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Research article

E_m-metrics: Defect profile representation, quotient geometry, and canonical topologies

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Abstract: We study a Riesz-space-valued analogue of an M-metric, called an E_m-metric. Such a distance splits into an E-valued defect pseudometric and a self-distance profile. From this decomposition, we obtain an associated E-metric, a separated defect quotient, and two generally different notions of convergence and completeness. When the range space has an order unit, the defect topology is pseudometrizable and may fail to be T_0, and the associated topology is metrizable and Hausdorff. We represent the associated geometry as an isometric subspace of the product of the separated defect quotient and the positive cone. This representation yields a closed-image criterion for associated completeness. Under an order unit, equality of the two canonical topologies is characterized by continuity of the profile. Finally, a coupled positive Fredholm system illustrates how the componentwise Picard estimates distinguish the full-state error from the mean-imbalance error.

Keywords: M-metric; Riesz space; vector metric; order convergence; order unit; associated metric; Fredholm integral system
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1. Introduction

Partial metrics, introduced by Matthews and developed further by O'Neill, allow nonzero self-distances [1, 2]. Asadi, Karapınar, and Salimi subsequently defined scalar M-metrics by removing the small self-distance axiom [3]. Further work established fixed-point results for Meir–Keeler type mappings [4]. For multivalued mappings, Nadler, Kannan, and Feng-Liu type conditions were considered [5, 6], together with F-contractions [7]. Later studies addressed Θ- and P-contractive mappings [8, 9]. Rectangular and quasi variants were studied in [10, 11]. Allowing self-distances to take values in an ordered vector space makes it possible to represent several components of self-distance without reducing them to a single scalar.

A related line of research concerns vector-valued distances. Cone metrics take values in ordered

Banach spaces [12, 13], partial cone metrics combine cone-valued distances with a small self-distance axiom [14], and the vector metrics of Çevik and Altun take values in a Riesz space [15, 16]. A scalar self-distance records one numerical characteristic of a point, whereas a Riesz-space-valued self-distance can retain several characteristics and compare them coordinatewise. This feature is illustrated in Section 8. Vector-valued pseudometrics were further studied in [17], and metrics taking values in ordered vector spaces were considered in [18].

Further developments of the E-metric framework addressed vectorial Cauchy completion without an order-boundedness assumption [19] and coupled fixed-point problems on ordered vector metric spaces [20]. These studies demonstrate the role of the order structure in convergence and fixed-point arguments. Here, we consider distances with nonzero Riesz-space-valued self-distances and study their defect and profile components separately.

Related fixed-point results have also been obtained in orthogonal metric spaces using nonlinear φ-contractions [21]. In that setting, an auxiliary orthogonality relation determines the sequences considered in completeness and iteration arguments. This provides further context for contraction principles involving additional structure; the present paper studies the structure supplied by an E-valued defect and a self-distance profile.

Quantale-valued metrics provide a more general categorical setting in which distances take values in a commutative quantale [22]. The present construction instead uses the Riesz-space operations $\wedge$, $\vee$, and the lattice modulus in the decomposition of the distance. Partial cone metrics, by contrast, impose a small self-distance axiom that is not assumed here. A direct comparison also depends on the order structure of the range space. A closely related notion is that of M-cone metrics over Banach algebras [23, 24]. Their algebraic assumptions and cone order differ from the lattice structure used here; see Remark 3.3 for a structural comparison. Here, the defect-profile pair determines both the quotient geometry and the relation between the defect and associated topologies; see Theorems 4.4 and 6.2. Standard terminology for Riesz spaces follows [25, 26].

The defect-profile decomposition makes explicit the information carried by the defect pseudometric and by the self-distance profile. The associated metric extends the scalar metric m^w recalled in Section 2. Theorem 4.3 is an elementary reformulation of the defining axioms, and Theorem 4.4 combines the standard separated pseudometric quotient with a product metric. These constructions provide the structural basis for the completeness and topology results that follow.

The principal conclusions concern the interaction between the two components. First, defect convergence, the defect-Cauchy property, and defect completeness correspond to their counterparts on the separated defect quotient. Under completeness of both ambient factors, associated completeness is characterized by sequential E-closedness of the quotient-profile image (Theorem 5.6). Second, when E has an order unit, the associated topology is finer than the defect topology, and equality holds exactly when the profile is continuous for the defect topology (Theorem 6.2).

In the scalar setting, the associated metric therefore recovers a known construction. The comparison with Assaf's topology distinguishes the defect, directed-profile, and absolute-profile information. For the compatible targets described in Remark 3.3, the distance axioms may also agree with those of an M-cone metric. The contribution here is the joint analysis of quotient geometry, sequential order completeness, and canonical order-unit topologies through the defect-profile pair.

The topological results apply to Archimedean Riesz spaces with an order unit, including finite-dimensional Riesz spaces and $C(K)$ for compact Hausdorff K. The relationship between order

convergence and order-unit norm convergence is explained at the start of Section 6.

Finally, the associated E-metric permits a direct application of the existing vector-metric contraction principle. The componentwise hypothesis additionally retains separate estimates for the defect and profile terms. The coupled positive Fredholm system illustrates these estimates through a full-state error bound and an exact mean-imbalance error identity.

2. Preliminaries

We begin with the definitions and notation used below. A real vector space E endowed with a partial order $\preceq$ is called an ordered vector space if the order is compatible with the vector-space operations; that is, for all $x, y, z \in E$ and every $\alpha \geq 0$,

$$x \preceq y \Longrightarrow x + z \preceq y + z, \quad \text{and} \quad \alpha x \preceq \alpha y.$$

An ordered vector space E is a Riesz space (or a vector lattice) if every pair $x, y \in E$ possesses a supremum $x \vee y$ and an infimum $x \wedge y$. The positive cone of E is denoted by $E_+ := \{x \in E : 0 \preceq x\}$. For any $x \in E$, its lattice modulus is defined by $|x| := x \vee (-x)$.

A sequence (x_n) in E is said to order converge to $x \in E$, denoted by $x_n \xrightarrow{o} x$, if there exists a sequence (u_n) in E such that $u_n \downarrow 0$, and $|x_n - x| \preceq u_n$ for all $n \in \mathbb{N}$. Here, $u_n \downarrow 0$ means that $u_{n+1} \preceq u_n$ for each n and $\inf_{n\geq 1} u_n = 0$. For a sequence in E_+, $x_n \xrightarrow{o} 0$ simply asserts that (x_n) is dominated by a sequence decreasing to zero.

We shall repeatedly use a simple consequence of this definition. If $u_n \downarrow 0$, and $0 \preceq v \preceq u_n$ for every n, then $v = 0$, as 0 is the greatest lower bound of (u_n). Similarly, $0 \preceq v_n \preceq u_n$ implies $v_n \xrightarrow{o} 0$. No Dedekind completeness is needed here: The required infimum is already contained in the notation $u_n \downarrow 0$.

Recall that E is an Archimedean Riesz space if $n^{-1}x \downarrow 0$ for every $x \in E_+$. An element $e \in E_+$ is an order unit if for every $x \in E$ there exists some $\lambda > 0$ such that $|x| \preceq \lambda e$. For a comprehensive treatment of Riesz space theory, we refer to [25] and [26]. Throughout this work, E denotes an Archimedean real Riesz space.

Definition 2.1 ([15]). *Let X be a nonempty set and E a Riesz space. A mapping $d\colon X \times X \to E_+$ is called an E-metric if the following properties hold for all $x, y, z \in X$:*

(1) $d(x, y) = 0 \iff x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, y) \preceq d(x, z) + d(z, y)$.

The triple (X, d, E) is then called an E-metric space.

In an E-metric space (X, d, E), a sequence (x_n) in X is said to E-converge to $x \in X$ if $d(x_n, x) \xrightarrow{o} 0$. It is called an E-Cauchy sequence if there exists a sequence (u_n) in E with $u_n \downarrow 0$ such that $d(x_n, x_{n+p}) \preceq u_n$ for all $n, p \in \mathbb{N}$. The space (X, d, E) is E-complete if every E-Cauchy sequence E-converges to a point in X. A subset Y of X is called E-closed if every sequence in Y that E-converges in X has its limit in Y.

We next recall the scalar M-metric structure that supplies the nonzero self-distance component of the present framework. Together with the Riesz-space-valued setting of an E-metric, it motivates the definition of an E_m-metric given in the next section.

