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*Research article*

# Blow-up and global well-posedness for a nonlinear Schrödinger equation at critical mass

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**Abstract:** This work investigates the Cauchy problem for an inhomogeneous, nonlinear Schrödinger equation featuring a competing nonlinear structure. The model comprises a focusing term at the mass-critical level and a defocusing contribution that is mass-subcritical. We establish a sharp mass threshold for unconditional global existence of energy solutions. The presence of spatially dependent coefficients breaks translation invariance, which allows us to dispense with the radial symmetry hypotheses commonly required in concentration-compactness-based blow-up arguments. Furthermore, the defocusing perturbation violates the scaling symmetry of the pure critical equation. This leads to a notable departure from the unperturbed dynamics: Solutions that blow up with minimal mass do not occur. Our findings complement previous findings by addressing the mass-subcritical perturbation regime, and extend previous results to the inhomogeneous setting while permitting nonradial initial data. Numerical simulations corroborate our theoretical findings.

**Keywords:** Schrödinger problem; nonlinear equations; blow-up; global existence; mass-critical
**Mathematics Subject Classification:** 35B44, 35Q55

## 1. Introduction

The purpose of this paper is to study the initial-value problem for a class of inhomogeneous nonlinear Schrödinger equations featuring a competing nonlinear structure. Specifically, we consider

$$\begin{cases} \mathrm{i}\partial_t u + \Delta u = -|x|^{-b_c}|u|^{p_c-1}u + \lambda|x|^{-b}|u|^{p-1}u, \\ u(0,x) = u_0(x), \end{cases} \tag{INLS}$$

where $u\colon \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}$ represents the unknown complex field. The coefficients $|x|^{-b_c}$ and $|x|^{-b}$ are singular, spatially dependent potentials, with strictly positive exponents $b_c, b > 0$. The parameter

$\lambda \in \mathbb{R} \setminus \{0\}$ controls the balance between the two nonlinear effects. The first term on the right-hand side is taken to be mass-critical, whereas the second is mass-subcritical. Thus, the model consists of a focusing critical nonlinearity subjected to a defocusing subcritical perturbation.

Inhomogeneous, nonlinear, Schrödinger-type equations serve as fundamental models for wave propagation phenomena in several branches of physics, most notably in nonlinear optics and plasma dynamics. In plasma settings, the creation of a low-density channel via an appropriately shaped initial laser pulse allows high-intensity beams to propagate over extended distances. This is achieved because the reduced electron concentration inside the channel weakens the local nonlinear response, thereby counteracting self-focusing effects [1–3]. For optical waveguides, a similar confinement mechanism enables the robust transport of beams carrying intricate spatial structures and supports the existence of localized soliton-type solutions [4, 5]. In particular, competing focusing and defocusing nonlinearities arise in cubic–quintic optical models, where the cubic term describes the Kerr response, and the quintic term represents a higher-order correction to the nonlinear refractive index. Such models have been considered in optical media exhibiting non-Kerr and saturable nonlinear responses [6, 7]. Similar competing nonlinear interactions also arise in matter-wave models of Bose–Einstein condensates, where nonlinear terms can represent effective many-body interactions.

Solutions to the initial-value problem (INLS) enjoy two fundamental conserved quantities, namely the mass

$$M(u)\colon t \mapsto \|u(t)\|^2_{L^2(\mathbb{R}^d)} \tag{Mass}$$

and the energy, which are formally preserved along the flow:

$$E(u)\colon t \mapsto \|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} - \frac{2}{1+p_c}\int_{\mathbb{R}^d} |u(t,x)|^{1+p_c}\,\frac{dx}{|x|^{b_c}} + \frac{2\lambda}{1+p}\int_{\mathbb{R}^d} |u(t,x)|^{1+p}\,\frac{dx}{|x|^{b}}. \tag{Energy}$$

The mass functional measures the total $L^2$-norm of the wave field, and the energy functional comprises the kinetic contribution together with the competing potential energies arising from the focusing critical term and the defocusing subcritical perturbation. In the purely focusing regime corresponding to $\lambda = 0$, the equation (INLS) enjoys a scaling symmetry. Indeed, if $u$ is a solution, then for any scaling parameter $\kappa > 0$, the rescaled family

$$u_\kappa(t,x) := \kappa^{\frac{2-b_c}{p_c-1}} u(\kappa^2 t, \kappa x) \tag{1.1}$$

also solves the same equation. This invariance induces the following behavior of homogeneous Sobolev norms:

$$\|u_\kappa(t)\|_{\dot{H}^\nu} = \kappa^{\nu-\left(\frac{d}{2}-\frac{2-b_c}{p_c-1}\right)}\|u(\kappa^2 t)\|_{\dot{H}^\nu} = \kappa^{\nu-s_c}\|u(\kappa^2 t)\|_{\dot{H}^\nu}, \tag{1.2}$$

where the critical exponent

$$s_c := \frac{d}{2} - \frac{2-b_c}{p_c-1} \tag{1.3}$$

characterizes the regularity level at which the norm is invariant under the scaling transformation.

In the unperturbed case $\lambda = 0$, the Schrödinger problem (INLS) is classified as mass-critical whenever $s_c = 0$, which is equivalent to

$$p_c = 1 + \frac{2(2 - b_c)}{d}. \tag{1.4}$$

It is said to be energy-critical when $s_c = 1$, namely *

$$p_c = 1 + \frac{2(2 - b_c)}{(d - 2)_+}. \tag{1.5}$$

For the focusing equation with $\lambda = 0$, a ground state is understood as the unique positive radial solution to the stationary elliptic problem

$$-\Delta\varphi_{b_c} + \varphi_{b_c} - |x|^{-b_c}|\varphi_{b_c}|^{p_c-1}\varphi_{b_c} = 0. \tag{1.6}$$

The existence and uniqueness of the ground state to (1.6) are guaranteed under the condition $0 < b_c < \min\{2, d\}$. Existence was first established in the works [8–10]: Genoud [8] developed the bifurcation and stability theory for Schrödinger equations with a compact nonlinearity in his doctoral thesis, and together with Stuart [9] established the existence and stability of standing waves for equations with a spatially decaying nonlinearity, while Genoud [10] later extended the analysis to travelling waves in self-focusing planar waveguides. The uniqueness property, by contrast, was obtained later and independently: Yanagida [11] proved uniqueness of positive radial solutions of $\Delta u + g(r)u + h(r)u^p = 0$ in $\mathbb{R}^d$, and Toland [12] established uniqueness of positive solutions for a class of semilinear Sturm–Liouville problems on the half-line, supplying the uniqueness ingredient in the radially symmetric setting.

