\\ -2I_{n+1}\\ -2S_{n+1}\\ -2W_{n+1}\\ 2N_{n+1}\\ 2Q_{n+1}\end{pmatrix} = L_3\begin{pmatrix} -C_n\\ -B_n\\ -2K_n\\ -2I_n\\ -2S_n\\ -2W_n\\ 2N_n\\ 2Q_n\end{pmatrix} + \begin{pmatrix} -u_2\\ -u_1\\ -2u_4\\ -2u_3\\ -2u_7\\ -2u_8\\ 2u_5\\ 2u_6\end{pmatrix}k_n(t)x,$$

where $L_3 = (l_{ij})_{8\times 8}$ and l_{ij} is

$$l_{11} = -\frac{1}{2}\partial+u_2\partial^{-1}u_1+u_7\partial^{-1}u_5+u_8\partial^{-1}u_6,\ l_{12} = -u_2\partial^{-1}u_2,$$
$$l_{14} = \frac{1}{2}l_{32} = -\frac{1}{2}(u_7\partial^{-1}u_8-u_8\partial^{-1}u_7),\ l_{15} = l_{63} = -\frac{1}{2}(u_2\partial^{-1}u_5+u_8\partial^{-1}u_3),$$
$$l_{16} = -l_{53} = -\frac{1}{2}(u_2\partial^{-1}u_6-u_7\partial^{-1}u_3),\ l_{17} = -\frac{1}{2}l_{52} = \frac{1}{2}(u_2\partial^{-1}u_7+u_7\partial^{-1}u_2),$$
$$l_{18} = -\frac{1}{2}l_{62} = \frac{1}{2}(u_2\partial^{-1}u_8+u_8\partial^{-1}u_2),\ l_{21} = u_1\partial^{-1}u_1,$$
$$l_{22} = \frac{1}{2}\partial-u_1\partial^{-1}u_2+u_5\partial^{-1}u_7+u_6\partial^{-1}u_8,\ l_{23} = \frac{1}{2}l_{41} = -\frac{1}{2}(u_5\partial^{-1}u_6-u_6\partial^{-1}u_5),$$
$$l_{25} = \frac{1}{2}l_{71} = -\frac{1}{2}(u_1\partial^{-1}u_5+u_5\partial^{-1}u_1),\ l_{26} = \frac{1}{2}l_{81} = -\frac{1}{2}(u_1\partial^{-1}u_6+u_6\partial^{-1}u_1),$$

$$l_{27} = -l_{84} = \frac{1}{2}(u_1\partial^{-1}u_7 + u_6\partial^{-1}u_4),\ l_{28} = l_{74} = \frac{1}{2}(u_1\partial^{-1}u_8 - u_5\partial^{-1}u_4),$$
$$l_{33} = -\frac{1}{2}\partial - u_4\partial^{-1}u_3 - \frac{1}{2}u_7\partial^{-1}u_5 - \frac{1}{2}u_8\partial^{-1}u_6,\ l_{34} = u_4\partial^{-1}u_4,$$
$$l_{35} = -\frac{1}{2}l_{61} = \frac{1}{2}(u_4\partial^{-1}u_5 - u_8\partial^{-1}u_1),\ l_{36} = \frac{1}{2}l_{51} = \frac{1}{2}(u_4\partial^{-1}u_6 + u_7\partial^{-1}u_1),$$
$$l_{37} = -l_{54} = -\frac{1}{2}(u_4\partial^{-1}u_7 + u_7\partial^{-1}u_4),\ l_{38} = -l_{64} = -\frac{1}{2}(u_4\partial^{-1}u_8 + u_8\partial^{-1}u_4),$$
$$l_{43} = -u_3\partial^{-1}u_3,\ l_{44} = \frac{1}{2}\partial + u_3\partial^{-1}u_4 - \frac{1}{2}u_6\partial^{-1}u_8 - \frac{1}{2}u_5\partial^{-1}u_7,$$
$$l_{45} = l_{73} = \frac{1}{2}(u_3\partial^{-1}u_5 + u_5\partial^{-1}u_3),\ l_{46} = l_{83} = \frac{1}{2}(u_3\partial^{-1}u_6 + u_6\partial^{-1}u_3),$$
$$l_{47} = \frac{1}{2}l_{82} = -\frac{1}{2}(u_3\partial^{-1}u_7 - u_6\partial^{-1}u_2),\ l_{48} = -\frac{1}{2}l_{72} = -\frac{1}{2}(u_3\partial^{-1}u_8 + u_5\partial^{-1}u_2),$$
$$l_{51} = u_4\partial^{-1}u_6 + u_7\partial^{-1}u_1,\ l_{52} = -u_2\partial^{-1}u_7 - u_7\partial^{-1}u_2,$$
$$l_{55} = -\frac{1}{2}\partial + \frac{1}{2}u_2\partial^{-1}u_1 - \frac{1}{2}u_4\partial^{-1}u_3 + \frac{1}{2}u_8\partial^{-1}u_6,\ l_{56} = -\frac{1}{2}u_7\partial^{-1}u_6,$$
$$l_{58} = -l_{67} = \frac{1}{2}(u_2\partial^{-1}u_4 + u_4\partial^{-1}u_2 + u_7\partial^{-1}u_8 - u_8\partial^{-1}u_7),$$
$$l_{65} = -\frac{1}{2}u_8\partial^{-1}u_5,\ l_{66} = -\frac{1}{2}\partial + \frac{1}{2}u_2\partial^{-1}u_1 - \frac{1}{2}u_4\partial^{-1}u_3 + \frac{1}{2}u_7\partial^{-1}u_5,$$
$$l_{76} = -l_{85} = -\frac{1}{2}(u_1\partial^{-1}u_3 + u_3\partial^{-1}u_1 - u_5\partial^{-1}u_6 + u_6\partial^{-1}u_5),$$
$$l_{77} = \frac{1}{2}\partial - \frac{1}{2}u_1\partial^{-1}u_2 + \frac{1}{2}u_3\partial^{-1}u_4 + \frac{1}{2}u_6\partial^{-1}u_8,\ l_{78} = -\frac{1}{2}u_5\partial^{-1}u_8,\ l_{87} = -\frac{1}{2}u_6\partial^{-1}u_7,$$
$$l_{88} = \frac{1}{2}\partial - \frac{1}{2}u_1\partial^{-1}u_2 + \frac{1}{2}u_3\partial^{-1}u_4 + \frac{1}{2}u_5\partial^{-1}u_7.$$

All other elements $l_{ij} = 0$ are identically equal to zero.

Taking $V_3^{(n)} = V_{3+}^{(n)}$, from the zero curvature equation $-V_{3x}^{(n)} + \frac{\partial U_3}{\partial u}u_t + \frac{\partial U_3}{\partial \lambda}\lambda_{t,+}^{(n)} + [U_3, V_3^{(n)}] = 0$, we can obtain the following supersoliton hierarchy:

$$u_{t_{3,n}} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix}_{t_{3,n}} = \begin{pmatrix} -2B_{n+1} \\ 2C_{n+1} \\ -2I_{n+1} \\ 2K_{n+1} \\ -2N_{n+1} \\ -2Q_{n+1} \\ 2S_{n+1} \\ 2W_{n+1} \end{pmatrix} = J_3 \begin{pmatrix} -C_{n+1} \\ -B_{n+1} \\ -2K_{n+1} \\ -2I_{n+1} \\ -2S_{n+1} \\ -2W_{n+1} \\ 2N_{n+1} \\ 2Q_{n+1} \end{pmatrix} = J_3L_3 \begin{pmatrix} -C_n \\ -B_n \\ -2K_n \\ -2I_n \\ -2S_n \\ -2W_n \\ 2N_n \\ 2Q_n \end{pmatrix} + J_3 \begin{pmatrix} -u_2 \\ -u_1 \\ -2u_4 \\ -2u_3 \\ -2u_7 \\ -2u_8 \\ 2u_5 \\ 2u_6 \end{pmatrix} k_n(t)x,$$

where the Hamiltonian operator is given by

$$J_3 = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

Next, we construct the super Hamiltonian structure. By direct calculation, we obtain

$$\left\langle V_3, \frac{\partial U_3}{\partial \lambda} \right\rangle = 2D + 2\mathcal{H} - 2A, \left\langle V_3, \frac{\partial U_3}{\partial u_1} \right\rangle = -C, \left\langle V_3, \frac{\partial U_3}{\partial u_2} \right\rangle = -B, \left\langle V_3, \frac{\partial U_3}{\partial u_3} \right\rangle = -2K,$$
$$\left\langle V_3, \frac{\partial U_3}{\partial u_4} \right\rangle = -2I, \left\langle V_3, \frac{\partial U_3}{\partial u_5} \right\rangle = -2S, \left\langle V_3, \frac{\partial U_3}{\partial u_6} \right\rangle = -2W, \left\langle V_3, \frac{\partial U_3}{\partial u_7} \right\rangle = 2N, \left\langle V_3, \frac{\partial U_3}{\partial u_8} \right\rangle = 2Q.$$

