AIMS Mathematics, 11(11): 39523–39547.
DOI: 10.3934/math.20261522
Received: 29 April 2026
Revised: 09 September 2026
Accepted: 21 September 2026
Published: 26 November 2026

https://www.aimspress.com/journal/Math

Research article

Geometric properties of a fractional integral operator associated with Ramanujan-type entire functions

Khaled Mehrez[1,2] **and Abdulaziz Alenazi**[3,4,*]

[1] Department of Mathematics, Kairouan Preparatory Institute for Engineering Studies, University of Kairouan, Kairouan, Tunisia

[2] Research Laboratory: Chemistry, Materials and Modeling (LR24ES02) Preparatory Institute for Engineering Studies of Kairouan, University of Kairouan Kairouan 3100, Tunisia

[3] Department of Mathematics, College of Science, Northern Border University, Arar, Saudi Arabia

[4] Center for Scientific Research and Entrepreneurship, Northern Border University, Arar 73213, Saudi Arabia

* **Correspondence:** Email: a.alenazi@nbu.edu.sa.

Abstract: In this paper, we investigate geometric function theoretic properties of a normalized fractional integral operator defined via the Riemann-Liouville fractional integral and a class of Ramanujan-type entire functions. Sufficient conditions are established to ensure that the operator is starlike and convex of order η $(0 \leq \eta < 1)$ in the unit disk. As applications of the main results, explicit ranges of parameters are determined under which the operator belongs to the Hardy spaces $\mathcal{H}^{\infty}$ and $\mathcal{H}^{r}$. Furthermore, convolution properties are studied, and inclusion results mapping the class $\mathcal{R}(\beta)$ into $\mathcal{R}(\sigma)$ are obtained.

Keywords: Ramanujan entire-type function; fractional integral; fractional calculus; univalent functions; analytic functions; starlike functions; convex functions
Mathematics Subject Classification: 30C45, 30C50, 30C80

1. Introduction and preliminaries

Fractional calculus has emerged as a significant and rapidly developing branch of mathematical analysis, owing to its solid theoretical foundations and its broad range of applications in science and engineering. Fractional integral and differential operators provide powerful tools for modeling processes that exhibit memory and hereditary properties. In recent years, fractional calculus has also been successfully integrated into complex analysis, particularly within geometric function theory,

where it offers an effective framework for constructing and analyzing new subclasses of analytic and univalent functions.

A major advancement in geometric function theory is the theory of differential subordination, which has proven to be a versatile and powerful method for establishing geometric properties of analytic functions, including univalence, starlikeness, convexity, and close-to-convexity. When combined with suitably defined integral and differential operators, differential subordination yields elegant and unified approaches to classical results while also enabling the derivation of new ones. Moreover, the theory of admissible functions has further strengthened this framework, significantly extending its applicability and effectiveness in the study of analytic and univalent function classes.

In this context, operators arising from fractional calculus have attracted considerable attention. In particular, the Riemann–Liouville fractional integral and derivative operators have been adapted to classes of analytic functions and combined with various special functions, including hypergeometric functions [1,2] and their q-analogues [3]. These developments have led to the introduction of numerous fractional integral operators whose geometric properties have been widely studied. Such investigations highlight that fractional calculus provides a flexible and effective framework for constructing operators that preserve or induce desirable geometric characteristics.

On the other hand, Ramanujan-type entire functions constitute an important class of special functions with deep connections to the theory of the Gamma function, q-series, and generalized hypergeometric functions. These functions exhibit rich analytical features, including growth properties, zero distributions, and functional inequalities, which make them particularly attractive for further investigation. Their structural similarity to classical special functions suggests that they can serve as effective kernels in the construction of new operators within geometric function theory.

From the perspective of geometric function theory, functions associated with special functions have been widely studied in recent years. In particular, several authors have investigated geometric properties of normalized forms of such functions, establishing sufficient conditions for their inclusion in subclasses of univalent functions, such as starlike and convex functions of order $\eta\,(0 \leq \eta < 1)$. These contributions highlight the importance of special functions as a source of meaningful examples and applications in geometric function theory.

Motivated by the above developments, in this paper we introduce a new fractional integral operator obtained by combining the Riemann–Liouville fractional integral of order $\lambda > 0$ with a class of Ramanujan-type entire functions. To the best of our knowledge, the geometric properties of such a normalized fractional integral operator associated with generalized Ramanujan-type entire functions have not been systematically investigated. The primary objective of this work is to analyze its geometric properties.

The investigation is carried out by employing techniques from the theory of differential subordination and admissible functions, together with classical tools of geometric function theory. This approach enables us to derive new sufficient conditions and extend existing results related to fractional integral operators associated with special functions. The results obtained herein contribute to the ongoing development of fractional calculus methods in geometric function theory and further emphasize the analytical richness of Ramanujan-type entire functions.

We now recall some basic definitions and notations that will be used throughout the paper. Let $\mathbb{H}$ denote the class of all analytic functions in the unit disk

$$\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}.$$

Let $\mathcal{A}$ denote the subclass of functions $\mathfrak{F} \in \mathbb{H}$ satisfying

$$\mathfrak{F}(0) = 0 \quad \text{and} \quad \mathfrak{F}'(0) = 1,$$

and having the Taylor expansion

$$\mathfrak{F}(\zeta) = \zeta + \sum_{\ell=2}^{\infty} a_\ell \zeta^\ell, \qquad \zeta \in \mathbb{D}.$$

A function $\mathfrak{F} \in \mathcal{A}$ is said to be starlike (with respect to the origin) if

$$\Re\left(\frac{\zeta\, \mathfrak{F}'(\zeta)}{\mathfrak{F}(\zeta)}\right) > 0, \qquad \zeta \in \mathbb{D}.$$

The class of all such functions is denoted by $\mathcal{S}^*$.

For a given $0 \leq \eta < 1$, a function $\mathfrak{F} \in \mathcal{A}$ is said to be starlike of order η, denoted by $\mathcal{S}^*(\eta)$, if

$$\Re\left(\frac{\zeta\, \mathfrak{F}'(\zeta)}{\mathfrak{F}(\zeta)}\right) > \eta, \qquad \zeta \in \mathbb{D}.$$

An analytic function $\mathfrak{F} \in \mathcal{A}$ is said to be convex in $\mathbb{D}$ if $\mathfrak{F}$ is univalent in $\mathbb{D}$ and $\mathfrak{F}(\mathbb{D})$ is a convex domain in $\mathbb{C}$. We denote the class of convex functions by C. It is well known (see [4]) that

$$\mathfrak{F} \in C \quad \Longleftrightarrow \quad \mathrm{Re}\left(1 + \frac{\zeta\, \mathfrak{F}''(\zeta)}{\mathfrak{F}'(\zeta)}\right) > 0, \qquad \zeta \in \mathbb{D}.$$

Moreover, if

$$\Re\left(1 + \frac{\zeta\, \mathfrak{F}''(\zeta)}{\mathfrak{F}'(\zeta)}\right) > \eta, \qquad \zeta \in \mathbb{D},$$

where $\eta \in [0, 1)$, then $\mathfrak{F}$ is called convex of order η. The class of such functions is denoted by $C(\eta)$.

We remark that, for every $\eta \in [0, 1)$, the following inclusions hold:

$$\mathcal{S}^*(\eta) \subseteq \mathcal{S}^*(0) = \mathcal{S}^* \quad \text{and} \quad C(\eta) \subseteq C(0) = C.$$

Additionally, for a given $0 \leq \eta < 1$, we introduce the classes of functions described in [4]:

$$\mathcal{P}(\eta) := \{p \in \mathbb{H} : p(0) = 1,\ \Re(p(\zeta)) > \eta,\ \zeta \in \mathbb{D}\}$$

and

$$\mathcal{R}(\eta) := \{\mathfrak{F} \in \mathcal{A} : \Re(\mathfrak{F}'(\zeta)) > \eta,\ \zeta \in \mathbb{D}\}.$$

For analytic functions $\mathfrak{F}$ and g in $\mathbb{D}$, we say that $\mathfrak{F}$ is subordinate to g, written as $\mathfrak{F} \prec g$, if there exists a Schwarz function ω analytic in $\mathbb{D}$, satisfying $\omega(0) = 0$ and $|\omega(\zeta)| < 1$, such that

$$\mathfrak{F}(\zeta) = g(\omega(\zeta)), \qquad \zeta \in \mathbb{D}.$$

Let $\mathcal{H}^\infty$ denote the space of all bounded analytic functions in $\mathbb{D}$. For any $\kappa \in (0, \infty]$, any function $\mathfrak{F} \in \mathbb{H}$, and $r \in [0, 1)$, we set

$$\mathcal{M}_\kappa(r; \mathfrak{F}) = \begin{cases} \left(\dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \left|\mathfrak{F}(re^{i\theta})\right|^\kappa \, d\theta\right)^{1/\kappa}, & \text{if } 0 < \kappa < \infty, \\ \displaystyle\max_{|\zeta| \leq r} |\mathfrak{F}(\zeta)|, & \text{if } \kappa = \infty. \end{cases}$$

It is well known that $\mathfrak{F} \in \mathbb{H}$ belongs to the Hardy space $\mathcal{H}^{\kappa}$, where $0 < \kappa \leq \infty$, if

$$\sup_{0 \leq r < 1} \mathcal{M}_{\kappa}(r; \mathfrak{F}) < \infty.$$

Moreover, for $0 < \nu \leq \mu \leq \infty$, we have

$$\mathcal{H}^{\infty} \subseteq \mathcal{H}^{\mu} \subseteq \mathcal{H}^{\nu}.$$

In recent decades, increasing attention has been devoted to the geometric analysis of special functions, due to their significant role in univalent function theory and various branches of mathematical analysis. For comprehensive accounts and recent developments in this area, the reader is referred to [5–7] and more recent studies seen in papers [8, 9].