Building on the abstract theory from [13], a local theory for the energy subcritical regime in $H^1(\mathbb{R}^N)$ was developed in [9], and a global theory was established for both the mass-subcritical and intercritical regimes for sufficiently small initial data. In [14], a global theory was further extended in $H^1(\mathbb{R}^N)$ below the ground state threshold, where an adapted inhomogeneous Gagliardo–Nirenberg type estimate was derived. Later, [15] revisited the local theory in Sobolev spaces, using the contraction mapping principle alongside Strichartz estimates. The distinction between global existence, scattering, and finite-time blowup below the ground state threshold was analyzed in [16, 17] through the concentration–compactness method developed in [18]. This method was expanded in [19] by incorporating the Dodson–Murphy approach [20], and in [21], the assumption of spherical symmetry was relaxed. Recently, the energy-critical regime was tackled, with local well-posedness results presented in [22–25]. In the three-dimensional case, scattering for spherically symmetric initial data was shown in [26, 27]. The radial symmetry assumption was removed in [28], and results were extended to higher dimensions in [29]. The finite-time blowup of solutions without spherically symmetric or time variance assumptions was examined in [30, 31].

In the context of the combined nonlinear Schrödinger equation, [32] explored both the local and global well-posedness issues, as well as the asymptotic behavior of blow-up solutions in energy space and pseudoconformal space. Later, [33] examined the existence and instability of standing waves, deriving the precise condition for global existence and blow-up through a concavity argument. The study in [34] established the existence of blow-up solutions and identified the sharp threshold mass

*We adopt the notation $\kappa_+ := \max\{\kappa, 0\}$ with the convention that $0^{-1} = \infty$.

for global existence and blow-up using a variational approach along with Hamiltonian conservation laws. Additionally, they investigated the dynamical properties of blow-up solutions. Furthermore, the work [35] focused on the existence of minimal mass blow-up solutions.

In the energy subcritical regime $1 < p < 1 + 2(2-b)/(d-2)^+$, it is known that for $0 < b, b_c < \min\{2, d\}$ and a datum with a finite energy, there is a unique local solution to (INLS) in the space $C([0,T]; H^1)$, see [9] and [15, Theorem 1.4].

**Proposition 1.1.** *Assume that $d \geq 1$, $1 < p < 1 + 2(2-b)/(d-2)_+$, $0 < b, b_c < \min\{2, d\}$, and $u_0 \in H^1$. Then, there exists a unique solution $u(t,x)$ for the Cauchy problem* (INLS) *in $C([0, T^{\max}); H^1)$ for some $T^{\max} \in (0, \infty]$. At the same time, we have the alternatives $T^{\max} = \infty$ (global existence) or else $T^{\max} < \infty$ and $\lim_{t \to T^{\max}} \|\nabla u(t)\|_{L^2} = \infty$ (blow-up). Moreover, for all $t \in [0, T^{\max})$, the solution $u$ satisfies the two conservation laws of the mass* (Mass) *and energy* (Energy).

In this note, we consider the focusing mass critical regime with a defocusing perturbation, so we assume that

$$\lambda = 1, \quad d \geq 1, \quad 0 < b, b_c < \min\{2, d\}, \quad 1 < p < 1 + \frac{2(2-b)}{d}, \quad p_c = 1 + \frac{2(2-b_c)}{d}. \tag{1.7}$$

The aim of this note is to explore the sharpness of the mass threshold for unconditional global existence of energy solutions, drawing inspiration from the unperturbed regime, that is, when $\lambda = 0$. For the mass-critical, inhomogeneous, nonlinear Schrödinger equation, namely (INLS) with $\lambda = 0$, it is known that the local energy solution becomes global if

$$\int_{\mathbb{R}^d} |u_0(x)|^2 \, dx < \int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^2 \, dx.$$

In the critical case where

$$\int_{\mathbb{R}^d} |u_0(x)|^2 \, dx = \int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^2 \, dx,$$

the standing wave $e^{it}\varphi_{b_c}$ serves as a global solution, though there are also nonglobal solutions. Specifically, there exists $\omega_0 > 0$ and $\theta_0 \in \mathbb{R}$ such that, for any $t \in [0, T)$,

$$u(t,x) = e^{i\theta_0} e^{i\frac{\omega_0^2}{T-t}} e^{-i\frac{|x|^2}{4(T-t)}} \Big(\frac{\omega_0}{T-t}\Big)^{\frac{d}{2}} \varphi_{b_c}\Big(\frac{\omega_0}{T-t} x\Big). \tag{1.8}$$

For such a result, we refer to [36, Theorem 1] and [37, Theorem 1.6]. The aim here is to extend the existing literature, specifically [38] to the nonradial regime and [39] to the mass-critical regime. The central result of this work is a sharp mass threshold for unconditional global existence for energy-class solutions of (INLS).

**Theorem 1.1.** *Assume that the hypotheses* (1.7) *are in force, and let $u \in C([0,T); H^1)$ denote the unique maximal solution to the Cauchy problem* (INLS)*. Then, the following sharp dichotomy holds:*

*(1) If the initial mass does not exceed the ground-state mass, that is,*

$$\int_{\mathbb{R}^d} |u_0(x)|^2 \, dx \leq \int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^2 \, dx, \tag{1.9}$$

*then, the solution exists globally in time, so that $T = +\infty$.*

*(2) Conversely, there exists initial data $u_0 \in H^1$ satisfying*

$$\int_{\mathbb{R}^d} |u_0(x)|^2 \, dx > \int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^2 \, dx, \tag{1.10}$$

*for which the corresponding solution blows up in finite time, that is, $T < \infty$.*

Some observations regarding Theorem 1.1 are warranted.