Substituting the above equations into the supertrace identity and comparing the coefficients of λ^{-n-1} on both sides, we obtain

$$\frac{\delta}{\delta u} \int (2D_{n+1} + 2\mathcal{H}_{n+1} - 2A_{n+1}) dx = (\gamma - n) \begin{pmatrix} -C_n \\ -B_n \\ -2K_n \\ -2I_n \\ -2S_n \\ -2W_n \\ 2N_n \\ 2Q_n \end{pmatrix}.$$

Setting $n = 1$ yields $\gamma = 0$. Thus, we have

$$u_{t_{3,n}} = J_3 \frac{\delta H_{3,n+1}}{\delta u}, \quad H_{3,n+1} = -2 \int \frac{(D_{n+2} + \mathcal{H}_{n+2} - A_{n+2})}{n+1} dx, \quad n \geq 0.$$

3.2. The second type of super integrable hierarchy

We consider a nonisospectral problem

$$\begin{cases} \varphi_x = U_4 \varphi, \\ \varphi_t = V_4 \varphi, \\ \lambda_t = \sum\limits_{i \geq 0} k_i(t) \lambda^{-i}. \end{cases}$$

The auxiliary matrix V_4 has the same matrix structure as V_3 in (3.1), but its coefficient matrices are determined independently by

$$V_{4,x} = \frac{\partial U_4}{\partial \lambda} \lambda_t + [U_4, V_4].$$

The formal Laurent expansion is given by

$$V_4 = \sum_{i=0}^{\infty} V_{4,i}\lambda^{-i},$$

and U_4 is defined as follows:

$$U_4 = \begin{pmatrix} \lambda & u_3 & 0 & u_5 & 0 & u_7 \\ u_4 & 0 & -u_5 & 0 & u_8 & u_9 \\ 0 & u_6 & -\lambda & -u_4 & u_{10} & 0 \\ -u_6 & 0 & -u_3 & 0 & u_{11} & u_{12} \\ 0 & -u_{12} & -u_7 & -u_9 & \lambda & u_1 \\ u_{10} & u_{11} & 0 & u_8 & u_2 & -\lambda \end{pmatrix},$$

where $u_7, u_8, u_9, u_{10}, u_{11}$, and u_{12} are Grassmann variables satisfying the following anticommutation and nilpotency relations:

$$\begin{aligned} &u_7^2 = u_8^2 = u_9^2 = u_{10}^2 = u_{11}^2 = u_{12}^2 = 0, \\ &u_i u_j + u_j u_i = 0, \quad i, j = 7, 8, 9, 10, 11, 12, \text{ and } i \neq j. \end{aligned}$$

The stationary zero curvature representation $V_{4,x} = \frac{\partial U_4}{\partial \lambda}\lambda_t + [U_4, V_4]$ gives

$$\begin{cases} A_{ix} = u_1 C_i - u_2 B_i - u_7 S_i - u_8 Z_i - u_9 W_i - u_{10} N_i - u_{11} Q_i - u_{12} P_i + k_i(t), \\ B_{ix} = 2B_{i+1} - 2u_1 A_i - 2u_7 T_i - 2u_9 Z_i - 2u_{12} Q_i, \\ C_{ix} = -2C_{i+1} + 2u_2 A_i + 2u_8 W_i + 2u_{10} M_i + 2u_{11} P_i, \\ D_{ix} = u_3 G_i - u_4 F_i - u_5 K_i + u_6 I_i + u_7 S_i + u_{10} N_i + k_i(t), \\ F_{ix} = F_{i+1} - u_3 D_i + u_3 \mathcal{H}_i + u_7 W_i + u_{11} N_i - u_{12} M_i, \\ G_{ix} = -G_{i+1} + u_4 D_i - u_4 \mathcal{H}_i - u_8 T_i + u_9 S_i + u_{10} Q_i, \\ \mathcal{H}_{ix} = -u_3 G_i + u_4 F_i - u_5 K_i + u_6 I_i - u_8 Z_i + u_9 W_i + u_{11} Q_i - u_{12} P_i, \\ I_{ix} = I_{i+1} - u_5 D_i - u_5 \mathcal{H}_i + u_7 P_i + u_8 N_i - u_9 M_i, \\ K_{ix} = -K_{i+1} + u_6 D_i + u_6 \mathcal{H}_i - u_{10} Z_i + u_{11} T_i - u_{12} S_i, \\ M_{ix} = -u_2 N_i + u_3 P_i + u_5 W_i + u_7 C_i - u_8 F_i - u_{11} I_i, \\ N_{ix} = 2N_{i+1} - u_1 M_i + u_3 Q_i + u_5 Z_i - u_7 A_i - u_7 D_i - u_9 F_i - u_{12} I_i, \\ P_{ix} = -P_{i+1} - u_2 Q_i + u_4 M_i - u_5 S_i + u_8 A_i - u_8 \mathcal{H}_i + u_9 C_i + u_{10} I_i, \\ Q_{ix} = Q_{i+1} - u_1 P_i + u_4 N_i - u_5 T_i - u_7 G_i + u_8 B_i - u_9 A_i - u_9 \mathcal{H}_i, \\ S_{ix} = -2S_{i+1} - u_2 T_i - u_4 W_i + u_6 P_i - u_8 K_i + u_{10} A_i + u_{10} D_i + u_{11} G_i, \\ T_{ix} = -u_1 S_i - u_4 Z_i + u_6 Q_i - u_9 K_i + u_{10} B_i + u_{12} G_i, \\ W_{ix} = -W_{i+1} - u_2 Z_i - u_3 S_i - u_6 M_i + u_{10} F_i + u_{11} A_i + u_{11} \mathcal{H}_i + u_{12} C_i, \\ Z_{ix} = Z_{i+1} - u_1 W_i - u_3 T_i - u_6 N_i + u_7 K_i + u_{11} B_i - u_{12} A_i + u_{12} \mathcal{H}_i. \end{cases} \tag{3.3}$$

We take the initial values

$$\begin{aligned} &A_0 = \alpha(t), D_0 = \beta(t), \mathcal{H}_0 = \eta(t), \\ &B_0 = C_0 = F_0 = G_0 = I_0 = K_0 = M_0 = N_0 = P_0 = Q_0 = S_0 = T_0 = W_0 = Z_0 = k_0(t) = 0. \end{aligned}$$