Special functions occupy a fundamental position in both pure and applied mathematics. They arise naturally in the analysis of differential equations, the theory of integral transforms, and a wide variety of classical and contemporary problems in mathematical physics. Their importance extends further to several applied disciplines, including engineering, quantum mechanics, and number theory, where they serve as essential analytical tools.

Among the most remarkable developments in this area are the Rogers-Ramanujan identities, which have had a profound impact on diverse fields such as combinatorics and statistical mechanics (see, for example, [10–12]). Inspired by these identities, Ramanujan introduced a noteworthy generalization that led to the definition of the so-called Ramanujan entire function, also known as the q-Airy function [13], given by

$$\mathbf{A}_q(\zeta) = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} q^{\ell^2}}{(q;q)_{\ell}} \zeta^{\ell}, \qquad (\zeta \in \mathbb{C},\ q \in (0,1)). \tag{1.1}$$

Here, $(a;q)_k$ denotes the q-shifted factorial defined by

$$(a;q)_0 = 1, \qquad (a;q)_{\ell} = \prod_{j=0}^{\ell-1} (1 - aq^j), \qquad \ell \geq 1. \tag{1.2}$$

A natural extension of the function $\mathbf{A}_q$ was later introduced in [14] through the generalized form

$$\mathbf{A}_q^{(\alpha)}(a;\zeta) = \sum_{\ell=0}^{\infty} \frac{(a;q)_{\ell} q^{\alpha \ell^2}}{(q;q)_{\ell}} \zeta^{\ell}, \qquad (a, \zeta \in \mathbb{C},\ \alpha > 0,\ q \in (0,1)). \tag{1.3}$$

It is readily seen that, for the particular choice $\alpha = 1$ and $a = 0$, the above definition reduces to the classical Ramanujan entire function in the form

$$\mathbf{A}_q^{(1)}(0;\zeta) = \mathbf{A}_q(-\zeta). \tag{1.4}$$

Various analytical and structural properties of the generalized function (1.3) have been investigated in [15, 16] and the references cited therein.

Fractional integral operators associated with special functions have been extensively investigated in the literature, particularly in connection with geometric function theory. The recent review study by Srivastava [17] highlighted the important role of fractional calculus in geometric function theory and

stimulated further investigations involving fractional operators and univalent functions. In addition, the theory of differential subordination developed by Miller and Mocanu [18] has provided an effective framework for deriving geometric properties of analytic functions when combined with suitable integral and differential operators. Within this framework, numerous studies have considered fractional integral operators associated with different special functions. The Riemann-Liouville fractional integral was adapted to the setting of analytic functions in [19] and subsequently combined with the generalized hypergeometric function [20], the Gauss hypergeometric function [2], the Libera integral operator [21], Mittag–Leffler–confluent hypergeometric functions [22], q-hypergeometric functions [3], Bessel function of the first kind [23] and the generalized Bessel-Maitland-Tremblay function [24]. These investigations demonstrate that the combination of fractional calculus with special functions provides a fruitful framework for constructing analytic functions and studying their geometric properties.

Despite these developments, the geometric theory of fractional integral operators associated with Ramanujan-type entire functions appears to be considerably less explored. In particular, while several geometric properties of generalized Ramanujan-type entire functions and their normalized forms have been investigated [14, 15], the corresponding Riemann–Liouville fractional-integral transforms have not been systematically studied from the viewpoint of geometric function theory. This observation motivates the present study. The motivation for the proposed construction also comes from the complementary roles of the two ingredients involved. The Riemann-Liouville fractional integral introduces the additional parameter $\lambda > 0$, which allows the construction to describe a family of fractional-order transforms, while the generalized Ramanujan-type function retains its characteristic q-dependent structure. Hence, the resulting operator simultaneously incorporates the flexibility of fractional calculus and the analytic structure of Ramanujan-type entire functions. Furthermore, after normalization, the resulting operator belongs to the standard class $\mathcal{A}$, which makes it suitable for the direct application of classical methods of geometric function theory. This combination provides a convenient framework for deriving new geometric criteria and studying properties that are not directly addressed by the existing fractional integral operators associated with other special functions.

The fractional integral of order λ ($\lambda > 0$) for an analytic function $\mathfrak{F}$ is defined as follows [19, 20]:

$$D_{\zeta}^{-\lambda}\mathfrak{F}(\zeta) = \frac{1}{\Gamma(\lambda)} \int_0^{\zeta} \frac{\mathfrak{F}(\xi)}{(\zeta - \xi)^{1-\lambda}}\, d\xi, \tag{1.5}$$

where $\mathfrak{F}$ is analytic in a simply-connected region of the ζ-plane containing the origin, and the multiplicity of $(\zeta - \xi)^{1-\lambda}$ is removed by requiring $\log(\zeta - \xi)$ to be real when $\zeta - \xi > 0$.

Applying this fractional integral of order λ to the function $\mathbf{A}_q^{(\alpha)}(a;\zeta)$ defined in (1.3), we obtain

$$D_{\zeta}^{-\lambda}\mathbf{A}_q^{(\alpha)}(a;\zeta) = \sum_{\ell=0}^{\infty} \frac{(a;q)_\ell\, \Gamma(\ell+1)\, q^{\alpha\ell^2}}{(q;q)_\ell \Gamma(\ell+\lambda+1)}\, \zeta^{\ell+\lambda}, \qquad (a, \zeta \in \mathbb{C},\ \alpha > 0,\ q \in (0,1)). \tag{1.6}$$

It is readily observed that the function defined above does not satisfy the standard normalization requirements in geometric function theory. Accordingly, to enable its treatment within the usual

framework, we introduce the following normalized function:

$$\begin{aligned}\mathfrak{A}_q^{(\alpha,\lambda)}(a;\zeta) &= \zeta^{1-\lambda} D_\zeta^{-\lambda} \mathbf{A}_q^{(\alpha)}(a;\zeta) \\ &= \sum_{\ell=0}^{\infty} \frac{(a;q)_\ell \Gamma(\ell+1)\Gamma(\lambda+1) q^{\alpha\ell^2}}{(q;q)_\ell \Gamma(\ell+\lambda+1)} \zeta^{\ell+1}, (a \in \mathbb{C},\ \lambda > 0,\ \zeta \in \mathbb{D},\ \alpha > 0,\ q \in (0,1)).\end{aligned} \tag{1.7}$$

Here, $D_\zeta^{-\lambda}\mathbf{A}_q^{(\alpha)}(a;\zeta)$ is defined by (1.6). Indeed, the factor $\zeta^{1-\lambda}$ converts the powers $\zeta^{\ell+\lambda}$ generated by the Riemann-Liouville fractional integral into the standard powers $\zeta^{\ell+1}$, thereby ensuring the normalization $\mathfrak{A}_q^{(\alpha,\lambda)}(a;0) = 0$ and $\left[\mathfrak{A}_q^{(\alpha,\lambda)}\right]'(a;0) = 1$. This normalization allows the resulting function to belong to the standard class $\mathcal{A}$ and makes it suitable for the application of classical tools of geometric function theory.

The main objective of this paper is to examine the geometric function-theoretic properties of the normalized fractional integral operator $\mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta)$. We derive sufficient conditions for its membership in subclasses of univalent functions, including starlike and convex functions of order η $(0 \le \eta < 1)$ and the class $\mathcal{R}(\eta)$, as well as sufficient conditions for $2\zeta - \mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta)$ to belong to these classes. Additionally, we establish sufficient conditions under which the normalized operator

$$\zeta \mapsto \frac{\mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta)}{\zeta}, \qquad (\zeta \in \mathbb{D})$$

belongs to the class $\mathcal{P}(\eta)$, and we further investigate its convolution properties with functions in $\mathcal{R}(\kappa)$. These findings highlight the interplay between fractional calculus, special functions, and geometric function theory, offering a systematic framework for constructing operators with desirable analytic and geometric characteristics.

2. Some useful lemmas

In order to prove our main results the following preliminary lemmas will be helpful.

Lemma 2.1. *[25] Let $\eta \in [0,1)$. A sufficient condition for*

$$\mathfrak{F}(\zeta) = \zeta + \sum_{\ell=2}^{\infty} \alpha_\ell \zeta^\ell,$$

to be in $S^*(\eta)$ and $C(\eta)$, respectively, is that

$$\sum_{\ell\ge2} (\ell-\eta)\,|\alpha_\ell| \le 1-\eta,$$

$$\sum_{\ell\ge2} \ell(\ell-\eta)\,|\alpha_\ell| \le 1-\eta,$$

respectively.

The following elementary estimate for the q-shifted factorial will be useful in establishing the coefficient bounds used below.

Lemma 2.2. *Let* $0 < q < 1$ *and* $\ell \in \mathbb{N}$. *Then*

$$\frac{1}{(q;q)_\ell} \leq \frac{1+q}{(1-q^2)^\ell}, \qquad (\ell \in \mathbb{N}). \tag{2.1}$$

Proof. For $k \geq 2$, we have $1 - q^k \geq 1 - q^2$. Hence

$$(q;q)_\ell = (1-q)\prod_{k=2}^{\ell}(1-q^k) \geq (1-q)(1-q^2)^{\ell-1}.$$

Consequently, we get

$$\frac{1}{(q;q)_\ell} \leq \frac{1}{(1-q)(1-q^2)^{\ell-1}} = \frac{1+q}{(1-q^2)^\ell},$$

where we have used $1 - q^2 = (1 - q)(1 + q)$. This completes the proof. □

Lemma 2.3. *[26] Let* $a > 0$ *and* $q \in (0, 1)$ *satisfies the following conditions*

$$a\log(q) \leq \log(1-q), \tag{2.2}$$

then the function $\mathfrak{f}_{q,a}(s)$ *defined on* $[0, \infty)$ *by*

$$\mathfrak{f}_{q,a}(s) = q^{as}\Gamma_q(s+a) \tag{2.3}$$

is decreasing on $[0, \infty)$.