Given a nonempty set X, let $m\colon X \times X \to [0, \infty)$ be a mapping. For $x, y \in X$, define

$$m_{x,y} := \min\{m(x, x), m(y, y)\} \qquad \text{and} \qquad M_{x,y} := \max\{m(x, x), m(y, y)\}.$$

Definition 2.2 ([3]). *The mapping m is an M-metric if the following conditions hold for all $x, y, z \in X$:*

(1) $m(x, x) = m(y, y) = m(x, y) \iff x = y$;
(2) $m_{x,y} \leq m(x, y)$;
(3) $m(x, y) = m(y, x)$;
(4) $m(x, y) - m_{x,y} \leq (m(x, z) - m_{x,z}) + (m(z, y) - m_{z,y})$.

Every scalar M-metric induces the standard metric

$$m^w(x, y) := m(x, y) - 2m_{x,y} + M_{x,y} = (m(x, y) - m_{x,y}) + |m(x, x) - m(y, y)|.$$

This identity motivates the associated E-metric introduced below.

3. E_m-metric spaces and examples

Throughout this paper, X denotes a nonempty set. For a mapping $\xi\colon X \times X \to E_+$, we introduce the following notation:

$$\xi^{\wedge}_{x,y} := \xi(x, x) \wedge \xi(y, y) \qquad \text{and} \qquad \xi^{\vee}_{x,y} := \xi(x, x) \vee \xi(y, y),$$

where $\wedge$ and $\vee$ denote the lattice infimum and supremum in E, respectively.

A lattice order, unlike the usual order on $\mathbb{R}$, need not be total. Two self-distances may therefore be incomparable even though their lattice infimum and supremum still exist. In coordinatewise ordered $\mathbb{R}^2$, for instance, $u = (0, 1)$ and $v = (1, 0)$ are incomparable, although $u \wedge v = (0, 0)$ and $u \vee v = (1, 1)$ are well-defined. Accordingly, the lattice infimum plays the role of the scalar minimum in the present setting.

Definition 3.1. *A mapping $\xi\colon X \times X \to E_+$ is called an E_m-metric if the following conditions hold for all $x, y, z \in X$:*

EM1. $\xi(x, x) = \xi(y, y) = \xi(x, y) \iff x = y$;
EM2. $\xi^{\wedge}_{x,y} \preceq \xi(x, y)$;
EM3. $\xi(x, y) = \xi(y, x)$;
EM4. $\xi(x, y) - \xi^{\wedge}_{x,y} \preceq \xi(x, z) - \xi^{\wedge}_{x,z} + \xi(z, y) - \xi^{\wedge}_{z,y}$.
Then, the pair (X, ξ) is called an E_m-metric space.

Remark 3.2. *(1) If $E = \mathbb{R}$ with usual order, then $\xi(x, x) \wedge \xi(y, y) = \min\{\xi(x, x), \xi(y, y)\}$. Thus, Definition 3.1 reduces to the classical notion of an M-metric.*
(2) If $\xi(x, x) = 0$ for all $x \in X$, then $\xi^{\wedge}_{x,y} = 0$ for all $x, y \in X$, and ξ reduces to an E-metric.
(3) Condition (EM2) implies neither $\xi(x, x) \preceq \xi(x, y)$ nor $\xi(y, y) \preceq \xi(x, y)$. It merely guarantees that the lattice infimum of the self-distances is bounded above by $\xi(x, y)$.

Remark 3.3. *The relation with M-cone metrics over Banach algebras is confined to compatible targets. If the cone order is also a lattice order, and the minimum in the M-cone definition is interpreted*

as the lattice infimum, its distance axioms agree with (EM1)–(EM4). *In general, however, a cone order need not admit pairwise infima or the lattice modulus used below, whereas the present framework assumes neither a norm nor multiplication. Thus, neither theory contains the other without additional compatibility assumptions.*

Example 3.4. *Standard constructions include the ray-valued metric $\xi(x,y) = m(x,y)e$, where m is an M-metric, $0 \neq e \in E_+$, and the product $\xi = (m_1, \dots, m_k)$ of finitely many M-metrics under the coordinatewise order. Likewise, let $(m_t)_{t\in[0,1]}$ be a family of scalar M-metrics on X such that $m_t(x,y)$ depends continuously on t for every $x, y \in X$. Then, $\xi(x,y)(t) := m_t(x,y)$ defines a $C([0,1])$-valued E_m-metric on X.*

Separation may depend on the coordinates jointly. The following example has no separating scalar coordinate, although the vector-valued mapping separates points.

Example 3.5. *Let $X = \{a, b, c\}$, and let $E = \mathbb{R}^2$ be equipped with the coordinatewise order. Define $f, g\colon X \to \mathbb{R}$ by*

$$f(a) = 0,\ f(b) = f(c) = 1, \qquad \text{and} \qquad g(a) = g(b) = 0,\ g(c) = 1.$$

Now, define $\xi\colon X \times X \to E_+$ by

$$\xi(x,y) := (1 + |f(x) - f(y)|,\ 1 + |g(x) - g(y)|)\,.$$

For all $x, y \in X$, we have $\xi(x,x) = (1,1)$ and

$$\xi(x,y) - \xi^{\wedge}_{x,y} = (|f(x) - f(y)|,\ |g(x) - g(y)|).$$

Thus, (EM2)–(EM4) follow from positivity, symmetry, and the coordinatewise triangle inequality. The map $x \mapsto (f(x), g(x))$ is injective, which gives (EM1).

Neither scalar coordinate is an M-metric: The first identifies b and c, and the second identifies a and b. Hence, joint vector separation need not come from separating scalar M-metrics in each coordinate. In particular, ray-valued E_m-metrics form a proper subclass of the present framework.

We shall use without further mention the lattice identities $u\vee v+u\wedge v = u+v$ and $u\vee v-u\wedge v = |u-v|$, together with the lattice triangle inequality.

4. Defect profile representation and associated structures

This section describes an E_m-metric in terms of its defect and self-distance profile. We first establish the corresponding representation and then examine the separated defect quotient and its product embedding.

4.1. The defect profile representation

We begin by introducing the defect mapping and the self-distance profile associated with an E_m-metric.

Definition 4.1. *A mapping $\delta\colon X \times X \to E_+$ is called an E-valued pseudometric if, for all $x, y, z \in X$,*

(1) $\delta(x, x) = 0$*;*
(2) $\delta(x, y) = \delta(y, x)$*;*
(3) $\delta(x, y) \preceq \delta(x, z) + \delta(z, y)$.

The pair (X, δ) *is then called an* E*-valued pseudometric space.*

Unlike an E-metric, an E-valued pseudometric need not separate points. Every E-metric is therefore an E-valued pseudometric, but the converse is generally false.

The triangle axiom is expressed in terms of the difference between $\xi(x, y)$ and the lattice infimum of the corresponding self-distances. We call this difference the defect of ξ.

Definition 4.2. *Let* (X, ξ) *be an* E_m*-metric space. The mappings* $\delta_\xi \colon X \times X \to E_+$ *and* $s_\xi \colon X \to E_+$ *defined by*

$$\delta_\xi(x, y) := \xi(x, y) - \xi^\wedge_{x,y} \qquad \text{and} \qquad s_\xi(x) := \xi(x, x)$$

are called the defect mapping of ξ *and its self-distance profile, respectively.*

We call a pair (δ, s) separating if $\delta(x, y) = 0$ and $s(x) = s(y)$ together imply $x = y$.

Theorem 4.3 (Structural representation and associated metric)**.** *Let* ξ *be an* E_m*-metric on* X*. Then,* δ_ξ *is an* E*-valued pseudometric, the pair* (δ_ξ, s_ξ) *is separating, and*

$$\xi(x, y) = \delta_\xi(x, y) + s_\xi(x) \wedge s_\xi(y).$$

Conversely, if δ *is an* E*-valued pseudometric on* X*,* $s \colon X \to E_+$*, and* (δ, s) *is separating, then*

$$\xi(x, y) := \delta(x, y) + s(x) \wedge s(y)$$

defines an E_m*-metric on* X*. The representing pair* (δ, s) *is uniquely determined by* ξ*. Moreover,*

$$\rho_\xi(x, y) := \delta_\xi(x, y) + |s_\xi(x) - s_\xi(y)| = \xi(x, y) - 2\xi^\wedge_{x,y} + \xi^\vee_{x,y}$$

is an E*-metric on* X*.*

Proof. Suppose first that ξ is an E_m-metric. Axiom (EM2) gives $\delta_\xi(x, y) \in E_+$, and $\xi^\wedge_{x,x} = \xi(x, x)$ gives $\delta_\xi(x, x) = 0$. Symmetry follows from (EM3), and (EM4) is precisely the triangle inequality for δ_ξ. If $\delta_\xi(x, y) = 0$, and $s_\xi(x) = s_\xi(y)$, then $\xi(x, y) = \xi(x, x) = \xi(y, y)$, so (EM1) yields $x = y$. Thus, (δ_ξ, s_ξ) is separating, and the displayed decomposition follows directly from the definitions.