From (3.3), we obtain

$$A_1 = D_1 = k_1(t)x, \mathcal{H}_1 = 0, B_1 = \alpha u_1, C_1 = \alpha u_2, F_1 = (\beta - \eta)u_3, G_1 = (\beta - \eta)u_4,$$
$$I_1 = (\beta + \eta)u_5, K_1 = (\beta + \eta)u_6, N_1 = \frac{\alpha + \beta}{2}u_7, P_1 = (\alpha - \eta)u_8, Q_1 = (\alpha + \eta)u_9,$$
$$S_1 = \frac{\alpha + \beta}{2}u_{10}, W_1 = (\alpha + \eta)u_{11}, Z_1 = (\alpha - \eta)u_{12},$$
$$M_1 = \frac{\alpha - \beta}{2}\partial^{-1}(u_2u_7) + (\alpha - \beta)\partial^{-1}(u_3u_8 + u_5u_{11}),$$
$$T_1 = \frac{\alpha - \beta}{2}\partial^{-1}(u_1u_{10}) + (\alpha - \beta)\partial^{-1}(u_6u_9 - u_4u_{12}),$$
$$B_2 = \frac{1}{2}B_{1x} + u_7T_1 - 2\eta u_9u_{12} + u_1k_1(t)x,\ C_2 = -\frac{1}{2}C_{1x} + u_{10}M_1 + 2\eta u_8u_{11} + u_2k_1(t)x,$$
$$F_2 = F_{1x} + u_{12}M_1 + \left(\frac{\beta - \alpha}{2} - \eta\right)u_7u_{11} + u_3k_1(t)x,$$
$$G_2 = -G_{1x} - u_8T_1 + \left(\frac{\beta - \alpha}{2} - \eta\right)u_9u_{10} + u_4k_1(t)x,$$
$$I_2 = I_{1x} + u_9M_1 + \left(\frac{\beta - \alpha}{2} + \eta\right)u_7u_8 + u_5k_1(t)x,$$
$$K_2 = -K_{1x} + u_{11}T_1 + \left(\frac{\beta - \alpha}{2} + \eta\right)u_{10}u_{12} + u_6k_1(t)x,$$
$$N_2 = \frac{1}{2}N_{1x} + \frac{1}{2}u_1M_1 + \left(\frac{\beta - \alpha}{2} - \eta\right)u_3u_9 + \left(\frac{\beta - \alpha}{2} + \eta\right)u_5u_{12} + u_7k_1(t)x,$$
$$P_2 = -P_{1x} + u_4M_1 - \eta u_2u_9 + \left(\frac{\beta - \alpha}{2} + \eta\right)u_5u_{10} + u_8k_1(t)x,$$
$$Q_2 = Q_{1x} + u_5T_1 - \eta u_1u_8 + \left(\frac{\beta - \alpha}{2} - \eta\right)u_4u_7 + u_9k_1(t)x,$$
$$S_2 = -\frac{1}{2}S_{1x} - \frac{1}{2}u_2T_1 + \left(\frac{\alpha - \beta}{2} - \eta\right)u_6u_8 + \left(\frac{\beta - \alpha}{2} - \eta\right)u_4u_{11} + u_{10}k_1(t)x,$$
$$W_2 = -W_{1x} - u_6M_1 + \eta u_2u_{12} + \left(\frac{\beta - \alpha}{2} - \eta\right)u_3u_{10} + u_{11}k_1(t)x,$$
$$Z_2 = Z_{1x} + u_3T_1 + \eta u_1u_{11} + \left(\frac{\alpha - \beta}{2} - \eta\right)u_6u_7 + u_{12}k_1(t)x,$$
$$\cdots.$$

According to (3.3), we obtain the following recursion relation:

$$\begin{pmatrix} -C_{n+1} \\ -B_{n+1} \\ 2G_{n+1} \\ 2F_{n+1} \\ -2K_{n+1} \\ -2I_{n+1} \\ -2S_{n+1} \\ 2Z_{n+1} \\ -2W_{n+1} \\ 2N_{n+1} \\ 2Q_{n+1} \\ -2P_{n+1} \end{pmatrix} = L_4 \begin{pmatrix} -C_n \\ -B_n \\ 2G_n \\ 2F_n \\ -2K_n \\ -2I_n \\ -2S_n \\ 2Z_n \\ -2W_n \\ 2N_n \\ 2Q_n \\ -2P_n \end{pmatrix} + \begin{pmatrix} -u_2 \\ -u_1 \\ 2u_4 \\ 2u_3 \\ -2u_6 \\ -2u_5 \\ -2u_{10} \\ 2u_{12} \\ -2u_{11} \\ 2u_7 \\ 2u_9 \\ -2u_8 \end{pmatrix} k_n(t)x,$$

where $L_4 = (l_{ij})_{12\times 12}$ and l_{ij} is

$$l_{11} = -\frac{1}{2}\partial + u_2\partial^{-1}u_1 + u_{10}\partial^{-1}u_7,\ l_{12} = -u_2\partial^{-1}u_2,\ l_{14} = \frac{1}{2}u_{10}\partial^{-1}u_8,\ l_{16} = -\frac{1}{2}u_{10}\partial^{-1}u_{11},$$
$$l_{17} = -\frac{1}{2}u_2\partial^{-1}u_7,\ l_{18} = \frac{1}{2}u_2\partial^{-1}u_8,\ l_{19} = -l_{75} = \frac{1}{2}(-u_2\partial^{-1}u_9 + u_8 + u_{10}\partial^{-1}u_5),$$
$$l_{1,10} = -\frac{1}{2}l_{72} = \frac{1}{2}(u_2\partial^{-1}u_{10} + u_{10}\partial^{-1}u_2),\ l_{1,11} = \frac{1}{2}u_2\partial^{-1}u_{11},\ l_{21} = u_1\partial^{-1}u_1,$$
$$l_{1,12} = -l_{73} = \frac{1}{2}(-u_2\partial^{-1}u_{12} + u_{10}\partial^{-1}u_3 + u_{11}),\ l_{22} = \frac{1}{2}\partial - u_1\partial^{-1}u_2 + u_7\partial^{-1}u_{10},$$
$$l_{23} = -\frac{1}{2}u_7\partial^{-1}u_{12},\ l_{25} = -\frac{1}{2}u_7\partial^{-1}u_9,\ l_{27} = \frac{1}{2}l_{10,1} = -\frac{1}{2}(u_1\partial^{-1}u_7 + u_7\partial^{-1}u_1),$$
$$l_{28} = -l_{10,4} = \frac{1}{2}(u_1\partial^{-1}u_8 + u_7\partial^{-1}u_4 - u_9),\ l_{29} = -\frac{1}{2}u_1\partial^{-1}u_9,\ l_{2,10} = \frac{1}{2}u_1\partial^{-1}u_{10},$$
$$l_{2,11} = l_{10,6} = \frac{1}{2}(u_1\partial^{-1}u_{11} - u_7\partial^{-1}u_6 - u_{12}),\ l_{2,12} = -\frac{1}{2}u_1\partial^{-1}u_{12},\ l_{32} = 2u_8\partial^{-1}u_{10},$$
$$l_{33} = -\partial + 2u_4\partial^{-1}u_3 - u_8\partial^{-1}u_{12},\ l_{34} = -2u_4\partial^{-1}u_4,\ l_{35} = -u_8\partial^{-1}u_9,$$
$$l_{37} = -\frac{1}{2}l_{12,1} = -u_4\partial^{-1}u_7 - u_8\partial^{-1}u_1 - u_9,\ l_{38} = l_{12,4} = u_4\partial^{-1}u_8 + u_8\partial^{-1}u_4,\ l_{39} = u_4\partial^{-1}u_9,$$
$$l_{3,10} = u_4\partial^{-1}u_{10},\ l_{3,11} = l_{12,6} = -u_4\partial^{-1}u_{11} - u_8\partial^{-1}u_6 + u_{10},\ l_{3,12} = -u_4\partial^{-1}u_{12},$$
$$l_{41} = -2u_{12}\partial^{-1}u_7,\ l_{43} = 2u_3\partial^{-1}u_3,\ l_{44} = \partial - 2u_3\partial^{-1}u_4 - u_{12}\partial^{-1}u_8,\ l_{46} = u_{12}\partial^{-1}u_{11},$$
$$l_{47} = -u_3\partial^{-1}u_7,\ l_{48} = u_3\partial^{-1}u_8,\ l_{49} = l_{85} = u_3\partial^{-1}u_9 + u_7 - u_{12}\partial^{-1}u_5,$$
$$l_{4,10} = -\frac{1}{2}l_{82} = u_3\partial^{-1}u_{10} - u_{11} - u_{12}\partial^{-1}u_2,\ l_{4,11} = -u_3\partial^{-1}u_{11},$$
$$l_{4,12} = -l_{83} = -u_3\partial^{-1}u_{12} - u_{12}\partial^{-1}u_3,\ l_{52} = 2u_{11}\partial^{-1}u_{10},\ l_{53} = -u_{11}\partial^{-1}u_{12},$$
$$l_{55} = -\partial - 2u_6\partial^{-1}u_5 - u_{11}\partial^{-1}u_9,\ l_{56} = 2u_6\partial^{-1}u_6,\ l_{57} = -\frac{1}{2}l_{91} = u_6\partial^{-1}u_7 - u_{11}\partial^{-1}u_1 - u_{12},$$
$$l_{58} = -l_{94} = u_6\partial^{-1}u_8 + u_{10} + u_{11}\partial^{-1}u_4,\ l_{59} = u_6\partial^{-1}u_9,\ l_{5,10} = -u_6\partial^{-1}u_{10},$$
$$l_{5,11} = -l_{96} = -u_6\partial^{-1}u_{11} - u_{11}\partial^{-1}u_6,\ l_{5,12} = -u_6\partial^{-1}u_{12},\ l_{61} = 2u_9\partial^{-1}u_7,\ l_{64} = u_9\partial^{-1}u_8,$$
$$l_{65} = -2u_5\partial^{-1}u_5,\ l_{66} = \partial + 2u_5\partial^{-1}u_6 - u_9\partial^{-1}u_{11},\ l_{67} = u_5\partial^{-1}u_7,\ l_{68} = u_5\partial^{-1}u_8,$$