Lemma 2.4. *[25] Let* $\eta \in [0, 1)$. *Suppose that*

$$\mathfrak{F}(\zeta) = \zeta - \sum_{\ell=2}^{\infty}\alpha_\ell\zeta^\ell, \qquad \alpha_\ell \geq 0.$$

Then a necessary and sufficient condition for $\mathfrak{F}$ *to be in* $S^*(\eta)$ *and* $C(\eta)$, *respectively, is that*

$$\sum_{\ell=2}^{\infty}(\ell-\eta)\,\alpha_\ell \leq 1-\eta,$$

$$\sum_{\ell=2}^{\infty}\ell(\ell-\eta)\,\alpha_\ell \leq 1-\eta,$$

respectively.

Lemma 2.5. *[27] Let* $\mathfrak{F} \in C(\eta)$, *where* $0 \leq \eta < 1$, *and suppose that* $\mathfrak{F}$ *is not of the following form:*

$$\begin{cases} n_1 + \dfrac{n_2\zeta}{(1-\zeta e^{i\omega})^{1-2\mu}}, & \mu \neq \frac{1}{2}, \\ n_1 + n_2\log(1-\zeta e^{i\omega}), & \mu = \frac{1}{2}, \end{cases}$$

where $n_1, n_2 \in \mathbb{C}$ *and* $\omega \in \mathbb{R}$. *Then the following assertions hold:*
(i) *If* $\mu \geq \frac{1}{2}$, *then* $\mathfrak{F} \in \mathcal{H}^\infty$.
(ii) *If* $\mu \in \left[0, \frac{1}{2}\right)$, *then there exists* $\delta = \delta(\mathfrak{F}) > 0$ *such that* $\mathfrak{F} \in \mathcal{H}^{\delta+\frac{1}{1-2\mu}}$.

The following lemma is a particular case of the more general result given in [28, Theorem 2.4].

Lemma 2.6. *Let ρ be analytic in $\mathbb{D}$ such that $\rho(0) = 1$ and*

$$\Re\,\rho(\zeta) > \frac{1}{2}, \qquad \zeta \in \mathbb{D}.$$

Then, for any function $\mathfrak{F}$ analytic in $\mathbb{D}$, we have

$$(\rho * \mathfrak{F})(\mathbb{D}) \subset \text{co}\ \mathfrak{F}(\mathbb{D}),$$

where co $\mathfrak{F}(\mathbb{D})$ *denotes the closed convex hull of* $\mathfrak{F}(\mathbb{D})$.

3. Starlikeness and convexity of the normalized fractional integral operator associated to the Ramanujan-type entire functions

In this section, we present a sufficient condition for $\mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta)$ to belong to the class $S^*(\eta)$, and a sufficient condition for

$$\mathfrak{B}_q^{(\alpha,\lambda)}(q^a;\zeta) = 2\zeta - \mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta), \tag{3.1}$$

to belong to the same class. The first main result is stated below.

Theorem 3.1. *Let $a, \lambda > 0$, $\alpha \geq a + 1$, $\eta \in [0, 1)$ and $q \in (0, 1)$ such that the inequality (2.2) holds. Assume that the following conditions hold:*

$$\frac{2q^{2a}\Gamma_q(a+2)\Gamma(\lambda+1)\left[q^2(2+q)(1+q) - \eta q^2\right]}{\Gamma_q(a)\Gamma(\lambda+3)} \leq 1 - \eta, \tag{3.2}$$

then the function $\mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta)$ belongs to the class $S^*(\eta)$. Furthermore, the inequality (3.2) is a sufficient condition to the function $\mathfrak{B}_q^{(\alpha,\lambda)}(q^a;\zeta)$ defined by (3.1) to be in $S^*(\eta)$.

Proof. According to Lemma 2.1, we need only to prove that

$$\sum_{\ell=2}^{\infty}(\ell - \eta)\beta_\ell \leq 1 - \eta,$$

where the sequence $\{\beta_\ell\}_{\ell\geq 0}$ is defined by

$$\beta_\ell = \frac{(q^a;q)_\ell\Gamma(\ell+1)\Gamma(\lambda+1)\,q^{\alpha\ell^2}}{(q;q)_\ell\Gamma(\ell+\lambda+1)}, \qquad (\ell \geq 0). \tag{3.3}$$

In view of the relation

$$\frac{\Gamma_q(a+\ell)}{\Gamma_q(a)} = \frac{(q^a;q)_\ell}{(1-q)^\ell} \tag{3.4}$$

and (2.1) together with (3.3), we establish that

$$\begin{aligned}\sum_{\ell=2}^{\infty}(\ell-\eta)\beta_\ell &= \sum_{\ell=2}^{\infty}\frac{(\ell-\eta)(q^a;q)_\ell\Gamma(\ell+1)\Gamma(\lambda+1)\,q^{\alpha\ell^2}}{(q;q)_\ell\Gamma(\ell+\lambda+1)}\\ &\leq \sum_{\ell=2}^{\infty}\frac{(\ell-\eta)\Gamma_q(a+\ell)\Gamma(\ell+1)\Gamma(\lambda+1)\,q^{\alpha\ell^2}}{\Gamma_q(a)(1+q)^{\ell-1}\Gamma(\ell+\lambda+1)}\\ &= \sum_{\ell=2}^{\infty}\frac{(\ell-\eta)\mathfrak{f}_{q,a}(\ell)\Gamma(\ell+1)\Gamma(\lambda+1)\,q^{\alpha\ell^2}}{q^{a\ell}\Gamma_q(a)(1+q)^{\ell-1}\Gamma(\ell+\lambda+1)}.\end{aligned} \tag{3.5}$$

where $\mathfrak{f}_{q,a}(s)$ is defined by (2.3). From Lemma 2.3, the function $s \mapsto \mathfrak{f}_{q,a}(s)$ is decreasing on $(0, \infty)$. Since $\ell \geq 2$, we have

$$\mathfrak{f}_{q,a}(\ell) \leq \mathfrak{f}_{q,a}(2) = q^{2a}\Gamma_q(a+2). \tag{3.6}$$

Furthermore, since the function $s \mapsto \frac{\Gamma(s+1)}{\Gamma(s+\lambda+1)}$ is decreasing on $(0, \infty)$ for each $\lambda > 0$, we infer that

$$\frac{\Gamma(\ell+1)}{\Gamma(\ell+\lambda+1)} \leq \frac{\Gamma(3)}{\Gamma(\lambda+3)}, \qquad (\forall \ell \geq 2). \tag{3.7}$$

Also, because $q \in (0,1)$ and $\ell^2 \geq \ell$, we have $q^{\alpha\ell^2} \leq q^{\alpha\ell}$. Consequently, we have

$$q^{\alpha\ell^2 - a\ell} \leq q^{(\alpha-a)\ell}.$$

Since $\alpha \geq a+1$ and $q \in (0,1)$ we conclude that

$$q^{(\alpha-a)\ell} \leq q^{\ell}. \tag{3.8}$$

Combining these estimates with (3.5) gives

$$\begin{aligned}
\sum_{\ell=2}^{\infty}(\ell-\eta)\beta_\ell &\leq \frac{2q^{2a}\Gamma_q(a+2)\Gamma(\lambda+1)}{\Gamma_q(a)\Gamma(\lambda+3)} \sum_{\ell=2}^{\infty} \frac{(\ell-\eta)q^{\alpha\ell^2-a\ell}}{(1+q)^{\ell-1}} \\
&\leq \frac{2q^{2a}\Gamma_q(a+2)\Gamma(\lambda+1)}{\Gamma_q(a)\Gamma(\lambda+3)} \sum_{\ell=2}^{\infty} \frac{(\ell-\eta)q^{(\alpha-a)\ell}}{(1+q)^{\ell-1}} \\
&\leq \frac{2q^{2a}(1+q)\Gamma_q(a+2)\Gamma(\lambda+1)}{\Gamma_q(a)\Gamma(\lambda+3)} \sum_{\ell=2}^{\infty}(\ell-\eta)\left(\frac{q}{1+q}\right)^{\ell}.
\end{aligned} \tag{3.9}$$

Moreover, by using the fact that for $\mathfrak{b} \in (0,1)$ we have

$$\begin{aligned}
\sum_{\ell=2}^{\infty}\ell\mathfrak{b}^{\ell} &= \mathfrak{b}^2\sum_{\ell=0}^{\infty}(\ell+2)\mathfrak{b}^{\ell} \\
&= \mathfrak{b}^3\sum_{\ell=0}^{\infty}\ell\mathfrak{b}^{\ell-1} + 2\mathfrak{b}^2\sum_{\ell=0}^{\infty}\mathfrak{b}^{\ell} \\
&= \frac{\mathfrak{b}^2(2-\mathfrak{b})}{(1-\mathfrak{b})^2}.
\end{aligned} \tag{3.10}$$

Hence, by (3.10), we conclude for $q \in (0,1)$ that

$$\sum_{\ell=2}^{\infty}\ell\left(\frac{q}{1+q}\right)^{\ell} = q^2(2+q).$$

Therefore, with the help of the above formula and (3.9) we conclude that

$$\sum_{\ell=2}^{\infty}(\ell-\eta)\beta_\ell \leq \frac{2q^{2a}\Gamma_q(a+2)\Gamma(\lambda+1)\left[q^2(2+q)(1+q)-\eta q^2\right]}{\Gamma_q(a)\Gamma(\lambda+3)}.$$

The upper bound $1-\eta$ for this sum is achieved if and only if (3.2) is fulfilled. Moreover, since

$$\mathfrak{B}_q^{(\alpha,\lambda)}(q^a;\zeta) = \zeta - \sum_{\ell=2}^{\infty} \beta_\ell \zeta^\ell,$$

it follows from Lemma 2.4 that the necessary and sufficient condition for the function $\mathfrak{B}_q^{(\alpha,\lambda)}(q^a;\zeta)$ to belong to $S^*(\eta)$ is given by (3.2). This completes the proof of Theorem 3.1. □

By setting $\eta = 0$ in Theorem 3.1, we obtain the following criterion for the functions $\mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta)$ and $\mathfrak{B}_q^{(\alpha,\lambda)}(q^a;\zeta)$ to be starlike.

