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Research article

Modified inertial viscosity and Mann-type Tseng methods for variational inclusion problems

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Abstract: In this paper, we propose two accelerated inertial Tseng-type iterative methods for solving variational inclusion problems. The first method incorporates an inertial term into a viscosity-based Tseng iteration, and the second combines an inertial term with a Mann-type approximation at the initial stage of the iterative process. This design distinguishes the proposed methods from conventional viscosity and Mann-type Tseng methods and provides an effective mechanism for exploiting previous iterates to enhance convergence. Under appropriate assumptions, we prove the strong convergence of the generated sequences to solutions of the variational inclusion problems. Applications to variational inequality and optimization problems are presented to illustrate the applicability of the proposed methods. Finally, numerical experiments demonstrate their effectiveness, enhanced computational performance, and practical potential.

Keywords: Mann; viscosity; inertial; algorithms; variational inclusion; strong convergence
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1. Introduction

Let $G : \mathbb{U} \rightrightarrows \mathbb{U}$ be a maximal monotone operator and $F : \mathbb{U} \to \mathbb{U}$ be a monotone operator. Our aim in this paper is to study the following monotone variational inclusion problem (MVIP):

$$Find\ q \in \mathbb{U}\ such\ that\ 0 \in (F + G)(q). \tag{1.1}$$

Iterative methods provide effective and computationally tractable approaches for MVIPs, particularly when explicit solutions are difficult to obtain. Various iterative techniques have been developed for variational inclusion and fixed-point problems, including viscosity-type, Mann-type and modified iterative methods [1–5]. Beyond variational inclusion and fixed-point problems, iterative techniques

have also been employed to approximate solutions of partial and fractional differential equations [6–8], demonstrating their applicability across a broad range of nonlinear problems.

One of the prominent and widely used methods for solving MVIPs is the forward-backward splitting method introduced by Lions and Mercier [9], defined as follows:

$$x_{m+1} = R^G_{\mu_m}(I - \mu_m F)x_m, \ \forall m \in \mathbb{N}, \tag{1.2}$$

where the operator $I-\mu_m F$ is referred as forward operator and the operator $R^G_{\mu_m} = (I+\mu_m G)^{-1}$ is known as the backward operator or the resolvent operator, $\mu_m \in (0, 2/L)$ and I is the identity operator on $\mathbb{U}$. The sequence $\{x_m\}$ converges weakly to the solution of an MVIP with the assumptions that F is L-inverse strongly monotone (or cocoercive) and that G is maximal monotone. Subsequently, the MVIP was analyzed by numerous authors; see, for example, [10–13].

Tseng [14] modified the forward-backward splitting method by imposing Lipschitz continuity on F in place of cocoercive assumption: For $t, \theta \in (0, 1), \rho > 0$, be given

$$\begin{aligned} w_m &= R^G_{\mu_m}(I - \mu_m F)x_m, \\ x_{m+1} &= w_m - \mu_m(Fw_m - Fx_m), \end{aligned} \tag{1.3}$$

where step sizes μ_m are automatically updated using the Armijo-type line method

$$\lambda\|F(x_{m+1}) - F(x_m)\| \leq \theta\|x_{m+1} - x_m\|, \tag{1.4}$$

where μ_m is the largest $\lambda \in \{\rho, \rho t, \rho t^2, \cdots\}$ satisfying (1.4). It is proved that if G is maximal monotone and that F is Lipschitz continuous and monotone, the sequence $\{x_m\}$ generated by (1.3) converges weakly to a solution of an MVIP in real Hilbert spaces. This is notable because the Lipschitz constant is frequently unknown or challenging to determine in practical applications. Tseng's method solves this challenge by using a line search technique, which eliminates the need for prior knowledge of the Lipschitz constant. However, it should be noted that using a line search technique can be computationally expensive, as it involves several additional evaluations at each iteration.

To overcome this difficulty, several researchers have proposed relaxed iterative methods for solving MVIPs. Cholamjiak et al. [15] investigated a relaxed forward–backward splitting approach and established a weak convergence theorem. Gibali and Thong [16] modified Tseng's method and presented an iterative method with a self-adaptive stepsize. Tang and Cho [17] further modified Tseng's method and proposed a self-adaptive viscosity approximation iterative method for MVIPs. Some other variants of iterative methods for solving MVIPs can also be found in [18–20] and the references therein.

Very recently, Reich and Taiwo [21] studied a modification of the Tseng method and presented the fast hybrid iterative methods for MVIPs. They applied the inertial extrapolation and viscosity approximation with Tseng's forward-backward splitting method. Let $x_0, x_1 \in \mathbb{U}$, $\lambda \in (0, 1)$, $\mu_1 > 0, \delta \in [0, 1)$. Compute x_{m+1} as follows:

$$u_m = \eta_m h(x_m) + (1 - \eta_m)[x_m + \delta_m(x_m - x_{m-1})], \tag{1.5}$$

$$v_m = R^G_{\mu_m}(I - \mu_m F)u_m, \tag{1.6}$$

$$x_{m+1} = v_m - \mu_m(Fv_m - Fu_m), \tag{1.7}$$

where $$\mu_{m+1} = \begin{cases} \min\left\{\frac{\lambda\|v_m - u_m\|}{\|Fv_m - Fu_m\|}, \quad \mu_m\right\}, & \text{if } Fv_m \neq Fu_m, \\ \mu_m & \text{otherwise,} \end{cases}$$

and $$\bar{\delta}_m = \begin{cases} \min\left\{\frac{\varepsilon_m}{\|x_m - x_{m-1}\|}, \quad \delta\right\}, & \text{if } x_m \neq x_{m-1}, \\ \delta, & \text{otherwise,} \end{cases}$$

where G is maximal monotone, F is monotone and τ-Lipschitz continuous, h is a contraction mapping, $\{\eta_m\}$ is a sequence in $(0, 1)$, and $\{\varepsilon_m\}$ is a positive sequence. One can replace

$$u_m = \eta_m h(x_m) + (1 - \eta_m)x_m + \delta_m(x_m - x_{m-1}) \tag{1.8}$$

in (1.5), another scheme that has also been discussed in [21]. The strong convergence theorems have been proved for both the methods.

To accelerate convergence, Polyak [22] introduced inertial terms into iterative schemes for smooth, convex optimization problems. Their effectiveness has subsequently motivated the incorporation of inertial techniques into various algorithms. In this direction, Alvarez and Attouch [23] developed an inertial iterative method for solving the null-point problem. Their methods incorporate arbitrary initial points x_0, x_1 and parameter $\delta_m \in [0, 1)$:

$$x_{m+1} = R^G_{\mu_m}[x_m + \delta_m(x_m - x_{m-1})], \quad m \in \mathbb{N}, \tag{1.9}$$

where $\delta_m(x_m - x_{m-1})$ is referred as the inertial term. Inertial techniques have been extensively adopted and employed by numerous authors to enhance the convergence of iterative methods for nonlinear problems; see, for example, [24–27].

Motivated by the work of Reich and Taiwo [21] and the works discussed above, we develop two modified inertial Tseng-type methods for solving MVIPs.

- The first method combines viscosity approximation with inertial extrapolation within Tseng's forward–backward–forward framework. The first method introduces the alternate construction

  $$u_m = (1 - \alpha_m - \beta_m)h(x_m) + \beta_m[x_m + \delta_m(x_m - x_{m-1})] \tag{1.10}$$

  by assigning an additional control parameter β_m, which independently determines the weight assigned to the inertial extrapolation. We also use the more restrictive parameter $0 < \delta_m < \bar{\delta}_m$, where

  $$\bar{\delta}_m = \begin{cases} \min\left\{\frac{\varepsilon_m}{\max\{\|x_m - x_{m-1}\|, \|x_m - x_{m-1}\|^2\}}, \quad \delta\right\}, & \text{if } x_m \neq x_{m-1}, \\ \delta, & \text{otherwise,} \end{cases} \tag{1.11}$$

  which is useful in controlling the inertial contribution in the convergence analysis.
- The second method is obtained from the first method by setting $h = I$, where I denotes the identity mapping. Consequently, the intermediate point in this method is given by

  $$u_m = (1 - \alpha_m - \beta_m)x_m + \beta_m[x_m + \delta_m(x_m - x_{m-1})]. \tag{1.12}$$

  It also retains the same adaptive restriction on the inertial parameter δ_m specified in (1.11).

Both methods use self-adaptive step sizes. Under suitable assumptions, we establish strong convergence results for both proposed methods, taking into account the incorporated modifications in the convergence analysis. Applications to variational inequality and optimization problems are also presented. Numerical experiments are conducted to investigate their effectiveness and computational performance, including comparisons with the method studied in [21].

2. Preliminaries

Throughout this paper, $\mathbb{U}$ denotes a real Hilbert space, and $\mathbb{C} \subseteq \mathbb{U}$ denotes a nonempty, closed, and convex set. For a sequence $\{z_m\} \subseteq \mathbb{U}$, we write $z_m \to z$ and $z_m \rightharpoonup z$ to indicate, respectively, strong and weak convergence to $z \in \mathbb{U}$.

Definition 2.1. [28, 29] A mapping $F : \mathbb{U} \to \mathbb{U}$ is said to be

(i) monotone if
$$\langle F(l_1) - F(l_2), l_1 - l_2\rangle \geq 0, \forall\ l_1, l_2 \in \mathbb{U};$$

(ii) τ-Lipchitz continuous if there exists a constant $\tau > 0$ such that
$$\|F(l_1) - F(l_2)\| \leq \tau\|l_1 - l_2\|, \forall\ l_1, l_2 \in \mathbb{U}.$$
If $\tau = 1$, then F is said to be nonexpansive;

(iii) a θ-contraction if there exists a constant $0 < \theta < 1$ such that
$$\|F(l_1) - F(l_2)\| \leq \theta\|l_1 - l_2\|, \forall\ l_1, l_2 \in \mathbb{U};$$

(iv) α-strongly monotone if there exists $\alpha > 0$ such that
$$\langle F(l_1) - F(l_2), l_1 - l_2\rangle \geq \alpha\|l_1 - l_2\|^2, \forall\ l_1, l_2 \in \mathbb{U}.$$

Definition 2.2. [30] Let $G : \mathbb{U} \to 2^{\mathbb{U}}$ be a set-valued mapping. Then,

(i) G is called monotone if $\langle l_1 - l_2, x_1 - x_2\rangle \geq 0, \forall\ l_1, l_2 \in \mathbb{U},\ x_1 \in G(l_1),\ x_2 \in G(l_2)$;
(ii) $Graph(G) = \{(l_1, x_1) \in \mathbb{U} \times \mathbb{U} : x_1 \in G(l_1)\}$;
(iii) G is called maximal monotone, if G is monotone, and the graph of G is not properly contained in the graph of any other monotone mapping;
(iv) the resolvent of G is defined by $R_\mu^G = [I + \mu G]^{-1}$, where I is the identity mapping, and $\mu > 0$.