Conversely, let (δ, s) be separating, and set $\xi(x, y) := \delta(x, y) + s(x) \wedge s(y)$. Because $\delta(x, x) = 0$, we have $\xi(x, x) = s(x)$, $\xi^\wedge_{x,y} = s(x) \wedge s(y)$, and $\xi(x, y) - \xi^\wedge_{x,y} = \delta(x, y)$. The pseudometric axioms for δ now give (EM2)–(EM4).

To verify (EM1), suppose $\xi(x, x) = \xi(y, y) = \xi(x, y)$. This implies $s(x) = s(y)$, so the identity above gives $\delta(x, y) = 0$. Because (δ, s) is separating, $x = y$; the converse implication is immediate.

The identities $s(x) = \xi(x, x)$ and $\delta(x, y) = \xi(x, y) - \xi^\wedge_{x,y}$ establish the uniqueness of the representation. The lattice identity $|s_\xi(x) - s_\xi(y)| = \xi^\vee_{x,y} - \xi^\wedge_{x,y}$ then gives

$$\begin{aligned} \rho_\xi(x, y) &= \delta_\xi(x, y) + |s_\xi(x) - s_\xi(y)| \\ &= \xi(x, y) - \xi^\wedge_{x,y} + \xi^\vee_{x,y} - \xi^\wedge_{x,y} \end{aligned}$$

$$= \xi(x,y) - 2\xi^{\wedge}_{x,y} + \xi^{\vee}_{x,y}.$$

By the lattice triangle inequality, $|s_\xi(x) - s_\xi(y)|$ defines an E-valued pseudometric. Consequently, ρ_ξ is an E-valued pseudometric. If $\rho_\xi(x,y) = 0$, positivity implies that $\delta_\xi(x,y) = 0$, and $s_\xi(x) = s_\xi(y)$. The separating property then gives $x = y$. Therefore, ρ_ξ is an E-metric. □

The theorem shows that an E_m-metric is completely determined by its defect pseudometric and self-distance profile. Although the defect may identify distinct points, such points must have different self-distances. This distinction will be used in the quotient representation below and later in the comparison of the defect and associated topologies.

We write

$$\omega_\xi(x,y) := |s_\xi(x) - s_\xi(y)| = \xi^{\vee}_{x,y} - \xi^{\wedge}_{x,y}$$

for the profile variation and refer to $\rho_\xi = \delta_\xi + \omega_\xi$ as the associated E-metric.

The profile in this representation need not be constant. For example, on $X = \mathbb{R}_+$, take $\delta(x,y) = (1-\alpha)|x-y|$ and $s(x) = \alpha x$, where $0 \le \alpha \le 1$. Then,

$$\xi(x,y) = (1-\alpha)|x-y| + \alpha \min\{x,y\}.$$

The pair (δ, s) is separating, so Theorem 4.3 applies; at $\alpha = 1$, separation is carried entirely by the profile.

4.2. The quotient-profile embedding

Define a relation as

$$x \sim_\delta y \quad \Longleftrightarrow \quad \delta_\xi(x,y) = 0.$$

The pseudometric axioms show that this is an equivalence relation. Write $X_\delta = X/\!\sim_\delta$ and $q(x) = [x]$. Define

$$\bar{\delta}_\xi([x],[y]) := \delta_\xi(x,y).$$

If $x \sim_\delta x'$, and $y \sim_\delta y'$, the triangle inequality gives $\delta_\xi(x,y) \le \delta_\xi(x',y')$; interchanging the representatives gives the reverse inequality. Thus, $\bar{\delta}_\xi$ is well-defined. It inherits symmetry and the triangle inequality, and

$$\bar{\delta}_\xi([x],[y]) = 0 \quad \Longleftrightarrow \quad [x] = [y].$$

Consequently, $(X_\delta, \bar{\delta}_\xi)$ is an E-metric space.

Theorem 4.4 (Quotient-profile representation)**.** *Equip $X_\delta \times E_+$ with the E-metric*

$$D(([x],u),([y],v)) := \bar{\delta}_\xi([x],[y]) + |u - v|.$$

Then, the function $J\colon X \longrightarrow X_\delta \times E_+$ defined as $J(x) := ([x], s_\xi(x))$ is injective and satisfies $D(Jx, Jy) = \rho_\xi(x,y)$ for all $x, y \in X$. Thus, (X, ρ_ξ) is E-isometric to the subspace $J(X)$.

Proof. The mapping D is an E-metric because it is the sum of the E-metrics on the two factors. If $J(x) = J(y)$, then $\delta_\xi(x,y) = 0$, and $s_\xi(x) = s_\xi(y)$; the separating property of (δ_ξ, s_ξ) gives $x = y$. Finally,

$$D(Jx, Jy) = \delta_\xi(x,y) + |s_\xi(x) - s_\xi(y)| = \rho_\xi(x,y),$$

which proves the asserted isometry. □

5. Convergence and completeness for the defect and associated metrics

The decomposition of ξ leads naturally to two notions of convergence. Defect convergence is defined through the pseudometric δ_ξ, whereas associated convergence also requires the self-distance profile to converge.

We shall use the elementary observation that for sequences (u_n) and (v_n) in E_+, $u_n + v_n \xrightarrow{o} 0$ implies both $u_n \xrightarrow{o} 0$ and $v_n \xrightarrow{o} 0$.

Definition 5.1. *A sequence (x_n) is said to converge to x with respect to the defect pseudometric δ_ξ, or briefly to defect converge to x, written $x_n \xrightarrow{\delta_\xi} x$, if $\delta_\xi(x_n, x) \xrightarrow{o} 0$. It converges to x in the associated sense, written $x_n \xrightarrow{\rho_\xi} x$, if $\rho_\xi(x_n, x) \xrightarrow{o} 0$.*

Because δ_ξ need not separate points, defect limits may be nonunique. Associated convergence also requires convergence of the profile variation and therefore has unique limits.

Proposition 5.2. *Let (x_n) be a sequence in X, and let $x \in X$. Then, $x_n \xrightarrow{\rho_\xi} x$ if and only if $x_n \xrightarrow{\delta_\xi} x$, and $\omega_\xi(x_n, x) \xrightarrow{o} 0$. In particular, associated convergence implies defect convergence. The converse holds when the self-distances also order converge to the self-distance of the proposed limit. Associated limits are unique.*

Proof. The equivalence follows from $\rho_\xi(x_n, x) = \delta_\xi(x_n, x) + \omega_\xi(x_n, x)$ and the elementary order-convergence observation above.

To prove uniqueness, suppose that $x_n \xrightarrow{\rho_\xi} x$ and $x_n \xrightarrow{\rho_\xi} y$. The triangle inequality gives $0 \preceq \rho_\xi(x, y) \preceq \rho_\xi(x, x_n) + \rho_\xi(x_n, y)$. Because the right-hand side order converges to zero, $\rho_\xi(x, y) = 0$. Because ρ_ξ is an E-metric, we obtain that $x = y$. □

Definition 5.3. *A sequence (x_n) in X is called δ_ξ-Cauchy, or briefly defect-Cauchy, if there exists (a_n) in E_+ such that $a_n \downarrow 0$, and $\delta_\xi(x_n, x_{n+p}) \preceq a_n$ for all $n, p \in \mathbb{N}$.*

The sequence is called associated Cauchy, or equivalently ρ_ξ-Cauchy, if the same condition holds with $\rho_\xi(x_n, x_{n+p}) \preceq c_n$ for some $c_n \downarrow 0$.