Corollary 3.2. *Let $a, \lambda > 0$, $\alpha \geq a+1$ and $q \in (0,1)$ such that the inequality (2.2) holds. Assume that the following conditions hold:*

$$\frac{2q^{2a}\Gamma_q(a+2)\Gamma(\lambda+1)\left[q^2(2+q)(1+q)\right]}{\Gamma_q(a)\Gamma(\lambda+3)} \leq 1, \tag{3.11}$$

then the function $\mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta)$ belongs to the class S^*. Furthermore, the inequality (3.11) is a necessary and sufficient to the function $\mathfrak{B}_q^{(\alpha,\lambda)}(q^a;\zeta)$ defined by (3.1) to be in S^*.

By setting $a = 1$ in the above corollary, we obtain the following result.

Corollary 3.3. *Let $\lambda > 0$, $\alpha \geq 2$ and $q \in (0,1/2)$. Assume that*

$$\frac{2q^4(1+q)^2(2+q)}{(\lambda+1)(\lambda+2)} \leq 1. \tag{3.12}$$

Then the function $\mathfrak{A}_q^{(\alpha,\lambda)}(q;\zeta)$ belongs to the class S^*. Furthermore, the inequality (3.12) is necessary and sufficient for the function $\mathfrak{B}_q^{(\alpha,\lambda)}(q;\zeta)$ defined by (3.1) to be in S^*.

Remark 3.4. *We note that the admissible parameter set defined by the conditions* (3.12) *is nonempty. To illustrate this, Figure 1 displays the function*

$$\varphi_q(\lambda) = (\lambda+1)(\lambda+2) - 2q^4(1+q)^2(2+q),$$

for the values $q = \frac{1}{2}$ and $q = \frac{1}{3}$. The regions where these functions are positive clearly show that there exist nonempty ranges of the parameter b satisfying inequality (3.2)*. This confirms that the hypotheses of Theorem 3.1 are meaningful and can be realized for concrete parameter values.*

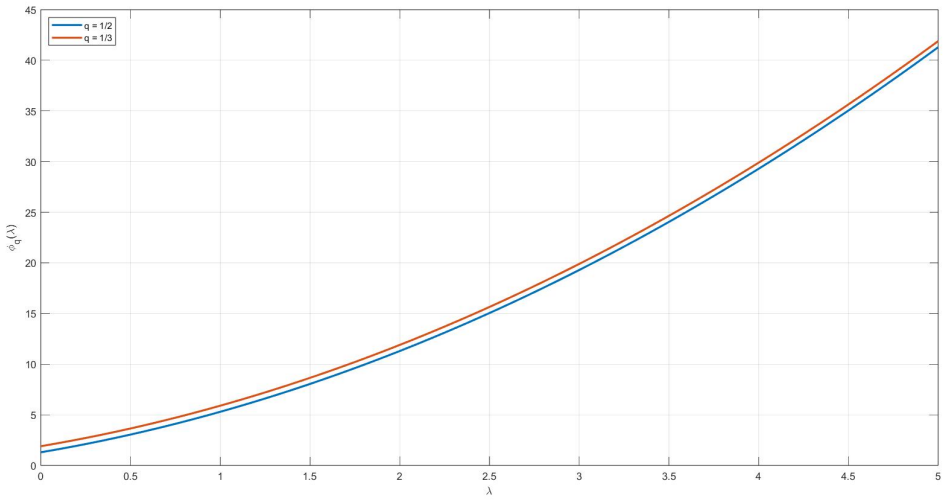


Figure 1. Graph of the function $\varphi_q(\lambda)$ for $q \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$.

Example 3.5. *To illustrate the applicability of Corollary 3.3, we consider two different sets of parameters:*

$$(a, \alpha, \lambda, q) = \left(1, 2, 1, \frac{1}{2}\right) \quad \text{and} \quad \left(1, \frac{5}{2}, \frac{1}{2}, \frac{1}{3}\right).$$

For both choices, all the assumptions of Corollary 3.3 are satisfied. Consequently, the corresponding functions $\mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta)$ and $\mathfrak{B}_q^{(\alpha,\lambda)}(q^a;\zeta)$ belong to the class S^*.

The corresponding images of the open unit disk are presented in Figures 2–5. Figures 2 and 3 illustrate the images under $\mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta)$ for the two parameter sets, respectively, whereas Figures 4 and 5 show the corresponding images under $\mathfrak{B}_q^{(\alpha,\lambda)}(q^a;\zeta)$. These numerical illustrations, obtained for distinct choices of the parameters, demonstrate the applicability of Corollary 3.3 and provide a visual representation of the starlike behavior of the functions under consideration.

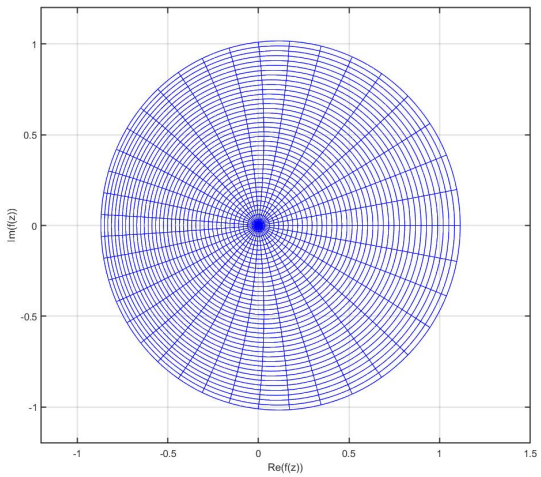


Figure 2. Image of the open unit disk under the function $\mathfrak{A}_{\frac{1}{2}}^{(2,1)}(\frac{1}{2};\zeta)$.

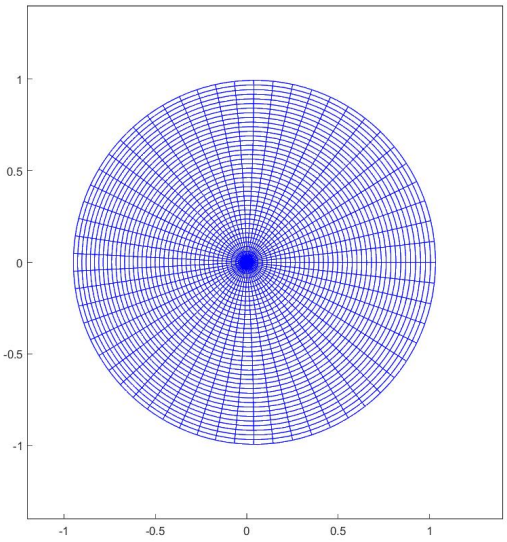


Figure 3. Image of the open unit disk under the function $\mathfrak{U}_{\frac{1}{3}}^{(\frac{5}{2},\frac{1}{2})}(\frac{1}{3};\zeta)$.

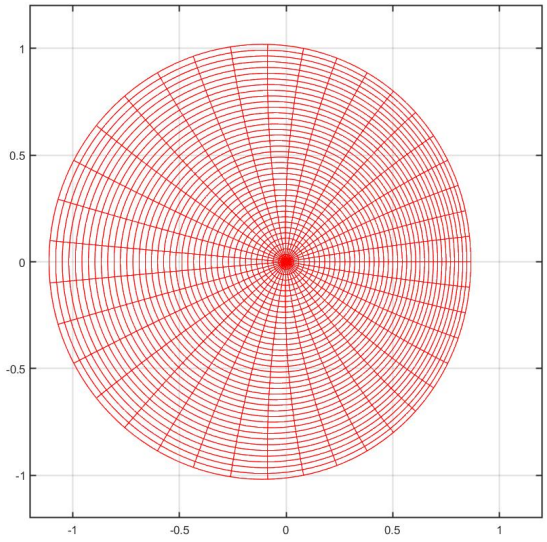


Figure 4. Image of the open unit disk under the function $\mathfrak{B}_{\frac{1}{2}}^{(2,1)}(\frac{1}{2};\zeta)$.

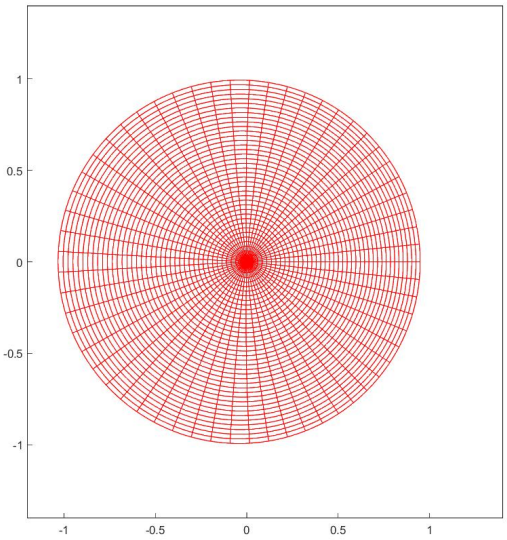


Figure 5. Image of the open unit disk under the function $\mathfrak{B}_{\frac{1}{3}}^{(\frac{5}{2},\frac{1}{2})}(\frac{1}{3};\zeta)$.