- ♠ The above result contrasts with the standard case $\lambda = 0$, where blowing-up solutions with minimal mass exist; see [36, Proposition 10].
- ♠ The bulk of the study of global existence of energy solutions concerns the limiting case $\|u_0\|_{L^2(\mathbb{R}^d)} = \|\varphi_{b_c}\|_{L^2(\mathbb{R}^d)}$, where we rely on the mass concentration in Proposition 1.2 and the characterization of blowing-up solutions in Proposition 1.3.
- ♠ The finite-time blow-up assertion is established by means of a localized Virial-type identity, following the strategy developed in [31]. This approach enables us to capture the collapse dynamics without requiring the initial datum to have finite variance, which is a standard limitation of the classical Virial argument.
- ♠ For the existence and uniqueness of ground states, see [14, Remark 1.3].
- ♠ In Section 5, we provide numerical simulations supporting Theorem 1.1.
- ♠ The case of a mass-critical source term with a perturbation by a mass-subcritical nonlinearity, considered in this paper, was not addressed in the pioneering work [32].
- ♠ The particular case $b = b_c = 0$ was treated in [35]. The present work differs from [35] not only in the technical complications stemming from the singular, inhomogeneous terms, but also in the blow-up analysis, where the finite-variance assumption made in [35] is completely removed here.

The proof of global existence in the critical case where the initial mass exactly equals the ground-state mass, that is,

$$\int_{\mathbb{R}^d} |u_0(x)|^2 \, dx = \int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^2 \, dx, \tag{1.11}$$

relies on two preparatory lemmas. The first of these provides a mass-concentration estimate for solutions that do not extend globally in time. Precisely, any finite-time blow-up solution must concentrate at least the ground-state mass in a ball of radius $v(t)$ as the blow-up time is approached.

**Proposition 1.2.** *Assume that the hypotheses* (1.7) *are satisfied, and let $u \in C([0, T); H^1)$ be a maximal solution to the Cauchy problem* (INLS) *that does not exist globally, that is, $T < \infty$. Suppose that there exists a positive function $v(t)$ such that*

$$\lim_{t \to T} v^2(t) \int_{\mathbb{R}^d} |\nabla u(t, x)|^2 \, dx = \infty. \tag{1.12}$$

*Then, the following mass-concentration lower bound holds:*

$$\liminf_{t \to T} \int_{\{|x| < v(t)\}} |u(t, x)|^2 \geq \int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^2 \, dx. \tag{1.13}$$

**Remark 1.1.** *Suppose that there exists $w \in L^2(\mathbb{R}^d)$ such that*

$$\lim_{t\to T} \int_{\mathbb{R}^d} |u(t,x) - w(x)|^2 \, dx = 0. \tag{1.14}$$

*Define*

$$v(t) := \frac{1}{\left( \int_{\mathbb{R}^d} |\nabla u(t,x)|^2 \, dx \right)^{\frac{1}{4}}}. \tag{1.15}$$

*Then, by the triangle inequality in integral form, we have*

$$\int_{\{|x|<v(t)\}} |u(t,x)|^2 \, dx \lesssim \int_{\mathbb{R}^d} |u(t,x) - w(x)|^2 \, dx + \int_{\{|x|<v(t)\}} |w(x)|^2 \, dx \to 0, \quad \text{as } t \to T. \tag{1.16}$$

*The estimate* (1.16) *directly contradicts the mass-concentration lower bound* (1.13)*. Consequently, the solution $u(t)$ cannot admit a limit in $L^2$ as $t \to T$. Moreover, no sequence $t_k \to T$ can exist such that $u(t_k)$ converges strongly in $L^2$. This observation implies that the maximal solution cannot be continued beyond the blow-up time in the strong $L^2$-topology.*

Now, any finite-time blow-up solution with mass exactly at the ground-state level must, after appropriate rescaling and phase adjustment, converge strongly to the ground state in the energy space.

**Proposition 1.3.** *Assume that the hypotheses* (1.7) *are satisfied. Let $u \in C([0,T); H^1)$ be a solution to* (INLS) *that blows up in finite time $T > 0$, with initial mass exactly equal to the ground-state mass:*

$$\int_{\mathbb{R}^d} |u_0(x)|^2 \, dx = \int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^2 \, dx. \tag{1.17}$$

*Then, there exists a phase parameter $\theta(t) \in \mathbb{R}$ such that, upon defining*

$$\omega(t) := \frac{\left( \int_{\mathbb{R}^d} |\nabla \varphi_{b_c}(x)|^2 \, dx \right)^{1/2}}{\left( \int_{\mathbb{R}^d} |\nabla u(t,x)|^2 \, dx \right)^{1/2}}, \tag{1.18}$$

*the following strong convergence in $H^1$ holds as $t \to T$:*

$$\omega^{d/2}(t) u(t, \omega(t)\,\cdot) \to \{ e^{i\theta} \varphi_{b_c}, \quad \theta \in [0, 2\pi] \}. \tag{1.19}$$

*Equivalently, this means that*

$$\int_{\mathbb{R}^d} \left| \omega^{d/2}(t) u(t, \omega(t)x) - e^{i\theta} \varphi_{b_c}(x) \right|^2 dx \to 0 \tag{1.20}$$

*and*

$$\int_{\mathbb{R}^d} \left| \nabla_x \left( \omega^{d/2}(t) u(t, \omega(t)x) \right) - e^{i\theta} \nabla \varphi_{b_c}(x) \right|^2 dx \to 0. \tag{1.21}$$

**Remark 1.2.** *The argument proceeds by combining the compactness property stated in Proposition 2.3 with the variational characterization of ground states recalled in Lemma 2.2.*

The structure of the paper is as follows. We begin in Section 2 by introducing the necessary functional framework and known results. The auxiliary compactness and convergence results, Propositions 1.2 and 1.3, are established in Section 3. With these in hand, we proceed in Section 4 to prove the main dichotomy stated in Theorem 1.1. Finally, in Section 5, we provide numerical simulations supporting Theorem 1.1.

## 2. Useful tools

Before proceeding to the proofs of the auxiliary results, we briefly collect several key analytical ingredients that will be used repeatedly throughout the paper. The first of these is an inhomogeneous Gagliardo–Nirenberg estimate, which was established in [14].