Proposition 5.4. *Every associated Cauchy sequence is defect-Cauchy. Conversely, a defect-Cauchy sequence (x_n) is associated Cauchy if and only if its self-distance sequence $(s_\xi(x_n))$ is order-Cauchy that is, if and only if there exists a sequence (b_n) in E_+ such that $b_n \downarrow 0$ and $\omega_\xi(x_n, x_{n+p}) \preceq b_n$ for all $n, p \in \mathbb{N}$.*

Proof. Because $0 \preceq \delta_\xi(x, y) \preceq \rho_\xi(x, y)$, every associated Cauchy sequence is defect-Cauchy.

Now, suppose that (x_n) is defect-Cauchy and that $(s_\xi(x_n))$ is order-Cauchy. For all n and p, $\delta_\xi(x_n, x_{n+p}) \preceq a_n$, and $\omega_\xi(x_n, x_{n+p}) \preceq b_n$. Therefore, $\rho_\xi(x_n, x_{n+p}) \preceq a_n + b_n$, and $a_n + b_n \downarrow 0$. Hence, (x_n) is associated Cauchy.

Conversely, $0 \preceq \omega_\xi(x_n, x_{n+p}) \preceq \rho_\xi(x_n, x_{n+p})$, so every associated Cauchy sequence has an order-Cauchy self-distance sequence. □

Definition 5.5. *The space (X, ξ) is called δ_ξ-complete, or defect complete, if every defect-Cauchy sequence defect converges to at least one point of X. It is called associated complete if every associated Cauchy sequence converges in the associated sense to a point of X.*

Equivalently, (X, ξ) is associated complete if and only if the associated E-metric space (X, ρ_ξ, E) is E-complete.

Associated Cauchy sequences form a smaller class than defect-Cauchy sequences, but associated completeness requires convergence in the stronger associated sense. Hence, neither completeness notion follows directly from the other without an additional compatibility condition on the profile.

The closed-image criterion below requires completeness of both ambient factors. Completeness of the separated defect quotient controls only the defect coordinate. To see why this alone is insufficient, take $X = E_+$, $\delta = 0$, and $s(x) = x$. Then, $\xi(x, y) = x \wedge y$, the defect quotient is a singleton, and $J(X)$ is the entire ambient product. Nevertheless, $\rho_\xi(x, y) = |x - y|$, whose E-completeness is precisely the E-completeness of $(E_+, |\cdot|)$. Remark 5.8 gives a target for which this additional completeness fails.

Theorem 5.6. *Let $q\colon X \to X_\delta$, given by $q(x) = [x]$, be the canonical quotient mapping. Then, the following statements hold.*

(1) A sequence (x_n) defect converges to x if and only if $([x_n])$ E-converges to $[x]$ in $(X_\delta, \overline{\delta}_\xi)$.
(2) A sequence (x_n) is defect-Cauchy if and only if $([x_n])$ is E-Cauchy in $(X_\delta, \overline{\delta}_\xi)$.
(3) The E_m-metric space (X, ξ) is defect complete if and only if $(X_\delta, \overline{\delta}_\xi)$ is E-complete.
(4) Suppose that $(X_\delta, \overline{\delta}_\xi)$ and $(E_+, |\cdot|)$ are E-complete. Then, (X, ξ) is associated complete if and only if $J(X)$ is E-closed in $X_\delta \times E_+$.

Proof. For all $x, y \in X$, $\overline{\delta}_\xi(qx, qy) = \overline{\delta}_\xi([x], [y]) = \delta_\xi(x, y)$. Hence, defect convergence and the defect-Cauchy property correspond exactly to E-convergence and the E-Cauchy property in the quotient. Because q is surjective, this correspondence also proves the equivalence of defect completeness and the E-completeness of $(X_\delta, \overline{\delta}_\xi)$.

Assume now that both $(X_\delta, \bar{\delta}_\xi)$ and $(E_+, |\cdot|)$ are E-complete. Their product, equipped with D, is E-complete: An E-Cauchy sequence is E-Cauchy in each coordinate, and the coordinate limits give an E-limit for D.

By Theorem 4.4, (X, ρ_ξ) is E-isometric to $J(X)$. It remains to justify the equivalence between E-completeness and sequential E-closedness of $J(X)$ in this complete ambient space.

Suppose first that $J(X)$ is E-complete, and let (z_n) be a sequence in $J(X)$ that E-converges in the product to z. Choose $a_n \downarrow 0$ such that $D(z_n, z) \preceq a_n$. Then,

$$D(z_n, z_{n+p}) \preceq D(z_n, z) + D(z, z_{n+p}) \preceq 2a_n.$$

Thus, (z_n) is E-Cauchy in $J(X)$ and has an E-limit $w \in J(X)$. If $D(z_n, w) \preceq b_n$ with $b_n \downarrow 0$, then

$$0 \preceq D(z, w) \preceq a_n + b_n \qquad (n \geq 1).$$

Because $a_n + b_n \downarrow 0$, we obtain $z = w \in J(X)$. Hence, $J(X)$ is sequentially E-closed.

Conversely, if $J(X)$ is sequentially E-closed, every E-Cauchy sequence in $J(X)$ has an E-limit in the complete ambient product, and that limit belongs to $J(X)$. Thus, $J(X)$ is E-complete. This proves the equivalence.

Here, E-closedness refers throughout to sequential order convergence. □

Corollary 5.7. *Suppose that E is finite dimensional. If $(X_\delta, \overline{\delta}_\xi)$ is E-complete, then (X, ρ_ξ) is E-complete if and only if $J(X)$ is E-closed in $X_\delta \times E_+$.*

Proof. By the finite-dimensional representation of Archimedean Riesz spaces [25, Theorem 26.11], E is lattice isomorphic to $\mathbb{R}^d$ with the coordinatewise order. Under this isomorphism, an E-Cauchy sequence in E_+ is Cauchy in each real coordinate and has a coordinatewise limit in $\mathbb{R}^d_+$. In finite dimensions, coordinatewise convergence is sequential order convergence. Thus, $(E_+, |\cdot|)$ is E-complete, and the conclusion follows from Theorem 5.6. □

Remark 5.8. *The completeness assumption on $(E_+, |\cdot|)$ in Theorem 5.6 cannot be omitted in general, as it does not follow from the Archimedean property.*

For example, let E be the vector lattice of continuous piecewise linear functions on $[0, 1]$, ordered pointwise. This space is Archimedean and has the order unit $\mathbf{1}$. Let (f_n) be a sequence of non-negative piecewise linear interpolants converging uniformly to $f(t) = t^2$. Then, (f_n) is E-Cauchy in $(E_+, |\cdot|)$, as its tails can be bounded by scalar multiples of $\mathbf{1}$ decreasing to zero.

The sequence cannot have an E-limit in E. Indeed, suppose that f_n E-converges to a piecewise linear function g. Because f_n converges uniformly to $f(t) = t^2$, we have $f \neq g$. Hence, there exist a closed interval $I \subseteq [0, 1]$ and a constant $\varepsilon > 0$ such that $|f - g| \geq 2\varepsilon$ on I. By uniform convergence, there is an $N \in \mathbb{N}$ such that $|f_n - g| \geq \varepsilon$ on I whenever $n \geq N$.

If $|f_n - g| \leq u_n$ for some $u_n \downarrow 0$ in E_+, choose a nonzero piecewise linear function $h \in E_+$ supported in I and satisfying $0 \leq h \leq \varepsilon$. Then, $h \leq u_n$ for every $n \geq N$. Because (u_n) is decreasing, $h \leq u_N \leq u_n$ also holds for $n < N$. Thus, h is a nonzero lower bound of (u_n), contradicting $\inf_n u_n = 0$. Therefore, (f_n) has no E-limit in E.

Corollary 5.7 gives a setting in which this condition holds automatically. In infinite-dimensional spaces, it must be verified separately.

Example 5.9. *Let $X = \{a, b\}$; let $E = \mathbb{R}$; and define $\xi(a, a) = 0$, $\xi(b, b) = 1$, $\xi(a, b) = \xi(b, a) = 0$. Then, ξ is an E_m-metric. Indeed, $\xi^{\wedge}_{a,b} = 0$, and its defect pseudometric vanishes identically: $\delta_\xi(x, y) = 0$. However, $s_\xi(a) = 0$, $s_\xi(b) = 1$, $\rho_\xi(a, b) = 1$.*

Consider the alternating sequence $x_{2n} = a$, $x_{2n+1} = b$. Because δ_ξ vanishes identically, (x_n) is defect-Cauchy. In fact, it defect converges to both a and b. On the other hand, $\omega_\xi(x_{2n}, x_{2n+1}) = 1$ for every n, so the self-distance sequence is not order-Cauchy. By Proposition 5.4, (x_n) is not associated Cauchy.