Theorem 3.6. *Let $a, \lambda > 0$, $\alpha \geq a+1$, $\eta \in [0,1)$ and $q \in (0,1)$ such that the inequality (2.2) holds. Assume that the following conditions hold:*

$$\frac{2q^{2a}(1+q)\Gamma_q(a+2)\Gamma(\lambda+1)}{\Gamma_q(a)\Gamma(\lambda+3)}\left[q(1+q)((1+2q)(1+q)-\eta)-(1-\eta)\frac{q}{1+q}\right] \leq 1-\eta, \tag{3.13}$$

then the function $\mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta)$ belongs to the class $C(\eta)$. Furthermore, the inequality (3.13) is a sufficient condition t to the function $\mathfrak{B}_q^{(\alpha,\lambda)}(q^a;\zeta)$ defined by (3.1) to be in $C(\eta)$.

Proof. According to Lemma 2.4, it suffices to prove that

$$\sum_{\ell=2}^{\infty} \ell(\ell-\eta)\beta_\ell \leq 1-\eta.$$

In view of (3.3), (3.4) and (2.1), we obtain

$$\begin{aligned}\sum_{\ell=2}^{\infty} \ell(\ell-\eta)\beta_\ell &= \sum_{\ell=2}^{\infty} \frac{\ell(\ell-\eta)(q^a;q)_\ell\Gamma(\ell+1)\Gamma(\lambda+1)\,q^{\alpha\ell^2}}{(q;q)_\ell\Gamma(\ell+\lambda+1)} \\ &\leq \sum_{\ell=2}^{\infty} \frac{\ell(\ell-\eta)\mathfrak{f}_{q,a}(\ell)\Gamma(\ell+1)\Gamma(\lambda+1)\,q^{\alpha\ell^2}}{q^{a\ell}\Gamma_q(a)(1+q)^{\ell-1}\Gamma(\ell+\lambda+1)}.\end{aligned} \tag{3.14}$$

where $\mathfrak{f}_{q,a}(s)$ is defined by (2.3). Combining (3.14) with the estimates established in (3.6)–(3.8), we

obtain, for $\alpha \geq a+1$ and $q \in (0,1)$, that

$$\begin{aligned}\sum_{\ell=2}^{\infty} \ell(\ell-\eta)\beta_\ell &\leq \frac{2q^{2a}(1+q)\Gamma_q(a+2)\Gamma(\lambda+1)}{\Gamma_q(a)\Gamma(\lambda+3)} \sum_{\ell=2}^{\infty} \ell(\ell-\eta)\left(\frac{q}{1+q}\right)^{\ell} \\ &= \frac{2q^{2a}(1+q)\Gamma_q(a+2)\Gamma(\lambda+1)}{\Gamma_q(a)\Gamma(\lambda+3)} \\ &\times \left[q(1+q)((1+2q)(1+q)-\eta) - (1-\eta)\frac{q}{1+q}\right].\end{aligned} \tag{3.15}$$

It follows that the right-hand side of (3.15) is bounded by $1-\eta$ whenever inequality (3.13) is satisfied. Finally, we observe that condition (3.13) is necessary for the function $\mathfrak{B}_q^{(\alpha,\lambda)}(q^a;\zeta)$ to belong to the class $C(\eta)$. □

As a direct consequence of Theorem 3.6 and Lemma 2.5, we obtain the following Hardy space inclusion results for the function $\mathfrak{U}_q^{(\alpha,\lambda)}(q^a;\zeta)$.

Corollary 3.7. *Under the assumptions of Theorem 3.6, the following assertions hold:*
(a) *If* $\mu \in \left[0, \frac{1}{2}\right)$, *then*

$$\mathfrak{U}_q^{(\alpha,\lambda)}(q^a;\zeta) \in \mathcal{H}^{\frac{1}{1-2\mu}}.$$

(b) *If* $\mu \geq \frac{1}{2}$, *then*

$$\mathfrak{U}_q^{(\alpha,\lambda)}(q^a;\zeta) \in \mathcal{H}^{\infty}.$$

Proof. We recall the series representation of the Gauss hypergeometric function (see [29, Eq (15.21)]):

$${}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}\,\middle|\,\zeta\right) = \sum_{\ell=0}^{\infty} \frac{(a)_\ell (b)_\ell}{(c)_\ell}\frac{\zeta^\ell}{\ell!},$$

where $(\lambda)_\ell$ denotes the Pochhammer symbol defined by

$$(\lambda)_0 = 1, \qquad (\lambda)_\ell = \lambda(\lambda+1)\cdots(\lambda+\ell-1), \qquad \ell \geq 1.$$

For $\mu \neq \frac{1}{2}$, we have

$$\begin{aligned}\frac{\zeta}{(1-\zeta e^{i\omega})^{1-2\mu}} &= \zeta\, {}_2F_1\left(\begin{matrix}1,\ 1-2\mu\\ 1\end{matrix}\,\middle|\,\zeta e^{i\omega}\right) \\ &= \zeta \sum_{\ell=0}^{\infty} \frac{(1)_\ell (1-2\mu)_\ell}{(1)_\ell}\frac{(\zeta e^{i\omega})^\ell}{\ell!} \\ &= \sum_{\ell=0}^{\infty} \frac{(1-2\mu)_\ell}{\ell!} e^{i\ell\omega}\zeta^{\ell+1}.\end{aligned}$$

Consequently,

$$n_1 + \frac{n_2\zeta}{(1-\zeta e^{i\omega})^{1-2\mu}} = n_1 + n_2 \sum_{\ell=0}^{\infty} \frac{(1-2\mu)_\ell}{\ell!} e^{i\ell\omega}\zeta^{\ell+1}.$$

In the limiting case $\mu = \frac{1}{2}$, we have

$$\frac{\zeta}{(1-\zeta e^{i\omega})^{1-2\mu}} = \zeta,$$

while the corresponding limiting expression involving the logarithm is given by

$$\log(1-\zeta e^{i\omega}) = -\sum_{\ell=0}^{\infty} \frac{e^{i\ell\omega}}{\ell+1}\zeta^{\ell+1}.$$

Thus, we have

$$n_1 + n_2 \log(1-\zeta e^{i\omega}) = n_1 - n_2 \sum_{\ell=0}^{\infty} \frac{e^{i\ell\omega}}{\ell+1}\zeta^{\ell+1}.$$

On the other hand, by Theorem 3.6, the function

$$\zeta \longmapsto \mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta)$$

is convex of order η in the unit disk $\mathbb{D}$ under the stated assumptions. Hence, applying Lemma 2.5, we obtain

$$\mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta) \in \begin{cases} \mathcal{H}^{\frac{1}{1-2\mu}}, & 0 \le \mu < \frac{1}{2}, \\ \mathcal{H}^{\infty}, & \mu \ge \frac{1}{2}. \end{cases}$$

This completes the proof. □

As an immediate consequence of Theorem 3.6, we state the corresponding Hardy space inclusion result for the function $\mathfrak{B}_q^{(\alpha,\lambda)}(q^a;\zeta)$. The proof follows along the same lines as that of the previous corollary and is therefore omitted.

Corollary 3.8. *Under the assumptions of Theorem 3.6, the following assertions hold:*
(a) *If* $\mu \in \left[0, \frac{1}{2}\right)$, *then* $\mathfrak{B}_q^{(\alpha,\lambda)}(q^a;\zeta) \in \mathcal{H}^{\frac{1}{1-2\mu}}$.
(b) *If* $\mu \ge \frac{1}{2}$, *then* $\mathfrak{B}_q^{(\alpha,\lambda)}(q^a;\zeta) \in \mathcal{H}^{\infty}$.

By taking the special case $\eta = 0$ in Theorem 3.6, we obtain the following result concerning the class of convex functions.

Corollary 3.9. *Let* $a, \lambda > 0$, $\alpha \ge a+1$ *and* $q \in (0,1)$ *such that condition* (2.2) *holds. Assume that*

$$\frac{2q^{2a}(1+q)\Gamma_q(a+2)\Gamma(\lambda+1)}{\Gamma_q(a)\Gamma(\lambda+3)}\left[q(1+q)(1+2q)(1+q) - \frac{q}{1+q}\right] \le 1, \tag{3.16}$$

then the function $\mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta)$ *belongs to the class* C. *Furthermore, inequality* (3.13) *is necessary and sufficient for the function* $\mathfrak{B}_q^{(\alpha,\lambda)}(q^a;\zeta)$ *defined by (3.1) to belong to* C.

By choosing $a = 1$ in the previous corollary, we obtain the following more explicit criterion for convexity in the class C.

Corollary 3.10. *Let* $\lambda > 0$, $\alpha \ge 2$ *and* $q \in (0, 1/2]$. *Assume that*

$$\frac{2q^4(1+q)^2}{(\lambda+1)(\lambda+2)}\left[q(1+q)((1+2q)(1+q)) - \frac{q}{1+q}\right] \le 1. \tag{3.17}$$

Then the function $\mathfrak{A}_q^{(\alpha,\lambda)}(q;\zeta)$ *belongs to the class* C. *Furthermore, inequality* (3.17) *is necessary and sufficient for the function* $\mathfrak{B}_q^{(\alpha,\lambda)}(q;\zeta)$ *defined by* (3.1) *to belong to* C.