$$l_{69} = l_{11,5} = u_5\partial^{-1}u_9 + u_9\partial^{-1}u_5,\ l_{6,10} = \frac{1}{2}l_{11,2} = -u_5\partial^{-1}u_{10} + u_8 + u_9\partial^{-1}u_2,$$
$$l_{6,11} = -u_5\partial^{-1}u_{11},\ l_{6,12} = -l_{11,3} = -u_5\partial^{-1}u_{12} - u_7 + u_9\partial^{-1}u_3,\ l_{71} = u_{10}\partial^{-1}u_1,$$
$$l_{74} = \frac{1}{2}u_{10}\partial^{-1}u_4,\ l_{76} = \frac{1}{2}u_{10}\partial^{-1}u_6,\ l_{77} = -\frac{1}{2}\partial + \frac{1}{2}u_2\partial^{-1}u_1,\ l_{78} = -\frac{1}{2}u_2\partial^{-1}u_4 + \frac{1}{2}u_{10}\partial^{-1}u_8,$$
$$l_{79} = -\frac{1}{2}u_4 - \frac{1}{2}u_{10}\partial^{-1}u_9,\ l_{7,11} = \frac{1}{2}u_2\partial^{-1}u_6 + \frac{1}{2}u_{10}\partial^{-1}u_{11},\ l_{7,12} = \frac{1}{2}u_6 - \frac{1}{2}u_{10}\partial^{-1}u_{12},$$
$$l_{81} = -2u_{12}\partial^{-1}u_1,\ l_{84} = -u_{12}\partial^{-1}u_4,\ l_{86} = u_{12}\partial^{-1}u_6,\ l_{87} = u_3\partial^{-1}u_1 + u_{12}\partial^{-1}u_7,$$
$$l_{88} = \partial - u_3\partial^{-1}u_4,\ l_{89} = -u_1 + 2u_{12}\partial^{-1}u_9,\ l_{8,10} = u_6 - u_{12}\partial^{-1}u_{10},\ l_{8,11} = u_3\partial^{-1}u_6 - 2u_{12}\partial^{-1}u_{11},$$
$$l_{92} = -2u_{11}\partial^{-1}u_2,\ l_{93} = u_{11}\partial^{-1}u_3,\ l_{95} = -u_{11}\partial^{-1}u_5,\ l_{97} = -u_3 - u_{11}\partial^{-1}u_7,$$
$$l_{98} = u_2 + 2u_{11}\partial^{-1}u_8,\ l_{99} = -\partial - u_6\partial^{-1}u_5,\ l_{9,10} = -u_6\partial^{-1}u_2 + u_{11}\partial^{-1}u_{10},$$
$$l_{9,12} = -u_6\partial^{-1}u_3 - 2u_{11}\partial^{-1}u_{12},\ l_{10,2} = u_7\partial^{-1}u_2,\ l_{10,3} = \frac{1}{2}u_7\partial^{-1}u_3,\ l_{10,5} = \frac{1}{2}u_7\partial^{-1}u_5,$$
$$l_{10,8} = -\frac{1}{2}u_5 - \frac{1}{2}u_7\partial^{-1}u_8,\ l_{10,9} = -\frac{1}{2}u_1\partial^{-1}u_5 + \frac{1}{2}u_7\partial^{-1}u_9,\ l_{10,10} = \frac{1}{2}\partial - \frac{1}{2}u_1\partial^{-1}u_2,$$
$$l_{10,11} = -\frac{1}{2}u_3 - \frac{1}{2}u_7\partial^{-1}u_{11},\ l_{10,12} = -\frac{1}{2}u_1\partial^{-1}u_3 + \frac{1}{2}u_7\partial^{-1}u_{12},\ l_{11,1} = -2u_9\partial^{-1}u_1,$$
$$l_{11,4} = u_9\partial^{-1}u_4,\ l_{11,6} = -u_9\partial^{-1}u_6,\ l_{11,7} = u_5\partial^{-1}u_1 + u_9\partial^{-1}u_7,\ l_{11,8} = -u_5\partial^{-1}u_4 - 2u_9\partial^{-1}u_8,$$
$$l_{11,10} = -u_4 - u_9\partial^{-1}u_{10},\ l_{11,11} = \partial + u_5\partial^{-1}u_6,\ l_{11,12} = -u_1 + 2u_9\partial^{-1}u_{12},\ l_{12,2} = -2u_8\partial^{-1}u_2,$$
$$l_{12,3} = -u_8\partial^{-1}u_3,\ l_{12,5} = u_8\partial^{-1}u_5,\ l_{12,7} = -u_5 - u_8\partial^{-1}u_7,\ l_{12,9} = u_4\partial^{-1}u_5 - 2u_8\partial^{-1}u_9,$$
$$l_{12,10} = u_4\partial^{-1}u_2 + u_8\partial^{-1}u_{10},\ l_{12,11} = u_2 + 2u_8\partial^{-1}u_{11},\ l_{12,12} = -\partial + u_4\partial^{-1}u_3.$$

All other elements $l_{ij} = 0$ are identically equal to zero.

Taking $V_4^{(n)} = V_{4+}^{(n)}$, from the zero curvature equation $-V_{4x}^{(n)} + \frac{\partial U_4}{\partial u}u_t + \frac{\partial U_4}{\partial \lambda}\lambda_{t,+}^{(n)} + [U_4, V_4^{(n)}] = 0$, we can obtain the following supersoliton hierarchy:

$$u_{t_{4,n}} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{pmatrix}_{t_{4,n}} = \begin{pmatrix} -2B_{n+1} \\ 2C_{n+1} \\ -F_{n+1} \\ G_{n+1} \\ -I_{n+1} \\ K_{n+1} \\ -2N_{n+1} \\ P_{n+1} \\ -Q_{n+1} \\ 2S_{n+1} \\ W_{n+1} \\ -Z_{n+1} \end{pmatrix} = J_4 \begin{pmatrix} -C_{n+1} \\ -B_{n+1} \\ 2G_{n+1} \\ 2F_{n+1} \\ -2K_{n+1} \\ -2I_{n+1} \\ -2S_{n+1} \\ 2Z_{n+1} \\ -2W_{n+1} \\ 2N_{n+1} \\ 2Q_{n+1} \\ -2P_{n+1} \end{pmatrix} = J_4L_4 \begin{pmatrix} -C_n \\ -B_n \\ 2G_n \\ 2F_n \\ -2K_n \\ -2I_n \\ -2S_n \\ 2Z_n \\ -2W_n \\ 2N_n \\ 2Q_n \\ -2P_n \end{pmatrix} + J_4 \begin{pmatrix} -u_2 \\ -u_1 \\ 2u_4 \\ 2u_3 \\ -2u_6 \\ -2u_5 \\ -2u_{10} \\ 2u_{12} \\ -2u_{11} \\ 2u_7 \\ 2u_9 \\ -2u_8 \end{pmatrix} k_n(t)x,$$

where the Hamiltonian operator is given by

$$J_4 = \begin{pmatrix} 2J_1' & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2}J_1' & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}J_1' & 0 & 0 \\ 0 & 0 & 0 & 0 & J_2' \\ 0 & 0 & 0 & J_2' & 0 \end{pmatrix},$$

where

$$J_1' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J_2' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}.$$

Next, we construct the super Hamiltonian structure. By direct calculation, we obtain

$$\left\langle V_4, \frac{\partial U_4}{\partial \lambda} \right\rangle = 2D - 2A, \left\langle V_4, \frac{\partial U_4}{\partial u_1} \right\rangle = -C, \left\langle V_4, \frac{\partial U_4}{\partial u_2} \right\rangle = -B, \left\langle V_4, \frac{\partial U_4}{\partial u_3} \right\rangle = 2G, \left\langle V_4, \frac{\partial U_4}{\partial u_4} \right\rangle = 2F,$$
$$\left\langle V_4, \frac{\partial U_4}{\partial u_5} \right\rangle = -2K, \left\langle V_4, \frac{\partial U_4}{\partial u_6} \right\rangle = -2I, \left\langle V_4, \frac{\partial U_4}{\partial u_7} \right\rangle = -2S, \left\langle V_4, \frac{\partial U_4}{\partial u_8} \right\rangle = 2Z, \left\langle V_4, \frac{\partial U_4}{\partial u_9} \right\rangle = -2W,$$
$$\left\langle V_4, \frac{\partial U_4}{\partial u_{10}} \right\rangle = 2N, \left\langle V_4, \frac{\partial U_4}{\partial u_{11}} \right\rangle = 2Q, \left\langle V_4, \frac{\partial U_4}{\partial u_{12}} \right\rangle = -2P.$$

Substituting the above equations into the supertrace identity and comparing the coefficients of λ^{-n-1} on both sides, we obtain

$$\frac{\delta}{\delta u} \int (2D_{n+1} - 2A_{n+1}) dx = (\gamma - n) \begin{pmatrix} -C_n \\ -B_n \\ 2G_n \\ 2F_n \\ -2K_n \\ -2I_n \\ -2S_n \\ 2Z_n \\ -2W_n \\ 2N_n \\ 2Q_n \\ -2P_n \end{pmatrix}.$$

Setting $n = 1$ yields $\gamma = 0$. Thus, we have

$$u_{t_{4,n}} = J_4 \frac{\delta H_{4,n+1}}{\delta u}, \quad H_{4,n+1} = -2 \int \frac{(D_{n+2} - A_{n+2})}{n+1} dx, \quad n \geq 0.$$

4. The isospectral (2 + 1)-dimensional integrable hierarchy based on Lie superalgebra $\mathcal{D}(2,1)$

The recursion operator R maps the n-th flow of the hierarchy into the $(n+1)$-th flow, namely,

$$K_{n+1} = RK_n.$$

Therefore, the higher-order integrable equations can be generated recursively from the initial flow.