The same example shows that defect limits need not be unique. As discussed in Section 6, its defect topology is indiscrete, whereas the associated metric induces the discrete topology.

The following examples show that neither completeness notion implies the other, even in the scalar setting.

Example 5.10. *Let $E = \mathbb{R}$, and $X = (0, 1)$. Set $\delta(x, y) = 0$, $s(x) = x$. The pair (δ, s) is separating, so Theorem 4.3 gives $\xi(x, y) = \min\{x, y\}$, $\rho_\xi(x, y) = |x - y|$. Every sequence defect converges to every point of X. Consequently, the space is defect complete. However, $x_n = 1/(n + 1)$ is associated Cauchy and has no associated limit in X. Thus the space is not associated complete.*

Example 5.11. *Let $E = \mathbb{R}$ and $X = (0, 1]$. Define $\delta(x, y) = |x - y|$, $s(x) = 1/x$. Because δ separates points, Theorem 4.3 yields*

$$\xi(x, y) = |x - y| + \min\left\{\frac{1}{x}, \frac{1}{y}\right\} \qquad \text{and} \qquad \rho_\xi(x, y) = |x - y| + \left|\frac{1}{x} - \frac{1}{y}\right|.$$

The sequence $x_n = 1/n$ is defect-Cauchy and has no defect limit in X, so the space is not defect complete.

Let (y_n) be ρ_ξ-Cauchy. Then, $(1/y_n)$ is a real Cauchy sequence. Because $1/y_n \geq 1$, it converges to some $L \geq 1$. Hence, $y_n \to 1/L \in X$, and

$$\rho_\xi(y_n, 1/L) = |y_n - 1/L| + |1/y_n - L| \longrightarrow 0.$$

Thus, the space is associated complete.

6. Defect and associated topologies

Throughout this section, E is an Archimedean Riesz space with an order unit e, and (X, ξ) is an E_m-metric space.

For $u, v \in E$, we write $u \ll_e v$ if there exists $\eta > 0$ such that $u \leq v - \eta e$. Accordingly, for $u \in E_+$ and $\varepsilon > 0$, the relation $u \ll_e \varepsilon e$ means that $u \leq \lambda e$ for some $\lambda < \varepsilon$.

For $u \in E$, the order-unit norm induced by e is $\|u\|_e := \inf\{\lambda > 0 : |u| \leq \lambda e\}$. Because E is Archimedean, $\|\cdot\|_e$ is a norm.

The topologies studied in this section use the order-unit norm. Their convergence must be distinguished from the sequential order convergence used in Sections 2 and 5.

Order-unit norm convergence implies sequential order convergence. Indeed, if $\|u_n\|_e \to 0$, put

$$a_n := \sup_{k \geq n} \|u_k\|_e.$$

Then, $a_n \downarrow 0$ in $\mathbb{R}$, and by the Archimedean property, $|u_n| \leq a_n e$, with $a_n e \downarrow 0$. Thus, $u_n \xrightarrow{o} 0$.

The converse can fail in infinite dimensions. In $E = C([0, 1])$ with order unit $\mathbf{1}$, consider $f_n(t) = t^n$ for $n \geq 1$. This decreasing sequence has infimum zero in the lattice $C([0, 1])$: Any continuous lower bound h satisfies $h(t) \leq 0$ for $t < 1$, and continuity also gives $h(1) \leq 0$. Consequently, $f_n \downarrow 0$ in $C([0, 1])$, whereas, $\|f_n\|_\infty = 1$. The lattice infimum here differs from the pointwise limit at $t = 1$.

In finite-dimensional Archimedean Riesz spaces, the two sequential convergence notions coincide by the coordinatewise representation in $\mathbb{R}^d$. The E-closedness in Theorem 5.6 therefore remains explicitly a sequential order notion; it is not identified in general with closedness for the order-unit topologies introduced below.

6.1. The two canonical topologies

Definition 6.1. *For $x \in X$ and $\varepsilon > 0$, define the defect ball and the associated ball, respectively, by $B^e_\delta(x, \varepsilon) := \{y \in X : \delta_\xi(x, y) \ll_e \varepsilon e\}$ and $B^e_\rho(x, \varepsilon) := \{y \in X : \rho_\xi(x, y) \ll_e \varepsilon e\}$. The topologies generated by these families are denoted by τ^e_δ and τ^e_ρ, respectively. We call τ^e_δ the defect topology and τ^e_ρ the associated order-unit topology.*

Theorem 6.2 (Comparison of the canonical topologies)**.** *For $x, y \in X$, define $\mathfrak{p}_e(x, y) := \|\delta_\xi(x, y)\|_e$ and $d_e(x, y) := \|\rho_\xi(x, y)\|_e$. Then, the following statements hold:*

(1) $\mathfrak{p}_e$ is a pseudometric on X, and its open balls are precisely the sets $B^e_\delta(x, \varepsilon)$. Hence, $\mathfrak{p}_e$ generates τ^e_δ.

(2) d_e *is a metric on* X*, and its open balls are precisely the sets* $B^e_\rho(x,\varepsilon)$*. Hence,* d_e *generates* τ^e_ρ*.*
(3) Both τ^e_δ *and* τ^e_ρ *are independent of the chosen order unit.*
(4) The associated topology is finer than the defect topology: $\tau^e_\delta \subseteq \tau^e_\rho$*. Equality holds if and only if*

$$s_\xi\colon (X,\tau^e_\delta) \longrightarrow (E,\|\cdot\|_e)$$

is continuous.

In particular, τ^e_δ *is pseudometrizable and is Hausdorff if and only if* δ_ξ *separates points, whereas* τ^e_ρ *is always metrizable and Hausdorff. If* $\tau^e_\delta = \tau^e_\rho$*, then* δ_ξ *separates points, and* τ^e_δ *is metrizable and Hausdorff.*

Proof. Because δ_ξ is an E-valued pseudometric, the positivity, homogeneity, and monotonicity of the order-unit norm show that $\mathfrak{p}_e$ is a pseudometric. Similarly, because ρ_ξ is an E-metric, d_e is a metric. In particular, if $d_e(x,y)=0$, then $\rho_\xi(x,y)=0$, and hence, $x=y$.

For every $u\in E_+$ and $\varepsilon>0$, we have $\|u\|_e<\varepsilon$ if and only if $u\preceq \lambda e$ for some $\lambda<\varepsilon$ if and only if $u\ll_e \varepsilon e$. Applying this equivalence to $\delta_\xi(x,y)$ and $\rho_\xi(x,y)$ shows that the open balls of $\mathfrak{p}_e$ and d_e coincide with the corresponding balls in Definition 6.1.

Let f be another order unit of E, and $u\in E$. Choose $a,b>0$ so that $e\preceq af$, and $f\preceq be$. Then, $b^{-1}\|u\|_e\le\|u\|_f\le a\|u\|_e$.

The pseudometrics obtained from δ_ξ using e and f therefore generate the same topology. The same argument applies to the metrics obtained from ρ_ξ. Hence, both topologies are independent of the order unit.

Because $0\preceq\delta_\xi(x,y)\preceq\rho_\xi(x,y)$, monotonicity gives $\mathfrak{p}_e(x,y)\le d_e(x,y)$. Therefore, $\tau^e_\delta\subseteq\tau^e_\rho$.

Suppose first that the two topologies coincide. Because

$$\|s_\xi(x)-s_\xi(y)\|_e=\|\omega_\xi(x,y)\|_e\le d_e(x,y),$$

the profile is continuous for the d_e-topology and hence for τ^e_δ.

Conversely, assume that $s_\xi\colon (X,\tau^e_\delta)\to(E,\|\cdot\|_e)$ is continuous. Fix $x\in X$ and $\varepsilon>0$. There exists $\eta_0>0$ such that

$$\mathfrak{p}_e(x,y)<\eta_0\Longrightarrow\|s_\xi(x)-s_\xi(y)\|_e<\frac{\varepsilon}{2}.$$

Set $\eta:=\min\{\eta_0,\varepsilon/2\}$. If $\mathfrak{p}_e(x,y)<\eta$, then

$$d_e(x,y)\le\mathfrak{p}_e(x,y)+\|s_\xi(x)-s_\xi(y)\|_e<\varepsilon.$$

Hence, the identity map from $(X,\mathfrak{p}_e)$ to (X,d_e) is continuous, and therefore, $\tau^e_\rho\subseteq\tau^e_\delta$.