**Proposition 2.1.** *Assume* $0 < b < \min\{2, d\}$, *and*

$$1 < p < 1 + \frac{2(2-b)}{(d-2)^+}. \tag{2.1}$$

*For every* $g \in H^1(\mathbb{R}^d)$, *the following estimate holds:*

$$\int_{\mathbb{R}^d} |g(x)|^{p+1} \frac{dx}{|x|^b} \leq \mathsf{C}_{\mathrm{opt}} \left( \int_{\mathbb{R}^d} |g(x)|^2 \, dx \right)^{\frac{1+p}{2} - \frac{d(p-1)+2b}{4}} \left( \int_{\mathbb{R}^d} |\nabla g(x)|^2 \, dx \right)^{\frac{d(p-1)+2b}{4}}. \tag{2.2}$$

*The optimal value of the constant in* (2.2) *is*

$$\mathsf{C}_{\mathrm{opt}} = \frac{p+1}{1+p-\frac{d(p-1)+2b}{2}} \left( \frac{1+p-\frac{d(p-1)+2b}{2}}{\frac{d(p-1)+2b}{2}} \right)^{\frac{d(p-1)+2b}{4}} \left( \int_{\mathbb{R}^d} |\varphi(x)|^2 \, dx \right)^{-\frac{p-1}{2}}, \tag{2.3}$$

*with* $\varphi \in H^1(\mathbb{R}^d) \setminus \{0\}$ *denoting a ground-state solution of the elliptic equation*

$$-\Delta \varphi + \varphi = |x|^{-b} |\varphi|^{p-1} \varphi. \tag{2.4}$$

*Furthermore, the ground state satisfies the Pohozaev relations*

$$\frac{1}{\frac{d(p-1)+2b}{2}} \int_{\mathbb{R}^d} |\nabla \varphi(x)|^2 \, dx = \frac{1}{1+p-\frac{d(p-1)+2b}{2}} \int_{\mathbb{R}^d} |\varphi(x)|^2 \, dx = \frac{1}{1+p} \int_{\mathbb{R}^d} |\varphi(x)|^{1+p} \frac{dx}{|x|^b}. \tag{2.5}$$

The following interpolation estimate, due to [31, Lemma 2.1], will be used below.

**Lemma 2.1.** *Let* $d \geqslant 1$, $0 < b < \min\{2, d\}$, *and let* $\varphi$ *be a positive real-valued function of the variable in* $\mathbb{R}^d$.

- *If* $d \neq 2$, *and* $\varphi^{\frac{1}{2-b}} \in W^{1,\infty}$, *then, for all* $f \in H^1$, *we have*

$$\int_{\mathbb{R}^d} \varphi |f|^{\frac{4-2b}{d}+2} \, dx \lesssim \left( \|\nabla(\varphi^{\frac{1}{2-b}}) f\|_{L^2}^2 + \|\varphi^{\frac{1}{2-b}} \nabla f\|_{L^2}^2 \right)^{\frac{2-b}{2}} \|f\|_{L^2}^{\frac{4+b(d-2)}{d}}. \tag{2.6}$$

- *If* $d = 2$, *and* $\varphi^{\frac{1}{2-b}} \in W^{1,\infty}$, *then, for all* $f \in H^1$, *we have*

$$\int_{\mathbb{R}^d} \varphi |f|^{4-b} \, dx \lesssim \left( \|\varphi^{\frac{1}{2-b/2}} f\|_{L^2}^2 + \|\nabla(\varphi^{\frac{1}{2-b/2}}) f\|_{L^2}^2 + \|\varphi^{\frac{1}{2-b/2}} \nabla f\|_{L^2}^2 \right)^{\frac{2-b/2}{2}} \|f\|_{L^2}^{2-\frac{b}{2}}. \tag{2.7}$$

Given a smooth function $\Phi \in C^\infty(\mathbb{R}^d, \mathbb{R})$ such that $|\nabla|^\alpha \Phi \in L^\infty$, for any $|\alpha| \leq 4$, we associate to it the following quantities. The variance potential is defined as

$$V_\Phi(t) := \int_{\mathbb{R}^d} |u(t,x)|^2 \, \Phi(x) \, dx, \tag{2.8}$$

while the Morawetz action takes the form

$$M_\Phi(t) := 2\mathfrak{I} \int_{\mathbb{R}^d} \overline{u(t,x)}\, u_j(t,x)\, \Phi_j(x)\, dx$$
$$= 2\mathfrak{I} \int_{\mathbb{R}^d} \overline{u(t,x)}\, \nabla u(t,x) \cdot \nabla\Phi(x)\, dx. \tag{2.9}$$

We adopt the Einstein summation convention for repeated indices throughout the sequel. We will employ the following Virial-type identity, which is inspired by the approaches developed in [19, 41].

**Proposition 2.2.** *Let $u \in C([0,T]; H^1(\mathbb{R}^d))$ be a local solution to the Cauchy problem* (INLS). *Then, for all $t \in [0,T]$, the variance functional $V_\Phi$ satisfies the second-order differential identity*

$$\frac{d^2}{dt^2} V_\Phi(t) = 4 \int_{\mathbb{R}^d} \mathfrak{R}(\partial_k u\, \partial_l \overline{u})\, \partial_l \partial_k \Phi\, dx - \int_{\mathbb{R}^d} |u|^2\, \Delta^2 \Phi\, dx$$
$$- 2\left(\frac{2}{1+p} - 1\right) \int_{\mathbb{R}^d} |u|^{1+p}\, \Delta\Phi\, \frac{dx}{|x|^b} - \frac{4}{1+p} \int_{\mathbb{R}^d} |u|^{1+p}\, \nabla\Phi \cdot \nabla(|x|^{-b})\, dx$$
$$+ 2\left(\frac{2}{1+p_c} - 1\right) \int_{\mathbb{R}^d} |u|^{1+p_c}\, \Delta\Phi\, \frac{dx}{|x|^{b_c}} + \frac{4}{1+p_c} \int_{\mathbb{R}^d} |u|^{1+p_c}\, \nabla\Phi \cdot \nabla(|x|^{-b_c})\, dx. \tag{2.10}$$

The following compactness result, due to [37, Theorem 1.3], will be needed in the sequel.