The recursion operator R is used to generate higher-order flows from lower-order ones. Its role in the present construction is to express the recursive structure of the hierarchy. The compatibility of the generated flows and any additional structural properties of R, such as the hereditary property, require separate justification and are not implied by the recursion relation alone.

The (2 + 1)-dimensional extensions considered in this section are constructed based on the two types of super integrable hierarchies introduced previously. The first and second types correspond to the lower-dimensional spectral representations with different dependent-variable spaces, and their higher-dimensional extensions are obtained by introducing additional spatial flows.

We define $\partial_y := \frac{\partial}{\partial_y}$ and introduce ξ as a differential operator satisfying

$$\xi f = f\xi + \partial_y f, \qquad f \in \mathcal{A},$$

where $\mathcal{A}$ denotes the underlying function algebra. Thus, ξ acts as a differential operator associated with the spatial variable y, and its commutation with coefficient functions generates the corresponding y-derivative terms in the zero-curvature equations.

The higher-dimensional hierarchies are constructed within the zero-curvature framework by incorporating an additional spatial dependence into the spectral representation. Substituting the corresponding spectral matrices into the zero-curvature equations and comparing the coefficients of the spectral parameter yields the evolution equations for the dependent variables. These coefficient equations provide the basis for deriving the higher-dimensional flows and their recursive representations. The compatibility of the resulting flows, however, must be established by the corresponding zero-curvature conditions rather than inferred solely from the recursive form.

4.1. The first type of super integrable hierarchy

We consider a (2+1)-dimensional isospectral problem

$$\begin{cases} \varphi_x = U_5\varphi, \\ \varphi_t = V_5\varphi, \\ \lambda_t = 0, \end{cases}$$

where

$$U_5 = \begin{pmatrix} \lambda+\xi & 0 & 0 & u_3 & 0 & u_5 \\ 0 & \lambda+\xi & -u_3 & 0 & 0 & u_6 \\ 0 & u_4 & -\lambda-\xi & 0 & u_7 & 0 \\ -u_4 & 0 & 0 & -\lambda-\xi & u_8 & 0 \\ 0 & 0 & -u_5 & -u_6 & \lambda+\xi & u_1 \\ u_7 & u_8 & 0 & 0 & u_2 & -\lambda-\xi \end{pmatrix},$$

$$V_5 = \begin{pmatrix} D & F & 0 & I & M & N \\ G & \mathcal{H} & -I & 0 & P & Q \\ 0 & K & -D & -G & S & T \\ -K & 0 & -F & -\mathcal{H} & W & Z \\ -T & -Z & -N & -Q & A & B \\ S & W & M & P & C & -A \end{pmatrix} = \sum_{i\geq 0}^{n} \begin{pmatrix} D_i & F_i & 0 & I_i & M_i & N_i \\ G_i & \mathcal{H}_i & -I_i & 0 & P_i & Q_i \\ 0 & K_i & -D_i & -G_i & S_i & T_i \\ -K_i & 0 & -F_i & -\mathcal{H}_i & W_i & Z_i \\ -T_i & -Z_i & -N_i & -Q_i & A_i & B_i \\ S_i & W_i & M_i & P_i & C_i & -A_i \end{pmatrix} \lambda^{-i}. \tag{4.1}$$

The stationary zero curvature representation $V_{5,x} = [U_5, V_5]$ gives

$$\begin{cases} A_{ix} = u_1 C_i - u_2 B_i - u_5 S_i - u_6 W_i - u_7 N_i - u_8 Q_i, \\ B_{ix} = 2B_{i+1} + 2B_i\xi + 2B_{iy} - 2u_1 A_i - 2u_5 T_i - 2u_6 Z_i, \\ C_{ix} = -2C_{i+1} - 2C_i\xi - 2C_{iy} + 2u_2 A_i + 2u_7 M_i + 2u_8 P_i, \\ D_{ix} = -u_3 K_i + u_4 I_i + u_5 S_i + u_7 N_i, \\ F_{ix} = u_5 W_i + u_8 N_i, \\ G_{ix} = u_6 S_i + u_7 Q_i, \\ \mathcal{H}_{ix} = -u_3 K_i + u_4 I_i + u_6 W_i + u_8 Q_i, \\ I_{ix} = 2I_{i+1} + 2I_i\xi + 2I_{iy} - u_3 D_i - u_3 \mathcal{H}_i - u_6 M_i + u_5 P_i, \\ K_{ix} = -2K_{i+1} - 2K_i\xi - 2K_{iy} + u_4 D_i + u_4 \mathcal{H}_i - u_7 Z_i + u_8 T_i, \\ M_{ix} = -u_2 N_i + u_3 W_i + u_5 C_i - u_8 I_i, \\ N_{ix} = 2N_{i+1} + 2N_i\xi + 2N_{iy} - u_1 M_i + u_3 Z_i - u_5 A_i - u_5 D_i - u_6 F_i, \\ P_{ix} = -u_2 Q_i - u_3 S_i + u_6 C_i + u_7 I_i, \\ Q_{ix} = 2Q_{i+1} + 2Q_i\xi + 2Q_{iy} - u_1 P_i - u_3 T_i - u_5 G_i - u_6 A_i - u_6 \mathcal{H}_i, \\ S_{ix} = -2S_{i+1} - 2S_i\xi - 2S_{iy} - u_2 T_i + u_4 P_i + u_7 A_i + u_7 D_i + u_8 G_i, \\ T_{ix} = -u_1 S_i + u_4 Q_i - u_6 K_i + u_7 B_i, \\ W_{ix} = -2W_{i+1} - 2W_i\xi - 2W_{iy} - u_2 Z_i - u_4 M_i + u_7 F_i + u_8 A_i + u_8 \mathcal{H}_i, \\ Z_{ix} = -u_1 W_i - u_4 N_i + u_5 K_i + u_8 B_i, \end{cases}$$

leading to the following isospectral supersoliton hierarchy:

$$u_{t_{5,n}} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix}_{t_{5,n}} = \begin{pmatrix} R(-2B_{n+1}) \\ R(2C_{n+1}) \\ R(-2I_{n+1}) \\ R(2K_{n+1}) \\ R(-2N_{n+1}) \\ R(-2Q_{n+1}) \\ R(2S_{n+1}) \\ R(2W_{n+1}) \end{pmatrix}.$$

To illustrate the explicit form of the higher-dimensional hierarchy, we consider its lowest-order nontrivial flow. Substituting the corresponding coefficient expressions into the recursive representation yields the following evolution equation:

$$\boldsymbol{u}_t = \boldsymbol{F}^{(1)}(\boldsymbol{u}, \boldsymbol{u}_x, \boldsymbol{u}_y, \ldots),$$

where $\boldsymbol{F}^{(1)}$ is obtained by evaluating the relevant flow in the hierarchy.