Remark 2.1. For all l_1, l_2, l_3 in $\mathbb{U}$; $\varrho \in \mathbb{R}$; and $\phi_1, \phi_2, \phi_3 \in [0, 1]$ satisfying $\phi_1 + \phi_2 + \phi_3 = 1$, the following equalities and inequality hold:

(i) $\|\phi_1 l_1 + \phi_2 l_2 + \phi_3 l_3\|^2 = \phi_1\|l_1\|^2 + \phi_2\|l_2\|^2 + \phi_3\|l_3\|^2 - \phi_1\phi_2\|l_1 - l_2\|^2 - \phi_2\phi_3\|l_2 - l_3\|^2 - \phi_3\phi_1\|l_3 - l_1\|^2$;
(ii) $\|l_1 \pm l_2\|^2 = \|l_1\|^2 \pm 2\langle l_1, l_2\rangle + \|l_2\|^2$;
(iii) $\|l_1 \pm l_2\|^2 \leq \|l_1\|^2 \pm 2\langle l_1, l_1 + l_2\rangle$;
(iv) $\|\varrho l_1 + (1 - \varrho)l_2\|^2 = \varrho\|l_1\|^2 + (1 - \varrho)\|l_2\|^2 - \varrho(1 - \varrho)\|l_1 - l_2\|^2$.

Definition 2.3. [30] The metric projection of $\mathbb{U}$ onto $\mathbb{C}$ is a mapping which assigns each point $l_1 \in \mathbb{U}$ to a unique nearest point in $\mathbb{C}$, denoted by $P_{\mathbb{C}} l_1$, such that
$$\|l_1 - P_{\mathbb{C}} l_1\| \leq \|l_1 - l_2\|,\ \forall\ l_2 \in \mathbb{C}.$$

For later use, we collect some fundamental properties of the metric projection $P_{\mathbb{C}}$:
$$\langle l_1 - l_2,\quad P_{\mathbb{C}} l_1 - P_{\mathbb{C}} l_2\rangle \geq \|P_{\mathbb{C}} l_1 - P_{\mathbb{C}} l_2\|^2,\quad l_1, l_2 \in \mathbb{U},$$

and

$$\langle l_1 - P_{\mathbb{C}} l_1, \quad l_2 - P_{\mathbb{C}} l_1 \rangle \leq 0, \quad \forall l_1 \in \mathbb{U}, l_2 \in \mathbb{C}.$$

The following remak provides an important characterization of the unique solution of a variational inequality, which will be used in establishing our main convergence theorem. For further details, we refer the reader to [30, 31].

Remark 2.2. Let $\mathbb{C}$ be a nonempty, closed, and convex subset of a real Hilbert space $\mathbb{H}$. Suppose that $h : \mathbb{H} \to \mathbb{H}$ is such that $I - h$ is θ-Lipschitz continuous and α-strongly monotone on $\mathbb{C}$, where $\theta, \alpha > 0$. Then, the variational inequality problem of finding $\Phi \in \mathbb{C}$ such that

$$\langle (I - h)(\Phi), \phi - \Phi \rangle \geq 0, \ \forall \ \phi \in \mathbb{C} \tag{2.1}$$

has a unique solution. Moreover, (2.1) is equivalent to the fixed-point equation $\Phi = P_{\mathbb{C}} h(\Phi)$.

Lemma 2.1. *[32] Let $\{a_m\}$ be a sequence of non-negative real numbers satisfying*

$$a_{m+1} \leq (1 - \gamma_m) a_m + \gamma_m d_m, \quad m \geq 0,$$

where $\{\gamma_m\}$ is a sequence in $(0, 1)$, and $\{d_m\}$ is a sequence of real numbers such that

(i) $\lim\limits_{m \to \infty} \gamma_m = 0$, and $\sum\limits_{m=1}^{\infty} \gamma_m = \infty$;
(ii) $\limsup\limits_{m \to \infty} d_m \leq 0$.

Then, $\lim\limits_{m \to \infty} a_m = 0$.

Lemma 2.2. *[33] Let $\{a_m\}$ be a sequence in a Hilbert space $\mathbb{U}$ such that there exists a nonempty, closed, and convex subset $\mathbb{C}$ of $\mathbb{U}$ satisfying*

(i) $\lim\limits_{m \to \infty} \|a_m - l\|$ *exists for all $l \in \mathbb{C}$;*
(ii) every weak cluster point of $\{a_m\}$ belongs to $\mathbb{C}$.

Then, there exists $l^* \in \mathbb{C}$ such that $a_m \rightharpoonup l^*$.

Lemma 2.3. *[34] Let $\{a_m\} \subset [0, +\infty)$ and $\{d_m\}$ be sequences such that the following conditions hold:*

(i) $a_{m+1} - a_m \leq \delta_m (a_m - a_{m-1}) + d_m$;
(ii) $\sum\limits_{n=1}^{\infty} d_m < \infty$;
(iii) $\delta_m \in [0, \delta]$, *where* $\delta \in (0, 1)$.

Then, $\{a_m\}$ is a convergent sequence, and $\sum\limits_{n=1}^{\infty} [a_{m+1} - a_m]_+ < \infty$, where $[s]_+ = \max(s, 0)$ for any real number $s \in \mathbb{R}$.

Lemma 2.4. *[34] Let $\{a_m\}$ be a real sequence which is not eventually decreasing in the sense that there exists a subsequence $\{a_{m_n}\}$ of $\{a_m\}$ such that $a_{m_n} < a_{m_n+1}$ for all $n \geq 0$. Also, consider the sequence of integers $\{\psi(m)\}_{m \geq m_0}$ defined by*

$$\psi(m) = \max\{k \leq m : a_k \leq a_{k+1}\}.$$

Then, $\{\psi(m)\}_{m \geq n_0}$ is a nondecreasing sequence satisfying $\lim_{m \to \infty} \psi(m) = \infty$, and for all $m \geq m_0$,

$$\max\{a_{\psi(m)}, a_m\} \leq a_{\psi(m)+1}.$$

3. Proposed methods and convergence theorems

This section is devoted to the development of modified inertial iterative methods for solving the MVIP. Let Ω denote the solution set of the MVIP such that $\Omega \neq \emptyset$.

3.1. Modified viscosity-type Tseng method

To solve the MVIP, we propose modified inertial viscosity-type Tseng iterative method. The corresponding iterative method is presented in Algorithm 3.1. The following assumptions are imposed to establish the convergence of the sequences generated by the proposed methods:

Assumption 3.1. (i) $F : \mathbb{U} \to \mathbb{U}$ is monotone and τ-Lipschitz continuous, and $G : \mathbb{U} \rightrightarrows \mathbb{U}$ is a maximal monotone operator;

(ii) $\{\eta_m\}, \{\zeta_m\}$ are sequences in $(0, 1)$ satisfying $\lim_{m\to\infty} \eta_m = 0$, $\sum_{m=1}^{\infty} \eta_m = \infty$ and $\inf_m \zeta_m(1 - \eta_m - \zeta_m) > 0$;

(iii) $\{\varepsilon_m\}$ is a positive sequence satisfying $\sum_{m=1}^{\infty} \varepsilon_m < \infty$ and $\lim_{m\to\infty} \frac{\varepsilon_m}{\eta_m} = 0$.

Algorithm 3.1. Modified inertial viscosity-type Tseng method

Initiation: Let $\lambda \in (0, 1), \mu_1 > 0, \delta \in [0, 1)$, and x_0 and x_1 are chosen arbitrary.
Iterative step. For iterates x_m and x_{m-1}, for $m \geq 1$, select $0 < \delta_m < \bar{\delta}_m$, where

$$\bar{\delta}_m = \begin{cases} \min\left\{\frac{\varepsilon_m}{\max\{\|x_m - x_{m-1}\|, \|x_m - x_{m-1}\|^2\}}, \quad \delta\right\}, & \text{if } x_m \neq x_{m-1}, \\ \delta, & \text{otherwise,} \end{cases} \tag{3.1}$$

compute

$$u_m = (1 - \eta_m - \zeta_m)h(x_m) + \zeta_m[x_m + \delta_m(x_m - x_{m-1})], \tag{3.2}$$

$$v_m = R^G_{\mu_m}(u_m - \mu_m F u_m), \tag{3.3}$$

If $v_m = u_m$, terminate the algorithm. Otherwise, set $m \leftarrow m + 1$, and proceed with the following iterative step:

$$x_{m+1} = v_m - \mu_m(Fv_m - Fu_m), \tag{3.4}$$

and update

$$\mu_{m+1} = \begin{cases} \min\left\{\frac{\lambda\|v_m - u_m\|}{\|Fv_m - Fu_m\|}, \quad \mu_m\right\}, & \text{if } Fv_m \neq Fu_m, \\ \mu_m & \text{otherwise.} \end{cases} \tag{3.5}$$

Remark 3.1. From the selection of δ_m, it follows immediately that

$$\sum_{m=1}^{\infty} \delta_m\|x_m - x_{m-1}\| < \infty, \quad \sum_{m=1}^{\infty} \delta_m\|x_m - x_{m-1}\|^2 < \infty,$$

which imply that

$$\lim_{m\to\infty} \delta_m\|x_m - x_{m-1}\| = 0, \quad \lim_{m\to\infty} \delta_m\|x_m - x_{m-1}\|^2 = 0.$$

Also from Assumption 3.1(*iii*), we find that

$$\lim_{m\to\infty} \tfrac{\delta_m}{\eta_m}\|x_m - x_{m-1}\| \leq \lim_{m\to\infty} \tfrac{\varepsilon_m}{\eta_m} = 0.$$

Hence, there exists a number $K_1 > 0$ such that $\frac{\delta_m}{\eta_m}\|x_m - x_{m-1}\| \leq K_1, \forall m \in \mathbb{N}$.