**Proposition 2.3.** *Let $d \geq 1$, $0 < b < \min\{2, d\}$, and suppose*

$$p = 1 + \frac{2(2-b)}{d}. \tag{2.11}$$

*Consider a bounded sequence $\{\phi_k\}_{k\in\mathbb{N}} \subset H^1(\mathbb{R}^d)$. Assume there exist positive constants $m$ and $M$ such that*

$$\limsup_{k\to\infty} \int_{\mathbb{R}^d} |\nabla\phi_k(x)|^2\, dx \leq M^2 \tag{2.12}$$

*and*

$$\limsup_{k\to\infty} \int_{\mathbb{R}^d} |\phi_k(x)|^{1+p}\, \frac{dx}{|x|^b} \geq m^{1+p}. \tag{2.13}$$

*Then, after passing to a subsequence, we have*

$$\phi_k \rightharpoonup \psi \quad \text{weakly in } H^1(\mathbb{R}^d), \tag{2.14}$$

*and the weak limit satisfies the lower bound*

$$\int_{\mathbb{R}^d} |\psi(x)|^2\, dx \geq \frac{m^{2(p+1)/(p-1)}\, d^{2/(p-1)}}{\big((d+2-b)M^2\big)^{2/(p-1)}} \int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^2\, dx. \tag{2.15}$$

Finally, we recall a characterization of the ground states to (1.6), which is due to [36, Proposition 2]; precisely, any function sharing the same mass and satisfying the virial-type identity must coincide, up to scaling and phase modulation, with the ground state.

**Lemma 2.2.** *Let $\varphi$ be a ground-state solution to* (2.4)*, and suppose $g \in H^1(\mathbb{R}^d)$ satisfies the two conditions*

$$\int_{\mathbb{R}^d} |g(x)|^2\,dx = \int_{\mathbb{R}^d} |\varphi(x)|^2\,dx \tag{2.16}$$

*and*

$$\int_{\mathbb{R}^d} |\nabla g(x)|^2\,dx - \frac{2}{1+p}\int_{\mathbb{R}^d} |g(x)|^{1+p}\,\frac{dx}{|x|^b} = 0. \tag{2.17}$$

*Then, there exist parameters $\gamma > 0$ and $\theta \in [0, 2\pi]$ such that*

$$g(x) = e^{i\theta}\gamma^{d/2}\varphi(\gamma x). \tag{2.18}$$

Having collected the necessary preliminary tools, we now proceed to establish the sharpness of the mass threshold for unconditional global existence stated in Theorem 1.1.

## 3. Auxiliary results

This section proves Propositions 1.2 and 1.3.

### *3.1. Mass concentration*

We begin this subsection with the proof of Proposition 1.2. Let $u$ be a maximal solution to (INLS) that does not exist globally, blowing up at finite time $T > 0$, and suppose it satisfies condition (1.12), namely

$$\lim_{t\to T} v(t)\left(\int_{\mathbb{R}^d} |\nabla u(t,x)|^2\,dx\right)^{1/2} = \infty. \tag{3.1}$$

Define the positive scaling function

$$\omega(t) := \frac{\left(\int_{\mathbb{R}^d} |\nabla \varphi_{b_c}(x)|^2\,dx\right)^{1/2}}{\left(\int_{\mathbb{R}^d} |\nabla u(t,x)|^2\,dx\right)^{1/2}}, \tag{3.2}$$

and introduce the rescaled profile

$$v(t,x) := \omega(t)^{d/2}u(t,\omega(t)x). \tag{3.3}$$

Consider an increasing sequence of times $\{t_k\}_{k=1}^{\infty}$ such that $t_k \uparrow T$ as $k \to \infty$. Set

$$\omega_k := \omega(t_k), \qquad v_k(x) := v(t_k, x). \tag{3.4}$$

From the blow-up criteria

$$\|\nabla u(t_k)\|_{L^2} \to 0, \quad \text{when} \quad k \to \infty, \tag{3.5}$$

it follows immediately that

$$\omega_k = \omega(t_k) \to 0 \quad \text{as } k \to \infty. \tag{3.6}$$

We now introduce the functional

$$\begin{aligned} F(v_k) &:= \int_{\mathbb{R}^d} |\nabla v_k(x)|^2 \, dx - \frac{2}{1+p_c} \int_{\mathbb{R}^d} |v_k(x)|^{1+p_c} \frac{dx}{|x|^{b_c}} \\ &= \omega_k^2 \int_{\mathbb{R}^d} |\nabla u(t_k, x)|^2 \, dx - \frac{2\omega_k^2}{1+p_c} \int_{\mathbb{R}^d} |u(t_k, x)|^{1+p_c} \frac{dx}{|x|^{b_c}} \\ &= \omega_k^2 \left( E(u_0) - \frac{2}{1+p} \int_{\mathbb{R}^d} |u(t_k, x)|^{1+p} \frac{dx}{|x|^b} \right). \end{aligned} \tag{3.7}$$

Making use of the inhomogeneous Gagliardo–Nirenberg inequality (2.2) in the right-hand side of Eq (3.7), we arrive at

$$\begin{aligned} |F(v_k)| &\lesssim \omega_k^2 \left( |E(u_0)| + \left( \int_{\mathbb{R}^d} |u(t_k, x)|^2 \, dx \right)^{\frac{1+p}{2} - \frac{d(p-1)+2b}{4}} \left( \int_{\mathbb{R}^d} |\nabla u(t_k, x)|^2 \, dx \right)^{\frac{d(p-1)+2b}{4}} \right) \\ &\lesssim \omega_k^2 + \left( \int_{\mathbb{R}^d} |\nabla u(t_k, x)|^2 \, dx \right)^{\frac{d(p-1)+2b}{4} - 1}. \end{aligned} \tag{3.8}$$