Next, we construct the super Hamiltonian structure. By direct calculation, we obtain

$$\left\langle V_5, \frac{\partial U_5}{\partial \lambda} \right\rangle = R(2D + 2\mathcal{H} - 2A), \left\langle V_5, \frac{\partial U_5}{\partial u_1} \right\rangle = R(-C), \left\langle V_5, \frac{\partial U_5}{\partial u_2} \right\rangle = R(-B),$$
$$\left\langle V_5, \frac{\partial U_5}{\partial u_3} \right\rangle = R(-2K), \left\langle V_5, \frac{\partial U_5}{\partial u_4} \right\rangle = R(-2I), \left\langle V_5, \frac{\partial U_5}{\partial u_5} \right\rangle = R(-2S),$$
$$\left\langle V_5, \frac{\partial U_5}{\partial u_6} \right\rangle = R(-2W), \left\langle V_5, \frac{\partial U_5}{\partial u_7} \right\rangle = R(2N), \left\langle V_5, \frac{\partial U_5}{\partial u_8} \right\rangle = R(2Q).$$

Substituting the above equations into the supertrace identity and comparing the coefficients of λ^{-n-1} on both sides, we obtain

$$\frac{\delta}{\delta u} \int R(2D_{n+1} + 2\mathcal{H}_{n+1} - 2A_{n+1})dx = (\gamma - n) \begin{pmatrix} R(-C_n) \\ R(-B_n) \\ R(-2K_n) \\ R(-2I_n) \\ R(-2S_n) \\ R(-2W_n) \\ R(2N_n) \\ R(2Q_n) \end{pmatrix}.$$

Setting $n = 1$ yields $\gamma = 0$. Thus, we have

$$u_{t_{5,n}} = J_5 \frac{\delta H_{5,n+1}}{\delta u}, \quad H_{5,n+1} = -2 \int \frac{R(D_{n+2} + \mathcal{H}_{n+2} - A_{n+2})}{n+1} dx, \quad n \geq 0.$$

4.2. The second type of super integrable hierarchy

We consider a (2+1)-dimension isospectral problem

$$\begin{cases} \varphi_x = U_6 \varphi, \\ \varphi_t = V_6 \varphi, \\ \lambda_t = 0. \end{cases}$$

The auxiliary matrix V_6 has the same matrix structure as V_5 in (4.1), but its coefficient matrices are determined independently by

$$V_{6,x} = [U_6, V_6].$$

The formal Laurent expansion is given by

$$V_6 = \sum_{i=0}^{\infty} V_{6,i} \lambda^{-i},$$

and U_6 is defined as follows:

$$U_6 = \begin{pmatrix} \lambda+\xi & u_3 & 0 & u_5 & 0 & u_7 \\ u_4 & 0 & -u_5 & 0 & u_8 & u_9 \\ 0 & u_6 & -\lambda-\xi & -u_4 & u_{10} & 0 \\ -u_6 & 0 & -u_3 & 0 & u_{11} & u_{12} \\ 0 & -u_{12} & -u_7 & -u_9 & \lambda+\xi & u_1 \\ u_{10} & u_{11} & 0 & u_8 & u_2 & -\lambda-\xi \end{pmatrix}.$$

The stationary zero curvature representation $V_{6,x} = [U_6, V_6]$ gives

$$\begin{cases} A_{ix} = u_1C_i - u_2B_i - u_7S_i - u_8Z_i - u_9W_i - u_{10}N_i - u_{11}Q_i - u_{12}P_i, \\ B_{ix} = 2B_{i+1} + 2B_i\xi + 2B_{iy} - 2u_1A_i - 2u_7T_i - 2u_9Z_i - 2u_{12}Q_i, \\ C_{ix} = -2C_{i+1} - 2C_i\xi - 2C_{iy} + 2u_2A_i + 2u_8W_i + 2u_{10}M_i + 2u_{11}P_i, \\ D_{ix} = u_3G_i - u_4F_i - u_5K_i + u_6I_i + u_7S_i + u_{10}N_i, \\ F_{ix} = F_{i+1} + F_i\xi + F_{iy} - u_3D_i + u_3\mathcal{H}_i + u_7W_i + u_{11}N_i - u_{12}M_i, \\ G_{ix} = -G_{i+1} - G_i\xi - G_{iy} + u_4D_i - u_4\mathcal{H}_i - u_8T_i + u_9S_i + u_{10}Q_i, \\ \mathcal{H}_{ix} = -u_3G_i + u_4F_i - u_5K_i + u_6I_i - u_8Z_i + u_9W_i + u_{11}Q_i - u_{12}P_i, \\ I_{ix} = I_{i+1} + I_i\xi + I_{iy} - u_5D_i - u_5\mathcal{H}_i + u_7P_i + u_8N_i - u_9M_i, \\ K_{ix} = -K_{i+1} - K_i\xi - K_{iy} + u_6D_i + u_6\mathcal{H}_i - u_{10}Z_i + u_{11}T_i - u_{12}S_i, \\ M_{ix} = -u_2N_i + u_3P_i + u_5W_i + u_7C_i - u_8F_i - u_{11}I_i, \\ N_{ix} = 2N_{i+1} + 2N_i\xi + 2N_{iy} - u_1M_i + u_3Q_i + u_5Z_i - u_7A_i - u_7D_i - u_9F_i - u_{12}I_i, \\ P_{ix} = -P_{i+1} - P_i\xi - P_{iy} - u_2Q_i + u_4M_i - u_5S_i + u_8A_i - u_8\mathcal{H}_i + u_9C_i + u_{10}I_i, \\ Q_{ix} = Q_{i+1} + Q_i\xi + Q_{iy} - u_1P_i + u_4N_i - u_5T_i - u_7G_i + u_8B_i - u_9A_i - u_9\mathcal{H}_i, \\ S_{ix} = -2S_{i+1} - 2S_i\xi - 2S_{iy} - u_2T_i - u_4W_i + u_6P_i - u_8K_i + u_{10}A_i + u_{10}D_i + u_{11}G_i, \\ T_{ix} = -u_1S_i - u_4Z_i + u_6Q_i - u_9K_i + u_{10}B_i + u_{12}G_i, \\ W_{ix} = -W_{i+1} - W_i\xi - W_{iy} - u_2Z_i - u_3S_i - u_6M_i + u_{10}F_i + u_{11}A_i + u_{11}\mathcal{H}_i + u_{12}C_i, \\ Z_{ix} = Z_{i+1} + Z_i\xi + Z_{iy} - u_1W_i - u_3T_i - u_6N_i + u_7K_i + u_{11}B_i - u_{12}A_i + u_{12}\mathcal{H}_i, \end{cases}$$

leading to the following isospectral supersoliton hierarchy:

$$u_{t_{6,n}} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{pmatrix}_{t_{6,n}} = \begin{pmatrix} R(-2B_{n+1}) \\ R(2C_{n+1}) \\ R(-F_{n+1}) \\ R(G_{n+1}) \\ R(-I_{n+1}) \\ R(K_{n+1}) \\ R(-2N_{n+1}) \\ R(P_{n+1}) \\ R(-Q_{n+1}) \\ R(2S_{n+1}) \\ R(W_{n+1}) \\ R(-Z_{n+1}) \end{pmatrix}.$$

To illustrate the explicit form of the higher-dimensional hierarchy, we consider its lowest-order nontrivial flow. Substituting the corresponding coefficient expressions into the recursive representation yields the following evolution equation:

$$\boldsymbol{u}_t = \boldsymbol{F}^{(1)}(\boldsymbol{u}, \boldsymbol{u}_x, \boldsymbol{u}_y, \ldots),$$

where $\boldsymbol{F}^{(1)}$ is obtained by evaluating the relevant flow in the hierarchy.

Next, we construct the super Hamiltonian structure. By direct calculation, we obtain

$$\begin{aligned}
&\left\langle V_6, \frac{\partial U_6}{\partial \lambda} \right\rangle = R(2D - 2A), \left\langle V_6, \frac{\partial U_6}{\partial u_1} \right\rangle = R(-C), \left\langle V_6, \frac{\partial U_6}{\partial u_2} \right\rangle = R(-B), \left\langle V_6, \frac{\partial U_6}{\partial u_3} \right\rangle = R(2G),\\
&\left\langle V_6, \frac{\partial U_6}{\partial u_4} \right\rangle = R(2F), \left\langle V_6, \frac{\partial U_6}{\partial u_5} \right\rangle = R(-2K), \left\langle V_6, \frac{\partial U_6}{\partial u_6} \right\rangle = R(-2I), \left\langle V_6, \frac{\partial U_6}{\partial u_7} \right\rangle = R(-2S),\\
&\left\langle V_6, \frac{\partial U_6}{\partial u_8} \right\rangle = R(2Z), \left\langle V_6, \frac{\partial U_6}{\partial u_9} \right\rangle = R(-2W), \left\langle V_6, \frac{\partial U_6}{\partial u_{10}} \right\rangle = R(2N), \left\langle V_6, \frac{\partial U_6}{\partial u_{11}} \right\rangle = R(2Q),\\
&\left\langle V_6, \frac{\partial U_6}{\partial u_{12}} \right\rangle = R(-2P).
\end{aligned}$$