Finally, a pseudometric topology is Hausdorff, equivalently T_0, precisely when its pseudometric separates points. Because d_e is a metric, τ^e_ρ is Hausdorff. □

Because the preceding theorem shows that neither topology depends on the particular order unit, we shall write τ_δ and τ_ρ whenever the choice of e is immaterial.

The global estimate $\omega_\xi(x,y)\preceq C\delta_\xi(x,y)$ is therefore only a convenient sufficient condition for equality: It makes the profile Lipschitz from $(X,\mathfrak{p}_e)$ to $(E,\|\cdot\|_e)$. The theorem above gives the exact local criterion.

Corollary 6.3. *Let $q\colon X \to X_\delta$ be the canonical projection from Theorem 4.4. Then, $\overline{p}_e([x],[y]) := \|\overline{\delta}_\xi([x],[y])\|_e$ is a metric on X_δ, and its topology is the Kolmogorov quotient of (X, τ_δ^e). Hence, the defect topology becomes Hausdorff after identifying those points whose defect distance is zero.*

Proof. For all $x, y \in X$, $\overline{p}_e(qx, qy) = \|\overline{\delta}_\xi([x],[y])\|_e = \|\delta_\xi(x,y)\|_e = \mathfrak{p}_e(x,y)$. Thus, the inverse image under q of each $\overline{p}_e$-ball is a $\mathfrak{p}_e$-ball. The fibers of q are exactly the topological indistinguishability classes determined by $\mathfrak{p}_e$, which proves the claim. □

6.2. The scalar case and Assaf's modified topology

Suppose in this subsection that $E = \mathbb{R}$, and $e = 1$. Then, the defect balls take the form $B_\delta(x,\varepsilon) = \{y \in X : \xi(x,y) - \xi^\wedge_{x,y} < \varepsilon\}$, and τ_δ is the original defect-ball topology associated with a scalar M-metric.

Following Assaf's construction [27, Section 4], let τ_{Assaf} denote the topology generated by the balls $B_{\text{Assaf}}(x,\varepsilon) := \{y \in X : \delta_\xi(x,y) + s_\xi(y) - s_\xi(x) < \varepsilon\}$. The associated topology τ_ρ is the metric topology generated by $m^w(x,y) = \rho_\xi(x,y) = \delta_\xi(x,y) + \omega_\xi(x,y)$.

The three topologies use different information: τ_δ depends only on the defect, τ_{Assaf} also involves the directed difference of self-distances, and τ_ρ uses their absolute difference.

Proposition 6.4. *For every scalar M-metric, $\tau_{\text{Assaf}} \subseteq \tau_\rho$. If, in addition, there exists a constant $C \in [0,1)$ such that for all $x, y \in X$, $\omega_\xi(x,y) \le C\delta_\xi(x,y)$ then $\tau_\delta = \tau_{\text{Assaf}} = \tau_\rho$. In particular, all three topologies coincide whenever s_ξ is constant.*

Proof. Put $h(x,y) := \delta_\xi(x,y) + s_\xi(y) - s_\xi(x)$. The triangle inequality for δ_ξ gives $h(x,z) \le h(x,y) + h(y,z)$, whereas $h(y,z) \le \delta_\xi(y,z) + \omega_\xi(y,z) = m^w(y,z)$. If $y \in B_{\text{Assaf}}(x,\varepsilon)$, set $\eta := \varepsilon - h(x,y) > 0$. Whenever $m^w(y,z) < \eta$, the preceding estimates yield $h(x,z) \le h(x,y) + h(y,z) \le h(x,y) + m^w(y,z) < \varepsilon$. Thus, $B_{m^w}(y,\eta) \subseteq B_{\text{Assaf}}(x,\varepsilon)$. Every τ_{Assaf}-open set is therefore τ_ρ-open, and hence, $\tau_{\text{Assaf}} \subseteq \tau_\rho$.

Under the additional hypothesis, $-C\delta_\xi(x,y) \le s_\xi(y) - s_\xi(x) \le C\delta_\xi(x,y)$. Therefore,

$$(1-C)\delta_\xi(x,y) \le \delta_\xi(x,y) + s_\xi(y) - s_\xi(x) \le (1+C)\delta_\xi(x,y).$$

Because $C < 1$, we have $\tau_{\text{Assaf}} = \tau_\delta$. Moreover, $\delta_\xi(x,y) \le m^w(x,y) \le (1+C)\delta_\xi(x,y)$, so $\tau_\rho = \tau_\delta$. The constant-profile case follows by taking $C = 0$. □

Remark 6.5. *The bound $C < 1$ in Proposition 6.4 is sufficient, but not necessary, for the three topologies to coincide. Its role is to provide the positive lower estimate $(1-C)\delta_\xi$ for Assaf's directed expression.*

Example 6.6. *Consider the two-point M-metric space from Example 5.9. Because δ_ξ vanishes identically, $\tau_\delta = \{\varnothing, X\}$. The defect topology is therefore indiscrete and fails to be T_0.*

For Assaf's modified expression, we have $\delta_\xi(a,b) + s_\xi(b) - s_\xi(a) = 1$ and $\delta_\xi(b,a) + s_\xi(a) - s_\xi(b) = -1$. It follows that $\tau_{\text{Assaf}} = \{\varnothing, \{a\}, X\}$, which is T_0 but not Hausdorff. By contrast, $\rho_\xi(a,b) = 1$, so the associated topology is discrete: $\tau_\rho = \{\varnothing, \{a\}, \{b\}, X\}$. Hence, $\tau_\delta \subsetneq \tau_{\text{Assaf}} \subsetneq \tau_\rho$. Thus, the three scalar topologies have different separation properties, even on a two-point space. The inclusion $\tau_\delta \subsetneq \tau_{\text{Assaf}}$ in this example is specific to this construction; it is not a general conclusion for scalar M-metrics.

Example 6.7 (Assaf's topology can be strictly coarser)**.** *Let $X = [0,\infty)$, $E = \mathbb{R}$, and set $\delta(x,y) = |x-y|$, $s(x) = x$. Then,*

$$\xi(x,y) = |x-y| + \min\{x,y\} = \max\{x,y\}, \qquad \rho_\xi(x,y) = 2|x-y|.$$

Thus, $\tau_\delta = \tau_\rho$ is the usual relative topology on $[0, \infty)$.

Assaf's directed expression is

$$\delta(x, y) + s(y) - s(x) = 2\max\{y - x, 0\}.$$

The corresponding balls are therefore $B_{\text{Assaf}}(x, \varepsilon) = [0, x + \varepsilon/2)$. These balls are open in the usual relative topology, so $\tau_{\text{Assaf}} \subseteq \tau_\delta$. Every nonempty τ_{Assaf}-open set contains 0, whereas $(1, \infty)$ is a usual open set that does not. Hence, $\tau_{\text{Assaf}} \subsetneq \tau_\delta = \tau_\rho$. Together with Example 6.6, this shows that neither inclusion between τ_δ and τ_{Assaf} holds universally. The universal inclusions established above are $\tau_\delta \subseteq \tau_\rho$ and $\tau_{\text{Assaf}} \subseteq \tau_\rho$.

7. A two-rate fixed point consequence

The next result is a consequence of the Banach contraction principle for the associated E-metric. Its purpose is not to replace that principle, but to retain the separate decay rates of the defect and profile components.

Definition 7.1. *Let (X, ξ) be an E_m-metric space. A mapping $T : X \to X$ is called a componentwise Banach-type E_m-contraction if there exist $q_1, q_2 \in [0, 1)$ such that $\delta_\xi(Tx, Ty) \le q_1\delta_\xi(x, y)$ and $\omega_\xi(Tx, Ty) \le q_2\omega_\xi(x, y)$ for all $x, y \in X$.*

The componentwise hypothesis implies contraction for ρ_ξ with constant $\max\{q_1, q_2\}$. It also imposes separate structural restrictions:

$$\delta_\xi(x, y) = 0 \implies \delta_\xi(Tx, Ty) = 0, \qquad \text{and} \qquad s_\xi(x) = s_\xi(y) \implies s_\xi(Tx) = s_\xi(Ty).$$

Thus, T preserves both the defect equivalence relation and the equivalence relation determined by equality of profiles. Contraction of ρ_ξ alone does not require these separate preservation properties. The componentwise condition supplies the information needed for the separate estimates below.