Now, using the conservation of mass

$$\int_{\mathbb{R}^d} |u(t_k, x)|^2 \, dx = \int_{\mathbb{R}^d} |u_0(x)|^2 \, dx \tag{3.9}$$

together with the facts that $\omega_k \to 0$ and, by the subcritical assumption on $p$,

$$\frac{d(p-1)+2b}{4} - 1 < 0 \tag{3.10}$$

and

$$\int_{\mathbb{R}^d} |\nabla u(t_k, x)|^2 \, dx \to \infty, \quad \text{as} \quad k \to \infty, \tag{3.11}$$

the estimate (3.8) implies that

$$F(v_k) \to 0, \quad \text{as} \quad k \to \infty. \tag{3.12}$$

Moreover, from the conservation of mass and the definition of $\omega_k$, we deduce the following identities:

$$\begin{aligned} \int_{\mathbb{R}^d} |v_k(x)|^2 \, dx &= \int_{\mathbb{R}^d} |u_0(x)|^2 \, dx, \\ \int_{\mathbb{R}^d} |\nabla v_k(x)|^2 \, dx &= \int_{\mathbb{R}^d} |\nabla \varphi_{b_c}(x)|^2 \, dx. \end{aligned} \tag{3.13}$$

Hence, by (3.7), (3.12), and (3.13), we obtain the convergence

$$\int_{\mathbb{R}^d} |v_k(x)|^{1+p_c} \frac{dx}{|x|^{b_c}} \to \frac{1+p_c}{2} \int_{\mathbb{R}^d} |\nabla \varphi_{b_c}(x)|^2 \, dx \quad \text{as } k \to \infty. \tag{3.14}$$

By (3.13), the sequence $(v_k)$ is bounded in $H^1(\mathbb{R}^d)$, so we deduce the existence of a function $\psi \in H^1(\mathbb{R}^d)$ such that, up to extraction of a subsequence,

$$v_k \rightharpoonup \psi \quad \text{weakly in } H^1(\mathbb{R}^d). \tag{3.15}$$

We now introduce the positive constants

$$m := \left(\frac{1+p_c}{2}\int_{\mathbb{R}^d} |\nabla\varphi_{b_c}(x)|^2\,dx\right)^{\frac{1}{1+p_c}}, \qquad M := \left(\int_{\mathbb{R}^d} |\nabla\varphi_{b_c}(x)|^2\,dx\right)^{\frac{1}{2}}. \tag{3.16}$$

Hence, by the second line in (3.14) via (3.16), the real numbers satisfy the assumptions in Proposition 2.3. Therefore, applying the lower bound from Proposition 2.3 with the choices of $m$ and $M$ given in (3.16), we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} |\psi(x)|^2\,dx &\geq \left(\frac{2}{(1+p_c)M^2}\right)^{\frac{2}{p_c-1}} m^{\frac{2(1+p_c)}{p_c-1}} \int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^2\,dx \\ &= \int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^2\,dx. \end{aligned}\tag{3.17}$$

From the blow-up condition (1.12), we deduce that

$$\frac{\nu(t_k)}{\omega_k} = \frac{\nu(t_k)\left(\int_{\mathbb{R}^d} |\nabla u(t_k,x)|^2\,dx\right)^{1/2}}{\left(\int_{\mathbb{R}^d} |\nabla\varphi_{b_c}(x)|^2\,dx\right)^{1/2}} \to \infty \quad \text{as } k\to\infty. \tag{3.18}$$

Consequently, for any fixed $R>0$, there exists $k_0\in\mathbb{N}$ such that for all $k\geq k_0$, we have $R\omega_k < \nu(t_k)$. It follows that

$$\begin{aligned}\liminf_{k\to\infty} \int_{|x|\leq \nu(t_k)} |u(t_k,x)|^2 dx &\geq \liminf_{k\to\infty} \int_{|x|\leq R\omega_k} |u(t_k,x)|^2 dx \\ &\geq \liminf_{k\to\infty} \int_{|x|\leq R} \omega_k^d |u(t_k,\omega_k x)|^2 dx \\ &\geq \liminf_{k\to\infty} \int_{|x|\leq R} |v_k(x)|^2 dx. \end{aligned}\tag{3.19}$$

Combining this with the weak convergence (3.15) and the lower semicontinuity of the $L^2$-norm, we obtain

$$\liminf_{k\to\infty} \int_{|x|\leq\nu(t_k)} |u(t_k,x)|^2\,dx \geq \liminf_{k\to\infty} \int_{|x|\leq R} |v_k(x)|^2\,dx \geq \int_{|x|\leq R} |\psi(x)|^2\,dx. \tag{3.20}$$

Since $R>0$ was chosen arbitrarily, we may pass to the limit $R\to\infty$ in (3.20), to get

$$\liminf_{k\to\infty} \int_{|x|\leq\nu(t_k)} |u(t_k,x)|^2\,dx \geq \int_{\mathbb{R}^d} |\psi(x)|^2\,dx. \tag{3.21}$$

Now, by virtue of (3.17) and the arbitrariness of the sequence $\{t_k\}$, we conclude that

$$\liminf_{t\to T} \int_{|x|\leq\nu(t)} |u(t,x)|^2\,dx \geq \int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^2\,dx.$$

This completes the proof of Proposition 1.2.

*3.2. Characterization of singular solutions at the ground-state mass level*

This subsection is devoted to the proof of Proposition 1.3. We retain the notation introduced in Subsection 3.1. Combining the conservation of mass with the lower bound obtained in (3.17) and (3.13) via (1.17), we derive the following chain of inequalities:

$$\begin{aligned}\int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^2\,dx &\leq \int_{\mathbb{R}^d} |\psi(x)|^2\,dx \leq \liminf_{k\to\infty} \int_{\mathbb{R}^d} |v_k(x)|^2\,dx \\ &= \int_{\mathbb{R}^d} |u_0(x)|^2\,dx = \int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^2\,dx. \end{aligned} \tag{3.22}$$

Particularly, (3.22) implies that

$$\int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^2\,dx = \int_{\mathbb{R}^d} |\psi(x)|^2\,dx. \tag{3.23}$$