Substituting the above equations into the supertrace identity and comparing the coefficients of λ^{-n-1} on both sides, we obtain

$$\frac{\delta}{\delta u} \int R(2D_{n+1} - 2A_{n+1}) dx = (\gamma - n) \begin{pmatrix} R(-C_n) \\ R(-B_n) \\ R(2G_n) \\ R(2F_n) \\ R(-2K_n) \\ R(-2I_n) \\ R(-2S_n) \\ R(2Z_n) \\ R(-2W_n) \\ R(2N_n) \\ R(2Q_n) \\ R(-2P_n) \end{pmatrix}.$$

Setting $n = 1$ yields $\gamma = 0$. Thus, we have

$$u_{t_{6,n}} = J \frac{\delta H_{6,n+1}}{\delta u}, \quad H_{6,n+1} = -2 \int \frac{R(D_{n+2} - A_{n+2})}{n+1} dx, \quad n \geq 0.$$

To examine the consistency of the higher-dimensional construction, we also consider its reduction to a lower-dimensional setting. Under an appropriate reduction of the additional spatial dependence, the higher-dimensional equations should recover the corresponding lower-dimensional flows, provided that the reduction is compatible with the governing equations. This comparison provides a consistency check on the derived hierarchies and clarifies the relationship between the lower- and higher-dimensional systems.

5. The nonisospectral (2 + 1)-dimensional integrable hierarchy based on Lie superalgebra $\mathcal{D}(2,1)$

Similar to the previous sections, the (2 + 1)-dimensional nonisospectral hierarchies are divided into two types according to the underlying spectral representations. The corresponding spectral matrices are denoted separately in this section to distinguish different construction procedures.

5.1. The first type of super integrable hierarchy

We consider a (2+1)-dimension nonisospectral problem

$$\begin{cases} \varphi_x = U_7\varphi, \\ \varphi_t = V_7\varphi, \\ \lambda_t = \sum\limits_{i\geq 0} k_i(t)\lambda^{-i}, \end{cases}$$

where

$$U_7 = \begin{pmatrix} \lambda+\xi & 0 & 0 & u_3 & 0 & u_5 \\ 0 & \lambda+\xi & -u_3 & 0 & 0 & u_6 \\ 0 & u_4 & -\lambda-\xi & 0 & u_7 & 0 \\ -u_4 & 0 & 0 & -\lambda-\xi & u_8 & 0 \\ 0 & 0 & -u_5 & -u_6 & \lambda+\xi & u_1 \\ u_7 & u_8 & 0 & 0 & u_2 & -\lambda-\xi \end{pmatrix},$$

$$V_7 = \begin{pmatrix} D & F & 0 & I & M & N \\ G & \mathcal{H} & -I & 0 & P & Q \\ 0 & K & -D & -G & S & T \\ -K & 0 & -F & -\mathcal{H} & W & Z \\ -T & -Z & -N & -Q & A & B \\ S & W & M & P & C & -A \end{pmatrix} = \sum_{i\geq 0}^{n} \begin{pmatrix} D_i & F_i & 0 & I_i & M_i & N_i \\ G_i & \mathcal{H}_i & -I_i & 0 & P_i & Q_i \\ 0 & K_i & -D_i & -G_i & S_i & T_i \\ -K_i & 0 & -F_i & -\mathcal{H}_i & W_i & Z_i \\ -T_i & -Z_i & -N_i & -Q_i & A_i & B_i \\ S_i & W_i & M_i & P_i & C_i & -A_i \end{pmatrix} \lambda^{-i}. \tag{5.1}$$

The stationary zero curvature representation $V_{7,x} = \frac{\partial U_7}{\partial \lambda}\lambda_t + [U_7, V_7]$ gives

$$\begin{cases} A_{ix} = u_1C_i - u_2B_i - u_5S_i - u_6W_i - u_7N_i - u_8Q_i + k_i(t), \\ B_{ix} = 2B_{i+1} + 2B_i\xi + 2B_{iy} - 2u_1A_i - 2u_5T_i - 2u_6Z_i, \\ C_{ix} = -2C_{i+1} - 2C_i\xi - 2C_{iy} + 2u_2A_i + 2u_7M_i + 2u_8P_i, \\ D_{ix} = -u_3K_i + u_4I_i + u_5S_i + u_7N_i + k_i(t), \\ F_{ix} = u_5W_i + u_8N_i, \\ G_{ix} = u_6S_i + u_7Q_i, \\ \mathcal{H}_{ix} = -u_3K_i + u_4I_i + u_6W_i + u_8Q_i + k_i(t), \\ I_{ix} = 2I_{i+1} + 2I_i\xi + 2I_{iy} - u_3D_i - u_3\mathcal{H}_i - u_6M_i + u_5P_i, \\ K_{ix} = -2K_{i+1} - 2K_i\xi - 2K_{iy} + u_4D_i + u_4\mathcal{H}_i - u_7Z_i + u_8T_i, \\ M_{ix} = -u_2N_i + u_3W_i + u_5C_i - u_8I_i, \\ N_{ix} = 2N_{i+1} + 2N_i\xi + 2N_{iy} - u_1M_i + u_3Z_i - u_5A_i - u_5D_i - u_6F_i, \\ P_{ix} = -u_2Q_i - u_3S_i + u_6C_i + u_7I_i, \\ Q_{ix} = 2Q_{i+1} + 2Q_i\xi + 2Q_{iy} - u_1P_i - u_3T_i - u_5G_i - u_6A_i - u_6\mathcal{H}_i, \\ S_{ix} = -2S_{i+1} - 2S_i\xi - 2S_{iy} - u_2T_i + u_4P_i + u_7A_i + u_7D_i + u_8G_i, \\ T_{ix} = -u_1S_i + u_4Q_i - u_6K_i + u_7B_i, \\ W_{ix} = -2W_{i+1} - 2W_i\xi - 2W_{iy} - u_2Z_i - u_4M_i + u_7F_i + u_8A_i + u_8\mathcal{H}_i, \\ Z_{ix} = -u_1W_i - u_4N_i + u_5K_i + u_8B_i, \end{cases}$$

leading to the following nonisospectral supersoliton hierarchy:

$$u_{t_{7,n}} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix}_{t_{7,n}} = \begin{pmatrix} R(-2B_{n+1}) \\ R(2C_{n+1}) \\ R(-2I_{n+1}) \\ R(2K_{n+1}) \\ R(-2N_{n+1}) \\ R(-2Q_{n+1}) \\ R(2S_{n+1}) \\ R(2W_{n+1}) \end{pmatrix}.$$

Next, we construct the super Hamiltonian structure. By direct calculation, we obtain

$$\left\langle V_7, \frac{\partial U_7}{\partial \lambda} \right\rangle = R(2D + 2\mathcal{H} - 2A), \left\langle V_7, \frac{\partial U_7}{\partial u_1} \right\rangle = R(-C), \left\langle V_7, \frac{\partial U_7}{\partial u_2} \right\rangle = R(-B),$$
$$\left\langle V_7, \frac{\partial U_7}{\partial u_3} \right\rangle = R(-2K), \left\langle V_7, \frac{\partial U_7}{\partial u_4} \right\rangle = R(-2I), \left\langle V_7, \frac{\partial U_7}{\partial u_5} \right\rangle = R(-2S),$$
$$\left\langle V_7, \frac{\partial U_7}{\partial u_6} \right\rangle = R(-2W), \left\langle V_7, \frac{\partial U_7}{\partial u_7} \right\rangle = R(2N), \left\langle V_7, \frac{\partial U_7}{\partial u_8} \right\rangle = R(2Q).$$

Substituting the above equations into the supertrace identity and comparing the coefficients of λ^{-n-1}

on both sides, we obtain

$$\frac{\delta}{\delta u}\int R(2D_{n+1}+2\mathcal{H}_{n+1}-2A_{n+1})dx=(\gamma-n)\begin{pmatrix}R(-C_n)\\R(-B_n)\\R(-2K_n)\\R(-2I_n)\\R(-2S_n)\\R(-2W_n)\\R(2N_n)\\R(2Q_n)\end{pmatrix}.$$

Setting $n=1$ yields $\gamma=0$. Thus, we have

$$u_{t_{7,n}}=J\frac{\delta H_{7,n+1}}{\delta u},\quad H_{7,n+1}=-2\int\frac{R(D_{n+2}+\mathcal{H}_{n+2}-A_{n+2})}{n+1}dx,\quad n\geq 0.$$