Remark 3.11. *We emphasize that the set of admissible parameters determined by condition (3.13) is nonempty. To illustrate this fact, Figure 6 depicts the function*

$$\phi_q(\lambda) = (1+q)(\lambda+1)(\lambda+2) - 2q^3(1+q)^4(2+q) - q,$$

for the representative values $q = \frac{1}{2}$ and $q = \frac{1}{3}$. The regions where $\varphi_q(\lambda) > 0$ indicate the existence of nontrivial intervals of the parameter λ for which inequality (3.13) is satisfied. This demonstrates that the assumptions of Theorem 3.6 are not merely theoretical, but are fulfilled for concrete choices of the parameters.

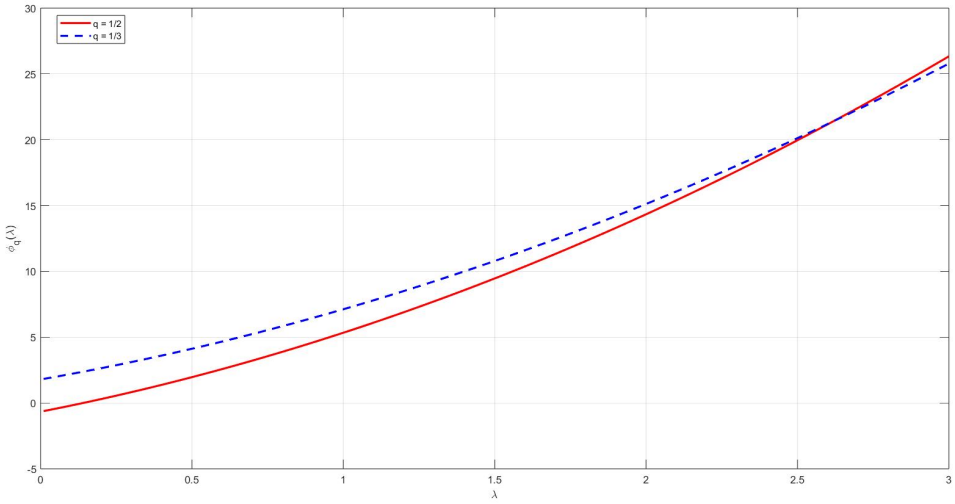


Figure 6. Graph of the function $\phi_q(\lambda)$ for $q \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$.

Example 3.12. *To illustrate the applicability of Corollary 3.3 and Remark 3.11, we consider the following two distinct sets of parameters:*

$$(a, \alpha, \lambda, q) = \left(1, 2, \frac{1}{2}, \frac{1}{2}\right) \quad \text{and} \quad \left(1, \frac{5}{2}, \frac{1}{3}, \frac{1}{3}\right).$$

For both choices, the assumptions of Corollary 3.3 and Remark 3.11 are satisfied. Hence, the corresponding functions $\mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta)$ and $\mathfrak{B}_q^{(\alpha,\lambda)}(q^a;\zeta)$ belong to the class C.

The corresponding images of the open unit disk are displayed in Figures 7–10. Figures 7 and 8 illustrate the images associated with $\mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta)$ for the first and second parameter sets, respectively. Similarly, Figures 9 and 10 present the corresponding images associated with $\mathfrak{B}_q^{(\alpha,\lambda)}(q^a;\zeta)$. These numerical illustrations, obtained for two different choices of the parameters, provide a visual demonstration of the convexity predicted by Corollary 3.3.

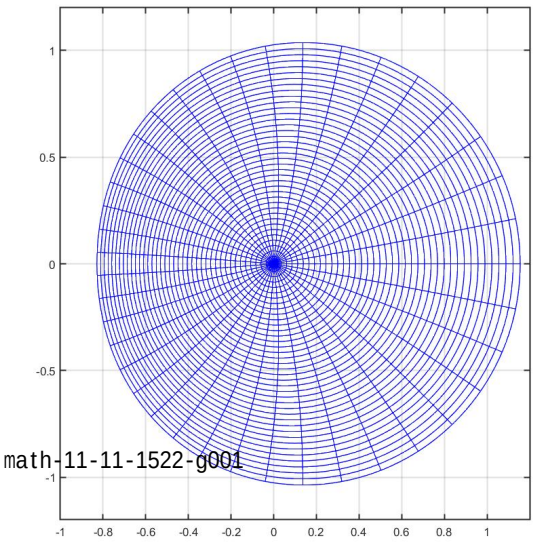


Figure 7. Image of the open unit disk under $\mathfrak{U}_{\frac{1}{2}}^{(2,\frac{1}{2})}!\left(\frac{1}{2};\zeta\right)$.

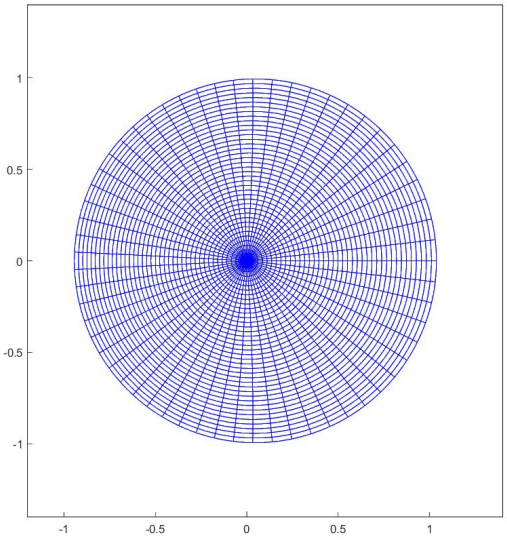


Figure 8. Image of the open unit disk under $\mathfrak{U}_{\frac{1}{3}}^{(\frac{5}{2},\frac{1}{3})}!\left(\frac{1}{3};\zeta\right)$.

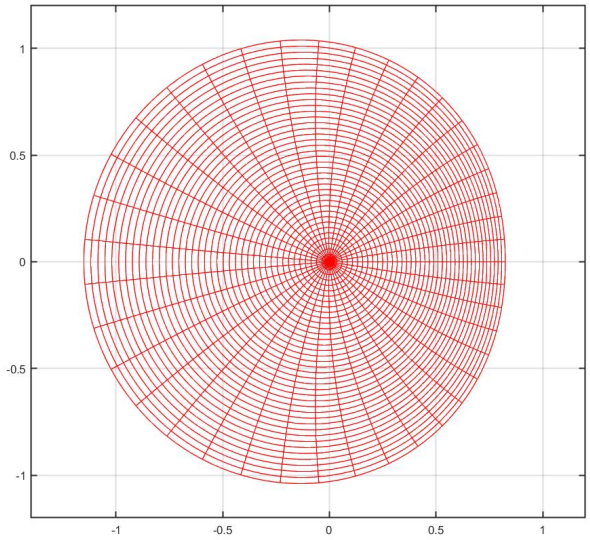


Figure 9. Image of the open unit disk under $\mathfrak{B}_{\frac{1}{2}}^{(2,\frac{1}{2})}!\left(\frac{1}{2};\zeta\right)$.

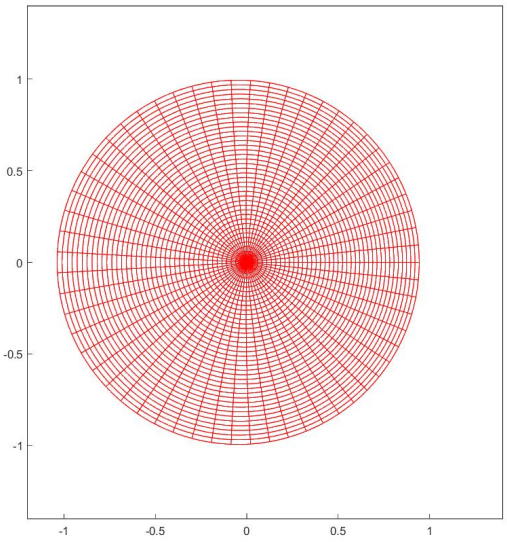


Figure 10. Image of the open unit disk under $\mathfrak{B}_{\frac{1}{3}}^{(\frac{5}{2},\frac{1}{3})}!\left(\frac{1}{3};\zeta\right)$.

4. Convolution properties of the normalized fractional integral operator associated with Ramanujan-type entire functions

Our first main result in this section is stated as follows.

Theorem 4.1. *Let $a, \lambda > 0$, $\alpha \geq a + 1$, $\eta \in [0, 1)$ and $q \in (0, 1)$ such that the inequality (2.2) holds.*

Assume that the following conditions hold:

$$\frac{q^{a+1}(1+q)(1-q^a)}{(\lambda+1)(1-q)} \leq 1-\eta. \tag{4.1}$$

Then

$$\frac{\mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta)}{\zeta} \in \mathcal{P}(\eta).$$

Proof. For convenience, we define

$$\mathfrak{H}(\zeta) = \frac{1}{1-\eta}\left[\frac{\mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta)}{\zeta} - \eta\right], \qquad (\zeta \in \mathbb{D}).$$