Remark 3.2. From (3.5), it can be easily seen that

$$\|Fv_m - Fu_m\| \leq \lambda \frac{\|v_m - u_m\|}{\mu_{m+1}},$$

and the sequence $\{\mu_m\}$ induced by (3.5) is a decreasing sequence satisfying

$$\lim_{m\to\infty} \mu_m = \mu \geq \min\Big\{\mu_1, \frac{\lambda}{\tau}\Big\}.$$

Lemma 3.1. *Suppose that $h : \mathbb{U} \to \mathbb{U}$ is a θ-contraction, and Assumption 3.1 is satisfied. Then, the sequence $\{x_m\}$ generated by Algorithm 3.1 is bounded.*

Proof. Let $\Phi \in \Omega$. Applying Remark 3.2(*ii*) to (3.4) and using (3.5), we have

$$\begin{aligned}
\|x_{m+1} - \Phi\|^2 &= \|v_m - \mu_m(Fv_m - Fu_m) - \Phi\|^2 \\
&= \|v_m - \Phi\|^2 - 2\mu_m\langle Fv_m - Fu_m, \quad v_m - \Phi\rangle + \mu_m^2\|Fv_m - Fu_m\|^2 \\
&= \|v_m - u_m + u_m - \Phi\|^2 - 2\mu_m\langle Fv_m - Fu_m, \quad v_m - \Phi\rangle + \mu_m^2\|Fv_m - Fu_m\|^2 \\
&= \|v_m - u_m\|^2 + \|u_m - \Phi\|^2 + 2\langle v_m - u_m,\ u_m - \Phi\rangle - 2\mu_m\langle Fv_m - Fu_m,\ v_m - \Phi\rangle \\
&\quad + \mu_m^2\|Fv_m - Fu_m\|^2 \\
&= \|v_m - u_m\|^2 + \|u_m - \Phi\|^2 - 2\langle v_m - u_m,\ v_m - u_m\rangle \\
&\quad + 2\langle v_m - u_m,\ v_m - \Phi\rangle - 2\mu_m\langle Fv_m - Fu_m,\ v_m - \Phi\rangle + \mu_m^2\|Fv_m - Fu_m\|^2 \\
&= \|v_m - u_m\|^2 + \|u_m - \Phi\|^2 - 2\langle v_m - u_m,\ v_m - u_m\rangle \\
&\quad - 2\mu_m\langle Fv_m - Fu_m - (v_m - u_m),\ v_m - \Phi\rangle + \mu_m^2\|Fv_m - Fu_m\|^2 \\
&= \|u_m - \Phi\|^2 - \|v_m - u_m\|^2 - 2\mu_m\langle Fv_m - Fu_m - (v_m - u_m),\ v_m - \Phi\rangle \\
&\quad + \mu_m^2\|Fv_m - Fu_m\|^2 \\
&\leq \|u_m - \Phi\|^2 - 2\mu_m\langle Fv_m - Fu_m - (v_m - u_m),\ v_m - \Phi\rangle + \Big(\frac{\mu_m^2}{\mu_{m+1}^2}\lambda^2 - 1\Big)\|v_m - u_m\|^2. \quad (3.6)
\end{aligned}$$

From (3.3), we see that $(u_m - \mu_m Fu_m) \in (I + \mu_m G)(v_m)$ implies that there exists $w_m \in G(v_m)$ such that $(u_m - \mu_m Fu_m) = v_m + \mu_m w_m$ or $(u_m - v_m) - \mu_m Fu_m = \mu_m w_m$.

Because $\Phi \in \Omega$, and $Fv_m + w_m \in (F + G)v_m$, the monotonicity of $F + G$ implies that

$$\langle Fv_m + \frac{(u_m - v_m)}{\mu_m} - Fu_m, v_m - \Phi\rangle \geq 0; \quad (3.7)$$

that is,

$$\Big\langle \mu_m(Fv_m - Fu_m) + (u_m - v_m), v_m - \Phi\Big\rangle \geq 0. \quad (3.8)$$

Consequently, (3.6) takes the following form:

$$\|x_{m+1} - \Phi\|^2 \leq \|u_m - \Phi\|^2 + \Big(\frac{\mu_m^2}{\mu_{m+1}^2}\lambda^2 - 1\Big)\|v_m - u_m\|^2. \tag{3.9}$$

In light of Remark 3.2, we obtain

$$\lim_{m\to\infty}\Big(\frac{\mu_m^2}{\mu_{m+1}^2}\lambda^2 - 1\Big) = \lambda^2 - 1 < 0, \tag{3.10}$$

which guarantees that there exists $k \in \mathbb{N}$ such that

$$\Big(\frac{\mu_m^2}{\mu_{m+1}^2}\lambda^2 - 1\Big) < 0, \forall m \geq k. \tag{3.11}$$

Hence, we conclude that

$$\|x_{m+1} - \Phi\| \leq \|u_m - \Phi\|, \forall m \geq k. \tag{3.12}$$

Let $z_m = x_m + \delta_m(x_m - x_{m-1})$; then, $\|z_m - \Phi\| \leq \|x_m - \Phi\| + \delta_m\|x_m - x_{m-1}\|$. From (3.2) and Remark 3.2, we have

$$\begin{aligned}
\|u_m - \Phi\| &= \|(1-\eta_m-\zeta_m)h(x_m) + \zeta_m z_m - \Phi\| \\
&= \|(1-\eta_m-\zeta_m)(h(x_m) - h(\Phi)) + (1-\eta_m-\zeta_m)(h(\Phi) - \Phi) + \zeta_m(z_m - \Phi) - \eta_m\Phi\| \\
&\leq (1-\eta_m-\zeta_m)\|h(x_m) - h(\Phi)\| + (1-\eta_m-\zeta_m)\|h(\Phi) - \Phi\| + \zeta_m\|z_m - \Phi\| + \eta_m\|\Phi\| \\
&\leq (1-\eta_m-\zeta_m)\theta\|x_m - \Phi\| + (1-\eta_m-\zeta_m)\|h(\Phi) - \Phi\| + \zeta_m(\|x_m - \Phi\| \\
&\quad +\delta_m\|x_m - x_{m-1}\|) + \eta_m\|\Phi\| \\
&\leq (1-\eta_m-\zeta_m)\theta\|x_m - \Phi\| + (1-\eta_m)\|h(\Phi) - \Phi\| + \zeta_m\|x_m - \Phi\| \\
&\quad +\delta_m\|x_m - x_{m-1}\| + \eta_m\|\Phi\| \\
&\leq (1-\eta_m-\zeta_m)\|x_m - \Phi\| + (1-\eta_m)\|h(\Phi) - \Phi\| + \zeta_m\|x_m - \Phi\| \\
&\quad +\delta_m\|x_m - x_{m-1}\| + \eta_m\|\Phi\| \\
&\leq (1-\eta_m)\Big[\|x_m - \Phi\| + \|h(\Phi) - \Phi\|\Big] + \eta_m\Big\{\|\Phi\| + \tfrac{\delta_m\|x_m - x_{m-1}\|}{\eta_m}\Big\} \\
&\leq \max\Big\{\|x_m - \Phi\| + \|h(\Phi) - \Phi\|, \quad \|\Phi\| + \frac{\delta_m\|x_m - x_{m-1}\|}{\eta_m}\Big\} \\
&\leq \max\Big\{\|u_{m-1} - \Phi\| + \|h(\Phi) - \Phi\|, \quad \|\Phi\| + K_1\Big\} \\
&\vdots \\
&\leq \max\Big\{\|u_0 - \Phi\| + \|h(\Phi) - \Phi\|, \quad \|\Phi\| + K_1\Big\},
\end{aligned}$$

which implies that the sequence $\{u_m\}$ is bounded. From (3.12), we conclude that the sequence $\{x_m\}$ is also bounded. □

Theorem 3.1. *Suppose that $h : \mathbb{U} \to \mathbb{U}$ is a θ-contraction, and Assumption 3.1 is satisfied. Then, the sequence $\{x_m\}$ generated by Algorithm 3.1 converges strongly to the unique point $\Phi \in \Omega$ satisfying $\Phi = P_\Omega h(\Phi)$.*

Proof. It is given that h is a θ-contraction. Then, for all $l_1, l_2 \in \mathbb{U}$, it is easy to see that

$$\|(I-h)(l_1)-(I-h)(l_2)\| \leq (1+\theta)\|l_1-l_2\|$$

and

$$\langle (I-h)(l_1)-(I-h)(l_2), l_1-l_2\rangle \geq (1-\theta)\|l_1-l_2\|^2;$$

that is, $(I-h)$ is $(1+\theta)$-Lipschitz continuous and $(1-\theta)$-strongly monotone. Hence, by Remark 2.2, there exists a unique $\Phi \in \Omega$ such that $\Phi = P_\Omega h(\Phi)$. Now, using $z_m = x_m + \delta_m(x_m - x_{m-1})$ and Remark 2.1(iv), we have

$$\begin{aligned}\|z_m-\Phi\|^2 &= \|x_m+\delta_m(x_m-x_{m-1})-\Phi\|^2\\ &= \|(1+\delta_m)(x_m-\Phi)-\delta_m(x_{m-1}-\Phi)\|^2\\ &= (1+\delta_m)\|x_m-\Phi\|^2-\delta_m\|x_{m-1}-\Phi\|^2+\delta_m(1+\delta_m)\|x_m-x_{m-1}\|^2. \end{aligned} \tag{3.13}$$

Now, from (3.2) and using (3.13) and Remark 2.1(i), we get

$$\begin{aligned}\|u_m-\Phi\|^2 &= \|(1-\eta_m-\zeta_m)h(x_m)+\zeta_m z_m-\Phi\|^2\\ &= \|(1-\eta_m-\zeta_m)(h(x_m)-\Phi)+\zeta_m(z_m-\Phi)+\eta_m(-\Phi)\|^2\\ &\leq \|(1-\eta_m-\zeta_m)\|h(x_m)-\Phi\|^2+\zeta_m\|z_m-\Phi\|^2+\eta_m\|\Phi\|^2\\ &\quad -\zeta_m(1-\eta_m-\zeta_m)\|h(x_m)-z_m\|^2\\ &\leq (1-\eta_m-\zeta_m)\theta^2\|x_m-\Phi\|^2+\zeta_m\Big((1+\delta_m)\|x_m-\Phi\|^2-\delta_m\|x_{m-1}-\Phi\|^2\\ &\quad +\delta_m(1+\delta_m)\|x_m-x_{m-1}\|^2\Big)+\eta_m\|\Phi\|^2-\zeta_m(1-\eta_m-\zeta_m)\|h(x_m)-z_m\|^2\\ &\leq (1-\eta_m-\zeta_m)\|x_m-\Phi\|^2+\zeta_m\Big((1+\delta_m)\|x_m-\Phi\|^2-\delta_m\|x_{m-1}-\Phi\|^2\\ &\quad +\delta_m(1+\delta_m)\|x_m-x_{m-1}\|^2\Big)+\eta_m\|\Phi\|^2-\zeta_m(1-\eta_m-\zeta_m)\|h(x_m)-z_m\|^2\\ &\leq (1-\eta_m)\|x_m-\Phi\|^2+\delta_m(\|x_m-\Phi\|^2-\|x_{m-1}-\Phi\|^2)+2\delta_m\|x_m-x_{m-1}\|^2\\ &\quad +\eta_m\|\Phi\|^2-\zeta_m(1-\eta_m-\zeta_m)\|h(x_m)-z_m\|^2. \end{aligned} \tag{3.14}$$

From (3.9) and (3.14), we obtain

$$\begin{aligned}\|x_{m+1}-\Phi\|^2 &\leq (1-\eta_m)\|x_m-\Phi\|^2+\delta_m(\|x_m-\Phi\|^2-\|x_{m-1}-\Phi\|^2)+2\delta_m\|x_m-x_{m-1}\|^2\\ &\quad +\eta_m\|\Phi\|^2-\zeta_m(1-\eta_m-\zeta_m)\|h(x_m)-z_m\|^2+\Big(\frac{\mu_m^2}{\mu_{m+1}^2}\lambda^2-1\Big)\|v_m-u_m\|^2\\ &\leq \|x_m-\Phi\|^2+\delta_m(\|x_m-\Phi\|^2-\|x_{m-1}-\Phi\|^2)+2\delta_m\|x_m-x_{m-1}\|^2\\ &\quad +\eta_m(\|\Phi\|^2-\|x_m-\Phi\|^2)-\zeta_m(1-\eta_m-\zeta_m)\|h(x_m)-z_m\|^2\\ &\quad +\Big(\frac{\mu_m^2}{\mu_{m+1}^2}\lambda^2-1\Big)\|v_m-u_m\|^2. \end{aligned} \tag{3.15}$$

There are two possibilities as follows.