For the iteration estimates below, let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, take $n \in \mathbb{N}_0$, and set $T^0 = \mathrm{id}_X$. For each geometric factor $r \in [0, 1)$, we interpret $r^0 = 1$, including when $r = 0$. For $n \ge 1$, the usual identity $0^n = 0$ applies. This convention also applies to Section 8.

Theorem 7.2 (Componentwise Picard estimates)**.** *Let (X, ξ) be an E_m-metric space such that (X, ρ_ξ, E) is E-complete, and let $T : X \to X$ be a componentwise Banach-type E_m-contraction with constants $q_1, q_2 \in [0, 1)$. Set $q := \max\{q_1, q_2\}$. Then, the following statements hold:*

(1) For every $n \in \mathbb{N}_0$ and all $x, y \in X$,

$$\delta_\xi(T^n x, T^n y) \le q_1^n \delta_\xi(x, y), \text{ and } \omega_\xi(T^n x, T^n y) \le q_2^n \omega_\xi(x, y).$$

(2) The associated E-metric satisfies

$$\rho_\xi(T^n x, T^n y) \le q_1^n \delta_\xi(x, y) + q_2^n \omega_\xi(x, y) \le q^n \rho_\xi(x, y).$$

(3) The mapping T has a unique fixed point $x^* \in X$, and for every $x_0 \in X$, the Picard sequence $x_{n+1} = Tx_n$ converges to x^* in the associated sense: $x_n \xrightarrow{\rho_\xi} x^*$.

(4) The two components converge at their respective rates: $\delta_\xi(x_n, x^*) \preceq q_1^n \delta_\xi(x_0, x^*)$*, and* $\omega_\xi(x_n, x^*) \preceq q_2^n \omega_\xi(x_0, x^*)$*. Moreover,*

$$\rho_\xi(x_n, x^*) \preceq \frac{q_1^n}{1-q_1}\delta_\xi(x_0, x_1) + \frac{q_2^n}{1-q_2}\omega_\xi(x_0, x_1) \preceq \frac{q^n}{1-q}\rho_\xi(x_0, x_1).$$

Proof. By iterating the componentwise inequalities, we obtain $\delta_\xi(T^n x, T^n y) \preceq q_1^n \delta_\xi(x, y)$ and $\omega_\xi(T^n x, T^n y) \preceq q_2^n \omega_\xi(x, y)$. Because $\rho_\xi = \delta_\xi + \omega_\xi$, it follows that

$$\begin{aligned}\rho_\xi(T^n x, T^n y) &= \delta_\xi(T^n x, T^n y) + \omega_\xi(T^n x, T^n y)\\ &\preceq q_1^n \delta_\xi(x, y) + q_2^n \omega_\xi(x, y)\\ &\preceq q^n \rho_\xi(x, y).\end{aligned}$$

In particular, by taking $n = 1$, we get $\rho_\xi(Tx, Ty) \preceq q\rho_\xi(x, y)$.

By hypothesis, (X, ρ_ξ, E) is E-complete. Moreover, $\rho_\xi(Tx, Ty) \preceq q\rho_\xi(x, y)$ with $q < 1$. Because E is Archimedean, $q^n u \downarrow 0$ for every $u \in E_+$ and every $0 < q < 1$; the case $q = 0$ is immediate. Thus, all the hypotheses of the Banach contraction principle for E-metric spaces [15, Theorem 3.1] are satisfied. That theorem yields a unique fixed point x^* and $x_n \xrightarrow{\rho_\xi} x^*$.

Because $T^n x^* = x^*$, by applying the componentwise iteration estimates to x_0 and x^*, we have

$$\delta_\xi(x_n, x^*) \preceq q_1^n \delta_\xi(x_0, x^*) \text{ and } \omega_\xi(x_n, x^*) \preceq q_2^n \omega_\xi(x_0, x^*).$$

Finally, for $p \geq 1$, the triangle inequality and the componentwise contraction estimates give

$$\begin{aligned}\rho_\xi(x_n, x_{n+p}) &\preceq \sum_{j=n}^{n+p-1} \left(\delta_\xi(x_j, x_{j+1}) + \omega_\xi(x_j, x_{j+1})\right)\\ &\preceq \sum_{j=n}^{n+p-1} \left(q_1^j \delta_\xi(x_0, x_1) + q_2^j \omega_\xi(x_0, x_1)\right)\\ &\preceq \frac{q_1^n}{1-q_1}\delta_\xi(x_0, x_1) + \frac{q_2^n}{1-q_2}\omega_\xi(x_0, x_1).\end{aligned}$$

Set

$$A_n := \frac{q_1^n}{1-q_1}\delta_\xi(x_0, x_1) + \frac{q_2^n}{1-q_2}\omega_\xi(x_0, x_1).$$

Because $x_k \xrightarrow{\rho_\xi} x^*$, there is a sequence (u_k) in E_+ with $u_k \downarrow 0$ and $\rho_\xi(x_k, x^*) \preceq u_k$. Hence, for every $p \geq 1$,

$$\rho_\xi(x_n, x^*) \preceq \rho_\xi(x_n, x_{n+p}) + \rho_\xi(x_{n+p}, x^*) \preceq A_n + u_{n+p}.$$

Hence, $(\rho_\xi(x_n, x^*) - A_n)^+ \preceq u_{n+p}$ for every $p \geq 1$. Because $u_{n+p} \downarrow 0$ as $p \to \infty$, the left-hand side must be zero. Therefore,

$$\rho_\xi(x_n, x^*) \preceq A_n \preceq \frac{q^n}{1-q}\rho_\xi(x_0, x_1),$$

which completes the proof. □

8. An illustrative coupled positive integral system

We use a symmetric two-state Fredholm system to illustrate the quantitative role of the defect profile decomposition. The example shows how the full-state error and a distinguished observable can converge at different rates within the same E_m-metric model. The common and imbalance modes evolve at different rates, with the profile recording the mean imbalance between the two components. Background on Fredholm integral operators can be found in [28]. Fix $M > 0$, and let

$$X_M := \{(u, v) \in C([0, 1], \mathbb{R}^2_+) : 0 \le u(t), v(t) \le M \text{ for all } t \in [0, 1]\}.$$

For $x = (u, v)$ and $y = (\widetilde{u}, \widetilde{v})$ in X_M, set

$$A(x, y) := \|u - \widetilde{u}\|_\infty, \qquad B(x, y) := \|v - \widetilde{v}\|_\infty.$$

On $E = \mathbb{R}^2$, endowed with the coordinatewise order, define

$$\delta(x, y) := \big(A(x, y) + B(x, y),\ \max\{A(x, y), B(x, y)\}\big).$$

Each coordinate of δ is a metric, so δ is an E-metric. Let

$$\mathcal{I}(u, v) := \int_0^1 \big(u(r) - v(r)\big)\, dr.$$

Fix $\gamma > 0$. Because $|\mathcal{I}(u, v)| \le M$ on X_M, choose $C \ge \gamma M$, and define

$$s(u, v) := \big(C + \gamma\mathcal{I}(u, v),\ C - \gamma\mathcal{I}(u, v)\big).$$

Then, $s(X_M) \subseteq E_+$, and the profile is nonconstant. Because δ separates points, (δ, s) is a separating pair. Theorem 4.3 therefore yields the E_m-metric

$$\xi(x, y) := \delta(x, y) + s(x) \wedge s(y). \tag{8.1}$$

Its defect mapping is $\delta_\xi = \delta$, whereas

$$\omega_\xi(x, y) = |s(x) - s(y)| = \big(\gamma|\mathcal{I}(u, v) - \mathcal{I}(\widetilde{u}, \widetilde{v})|, \gamma|\mathcal{I}(u, v) - \mathcal{I}(\widetilde{u}, \widetilde{v})|\big).$$