5.2. *The second type of super integrable hierarchy*

We consider a (2+1)-dimension nonisospectral problem

$$\begin{cases}\varphi_x=U_8\varphi,\\ \varphi_t=V_7\varphi,\\ \lambda_t=\sum\limits_{i\geq 0}k_i(t)\lambda^{-i}.\end{cases}$$

The auxiliary matrix V_8 has the same matrix structure as V_7 in (5.1), but its coefficient matrices are determined independently by

$$V_{8,x}=\frac{\partial U_8}{\partial\lambda}\lambda_t+[U_8,V_8].$$

The formal Laurent expansion is given by

$$V_8=\sum_{i=0}^{\infty}V_{8,i}\lambda^{-i},$$

and U_8 is defined as follows:

$$U_8=\begin{pmatrix}\lambda+\xi & u_3 & 0 & u_5 & 0 & u_7\\ u_4 & 0 & -u_5 & 0 & u_8 & u_9\\ 0 & u_6 & -\lambda-\xi & -u_4 & u_{10} & 0\\ -u_6 & 0 & -u_3 & 0 & u_{11} & u_{12}\\ 0 & -u_{12} & -u_7 & -u_9 & \lambda+\xi & u_1\\ u_{10} & u_{11} & 0 & u_8 & u_2 & -\lambda-\xi\end{pmatrix}.$$

The stationary zero curvature representation $V_{8,x} = \frac{\partial U_8}{\partial \lambda}\lambda_t + [U_8, V_8]$ gives

$$\begin{cases} A_{ix} = u_1C_i - u_2B_i - u_7S_i - u_8Z_i - u_9W_i - u_{10}N_i - u_{11}Q_i - u_{12}P_i + k_i(t), \\ B_{ix} = 2B_{i+1} + 2B_i\xi + 2B_{iy} - 2u_1A_i - 2u_7T_i - 2u_9Z_i - 2u_{12}Q_i, \\ C_{ix} = -2C_{i+1} - 2C_i\xi - 2C_{iy} + 2u_2A_i + 2u_8W_i + 2u_{10}M_i + 2u_{11}P_i, \\ D_{ix} = u_3G_i - u_4F_i - u_5K_i + u_6I_i + u_7S_i + u_{10}N_i + k_i(t), \\ F_{ix} = F_{i+1} + F_i\xi + F_{iy} - u_3D_i + u_3\mathcal{H}_i + u_7W_i + u_{11}N_i - u_{12}M_i, \\ G_{ix} = -G_{i+1} - G_i\xi - G_{iy} + u_4D_i - u_4\mathcal{H}_i - u_8T_i + u_9S_i + u_{10}Q_i, \\ \mathcal{H}_{ix} = -u_3G_i + u_4F_i - u_5K_i + u_6I_i - u_8Z_i + u_9W_i + u_{11}Q_i - u_{12}P_i, \\ I_{ix} = I_{i+1} + I_i\xi + I_{iy} - u_5D_i - u_5\mathcal{H}_i + u_7P_i + u_8N_i - u_9M_i, \\ K_{ix} = -K_{i+1} - K_i\xi - K_{iy} + u_6D_i + u_6\mathcal{H}_i - u_{10}Z_i + u_{11}T_i - u_{12}S_i, \\ M_{ix} = -u_2N_i + u_3P_i + u_5W_i + u_7C_i - u_8F_i - u_{11}I_i, \\ N_{ix} = 2N_{i+1} + 2N_i\xi + 2N_{iy} - u_1M_i + u_3Q_i + u_5Z_i - u_7A_i - u_7D_i - u_9F_i - u_{12}I_i, \\ P_{ix} = -P_{i+1} - P_i\xi - P_{iy} - u_2Q_i + u_4M_i - u_5S_i + u_8A_i - u_8\mathcal{H}_i + u_9C_i + u_{10}I_i, \\ Q_{ix} = Q_{i+1} + Q_i\xi + Q_{iy} - u_1P_i + u_4N_i - u_5T_i - u_7G_i + u_8B_i - u_9A_i - u_9\mathcal{H}_i, \\ S_{ix} = -2S_{i+1} - 2S_i\xi - 2S_{iy} - u_2T_i - u_4W_i + u_6P_i - u_8K_i + u_{10}A_i + u_{10}D_i + u_{11}G_i, \\ T_{ix} = -u_1S_i - u_4Z_i + u_6Q_i - u_9K_i + u_{10}B_i + u_{12}G_i, \\ W_{ix} = -W_{i+1} - W_i\xi - W_{iy} - u_2Z_i - u_3S_i - u_6M_i + u_{10}F_i + u_{11}A_i + u_{11}\mathcal{H}_i + u_{12}C_i, \\ Z_{ix} = Z_{i+1} + Z_i\xi + Z_{iy} - u_1W_i - u_3T_i - u_6N_i + u_7K_i + u_{11}B_i - u_{12}A_i + u_{12}\mathcal{H}_i, \end{cases}$$

leading to the following nonisospectral supersoliton hierarchy:

$$u_{t_{8,n}} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{pmatrix}_{t_{8,n}} = \begin{pmatrix} R(-2B_{n+1}) \\ R(2C_{n+1}) \\ R(-F_{n+1}) \\ R(G_{n+1}) \\ R(-I_{n+1}) \\ R(K_{n+1}) \\ R(-2N_{n+1}) \\ R(P_{n+1}) \\ R(-Q_{n+1}) \\ R(2S_{n+1}) \\ R(W_{n+1}) \\ R(-Z_{n+1}) \end{pmatrix}.$$

Next, we construct the super Hamiltonian structure. By direct calculation, we obtain

$$\left\langle V_8, \frac{\partial U_8}{\partial \lambda} \right\rangle = R(2D - 2A), \left\langle V_8, \frac{\partial U_8}{\partial u_1} \right\rangle = R(-C), \left\langle V_8, \frac{\partial U_8}{\partial u_2} \right\rangle = R(-B), \left\langle V_8, \frac{\partial U_8}{\partial u_3} \right\rangle = R(2G),$$
$$\left\langle V_8, \frac{\partial U_8}{\partial u_4} \right\rangle = R(2F), \left\langle V_8, \frac{\partial U_8}{\partial u_5} \right\rangle = R(-2K), \left\langle V_8, \frac{\partial U_8}{\partial u_6} \right\rangle = R(-2I), \left\langle V_8, \frac{\partial U_8}{\partial u_7} \right\rangle = R(-2S),$$
$$\left\langle V_8, \frac{\partial U_8}{\partial u_8} \right\rangle = R(2Z), \left\langle V_8, \frac{\partial U_8}{\partial u_9} \right\rangle = R(-2W), \left\langle V_8, \frac{\partial U_8}{\partial u_{10}} \right\rangle = R(2N), \left\langle V_8, \frac{\partial U_8}{\partial u_{11}} \right\rangle = R(2Q),$$
$$\left\langle V_8, \frac{\partial U_8}{\partial u_{12}} \right\rangle = R(-2P).$$

Substituting the above equations into the supertrace identity and comparing the coefficients of λ^{-n-1} on both sides, we obtain

$$\frac{\delta}{\delta u} \int R(2D_{n+1} - 2A_{n+1})dx = (\gamma - n) \begin{pmatrix} R(-C_n) \\ R(-B_n) \\ R(2G_n) \\ R(2F_n) \\ R(-2K_n) \\ R(-2I_n) \\ R(-2S_n) \\ R(2Z_n) \\ R(-2W_n) \\ R(2N_n) \\ R(2Q_n) \\ R(-2P_n) \end{pmatrix}.$$

Setting $n = 1$ yields $\gamma = 0$. Thus, we have

$$u_{t_{8,n}} = J\frac{\delta H_{8,n+1}}{\delta u}, \quad H_{8,n+1} = -2\int \frac{R(D_{n+2} - A_{n+2})}{n+1}dx, \quad n \geq 0.$$

6. Conclusions

In this paper, we have systematically investigated the super integrable hierarchies associated with the Lie superalgebra $\mathcal{D}(2, 1)$. By employing the zero curvature equation and the supertrace identity, we have constructed both isospectral and nonisospectral hierarchies in (1 + 1) and (2 + 1) dimensions, together with their corresponding super Hamiltonian structures.

The results obtained in this work demonstrate that the Lie superalgebra $\mathcal{D}(2, 1)$ provides a rich algebraic framework for constructing generalized super integrable systems. These findings offer a foundation for further studies of supersymmetric integrable models and related nonlinear phenomena.

Author contributions

Shuqiao Li: Writing–original draft preparation, writing; Baiying He: Methodology, validation, reviewing, editing; Jialin Xu: Software, supervision, validation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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