Case I. Suppose that the sequence $\{\|x_m-\Phi\|\}$ is nonincreasing; in this case, there exist a number $N_1 \in \mathbb{N}$

such that $\|x_{m+1} - \Phi\| \leq \|x_m - \Phi\|$ for all $m \geq N_1$. Because $\{\|x_m - \Phi\|\}$ is bounded, this implies that $\{\|x_m - \Phi\|\}$ is convergent and that

$$\lim_{m\to\infty} (\|x_{m+1} - \Phi\|^2 - \|x_m - \Phi\|^2) = 0. \tag{3.16}$$

Because $\eta_m \to 0$, $\inf \zeta_m(1 - \eta_m - \zeta_m) > 0$, and $\left(\frac{\mu_m^2}{\mu_{m+1}^2}\lambda^2 - 1\right) < 0$. From (3.15), we get

$$\lim_{m\to\infty} \|h(x_m) - z_m\|^2 = 0, \quad \lim_{m\to\infty} \|v_m - u_m\|^2 = 0. \tag{3.17}$$

Therefore, the θ-Lipschitz continuity of F implies that

$$\lim_{m\to\infty} \|Fv_m - Fu_m\| \leq \lim_{m\to\infty} \theta\|v_m - u_m\| = 0, \tag{3.18}$$

and hence, from (3.4), we have

$$\lim_{m\to\infty} \|x_{m+1} - v_m\| = 0. \tag{3.19}$$

By using triangle inequality, we get

$$\|x_{m+1} - u_m\| \leq \|x_{m+1} - v_m\| + \|v_m - u_m\| \to 0, \ as\ m \to \infty. \tag{3.20}$$

From (3.2), using boundedness of $\{x_m\}$, and the assumption on $\{\eta_m\}$, we get

$$\|u_m - z_m\| \quad \leq (1 - \zeta_m)\|h(x_m) - z_m\| + \eta_m\|h(x_m)\| \to 0\ as\ m \to \infty. \tag{3.21}$$

From (3.20) and (3.21), we obtain

$$\begin{aligned} \|x_{m+1} - x_m\| &\leq \|x_m - u_m\| + \|u_m - z_m\| + \|z_m - x_m\| \\ &\leq \|x_{m+1} - u_m\| + \|u_m - z_m\| + \delta_m\|x_m - x_{m-1}\| \to 0\ as\ m \to \infty. \end{aligned} \tag{3.22}$$

Because $\{x_m\}$ is bounded, there exists a subsequence $\{x_{m_n}\}$ such that $x_{m_n} \rightharpoonup \Phi^\star$. Similarly, the sequences $\{u_m\}$ and $\{v_m\}$ also possess weakly convergent subsequences $\{u_{m_n}\}$, $\{v_{m_n}\}$, respectively. On the other hand, (3.3) can be equivalently expressed as

$$\frac{u_m - v_m}{\mu_m} + (Fu_m - Fv_m) \in (F + G)v_m,$$

as F is Lipschitz continuous, and $\|u_m - v_m\| \to 0$ as $m \to \infty$. Taking the limit $m \to \infty$ and considering that $F + G$ is maximal monotone and using the fact that its graph is weakly-strongly closed, we obtain that

$$0 \in (F + G)(\Phi^\star),$$

which yields $\Phi^\star \in \Omega$.

Furthermore, let $s_m = (1 - \zeta_m)h(x_m) + \zeta_m z_m$; then, $u_m = (1 - \eta_m)s_m - \eta_m\zeta_m(h(x_m) - z_m)$. Because h is a contraction, using Remark 2.1(iv) and (3.13), we have

$$\|s_m - \Phi\|^2 = \|(1 - \zeta_m)h(x_m) + \zeta_m z_m - \Phi\|^2$$

$$
\begin{aligned}
&= \|(1-\zeta_m)(h(x_m)-\Phi)+\zeta_m(z_m-\Phi)\|^2\\
&= (1-\zeta_m)\|h(x_m)-\Phi\|^2+\zeta_m\|z_m-\Phi\|^2-\zeta_m(1-\zeta_m)\|h(x_m)-z_m\|^2\\
&= (1-\zeta_m)\theta^2\|x_m-\Phi\|^2+\zeta_m\|z_m-\Phi\|^2\\
&\leq \|(1-\zeta_m)\|x_m-\Phi\|^2+\zeta_m[(1+\delta_m)\|x_m-\Phi\|^2-\delta_m\|x_{m-1}-\Phi\|^2\\
&\quad+\delta_m(1+\delta_m)\|x_m-x_{m-1}\|^2]\\
&\leq \|x_m-\Phi\|^2+\delta_m(\|x_m-\Phi\|^2-\|x_{m-1}-\Phi\|^2)+\delta_m(1+\delta_m)\|x_m-x_{m-1}\|^2],\\
&= (1+\delta_m)\|x_m-\Phi\|^2-\delta_n\|x_{m-1}-\Phi\|^2+\delta_m(1+\delta_m)\|x_m-x_{m-1}\|, \qquad (3.23)
\end{aligned}
$$

and hence, applying Remark 3.2(*iii*), we get

$$
\begin{aligned}
\|u_m-\Phi\|^2 &= \|(1-\eta_m)s_m-\eta_m\zeta_m(h(x_m)-z_m)-\Phi\|^2\\
&= \|(1-\eta_m)(s_m-\Phi)-\eta_m\zeta_m(h(x_m)-z_m)-\eta_m\Phi\|^2\\
&\leq \|(1-\eta_m)\|s_m-\Phi\|^2-2\langle\eta_m\zeta_m(h(x_m)-z_m)+\eta_m\Phi,\ u_m-\Phi\rangle\\
&\leq (1-\eta_m)\|s_m-\Phi\|^2-2\eta_m[\zeta_m\langle h(x_m)-z_m,\quad u_m-\Phi\rangle+\langle\Phi,\quad u_m-\Phi\rangle]\\
&\leq (1-\eta_m)\|s_m-\Phi\|^2-2\eta_m[\zeta_m\langle h(x_m)-z_m,\quad u_m-\Phi\rangle+\langle\Phi,\quad u_m-\Phi\rangle]\\
&\leq (1-\eta_m)[(1+\delta_m)\|x_m-\Phi\|^2-\delta_m\|x_{m-1}-\Phi\|^2+\delta_m(1+\delta_m)\|x_m-x_{m-1}\|^2]\\
&\quad-2\eta_m[\zeta_m\langle h(x_m)-z_m,\quad u_m-\Phi\rangle+\langle\Phi,\quad u_m-\Phi\rangle]\\
&\leq (1-\eta_m)\|x_m-\Phi\|^2+\delta_m(\|x_m-\Phi\|^2-\|x_{m-1}-\Phi\|^2)+\delta_m(1+\delta_m)\|x_m-x_{m-1}\|^2\\
&\quad-2\eta_m[\zeta_m\langle h(x_m)-z_m,\quad u_m-\Phi\rangle+\langle\Phi,\quad u_m-\Phi\rangle]. \qquad (3.24)
\end{aligned}
$$

Hence, from (3.12) and (3.24), we that conclude for some $m \geq k$,

$$
\begin{aligned}
\|x_{m+1}-\Phi\|^2 &\leq (1-\eta_m)\|x_m-\Phi\|^2+\delta_m(\|x_m-\Phi\|^2-\|x_{m-1}-\Phi\|^2)+\delta_m(1+\delta_m)\|x_m-x_{m-1}\|^2\\
&\quad-2\eta_m[\zeta_m\langle h(x_m)-z_m,\quad u_m-\Phi\rangle+\langle\Phi,\quad u_m-\Phi\rangle]. \qquad (3.25)
\end{aligned}
$$

Because $\Phi = P_\Omega h(\Phi)$, using the property of the projection operator, we get

$$
\limsup_{m\to\infty}\langle h(\Phi)-\Phi, x_m-\Phi\rangle = \limsup_{m\to\infty}\langle h(\Phi)-\Phi, x_{m_n}-\Phi\rangle = \langle h(\Phi)-\Phi, q-\Phi\rangle \leq 0.
$$

By applying Lemma 2.1 in (3.25), we infer $\|x_m-\Phi\| \to 0$, which implies that the sequence $\{x_m\}$ converges strongly to $\Phi = P_\Omega h(\Phi)$.

Case II. If the sequence $\{\|x_m-\Phi\|\}$ is not nonincreasing, then there exists a subsequence $\{\|x_{m_n}-\Phi\|\}$ of $\{\|x_m-\Phi\|\}$ satisfying $\|x_{m_n}-\Phi\| \leq \|x_{m_n+1}-\Phi\|$ for each $n \in \mathbb{N}$. In this case, we define a function

$$
\psi(m) = \max\{n \leq m : \|x_n-\Phi\| \leq \|x_{n+1}-\Phi\|\};
$$

then, $\{\psi(m)\} \to \infty$ as $m \to \infty$, and $\|x_{\psi(m)}-\Phi\| \leq \|x_{\psi(m)+1}-\Phi\|$. Because (3.15) holds for $\psi(m)$, we have

$$
\begin{aligned}
\|x_{\psi(m)+1}-\Phi\|^2 &\leq \|x_{\psi(m)}-\Phi\|^2+\delta_{\psi(m)}(\|x_{\psi(m)}-\Phi\|^2-\|x_{\psi(m)-1}-\Phi\|^2)+2\delta_{\psi(m)}\|x_{\psi(m)}-x_{\psi(m)-1}\|^2\\
&\quad+\eta_{\psi(m)}(\|\Phi\|^2+\|x_{\psi(m)}-\Phi\|^2)-\zeta_{\psi(m)}(1-\eta_{\psi(m)}-\zeta_{\psi(m)})\|h(x_{\psi(m)})-z_{\psi(m)}\|^2\\
&\quad+\Big(\frac{\mu^2_{\psi(m)}}{\mu^2_{\psi(m)+1}}\lambda^2-1\Big)\|v_{\psi(m)}-u_{\psi(m)}\|^2 \qquad (3.26)
\end{aligned}
$$

or

$$\begin{aligned}
&\|x_{\psi(m)+1}-\Phi\|^2-\|x_{\psi(m)}-\Phi\|^2+\zeta_{\psi(m)}(1-\eta_{\psi(m)}-\zeta_{\psi(m)})\|h(x_{\psi(m)})-z_{\psi(m)}\|^2\\
&\quad+\Big(1-\frac{\mu^2_{\psi(m)}}{\mu^2_{\psi(m)+1}}\lambda^2\Big)\|v_{\psi(m)}-u_{\psi(m)}\|^2\\
&\le \delta_{\psi(m)}(\|x_{\psi(m)}-\Phi\|^2-\|x_{\psi(m)-1}-\Phi\|^2)+2\delta_{\psi(m)}\|x_{\psi(m)}-x_{\psi(m)-1}\|^2\\
&\quad+\eta_{\psi(m)}(\|\Phi\|^2+\|x_{\psi(m)}-\Phi\|^2). \qquad (3.27)
\end{aligned}$$