Let $P\colon C([0, 1]) \to C([0, 1])$ be the positive integral operator

$$(Pf)(t) := \int_0^1 K(t, r) f(r)\, dr,$$

where $K\colon [0, 1]^2 \to \mathbb{R}_+$ is continuous and satisfies

$$\int_0^1 K(t, r)\, dr = 1 \quad \text{and} \quad \int_0^1 K(t, r)\, dt = 1. \tag{8.2}$$

The first identity implies $\|Pf\|_\infty \le \|f\|_\infty$, and the second gives

$$\int_0^1 (Pf)(t)\, dt = \int_0^1 f(t)\, dt.$$

Let $a, b > 0$ with $a + b < 1$, and consider

$$\begin{aligned} u(t) &= g_1(t) + a(Pu)(t) + b(Pv)(t), \\ v(t) &= g_2(t) + b(Pu)(t) + a(Pv)(t), \qquad 0 \le t \le 1, \end{aligned} \tag{8.3}$$

where $g_1, g_2 \in C([0,1], \mathbb{R}_+)$ satisfy

$$0 \le g_i(t) \le (1 - a - b)M \qquad (t \in [0,1],\ i = 1, 2). \tag{8.4}$$

Because $b > 0$, the two equations are coupled. Introducing $w := u + v$ and $z := u - v$ separates the system into its common and imbalance modes:

$$w = g_1 + g_2 + (a + b)Pw, \qquad z = g_1 - g_2 + (a - b)Pz.$$

The profile records the mean of the imbalance mode, whereas the defect measures the full pair in the supremum norm.

Theorem 8.1. *System* (8.3) *has a unique solution $x^* = (u^*, v^*) \in X_M$. Moreover, the successive approximations $x_{n+1} = Tx_n$, starting from any $x_0 \in X_M$, converge to x^* in the associated sense. Set*

$$q_1 := a + b, \qquad q_2 := |a - b|.$$

The componentwise estimates are

$$\delta_\xi(x_n, x^*) \le q_1^n \delta_\xi(x_0, x^*), \tag{8.5}$$

$$\omega_\xi(x_n, x^*) = q_2^n \omega_\xi(x_0, x^*). \tag{8.6}$$

Moreover,

$$\mathcal{I}(x^*) = \frac{\displaystyle\int_0^1 (g_1(r) - g_2(r))\, dr}{1 - a + b}$$

and hence,

$$|\mathcal{I}(x_n) - \mathcal{I}(x^*)| = |a - b|^n \, |\mathcal{I}(x_0) - \mathcal{I}(x^*)| \,.$$

Because $a, b > 0$, we have $0 \le q_2 < q_1 < 1$. Estimate (8.5) *is an upper bound for the full-state error measured by the defect, whereas* (8.6) *and the mean-imbalance formula are exact error identities.*

If $a \ne b$, and $\mathcal{I}(x_0) \ne \mathcal{I}(x^*)$, the successive nonzero mean-imbalance errors have the exact ratio $q_2 = |a-b|$. If $a = b$, the mean-imbalance and profile errors vanish after one iteration. If $\mathcal{I}(x_0) = \mathcal{I}(x^*)$, they vanish for every iteration. Thus, the phrase "exact decay rate" applies to the nondegenerate geometric-error case.

Proof. Define $T : X_M \to X_M$ by the right-hand side of (8.3). Positivity, the estimate $\|Pf\|_\infty \le \|f\|_\infty$, and (8.4) show that $T(X_M) \subseteq X_M$. If $x = (u, v)$, and $y = (\widetilde{u}, \widetilde{v})$, then

$$\|(Tx)_1 - (Ty)_1\|_\infty \le aA(x, y) + bB(x, y),$$

whereas

$$\|(Tx)_2 - (Ty)_2\|_\infty \le bA(x, y) + aB(x, y).$$

We therefore have

$$\delta_\xi(Tx, Ty)_1 \le q_1(A(x, y) + B(x, y)),$$
$$\delta_\xi(Tx, Ty)_2 \le q_1 \max\{A(x, y), B(x, y)\}.$$

Hence, $\delta_\xi(Tx, Ty) \preceq q_1\delta_\xi(x, y)$.

The second identity in (8.2) gives

$$\mathcal{I}(Tx) = \mathcal{I}(g_1, g_2) + (a - b)\mathcal{I}(x).$$

Taking differences and then the lattice modulus yields

$$\omega_\xi(Tx, Ty) = |a - b|\, \omega_\xi(x, y).$$

Thus, T is a componentwise Banach-type E_m-contraction with rates $q_1 = a + b$ and $q_2 = |a - b|$. At the fixed-point, this identity gives $(1 - a + b)\mathcal{I}(x^*) = \mathcal{I}(g_1, g_2)$, which proves the stated formula for $\mathcal{I}(x^*)$ and its exact iterative error relation.

It remains to prove that (X_M, ρ_ξ, E) is E-complete. Let (x_n) be ρ_ξ-Cauchy. Because $\delta_\xi \preceq \rho_\xi$, the sequence is δ-Cauchy. Its two components are therefore Cauchy in the supremum norm and converge to some $x \in X_M$. Moreover,

$$|\mathcal{I}(u, v) - \mathcal{I}(\widetilde{u}, \widetilde{v})| \le \|u - \widetilde{u}\|_\infty + \|v - \widetilde{v}\|_\infty,$$

so $\omega_\xi(x_n, x) \xrightarrow{o} 0$, and $\delta_\xi(x_n, x) \xrightarrow{o} 0$. Thus, (X_M, ρ_ξ, E) is E-complete. Theorem 7.2 therefore applies and yields the unique fixed point together with the associated convergence of the Picard iterates and Estimate (8.5). Iterating the exact profile identity gives (8.6). □

Example 8.2. *Take*

$$K(t, r) = 1 + \frac{1}{2}(2t - 1)(2r - 1).$$

This kernel is continuous, non-negative, and satisfies (8.2)*. Choose* $a = 1/4, b = 1/8$*, and choose the constant data* $g_1(t) = M/4$ *and* $g_2(t) = M/8$*. These functions satisfy* (8.4)*. Then,*

$$q_1 = a + b = \frac{3}{8}, \qquad q_2 = |a - b| = \frac{1}{8}.$$

Because P fixes constant functions, the unique solution is

$$u^*(t) = \frac{13M}{35}, \qquad v^*(t) = \frac{8M}{35}.$$

In particular, Theorem 8.1 gives

$$\mathcal{I}(x^*) = \frac{M/4 - M/8}{1 - 1/4 + 1/8} = \frac{M/8}{7/8} = \frac{M}{7}.$$

This agrees with the explicit solution, as

$$\mathcal{I}(x^*) = \frac{13M}{35} - \frac{8M}{35} = \frac{M}{7}.$$

If the iteration starts from $x_0 = (0, 0)$, the full-state error has a geometric upper bound with ratio $3/8$. By contrast, the mean-imbalance error satisfies the exact identity

$$\left|\mathcal{I}(x_n) - \frac{M}{7}\right| = \frac{M}{7}\left(\frac{1}{8}\right)^n.$$

The associated-metric estimate gives the ratio $3/8$, and the profile component gives the exact ratio $1/8$ for the mean imbalance.

The system can also be solved by the classical Banach contraction principle after diagonalizing its common and imbalance modes. The defect profile formulation organizes the resulting estimates within a single associated E-metric: The defect controls the full state, and the profile records a distinguished observable.

Because $\mathcal{I}(x^*)$ is determined directly by the forcing data, its exact error identity allows a prescribed mean-imbalance tolerance to be checked without first computing the full solution. The full-state guarantee remains the bound involving q_1. Moreover, Theorem 7.2 supplies an associated-error bound in terms of the computable initial increments $\delta_\xi(x_0, x_1)$ and $\omega_\xi(x_0, x_1)$.

9. Conclusions

We introduced the defect profile decomposition of an E_m-metric and used it to study the associated quotient, completeness, and topology. The associated E-metric admits an isometric quotient-profile representation, and its completeness can be characterized through the sequential closedness of the image under suitable assumptions. When the range space has an order unit, the defect and associated topologies coincide exactly when the profile is continuous with respect to the defect topology.

The componentwise contraction condition gives separate convergence rates for the defect and profile terms. In the Fredholm system, these rates correspond to the full-state error and the mean-imbalance error, respectively. Extending the topological theory to Riesz spaces without an order unit and constructing completions that preserve the defect profile decomposition remain topics for further study.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest.

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