Thus, (3.15) and (3.23) give that when $k \to \infty$,

$$v_k \to \psi, \quad \text{in} \quad L^2. \tag{3.24}$$

Moreover, by Gagliardo–Nirenberg estimate (2.2), we have

$$\begin{aligned}\int_{\mathbb{R}^d} |v_k(x) - \psi(x)|^{1+p_c}\,\frac{dx}{|x|^{b_c}} &\lesssim \left(\int_{\mathbb{R}^d} |v_k(x) - \psi(x)|^2\,dx\right)^{\frac{p_c-1}{2}} \left(\int_{\mathbb{R}^d} |\nabla v_k(x) - \nabla\psi(x)|^2\,dx\right) \\ &\to 0. \end{aligned} \tag{3.25}$$

Therefore, by (3.25) via (3.14), it follows that

$$\frac{1+p_c}{2}\int_{\mathbb{R}^d} |\nabla\varphi_{b_c}|^2\,dx = \lim_{k\to\infty}\int_{\mathbb{R}^d} |v_k|^{1+p_c}\,\frac{dx}{|x|^{b_c}} = \int_{\mathbb{R}^d} |\psi|^{1+p_c}\,\frac{dx}{|x|^{b_c}}. \tag{3.26}$$

Combining (3.26) with (2.2) and (3.23) then, yields

$$\begin{aligned}&\frac{1+p_c}{2}\int_{\mathbb{R}^d} |\nabla\varphi_{b_c}(x)|^2\,dx \\ &\leq \mathsf{C}_{\text{opt}}\left(\int_{\mathbb{R}^d} |\psi(x)|^2\,dx\right)^{\frac{p_c-1}{2}}\left(\int_{\mathbb{R}^d} |\nabla\psi(x)|^2\,dx\right) \\ &\leq \frac{1+p_c}{p_c-1}\left(\frac{p_c-1}{2}\right)\left(\int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^2\,dx\right)^{\frac{-(p_c-1)}{2}}\left(\int_{\mathbb{R}^d} |\psi(x)|^2\,dx\right)^{\frac{p_c-1}{2}}\left(\int_{\mathbb{R}^d} |\nabla\psi(x)|^2\,dx\right) \\ &= \frac{1+p_c}{2}\int_{\mathbb{R}^d} |\nabla\psi(x)|^2\,dx. \end{aligned} \tag{3.27}$$

Thus, (3.13) and (3.27) imply that

$$\|\nabla v_k\|_{L^2} \leq \|\nabla\psi\|_{L^2}. \tag{3.28}$$

Hence, (3.28) coupled with the lower semicontinuity of the $L^2$-norm via (3.15) gives

$$\|\nabla\psi\|_{L^2} \leq \lim_k \|\nabla v_k\|_{L^2} \leq \|\nabla\psi\|_{L^2}. \tag{3.29}$$

Consequently,

$$\|\nabla v_k\|_{L^2} = \|\nabla \psi\|_{L^2}. \tag{3.30}$$

Therefore, the weak convergence (3.15) via (3.30) gives that

$$\lim_{k\to\infty} \|v_k - \psi\|_{H^1} = 0. \tag{3.31}$$

Furthermore, by (3.26), we have

$$\begin{aligned} F(\psi) &= \int_{\mathbb{R}^d} |\nabla \psi(x)|^2 \, dx - \frac{2}{1+p_c} \int_{\mathbb{R}^d} |\psi(x)|^{1+p_c} \frac{dx}{|x|^{b_c}} \\ &= \lim_{k\to\infty} \left( \int_{\mathbb{R}^d} |\nabla \varphi_{b_c}(x)|^2 \, dx - \frac{2}{1+p_c} \int_{\mathbb{R}^d} |v_k(x)|^{1+p_c} \frac{dx}{|x|^{b_c}} \right) \\ &= 0. \end{aligned} \tag{3.32}$$

Consequently, by Lemma 2.2, there exist parameters $\lambda > 0$ and $\theta \in [0, 2\pi]$ such that

$$\psi(x) = \lambda^{d/2} e^{i\theta} \varphi_{b_c}(\lambda x). \tag{3.33}$$

Now, thanks to (3.30) and (3.13), it follows that

$$\int_{\mathbb{R}^d} |\nabla \psi(x)|^2 \, dx = \int_{\mathbb{R}^d} |\nabla \varphi_{b_c}(x)|^2 \, dx, \tag{3.34}$$

so, using (3.33), yields

$$\lambda^2 \int_{\mathbb{R}^d} |\nabla \varphi_{b_c}(x)|^2 \, dx = \int_{\mathbb{R}^d} |\nabla \psi(x)|^2 \, dx = \int_{\mathbb{R}^d} |\nabla \varphi_{b_c}(x)|^2 \, dx. \tag{3.35}$$

Therefore, because $\lambda > 0$, we deduce from (3.35) that $\lambda = 1$. Hence,

$$\psi(x) = e^{i\theta} \varphi_{b_c}(x). \tag{3.36}$$

Finally, combining (3.31) and (3.36), we obtain the strong convergence in $H^1(\mathbb{R}^d)$:

$$\omega(t)^{d/2} u(t, \omega(t)\cdot) \to \{e^{i\theta} \varphi_{b_c}(\cdot), \quad \theta \in [0, 2\pi]\} \quad \text{as } t \to T.$$

This completes the proof of Proposition 1.3.

## 4. Proof of the main result

We now turn to the proof of our main result, Theorem 1.1. The demonstration proceeds in two distinct stages.

### 4.1. Global existence

The first step in the proof of Theorem 1.1 consists of showing that any solution with initial mass at or below the ground-state mass exists for all positive times. For simplicity, we take two separate cases.