From Assumption 3.1(*ii*) and (*iii*), it follows from (3.27) that

$$\lim_{m\to\infty}\|h(x_{\psi(m)})-z_{\psi(m)}\|^2=0,\ \lim_{m\to\infty}\|v_{\psi(m)}-u_{\psi(m)}\|^2=0. \qquad (3.28)$$

From (3.25), we have

$$\begin{aligned}
\|x_{\psi(m)+1}-\Phi\|^2 &\le (1-\eta_{\psi(m)})\|x_{\psi(m)}-\Phi\|^2+\delta_{\psi(m)}(\|x_{\psi(m)}-\Phi\|^2-\|x_{\psi(m)-1}-\Phi\|^2)\\
&\quad+\delta_{\psi(m)}(1+\delta_{\psi(m)})\|x_{\psi(m)}-x_{\psi(m)-1}\|^2\\
&\quad-2\eta_{\psi(m)}[\zeta_{\psi(m)}\langle h(x_{\psi(m)})-z_{\psi(m)},\quad u_{\psi(m)}-\Phi\rangle+\langle\Phi,\quad u_{\psi(m)}-\Phi\rangle]. \qquad (3.29)
\end{aligned}$$

From (3.29) and using $\|x_{\psi(m)}-\Phi\|\le\|x_{\psi(m)+1}-\Phi\|$, it follows that

$$\begin{aligned}
\|x_{\psi(m)}-\Phi\|^2 &\le \frac{\delta_{\psi(m)}}{\eta_{\psi(m)}}(\|x_{\psi(m)}-\Phi\|^2-\|x_{\psi(m)-1}-\Phi\|^2)+2\frac{\delta_{\psi(m)}}{\eta_{\psi(m)}}\|x_{\psi(m)}-x_{\psi(m)-1}\|^2\\
&\quad-2[\zeta_{\psi(m)}\langle h(x_{\psi(m)})-z_{\psi(m)},\quad u_{\psi(m)}-\Phi\rangle+\langle\Phi,\quad u_{\psi(m)}-\Phi\rangle]. \qquad (3.30)
\end{aligned}$$

From the selection of η_m and ε_m, for each $\epsilon>0$, we have $\frac{\varepsilon_{\psi(m)}}{\eta_{\psi(m)}}<\epsilon$ and

$$\frac{\delta_{\psi(m)}}{\eta_{\psi(m)}}\|x_{\psi(m)}-x_{\psi(m)-1}\|^2\le\frac{\epsilon\delta_{\psi(m)}}{\varepsilon_{\psi(m)}}\|x_{\psi(m)}-x_{\psi(m)-1}\|^2\le\frac{\epsilon\varepsilon_{\psi(m)}}{\varepsilon_{\psi(m)}}\le\epsilon. \qquad (3.31)$$

Furthermore, by applying Remark 2.1(*i*) to $\|l_1-l_2+l_1\|^2=2\|l_1\|^2-\|l_2\|^2+2\|l_1-l_2\|^2$ and substituting $l_1=x_{\psi(m)}-\Phi$ and $l_2=x_{\psi(m)-1}-\Phi$, we have

$$\|(x_{\psi(m)}-x_{\psi(m)-1})+(x_{\psi(m)}-\Phi)\|^2=2\|x_{\psi(m)}-\Phi\|^2-\|x_{\psi(m)-1}-\Phi\|^2+2\|x_{\psi(m)}-x_{\psi(m)-1}\|^2, \qquad (3.32)$$

and we know that

$$\Big(\|x_{\psi(m)}-x_{\psi(m)-1}\|+\|x_{\psi(m)}-\Phi\|\Big)^2=\|x_{\psi(m)}-x_{\psi(m)-1}\|^2+\|x_{\psi(m)}-\Phi\|^2+2\|x_{\psi(m)}-x_{\psi(m)-1}\|\|x_{\psi(m)}-\Phi\|. \qquad (3.33)$$

Using $\|l_1+l_2\|^2\le\Big(\|l_1\|+\|l_2\|\Big)^2$, from (3.32) and (3.33), we have

$$\begin{aligned}
&2\|x_{\psi(m)}-\Phi\|^2-\|x_{\psi(m)-1}-\Phi\|^2+2\|x_{\psi(m)}-x_{\Psi(m)-1}\|^2\\
&\qquad\le\|x_{\Psi(m)}-x_{\Psi(m)-1}\|^2+\|x_{\Psi(m)}-\Phi\|^2+2\|x_{\Psi(m)}-x_{\Psi(m)-1}\|\|x_{\Psi(m)}-\Phi\|,
\end{aligned}$$

or equivalently,

$$\|x_{\Psi(m)}-\Phi\|^2-\|x_{\Psi(m)-1}-\Phi\|^2\le-\|x_{\Psi(m)}-x_{\Psi(m)-1}\|^2+2\|x_{\Psi(m)}-x_{\Psi(m)-1}\|\|x_{\Psi(m)}-\Phi\|. \qquad (3.34)$$

Because $\|x_{\Psi(m)} - \Phi\|$ is bounded, there exists a number $K_2 > 0$ such that $\|x_{\Psi(m)} - \Phi\| \leq K_2$. Using Assumption 3.1(*iii*), we have

$$\begin{aligned}\frac{\delta_{\Psi(m)}}{\eta_{\Psi(m)}}\Big(2\|x_{\Psi(m)} - x_{\Psi(m)-1}\|\|x_{\Psi(m)} - \Phi\|\Big) &\leq 2\frac{\delta_{\Psi(m)}}{\eta_{\Psi(m)}}\Big(\|x_{\Psi(m)} - x_{\Psi(m)-1}\|K_2\Big),\\ &\leq 2\frac{\varepsilon\delta_{\Psi(m)}}{\varepsilon_{\Psi(m)}}\Big(\|x_{\Psi(m)} - x_{\Psi(m)-1}\|K_2\Big) \leq 2\varepsilon K_2.\end{aligned}$$

Consequently, taking the limit as $m \to \infty$ in (3.34), we deduce that

$$\frac{\delta_{\Psi(m)}}{\eta_{\Psi(m)}}\Big(\|x_{\Psi(m)} - \Phi\|^2 - \|x_{\Psi(m)-1} - \Phi\|^2\Big) \to 0.$$

Now, keeping in mind (3.28) and using (3.30), we have

$$\limsup_{m\to\infty} \|x_{\Psi(m)} - \Phi\|^2 = 0,$$

and hence,

$$\lim_{m\to\infty} \|x_{\Psi(m)} - \Phi\|^2 = 0.$$

From (3.29), we infer that

$$\limsup_{m\to\infty} \|x_{\Psi(m)+1} - \Phi\|^2 = \limsup_{m\to\infty} \|x_{\Psi(m)} - \Phi\|^2 = 0.$$

Hence, we have $\lim_{m\to\infty} \|x_{\Psi(m)+1} - \Phi\| = 0$. According to Lemma 2.4, we have

$$0 \leq \|x_m - \Phi\|^2 \leq \max\{\|x_{\Psi(m)} - \Phi\|^2, \quad \|x_m - \Phi\|^2\} \leq \|x_{\Psi(m)+1} - \Phi\|^2 \to 0.$$

Thus, we get $\|x_m - \Phi\| \to 0$ as $m \to \infty$. This completes the proof. □

By substituting $h = I$, where I is identity mapping, we obtain the modified Mann-Tseng iterative method for solving the MVIP. The corresponding iterative method is presented in Algorithm 3.2.

Algorithm 3.2. Modified inertial Mann-type Tseng method

Initiation: Let $\lambda \in (0, 1), \mu_1 > 0, \delta \in [0, 1)$ and x_0 and x_1 be chosen arbitrary.
Iterative step. For iterates x_m, and x_{m-1}, for $m \geq 1$, select $0 < \delta_m < \bar{\delta}_m$, where we compute

$$u_m = (1 - \eta_m - \zeta_m)x_m + \zeta_m[x_m + \delta_m(x_m - x_{m-1})], \tag{3.35}$$
$$v_m = R^G_{\mu_m}(u_m - \mu_m F u_m), \tag{3.36}$$

if $v_m = u_m$, terminate the algorithm. Otherwise, set $m \leftarrow m + 1$, and proceed with the following iterative step:

$$x_{m+1} = v_m - \mu_m(Fv_m - Fu_m), \tag{3.37}$$

and update δ_m and μ_m from (3.1) and (3.5), respectively.

Theorem 3.2. *If the Assumption 3.1 holds. Then, the sequence* $\{x_m\}$ *generated by Algorithm 3.2 converges strongly to* $\Phi \in \Omega$*, where* Φ *is characterized by* $\Phi = P_\Omega(0)$.

Proof. First, we show that the sequence $\{x_m\}, \{u_m\}$ and $\{v_m\}$ are bounded. Let $\Phi \in \Omega$; from (3.35), and using (3.12) and Remark 3.2, we estimate

$$\begin{aligned}
\|u_m - \Phi\| &= \|(1-\eta_m-\zeta_m)x_m + \zeta_m(x_m + \delta_m(x_m - x_{m-1})) - \Phi\| \\
&\leq \|(1-\eta_m-\zeta_m)\|x_m - \Phi\| + \zeta_m(\|x_m - \Phi\| + \delta_m\|x_m - x_{m-1}\|) + \eta_m\|\Phi\| \\
&\leq \|(1-\eta_m)\|x_m - \Phi\| + \delta_m\|x_m - x_{m-1}\| + \eta_m\|\Phi\| \\
&\leq \|(1-\eta_m)\|x_m - \Phi\| + \eta_m\Big\{\|\Phi\| + \frac{\delta_m\|x_m - x_{m-1}\|}{\eta_m}\Big\} \\
&\leq \|(1-\eta_m)\|x_m - \Phi\| + \eta_m\Big\{\|\Phi\| + K_1\Big\} \\
&\leq \max\Big\{\|x_m - \Phi\|, \|\Phi\| + K_1\Big\} \\
&\leq \max\Big\{\|u_{m-1} - \Phi\|, \|\Phi\| + K_1\Big\} \\
&\vdots \\
&\leq \max\Big\{\|u_0 - \Phi\|, \|\Phi\| + K_1\Big\},
\end{aligned}$$

which implies that $\{u_m\}$ is bounded. From (3.12), we conclude that $\{x_m\}$ is also bounded. By using (3.13) and substituting $h = I$ identity mappings in (3.14), we can easily obtain

$$\begin{aligned}
\|u_m - \Phi\|^2 \leq \|x_m - \Phi\|^2 &+ \delta_m(\|x_m - \Phi\|^2 - \|x_{m-1} - \Phi\|^2) + \delta_m(1+\delta_m)\|x_m - x_{m-1}\|^2 \\
&+ \eta_m\|\Phi\|^2 - \zeta_m(1-\eta_m-\zeta_m)\|x_m - z_m\|^2. \qquad (3.38)
\end{aligned}$$

Thus, from (3.9) and (3.38), we obtain

$$\begin{aligned}
\|x_{m+1} - \Phi\|^2 \leq \|x_m - \Phi\|^2 &+ \delta_m(\|x_m - \Phi\|^2 - \|x_{m-1} - \Phi\|^2) + 2\delta_m\|x_m - x_{m-1}\|^2 \\
&+ \eta_m\|\Phi\|^2 - \zeta_m(1-\eta_m-\zeta_m)\|x_m - z_m\|^2 + \Big(\frac{\mu_m^2}{\mu_{m+1}^2}\lambda^2 - 1\Big)\|v_m - u_m\|^2.
\end{aligned}$$