To prove that

$$\frac{\mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta)}{\zeta} \in \mathcal{P}(\eta), \qquad (\forall \zeta \in \mathbb{D}),$$

it is sufficient to show that

$$|\mathfrak{H}(\zeta) - 1| < 1, \qquad (\zeta \in \mathbb{D}).$$

By using (1.7) together with (3.4) and (2.1), it follows for every $\zeta \in \mathbb{D}$ that

$$\begin{aligned} |\mathfrak{H}(\zeta)-1| &= \left|\frac{1}{1-\eta}\left[1+\sum_{\ell=1}^{\infty}\frac{(q^a;q)_\ell\Gamma(\ell+1)\Gamma(\lambda+1)\,q^{\alpha\ell^2}}{(q;q)_\ell\Gamma(\ell+\lambda+1)}\zeta^\ell - \eta\right] - 1\right| \\ &= \left|\frac{1}{1-\eta}\sum_{\ell=1}^{\infty}\frac{(q^a;q)_\ell\Gamma(\ell+1)\Gamma(\lambda+1)\,q^{\alpha\ell^2}}{(q;q)_\ell\Gamma(\ell+\lambda+1)}\zeta^\ell\right| \\ &\leq \frac{\Gamma(\lambda+1)}{(1-\eta)\Gamma_q(a)}\sum_{\ell=1}^{\infty}\frac{\mathfrak{f}_{q,a}(\ell)\Gamma(\ell+1)q^{\alpha\ell^2}}{q^{a\ell}(1+q)^{\ell-1}\Gamma(\ell+\lambda+1)}, \end{aligned} \tag{4.2}$$

where $\mathfrak{f}_{q,a}(s)$ is defined by (2.3). Combining (4.2) with (3.6)–(3.8), we deduce, for $\alpha \geq a+1$ and $q \in (0,1)$, that

$$\begin{aligned} |\mathfrak{H}(\zeta)-1| &< \frac{q^a(1+q)\Gamma_q(a+1)}{(1-\eta)(\lambda+1)\Gamma_q(a)}\sum_{\ell=1}^{\infty}\left(\frac{q}{1+q}\right)^\ell \\ &= \frac{q^{a+1}(1+q)(1-q^a)}{(1-\eta)(\lambda+1)(1-q)} \\ &\leq 1, \end{aligned} \tag{4.3}$$

where the last inequality follows from (4.1). The proof of Theorem 4.1 is thus completed. □

As a particular case of Theorem 4.1, by taking $\eta = 0$, we obtain the following result.

Corollary 4.2. *Let $a, \lambda > 0$, $\alpha \geq a+1$, and $q \in (0,1)$ such that the inequality (2.2) holds. Assume that*

$$\frac{q^{a+1}(1+q)(1-q^a)}{(\lambda+1)(1-q)} \leq 1.$$

Then

$$\Re\left(\frac{\mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta)}{\zeta}\right) > 0.$$

In the following theorem, we establish sufficient conditions under which a function $f \in \mathcal{R}(\beta)$, $(0 \leq \beta < 1)$, ensures that the convolution operator associated with $\mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta)$ belongs to the class $\mathcal{R}(\sigma)$, where $\sigma \in [-1,1]$ is a constant depending on η and κ.

Theorem 4.3. *Let $a, \lambda > 0$, $\alpha \geq a+1$, $\eta \in [0,1)$, and $q \in (0,1)$ satisfy the conditions* (2.2) *and* (4.1). *If $\mathfrak{F} \in \mathcal{R}(\kappa)$ for some fixed $\kappa \in [0,1)$, then*

$$\mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta) * \mathfrak{F}(\zeta) \in \mathcal{R}(\sigma),$$

where

$$\sigma = \sigma(\eta,\kappa) = 1 - 2(1-\kappa)(1-\eta), \qquad -1 \leq \sigma < 1.$$

Proof. Let $\mathfrak{F} \in \mathcal{A}(\mathbb{D})$ be of the form

$$\mathfrak{F}(\zeta) = \zeta + \sum_{\ell=2}^{\infty} \alpha_\ell \zeta^\ell.$$

Then $\mathfrak{F} \in \mathcal{R}(\kappa)$ if and only if

$$\Re\left\{1 + \frac{1}{2(1-\kappa)} \sum_{\ell=2}^{\infty} \ell \alpha_\ell \zeta^{\ell-1}\right\} > \frac{1}{2}, \quad \zeta \in \mathbb{D}.$$

By the principle of differential subordination, it suffices to show that

$$\rho(\zeta) \prec \frac{1}{1-\zeta}, \quad \text{where} \quad \rho(\zeta) := 1 + \frac{1}{2(1-\kappa)} \sum_{\ell=2}^{\infty} \ell \alpha_\ell \zeta^{\ell-1}.$$

Then ρ is analytic in $\mathbb{D}$, $\rho(0) = 1$, and

$$\Re\rho(\zeta) > \frac{1}{2}, \qquad \zeta \in \mathbb{D}.$$

Furthermore, in view of Theorem 4.1, we can further write the following property

$$\frac{\mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta)}{\zeta} \in \mathcal{P}(\eta)$$

in the equivalent form

$$\Re\left\{1 + \frac{1}{2(1-\eta)} \sum_{\ell=1}^{\infty} \frac{(q^a;q)_\ell \Gamma(\lambda+1)\Gamma(\ell+1)\, q^{\alpha\ell^2}}{(q;q)_\ell \Gamma(\ell+\lambda+1)} \zeta^\ell\right\} > \frac{1}{2},$$

for all $\zeta \in \mathbb{D}$. Therefore, we get

$$\mathfrak{C}_q^{(\alpha,\lambda,\eta)}(q^a;\zeta) \prec \frac{1}{1-\zeta}, \quad \zeta \in \mathbb{D},$$

where

$$\mathfrak{C}_q^{(\alpha,\lambda,\eta)}(q^a;\zeta) := 1 + \frac{1}{2(1-\eta)} \sum_{\ell=1}^{\infty} \frac{(q^a;q)_\ell \Gamma(\lambda+1)\Gamma(\ell+1)\, q^{\alpha\ell^2}}{(q;q)_\ell \Gamma(\ell+\lambda+1)} \zeta^\ell.$$

Since $\rho(0) = 1$ and $\Re\rho(\zeta) > 1/2$ in $\mathbb{D}$, all the hypotheses of Lemma 2.6 are satisfied. Applying this lemma to ρ and $\mathfrak{C}_q^{(\alpha,\lambda,\eta)}$ yields

$$\rho * \mathfrak{C}_q^{(\alpha,\lambda,\eta)}(q^a;\zeta) \in \operatorname{co}\mathfrak{C}_q^{(\alpha,\lambda,\eta)}(q^a;\mathbb{D}).$$

Since

$$\mathfrak{C}_q^{(\alpha,\lambda,\eta)}(q^a;\mathbb{D}) \subset \left\{w : \Re w > \frac{1}{2}\right\},$$

and this half-plane is convex, it follows that

$$\Re\left\{\rho * \mathfrak{C}_q^{(\alpha,\lambda,\eta)}(q^a;\zeta)\right\} > \frac{1}{2}, \qquad \zeta \in \mathbb{D}.$$

By the definition of convolution, we have

$$\begin{aligned}
&\rho(\zeta) * \mathfrak{C}_q^{(\alpha,\lambda,\eta)}(q^a;\zeta)\\
&= 1 + \frac{1}{4(1-\eta)(1-\kappa)}\sum_{\ell=2}^{\infty}\frac{\ell\alpha_\ell(q^a;q)_{\ell-1}\Gamma(\lambda+1)\Gamma(\ell)q^{\alpha(\ell-1)^2}}{(q;q)_{\ell-1}\Gamma(\ell+\lambda)}\zeta^{\ell-1}\\
&\prec \frac{1}{1-\zeta}, \qquad (\zeta \in \mathbb{D}).
\end{aligned}$$

Applying Lemma 2.6 to the functions $\rho(\zeta)$ and $\mathfrak{C}_q^{(\alpha,\lambda,\eta)}(q^a;\zeta)$ defined above, we get

$$\begin{aligned}
\mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta) * \rho(\zeta) &= 1 + \frac{1}{4(1-\eta)(1-\kappa)}\sum_{\ell=2}^{\infty}\frac{\ell(q^a;q)_{\ell-1}\Gamma(\lambda+1)\Gamma(\ell)\,q^{\alpha(\ell-1)^2}\alpha_\ell}{(q;q)_{\ell-1}\Gamma(\ell+\lambda)}\zeta^{\ell-1}\\
&\prec \frac{1}{1-\zeta}, \qquad (\zeta \in \mathbb{D}).
\end{aligned}$$

Since $1 - \sigma = 2(1-\kappa)(1-\eta)$, we get

$$\frac{1}{2(1-\sigma)} = \frac{1}{4(1-\eta)(1-\kappa)}.$$

Furthermore,

$$\mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta) * \mathfrak{F}(\zeta) = \zeta + \sum_{\ell=2}^{\infty}\frac{(q^a;q)_{\ell-1}\Gamma(\lambda+1)\Gamma(\ell)q^{\alpha(\ell-1)^2}}{(q;q)_{\ell-1}\Gamma(\ell+\lambda)}\alpha_\ell\zeta^\ell.$$

Therefore, we have

$$\Re\left\{1 + \frac{1}{2(1-\sigma)}\left[(\mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta) * \mathfrak{F}(\zeta))' - 1\right]\right\} > \frac{1}{2}.$$

Equivalently,

$$\Re\left[(\mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta) * \mathfrak{F}(\zeta))'\right] > \sigma, \qquad \zeta \in \mathbb{D}.$$

Thus

$$\mathfrak{A}_q^{(\alpha,\lambda)}(q^a;\zeta) * \mathfrak{F}(\zeta) \in \mathcal{R}(\sigma),$$

where $\sigma = 1 - 2(1-\kappa)(1-\eta)$. Since $\eta, \kappa \in [0,1)$, we have $-1 \le \sigma < 1$. This completes the proof. □