Following the similar steps as in the Case I of the proof of Theorem 3.1, we get

$$\lim_{m\to\infty}\|x_m - z_m\| = 0, \quad \lim_{m\to\infty}\|v_m - u_m\| = 0$$

and

$$\lim_{m\to\infty}\|x_{m+1} - v_m\| = 0, \quad \lim_{m\to\infty}\|x_{m+1} - u_m\| = 0, \lim_{m\to\infty}\|x_{m+1} - x_m\| = 0.$$

Further, use $h = I$ in the estimation of (3.25), we obtain

$$\begin{aligned}
\|x_{m+1} - \Phi\|^2 \leq \|(1-\eta_m)\|x_m - \Phi\|^2 &+ \delta_m(\|x_m - \Phi\|^2 - \|x_{m-1} - \Phi\|^2) + 2\delta_m\|x_m - x_{m-1}\|^2 \\
&- 2\eta_m[\zeta_m\langle x_m - z_m, u_m - p\rangle + \langle\Phi,\ u_m - p\rangle]. \qquad (3.39)
\end{aligned}$$

Because $\Phi \in \Omega$, from the property of the projection operator,

$$\liminf_{m\to\infty}\langle\Phi, \quad u_m - \Phi\rangle \geq \min_{\bar{p}\in\Omega}\langle\Phi, \quad \bar{p} - \Phi\rangle \geq 0.$$

Applying Lemma 2.1 in (3.39), we infer $\|x_m - \Phi\| \to 0$, and we conclude that the sequence $\{x_m\}$ converges strongly to $\Phi = P_\Omega(0)$. Again, from the property of metric projection, we have

$$\langle \Phi, \bar{p} - \Phi \rangle \geq 0, \forall \bar{p} \in \Omega;$$

that is, $\langle \Phi, \bar{p} \rangle \geq \|\Phi\|^2$, or $\|\Phi\| \leq \|\bar{p}\|$, which implies that Φ is the minimum norm solution in Ω. The rest of the proof can be completed following similar steps as the proof of Theorem 3.1. □

4. Applications

Now, we discuss theoretical applications of our iterative methods for solving variational inequality and optimization problems.

4.1. Variational inequality problem

Let $\mathbb{C} \subseteq \mathbb{U}$, and let $F : \mathbb{U} \to \mathbb{U}$ be a monotone operator. The variational inequality problem (VIP) is to find $\Phi \in \mathbb{C}$ such that

$$\langle F(\Phi),\ v - \Phi \rangle \geq 0,\ \forall\ v \in \mathbb{C}.$$

The normal cone to $\mathbb{C}$ is defined by

$$N_{\mathbb{C}}(v) = \{x \in \mathbb{U} : \langle x, w - v \rangle \leq 0, \forall w \in \mathbb{C}\}.$$

We recall that Φ solves the VIP if and only if

$$0 \in F(\Phi) + N_{\mathbb{C}}(\Phi). \tag{4.1}$$

It is known to us that the indicator function $I_{\mathbb{C}}$ of $\mathbb{C}$ is a proper, lower-semicontinuous, convex function on $\mathbb{U}$ and that the subdifferential of $I_{\mathbb{C}}$ is defined by

$$\partial I_{\mathbb{C}}(v) = \{x \in \mathbb{U} : \langle x, y - v \rangle \leq 0, \forall y \in \mathbb{C}\},$$

which is maximal monotone [35]. Hence, (4.1) is equivalent to the following inclusion problem:

$$0 \in F(\Phi) + \partial I_{\mathbb{C}}(\Phi).$$

Denoting the solution set of the VIP by Ω such that $\Omega \neq \emptyset$ and replacing G by $\partial I_{\mathbb{C}}$, we can obtain the following strong convergence theorems to approximate the solution of the VIP.

Corollary 4.1. *Let $\mathbb{C}$ be a nonempty, closed, and convex subset of $\mathbb{U}$, and $F : \mathbb{U} \to \mathbb{U}$ be a τ-Lipschitz continuous operator, and $h : \mathbb{U} \to \mathbb{U}$ be a θ-contraction. Compute the sequence $\{x_n\}$ as follows:*

$$\begin{aligned} u_m &= (1 - \eta_m - \zeta_m)h(x_m) + \zeta_m[x_m + \delta_m(x_m - x_{m-1})], \\ v_m &= R^{\partial I_{\mathbb{C}}}_{\mu_m}(u_m - \mu_m F u_m); \end{aligned}$$

if $v_m = u_m$, terminate the algorithm. Otherwise, set $m \leftarrow m + 1$, and proceed with the following iterative step:

$$x_{m+1} = v_m - \mu_m(Fv_m - Fu_m).$$

The parameters δ_m, μ_m are the same as in Algorithm 3.1. If Assumptions 3.1(ii) and (iii) hold, then the sequence $\{x_m\}$ converges strongly to a point Φ such that $\Phi = P_\Omega h(\Phi)$.

Corollary 4.2. *Let $\mathbb{C}$ be a nonempty, closed, and convex subset of $\mathbb{U}$ and $F : \mathbb{U} \to \mathbb{U}$ be a τ-Lipschitz continuous operator. Compute the sequence $\{x_n\}$ as follows:*

$$\begin{aligned} u_m &= (1 - \eta_m - \zeta_m)x_m + \zeta_m[x_m + \delta_m(x_m - x_{m-1})], \\ v_m &= R^{\partial I_{\mathbb{C}}}_{\mu_m}(u_m - \mu_m F u_m); \end{aligned}$$

if $v_m = u_m$, terminate the algorithm. Otherwise, set $m \leftarrow m + 1$, and proceed with the following iterative step:

$$x_{m+1} = v_m - \mu_m(Fv_m - Fu_m).$$

The parameters δ_m, μ_m are the same as in Algorithm 3.1. If Assumptions 3.1(ii) and (iii) hold, then the sequence $\{x_m\}$ converges strongly to a point Φ such that $\Phi = P_\Omega(0)$.

4.2. Optimization problem

Let $\mathbb{C} \subseteq \mathbb{U}$ be a nonempty, closed, convex subset and B_1, B_2 be proper, lower-semicontinuous functions. Assume that B_1 is differentiable and Lipschitz continuous. The following convex minimization problem (COP) is taken into consideration:

$$\min_{w \in \mathbb{C}}\{Q(w)\} = \min_{x \in \mathbb{C}}\{B_1(x) + B_2(x)\}. \tag{4.2}$$

The subdifferential of B_2 is maximal monotone [35] and is defined by

$$\partial B_2(x) = \{z \in \Omega_2 : B_2(w) - B_2(x) \geq \langle w - x, z\rangle, \forall w \in \mathbb{U}\}.$$

Denoting the solution set of (4.2) by Ω such that $\Omega \neq \emptyset$ and by taking $F = \nabla B_1$ and $G = \partial B_2$ in Algorithms 3.1 and 3.2, we obtain the following version of our methods to solve the optimization problem (4.2).

Corollary 4.3. *Let B_1 and B_2 be two proper, convex, and lower-semicontinuous functions such that ∇B_1 and ∂B_2 are maximal monotone operators such that ∇B_1 is τ-Lipschitz continuous and that h is a θ-contraction. Compute the sequence $\{x_m\}$ as follows:*

$$\begin{aligned} u_m &= (1 - \eta_m - \zeta_m)h(x_m) + \zeta_m[x_m + \delta_m(x_m - x_{m-1})], \\ v_m &= R^{\partial B_2}_{\mu_m}(u_m - \mu_m \nabla B_1 u_m); \end{aligned}$$

if $v_m = u_m$, terminate the algorithm. Otherwise, set $m \leftarrow m + 1$, and proceed with the following iterative step:

$$x_{m+1} = v_m - \mu_m(\nabla B_1 v_m - \nabla B_1 u_m).$$

The parameters δ_m, μ_m are the same as in Algorithm 3.1. If Assumptions 3.1(ii) and (iii) hold, then the sequence $\{x_m\}$ converges strongly to a point Φ such that $\Phi = P_\Omega h(\Phi)$.

Corollary 4.4. *Let B_1 and B_2 be two proper, convex, and lower-semicontinuous functions such that ∇B_1 and ∂B_2 are maximal monotone operators such that ∇B_1 is τ-Lipschitz continuous. Compute the sequence $\{x_m\}$ as follows:*

$$u_m = (1 - \eta_m - \zeta_m)x_m + \zeta_m[x_m + \delta_m(x_m - x_{m-1})],$$

$$v_m = R^{\partial B_2}_{\mu_m}(u_m - \mu_m \nabla B_1 u_m);$$

if $v_m = u_m$, terminate the algorithm. Otherwise, set $m \leftarrow m + 1$, and proceed with the following iterative step:

$$x_{m+1} = v_m - \mu_m(\nabla B_1 v_m - \nabla B_1 u_m).$$

The parameters δ_m, μ_m are the same as in Algorithm 3.1. If the Assumptions 3.1(ii) and (iii) hold, then the sequence $\{x_m\}$ converges strongly to a point Φ such that $\Phi = P_\Omega(0)$.

5. Numerical experiments

In this section, we present some numerical examples to analyze the computing performance and efficiency of the proposed methods. All the methods were run by Matlab (version R2026b) on an HP Envoy 11th Gen Intel(R) Core(TM) i7-1165G7 @ 2.80GHz (2.80 GHz), RAM 16.0 GB.

Example 5.1. Let $\mathbb{U} = \mathbb{R}^5$ be a real Hilbert space equipped with the standard inner product and induced norm $\|\cdot\|$. Our objective is to solve the following monotone inclusion problem:

$$\text{Find } x \in \mathbb{U} \text{ such that } 0 \in (F + G)(x),$$

where the operators $F, G, h : \mathbb{U} \to \mathbb{U}$ are defined in matrix form by

$$F(x) = \mathbf{A}x, \quad G(v) = \mathbf{M}x, \quad h(v) = \mathbf{C}x,$$

with

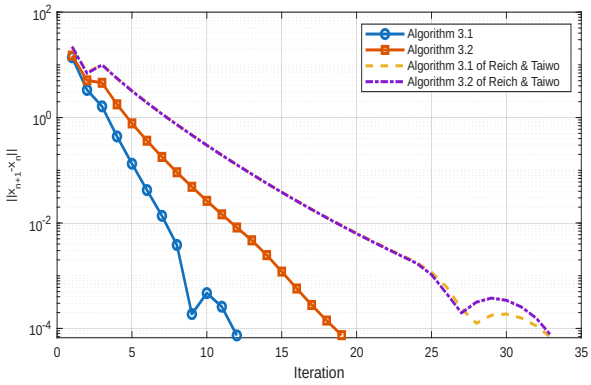
$$\mathbf{A} = \begin{bmatrix} 10 & 5 & 0 & 0 & 0 \\ 5 & 8 & 3 & 0 & 0 \\ 0 & 3 & 6 & 2 & 0 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 5 & -2 & 0 & 0 & 0 \\ -2 & 4 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}.$$

Here, F is Lipschitz continuous, G is maximal monotone, and h is a contraction with constant $\theta = 0.5$. We can see that $\Omega = \{0\}$.

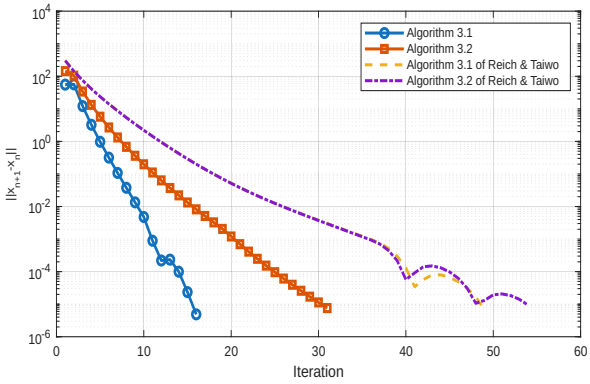
We select $\mu_1 = 0.5, \lambda = 0.5, \delta = 0.7$, and the iterative schemes employ the adaptive control sequences $\delta_m \in (0, \bar{\delta}_m)$, where $\bar{\delta}_m$ is given by (3.1); μ_{m+1} is given by and (3.5), respectively; and $\varepsilon_m = \frac{1}{(m+1)^2}$. The parameter sequences are chosen as $\eta_m = \frac{1}{(m+3)^{0.6}}$, $\zeta_m = \frac{1}{3} - \frac{1}{3m}$. The stopping criterion is given by $\|x_{m+1} - x_m\| \le d_m$; d_m are listed in the Table 1. We test the algorithms using the following four initial values:
Case (C_1): $x_0 = [10; -8; 15; -12; 6]^T$, $x_1 = [-5; 9; -7; 8; -4]^T$;
Case (C_2): $x_0 = [20; 20; 20; 20; 20]^T$, $x_1 = [-10; -10; -10; -10; -10]^T$;
Case (C_3): $x_0 = [50; -40; 30; -20; 10]^T$, $x_1 = [-25; 20; -15; 10; -5]^T$;
Case (C_4): $x_0 = [1; 2; 3; 4; 5]^T$, $x_1 = [-1; -2; -3; -4; -5]^T$.

Figure 1 and Table 1 illustrate the convergence behavior of the proposed methods. The numerical experiments confirm that all iterative methods converge reliably to a solution of the MVIP. In particular,

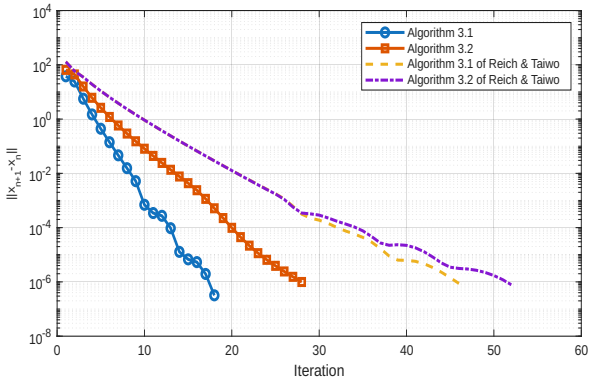
the adaptive parameters significantly enhance convergence speed and stability across all tested initial conditions, while maintaining low computational cost.



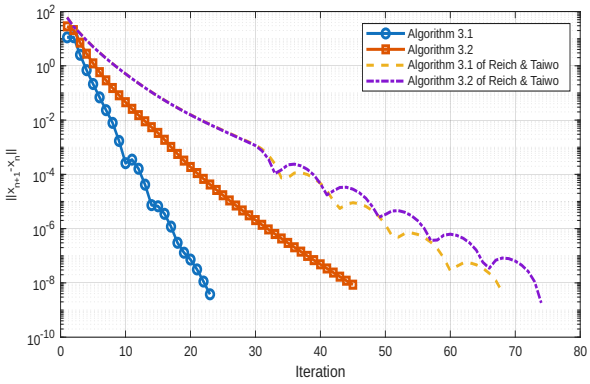
(a) Case (C_1) with $d_m = 10^{-4}$.



(b) Case (C_1) with $d_m = 10^{-5}$.



(c) Case (C_1) with $d_m = 10^{-6}$.



(d) Case (C_1) with $d_m = 10^{-8}$.

Figure 1. Convergence behavior of algorithms under different initial value cases and tolerances.

Table 1. Performance comparison of algorithms under different initial value cases and tolerances.

Cases	(C_1) with $d_m = 10^{-4}$			(C_2) with $d_m = 10^{-5}$			(C_3) with $d_m = 10^{-6}$			(C_4) with $d_m = 10^{-8}$		
Algorithm	Iter	CPU	Error	Iter	CPU	Error	Iter	CPU	Error	Iter	CPU	Error
Alg. 3.1	12	0.0117	7.368e-05	16	0.0032	4.87e-06	18	0.0003	3.2e-07	23	0.0002	0.0e+00
Alg. 3.2	19	0.0049	7.470e-05	31	0.0027	7.60e-06	28	0.0003	9.9e-07	45	0.0003	1.0e-08
Alg.3.1 [21]	33	0.0042	6.664e-05	49	0.0027	6.69e-06	46	0.0004	8.8e-07	68	0.0003	0.0e+00
Alg.3.2 [21]	33	0.0041	7.254e-05	54	0.0029	9.27e-06	52	0.0003	7.8e-07	74	0.0003	0.0e+00

Example 5.2. Let $\mathbb{U} = L^2[0, 1]$ be the real Hilbert space endowed with the inner product

$$\langle x_1, x_2 \rangle = \int_0^1 x_1(t)x_2(t)\,dt, \ \forall x_1(t), x_2(t) \in \mathbb{U},$$

and the induced norm $\|\cdot\|$. We consider the following MVIP:

$$\text{Find } x \in \mathbb{U} \text{ such that } 0 \in (F + G)(x),$$

where the operators $F, G, h : \mathbb{U} \to \mathbb{U}$ are defined by

$$(Fx)(t) = \frac{1}{2}x(t), \qquad (Gx)(t) = (2+t)x(t), \qquad (hx)(t) = \frac{1}{3}x(t), \quad t \in [0,1].$$

Here, F is a bounded linear and Lipschitz continuous operator with constant $\frac{1}{2}$, G is maximal monotone, and h is a contraction with contraction constant $\theta = \frac{1}{3}$. We observe that $\Omega = \{0\}$.

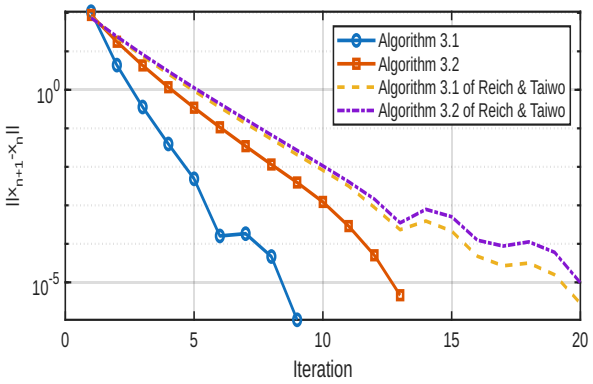
We select $\mu_1 = 0.5, \lambda = 0.5, \delta = 0.7$, and the iterative methods employ the adaptive control sequences $\delta_m \in (0, \bar{\delta}_m)$, where $\bar{\delta}_m$ is given by (3.1), and μ_{m+1} is given by (3.5). Moreover, $\varepsilon_m = \frac{1}{2(m+1)^2}$. The parameter sequences are chosen as $\eta_m = \frac{1}{(m+2)^{1/2}}, \zeta_m = \frac{1}{4} - \frac{1}{m+2}$. The stopping criterion is given by $\|x_{m+1} - x_m\| \leq d_m$, where the tolerances d_m are listed in Table 2. We test the algorithms using the following four initial values:

Case $(C_1^{'})$: $x_0(t) = 10\sin(\pi t), \quad x_1(t) = 5\cos(\pi t)$;
Case $(C_2^{'})$: $x_0(t) = 20(1-t), \quad x_1(t) = -10(1-t)$;
Case $(C_3^{'})$: $x_0(t) = 50\sin(2\pi t), \quad x_1(t) = -25\sin(2\pi t)$;
Case $(C_4^{'})$: $x_0(t) = 1 + t, \quad x_1(t) = -(1+t)$.

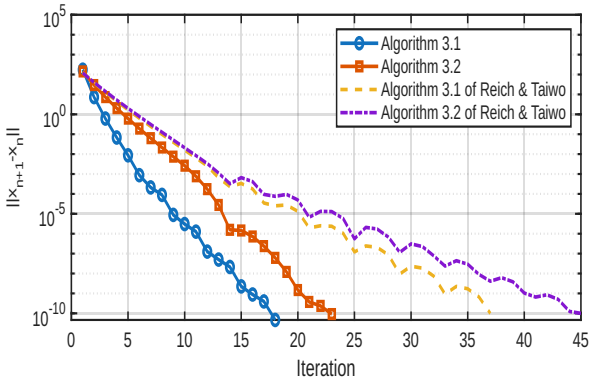
The convergence behavior of all algorithms is illustrated in Figure 2. The numerical results are summarized in Table 2. The experiments demonstrate that all iterative methods converge successfully to the unique solution of the monotone inclusion problem. Furthermore, the adaptive inertial parameters improve the convergence performance and numerical stability of the methods under different initial conditions and stopping tolerances. The iteration numbers remains relatively small for all methods, confirming their effectiveness for solving monotone inclusion problems in infinite-dimensional Hilbert spaces.

Table 2. Performance comparison of the algorithms under different stopping tolerances.

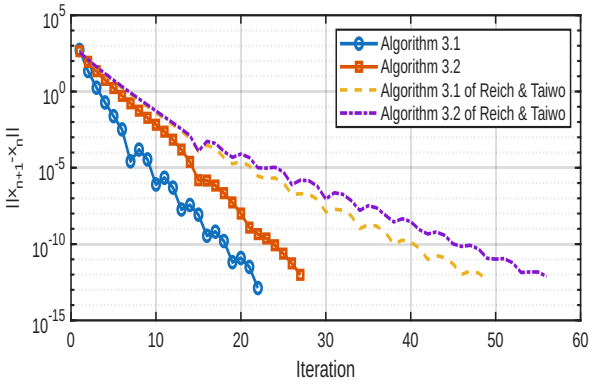
Cases	$(C_1^{'})$ with $d_m = 10^{-5}$			$(C_2^{'})$ with $d_m = 10^{-10}$			$(C_3^{'})$ with $d_m = 10^{-12}$			$(C_4^{'})$ with $d_m = 10^{-16}$		
Algorithm	Iter	CPU	Error	Iter	CPU	Error	Iter	CPU	Error	Iter	CPU	Error
Alg. 3.1	9	0.0066	1.06e-06	18	0.0017	4.64e-11	22	0.0004	1.32e-13	28	0.0007	6.67e-17
Alg. 3.2	13	0.0041	4.62e-06	23	0.0015	9.15e-11	27	0.0004	9.70e-13	32	0.0005	4.37e-17
Alg.3.1 [21]	20	0.0041	2.83e-06	37	0.0017	9.86e-11	49	0.0006	4.60e-13	61	0.0008	8.57e-17
Alg.3.2 [21]	20	0.0043	9.74e-06	45	0.0016	9.95e-11	56	0.0007	7.54e-13	70	0.0009	9.76e-17



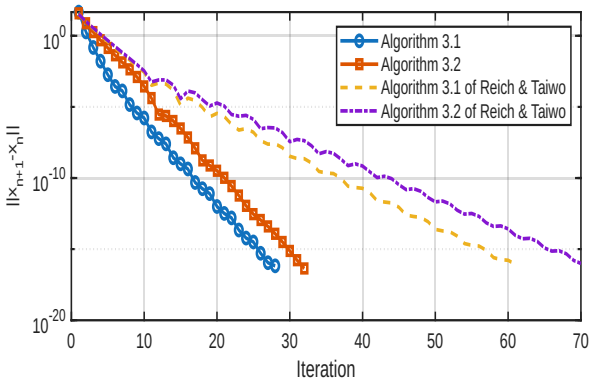
(a) Case $(C_1^{'})$ with $d_m = 10^{-5}$.