5. Discussion of the parameter conditions

It is worth emphasizing the role and nature of the parameter restrictions imposed in the main results. The conditions considered in this paper arise naturally from the coefficient estimates used throughout the proofs and are introduced to obtain explicit and readily verifiable sufficient conditions for the geometric properties under consideration. In particular, the condition

$$a \log q \leq \log(1-q)$$

is used in accordance with Lemma 2.3 (see [26]), which guarantees that the auxiliary function

$$\mathfrak{f}_{q,a}(s) = q^{as}\Gamma_q(s+a), \qquad s \geq 0,$$

is decreasing on $[0, \infty)$. This monotonicity plays a key role in deriving the coefficient estimates (3.6), which are subsequently employed in the proofs of the main theorems. On the other hand, the assumption $\alpha \geq a+1$ is motivated by the factor

$$q^{\alpha\ell^2 - a\ell}$$

appearing in the coefficient estimates. Since $0 < q < 1$ and $\ell \geq 1$, this assumption yields

$$\alpha\ell^2 - a\ell \geq (a+1)\ell^2 - a\ell = \ell^2 + a\ell(\ell-1) \geq \ell^2,$$

and consequently

$$q^{\alpha\ell^2 - a\ell} \leq q^{\ell^2} \leq q^{\ell}.$$

Together with (3.6)–(3.8), this provides the decay needed to dominate the relevant coefficient sums by convergent geometric series involving

$$\sum_{\ell \geq 1} \left(\frac{q}{1+q}\right)^{\ell}, \qquad 0 < \frac{q}{1+q} < 1.$$

Thus, both restrictions have a clear mathematical role in the present approach: the first provides the required monotonicity, while the second ensures sufficient decay of the coefficients. We stress, however, that the resulting parameter ranges are intended as sufficient conditions and are not claimed to be sharp or optimal in general. The successive majorizations used in our estimates may leave room for improvement, and sharper bounds for the q-gamma function, the auxiliary function $\mathfrak{f}_{q,a}$, or the coefficients of $\mathfrak{U}_q^{(\alpha,\lambda)}(q^a;\zeta)$ could potentially lead to weaker restrictions.

6. Conclusions

In this paper, we introduced a new normalized fractional integral operator associated with a class of Ramanujan-type entire functions by employing the Riemann-Liouville fractional integral. The study was carried out within the framework of geometric function theory, with particular emphasis on starlikeness, convexity, and related analytic properties.

In the first part, we established sufficient conditions ensuring that the operator $\mathfrak{U}_q^{(\alpha,\lambda)}(q^a;z)$ belongs to the classes of starlike and convex functions of order δ. Moreover, we obtained necessary and sufficient

conditions for the associated function $\mathfrak{B}_q^{(\alpha,\lambda)}(q^a; z)$ to belong to the same classes. Several corollaries and special cases were derived, providing more explicit criteria and illustrating the applicability of the main results. In addition, Hardy space inclusion results were obtained, further highlighting the analytic significance of the introduced operator.

In particular, the Hardy space inclusion results complement the obtained geometric properties by providing information on the growth and analytic behavior of the introduced operator. They establish a connection between the geometric properties of the introduced functions and their membership in appropriate Hardy spaces.

In the second part, we investigated inclusion properties related to the class $\mathcal{P}(\eta)$ and examined convolution properties with functions belonging to the class $\mathcal{R}(\kappa)$. These results demonstrate that the operator preserves important geometric properties under suitable conditions and provides a natural extension of existing results involving fractional integral operators and special functions. Moreover, the convolution results show that the geometric properties established for the introduced operator can be combined with those of other classes of analytic functions through the convolution operation, thereby providing an additional structural interpretation of the obtained results.

Overall, the results obtained in this work reveal that fractional integral operators associated with Ramanujan-type entire functions constitute a rich and flexible framework for generating new subclasses of analytic functions with desirable geometric properties.

Several natural questions remain open for future investigation. In particular, it would be interesting to determine sharp coefficient estimates for the normalized operator and to investigate whether the sufficient parameter conditions obtained in this paper can be improved to optimal ones. Another direction is to study the limiting behavior of the introduced operator and its geometric properties as $q \to 1^-$, which may lead to corresponding results for related classical special functions. It would also be worthwhile to extend the present approach to fractional integral operators associated with other special function kernels, such as q-Bessel, q-Mittag-Leffler and hypergeometric-type functions.

Author contributions

Khaled Mehrez and Abdulaziz Alenazi: Writing-original draft; writing-review & editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at Northern Border University, Arar, KSA for funding this research work through the project number "NBU-FPEJ-2026-970-03".

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. M. Acu, G. Oros, A. M. Rus, Fractional integral of the confluent hypergeometric function related to fuzzy differential subordination theory, *Fractal Fract.*, **6** (2022), 413. https://doi.org/10.3390/fractalfract6080413
2. H. M. Srivastava, M. Saigo, S. Owa, A class of distortion theorems involving certain operators of fractional calculus, *J. Math. Anal. Appl.*, **131** (1988), 412–420. https://doi.org/10.1016/0022-247X(88)90215-6
3. A. A. Lupas, G. I. Oros, Differential sandwich theorems involving Riemann–Liouville fractional integral of q-hypergeometric function, *AIMS Math.*, **8** (2023), 4930–4943. https://doi.org/10.3934/math.2023246
4. P. L. Duren, *Univalent functions*, New York: Springer, 1983.
5. İ. Aktas, On some geometric properties and Hardy class of q-Bessel functions, *AIMS Math.*, **5** (2020), 3156–3168. https://doi.org/10.3934/math.2020203
6. A. Alenazi, K. Mehrez, Geometric characterizations of some analytic functions related to the generalized Marcum Q-function, *Quaest. Math.*, **48** (2025), 1823–1844. https://doi.org/10.2989/16073606.2025.2549939
7. D. Bansal, J. K. Prajapat, Certain geometric properties of the Mittag-Leffler functions, *Complex Var. Elliptic*, **61** (2016), 338–350. https://doi.org/10.1080/17476933.2015.1079628
8. Á. Baricz, *Generalized Bessel functions of the first kind*, Berlin: Springer, 2010.
9. S. Noreen, M. Raza, M. U. Din, S. Hussain, On certain geometric properties of normalized Mittag-Leffler functions, *UPB Sci. Bull. Ser. A Appl. Math. Phys.*, **81** (2019), 167–174.
10. F. D. M. Haldane, Fractional statistics in arbitrary dimensions: A generalization of the Pauli principle, *Phys. Rev. Lett.*, **67** (1991), 937–940. https://doi.org/10.1103/PhysRevLett.67.937
11. R. Kedem, T. R. Klassen, B. M. McCoy, E. Melzer, Fermionic quasiparticle representations for conformal field theory characters, *Phys. Lett. B*, **304** (1993), 263–270. https://doi.org/10.1016/0370-2693(93)90292-P
12. G. E. Andrews, *q-Series: Their development and application in analysis, number theory, combinatorics, physics and computer algebra*, American Mathematical Soc., **66** (1986).
13. S. Ramanujan, *The lost notebook and other unpublished papers*, Springer, 1998.
14. M. E. H. Ismail, R. Zhang, q-Bessel functions and Rogers–Ramanujan type identities, *P. Am. Math. Soc.*, **146** (2018), 3633–3646. https://doi.org/10.1090/proc/13078
15. Y. Özkan, K. Mehrez, E. Deniz, Some Turán-type inequalities for the Ramanujan-type entire function, *Ramanujan J.*, **67** (2025), 17. https://doi.org/10.1007/s11139-025-01056-z

16. B. Khan, T. G. Shaba, S. Araci, B. O. Adebesin, F. Usta, A q-Rogers–Ramanujan-based characterization of q-bi-bounded turning functions, *Demonstr. Math.*, **59** (2026), 20250225. https://doi.org/10.1515/dema-2025-0225
17. H. M. Srivastava, Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis, *Iran. J. Sci. Technol. A*, **44** (2020), 327–344. https://doi.org/10.1007/s40995-019-00815-0
18. S. S. Miller, P. T. Mocanu, Differential subordinations and univalent functions, *Mich. Math. J.*, **28** (1981), 157–172. https://doi.org/10.1307/mmj/1029002507
19. S. Owa, On the distortion theorems I, *Kyungpook Math. J.*, **18** (1978), 53–59.
20. S. O. Owa, H. M. Srivastava, Univalent and starlike generalized hypergeometric functions, *Can. J. Math.*, **39** (1987), 1057–1077.
21. G. I. Oros, G. Oros, S. Owa, Subordination properties of certain operators concerning fractional integral and Libera integral operator, *Fractal Fract.*, **7** (2023), 42. https://doi.org/10.3390/fractalfract7010042
22. F. Ghanim, S. Bendak, A. Al Hawarneh, Certain implementations in fractional calculus operators involving Mittag-Leffler-confluent hypergeometric functions, *P. Roy. Soc. A-Math. Phy.*, **478** (2022), 20210839. https://doi.org/10.1098/rspa.2021.0839
23. G. I. Oros, G. Oros, D. A. Bardac-Vlada, Certain geometric properties of the fractional integral of the Bessel function of the first kind, *AIMS Math.*, **9** (2024), 7095–7110. https://doi.org/10.3934/math.2024346
24. A. Alameer, Geometric properties and applications of a generalized Bessel–Maitland–Tremblay function, *Fractal Fract.*, **10** (2026), 582. https://doi.org/10.3390/fractalfract10080582
25. H. Silverman, Univalent functions with negative coefficients, *P. Am. Math. Soc.*, **51** (1975), 109–116. https://doi.org/10.1090/S0002-9939-1975-0367276-1
26. K. Mehrez , A. Alenazi, The geometric characterizations of the Ramanujan-type entire function, *Mathematics*, **13** (2025), 2301. https://doi.org/10.3390/math13142301
27. P. J. Eenigenburg, F. R. Keogh, The Hardy class of some univalent functions and their derivatives, *Mich. Math. J.*, **17** (1970), 335–346.
28. S. Ruscheweyh, *Convolutions in geometric function theory*, Les Presses de l'Université de Montréal, 1982.
29. F. W. J. Olver, *NIST handbook of mathematical functions hardback and CD-ROM*, Cambridge Univ. Press, 2010. https://doi.org/10.1017/CBO9780511920944



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)