(b) Case $(C_2^{'})$ with $d_m = 10^{-10}$.



(c) Case $(C_3^{'})$ with $d_m = 10^{-14}$.



(d) Case $(C_4^{'})$ with $d_m = 10^{-16}$.

Figure 2. Convergence behavior of algorithms under different initial value cases and tolerances.

Example 5.3. (Nonsmooth convex minimization problem in $\mathbb{R}^3$) Consider

$$\min_{x\in\mathbb{R}^3}\{\|x\|_2^2 + (1,1,1)x + \|x\|_1\}, \tag{5.1}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\|x\|_1 = |x_1| + |x_2| + |x_3|$, and $\|x\|_2^2 = x_1^2 + x_2^2 + x_3^2$. Then, the objective function is

$$Q(x) = x_1^2 + x_2^2 + x_3^2 + x_1 + x_2 + x_3 + |x_1| + |x_2| + |x_3|. \tag{5.2}$$

Let $B_1 = \|x\|_2^2 + (1,1,1)x$, and $B_2 = \|x\|_1$. Then, B_1 is convex and differentiable with $\nabla B_1 = 2x + (1,1,1)$ being τ-Lipschitz continuous with constant $\tau = 2$ and monotone. We know that $(I + \partial B_2)^{-1} = (y_1, y_2, y_3)$, where $y_i = sign(x_i)\max\{|x_i| - \mu,\ 0\}, i = 1,2,3$, and ∂B_2 is maximal monotone. Let Ω be the solution set of the optimization problem (5.1). Then, the optimization problem is equivalent to the following inclusion problem: Find $\Phi \in \Omega$ such that

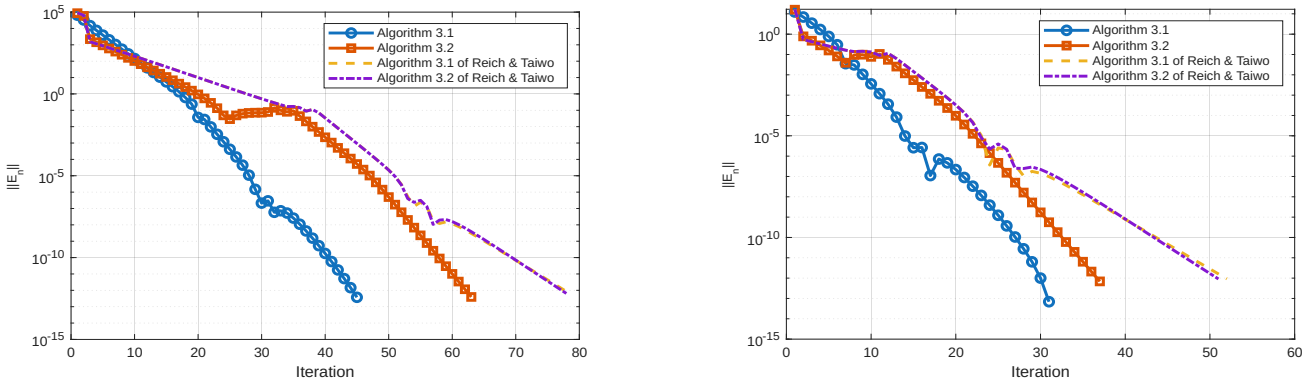
$$0 \in \nabla B_1(\Phi) + \partial B_2(\Phi). \tag{5.3}$$

Now, we apply our proposed methods to solve the problem (5.3) and compare their performance with [21, Algorithms 3.1 and 3.2]. We choose $\eta_m = \frac{1}{(m+33)^{0.7}}$, $\zeta_m = \frac{1-\eta_m}{2}$, $h(x) = \frac{x}{2}$, $\varepsilon_m = \frac{1}{(m+1)^3}$,

$\mu_m = 0.25$, $\delta = 0.3$. The value of $E_n = 0.5\|x_m - R^G_\mu(x_m - \mu F x_m)\|$ for Cases $(C_1^{''}) - (C_2^{''})$ and different values of λ are shown in Table 3 and Figure 3:

Table 3. Performance comparison of our Algorithms 3.1 and 3.2 with [21, Algorithms 3.1 and 3.2].

Cases	$(C_1^{''})$			$(C_2^{''})$		
Algorithm	Iter	CPU	Error	Iter	CPU	Error
Alg. 3.1	45	0.000273	3.747e-13	31	0.000133	6.8964e-14
Alg. 3.2	63	0.000249	3.9411e-13	37	0.000162	6.9022e-13
Alg.3.1 [21]	78	0.000248	8.3517e-13	52	0.000229	9.1989e-13
Alg.3.2 [21]	78	0.000252	6.305e-13	50	0.000204	9.0867e-13



(a) Case $(C_1^{''})$ **(b)** Case $(C_2^{''})$

Figure 3. Convergence behavior of algorithms with tolerance $0.5\|x_m - R^G_\mu(x_m - \mu F x_m)\| \leq 10^{-12}$.

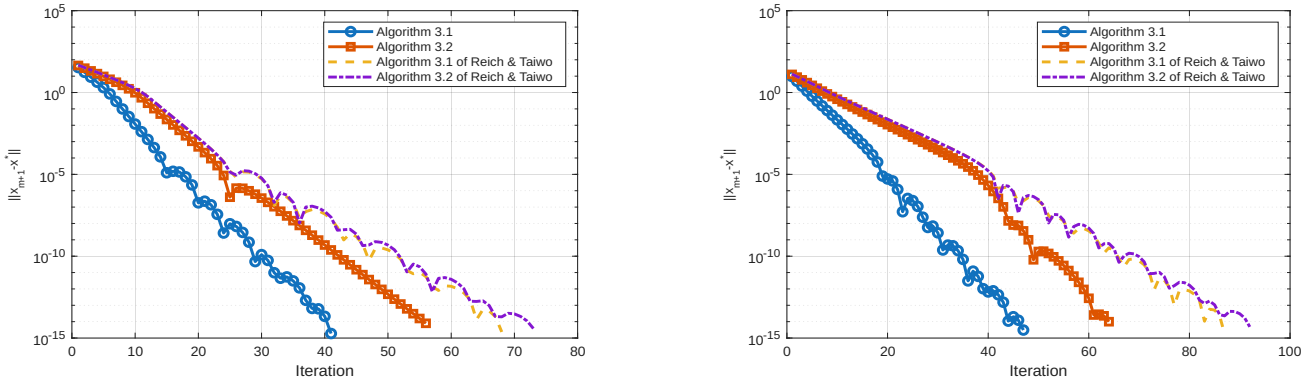
Case $(C_1^{''})$: $x_0 = [50; -40; 30]^T$, $x_1 = [65; 30; 25]^T$, $\lambda = 0.60$;
Case $(C_2^{''})$: $x_0 = [10; 8; 50]^T$, $x_1 = [51; 17; 27]^T$, $\lambda = 0.70$.

Notice that the solution of the problem (5.1) is $\Omega = \{(0, 0, 0)\}$, and the minimum value is $Q(0, 0, 0) = 0$. Select $\eta_m = \frac{1}{(m+44)}$, $\zeta_m = \frac{(1-\eta_m)}{2}$, $h(x) = \frac{x}{2}$, $\varepsilon_m = \frac{1}{(m+1)^{1.5}}$, $\lambda = 0.5$, and $\mu_m = 0.25$. With he stopping condition $Tol_m = 0.5\|x_m - (0, 0, 0)\|_2 \leq 10^{-14}$, we compute the performance of of all algorithms. We plot the graphs of Tol_m and number of iterations for the following choices of initial values and different values of δ:

Case $(C_3^{''})$: $x_0 = [27; 51; 16]^T$, $x_1 = [-41; -15; -29]^T$, $\delta = 0.6$;
Case $(C_4^{''})$: $x_0 = [2; 9; 8]^T$, $x_1 = [1; 0; 10]^T$, $\delta = 0.7$.
The value of $Tol_m = 0.5\|x_m - (0, 0, 0)\|_2$ for Cases $(C_3^{''}) - (C_4^{''})$ are shown in Table 4 and Figure 4.

Table 4. Performance comparison of our Algorithms 3.1 and 3.2 with [21, Algorithms 3.1 and 3.2].

Cases	(C_3'')			(C_4'')		
Algorithm	Iter	CPU	Error	Iter	CPU	Error
Alg. 3.1	44	0.00016	1.2986e-15	46	0.000256	3.287e-15
Alg. 3.2	68	0.000187	7.7373e-15	59	0.000264	8.6292e-15
Alg.3.1 [21]	74	0.000176	1.8631e-15	86	0.000345	8.6587e-15
Alg.3.2 [21]	74	0.000171	7.3359e-15	89	0.000341	2.7751e-15



(a) Case (C_3''). **(b)** Case (C_4'').

Figure 4. Convergence behavior of algorithms with tolerance $0.5\|x_{m+1} - \Phi\| \leq 10^{-14}$.

6. Conclusions

In this article, we propose two modified inertial methods for solving variational inclusion problems. The first is a viscosity-type Tseng method, and the second is a Mann-type Tseng method. The proposed methods incorporate inertial extrapolation together with viscosity approximation and Mann-type iteration, respectively, at the initial stage of the iterative process. Under suitable assumptions, we establish two strong convergence theorems for the proposed methods. Applications to variational inequality and optimization problems are also discussed. Finally, it is shown in the numerical examples that our proposed methods showed superior performance in the number of iterations.

Further research may focus on establishing and comparing the convergence rates of the proposed methods with those of other contemporary iterative methods for solving variational inclusion problems.

Author contributions

Mohammad Dilshad: Conceptualization, methodology, writing-original draft; Fahad M. Alamrani: Methodology, investigation, supervision, project administration, funding acquisition; Yasser Alzahrani: Investigation, project administration, funding acquisition; Nifeen H. Altaweel: Validation, funding acquisition; Mona Y. Alfef: Validation, funding acquisition. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflicts of interest

The authors declare no conflicts of interest.

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