(1) Assume now that the initial mass lies strictly below the ground-state level:

$$\int_{\mathbb{R}^d} |u_0(x)|^2\,dx < \int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^2\,dx. \tag{4.1}$$

Applying the inhomogeneous Gagliardo–Nirenberg inequality from Proposition 2.1, we obtain the following estimate for the kinetic energy:

$$\begin{aligned}\int_{\mathbb{R}^d} |\nabla u(t,x)|^2\,dx &= E(u_0) + \frac{2}{1+p_c}\int_{\mathbb{R}^d} |u(t,x)|^{1+p_c}\,|x|^{-b_c}\,dx - \frac{2}{1+p}\int_{\mathbb{R}^d} |u(t,x)|^{1+p}\,|x|^{-b}\,dx\\ &\le E(u_0) + \frac{2}{1+p_c}\mathsf{C}_{\mathrm{opt}}\left(\int_{\mathbb{R}^d} |u_0(x)|^2\,dx\right)^{\frac{p_c-1}{2}}\left(\int_{\mathbb{R}^d} |\nabla u(t,x)|^2\,dx\right)\\ &\le E(u_0) + \left(\frac{\int_{\mathbb{R}^d} |u_0(x)|^2\,dx}{\int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^2\,dx}\right)^{\frac{p_c-1}{2}}\int_{\mathbb{R}^d} |\nabla u(t,x)|^2\,dx.\end{aligned} \tag{4.2}$$

From (4.2) and the strict mass assumption

$$\int_{\mathbb{R}^d} |u_0(x)|^2\,dx < \int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^2\,dx, \tag{4.3}$$

we deduce that the coefficient multiplying the gradient norm on the right-hand side is strictly less than one. Therefore, rearranging (4.2) yields the uniform bound

$$\int_{\mathbb{R}^d} |\nabla u(t,x)|^2\,dx \le \frac{E(u_0)}{1-\left(\frac{\int_{\mathbb{R}^d} |u_0(x)|^2\,dx}{\int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^2\,dx}\right)^{\frac{p_c-1}{2}}} = \frac{E(u_0)}{1-\frac{2}{1+p_c}\mathsf{C}_{\mathrm{opt}}\left(\int_{\mathbb{R}^d} |u_0(x)|^2\,dx\right)^{\frac{p_c-1}{2}}}. \tag{4.4}$$

Finally, the inequality (4.4) gives $\sup_t \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} < \infty$. Thus, $u$ is global.

(2) We now consider the case where the initial mass equals the ground-state mass:

$$\int_{\mathbb{R}^d} |u_0(x)|^2\,dx = \int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^2\,dx. \tag{4.5}$$

We argue by contradiction to show that the maximal solution to the Cauchy problem (INLS) exists globally. To this end, we assume the existence of a sequence $t_k \to T^{\max}$ such that the gradient norm becomes unbounded:

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} |\nabla u(t_k,x)|^2\,dx = \infty. \tag{4.6}$$

We will employ the same notation as in Subsection 3.1. By the energy conservation law, we write

$$E(u_0) = E(u(t_k)) \ge F(u(t_k)) = \omega_k^{-2} F(v_k). \tag{4.7}$$

Because $\omega_k \to 0$, when $k \to \infty$, by (4.7), it follows that

$$\limsup_{k\to\infty} F(v_k) \leq 0. \tag{4.8}$$

Now, by Proposition 1.3, we write

$$v_k \to e^{i\theta}\varphi_{b_c}, \text{ strongly in } H^1. \tag{4.9}$$

Thus, (4.9) gives

$$\int_{\mathbb{R}^d} |u(t_k)|^{1+p} \frac{dx}{|x|^b} = \omega_k^{-\left(b+\frac{d(p-1)}{2}\right)} \int_{\mathbb{R}^d} |v_k|^{1+p} \frac{dx}{|x|^b} \to \infty, \quad \text{when} \quad k \to \infty. \tag{4.10}$$

Now, by (2.2) and (2.3), we have

$$\begin{aligned} F(u) &= \int_{\mathbb{R}^d} |\nabla u|^2\, dx - \frac{2}{1+p_c} \int_{\mathbb{R}^d} |u|^{1+p_c} \frac{dx}{|x|^{b_c}} \\ &\geq \int_{\mathbb{R}^d} |\nabla u|^2\, dx - \frac{2}{1+p_c} \mathsf{C}_{\text{opt}} \left(\int_{\mathbb{R}^d} |u|^2\, dx\right)^{\frac{1+p_c}{2} - \frac{d(p_c-1)+2b_c}{4}} \left(\int_{\mathbb{R}^d} |\nabla u|^2\, dx\right)^{\frac{d(p_c-1)+2b_c}{4}} \\ &= \int_{\mathbb{R}^d} |\nabla u|^2 \left[1 - \left(\frac{\int_{\mathbb{R}^d} |u_0|^2\, dx}{\int_{\mathbb{R}^d} |\varphi_{b_c}|^2\, dx}\right)^{\frac{p_c-1}{2}}\right] dx \\ &\geq 0, \end{aligned} \tag{4.11}$$

so, by (4.10) via (4.11), we write

$$E(u_0) = E(u(t_k)) \gtrsim \int_{\mathbb{R}^d} |u(t_k)|^{1+p} \frac{dx}{|x|^b} \to \infty, \quad \text{when} \quad k \to \infty. \tag{4.12}$$

The contradiction (4.12) ends the proof of the global existence.

### *4.2. Finite–time blow-up*

Assume that $\|u_0\|_{L^2(\mathbb{R}^d)} > \|\varphi_{b_c}\|_{L^2(\mathbb{R}^d)}$. First, let us prove the claim

$$\text{If} \quad E(u_0) < 0, \quad \text{then,} \quad T^{\max} < \infty. \tag{4.13}$$

*Proof of the claim.* We now apply the Virial identity from Proposition 2.2 with a carefully chosen test function $\Phi$. For a fixed parameter $R > 0$, we introduce the smooth radial function

$$\Phi(r) := \Phi_R(r) := R^2 \int_0^{\frac{r}{R}} \Upsilon(s)\, ds. \tag{4.14}$$

Inspired by the approach developed in [40], we define, for large $m \in \mathbb{N}$, the auxiliary function $\Upsilon_m$ as follows:

$$\Upsilon(r) := \begin{cases} 2r, & 0 \leq r \leq 1, \\ 2r - 2(r-1)^m, & 1 < r \leq 1 + m^{-\frac{1}{m-1}}, \\ \text{smooth and,} \quad \Upsilon' < 0 & 1 + m^{-\frac{1}{m-1}} < r < 2, \\ 0, & r \geq 2. \end{cases} \tag{4.15}$$

Following the construction in [31], we assume that the function $\Phi$ satisfies the following pointwise estimates:

$$\min\Big\{2-\Phi''(r), 2-\frac{\Phi'(r)}{r}, \Phi'(r)-r\Phi''(r)\Big\}\geq 0. \tag{4.16}$$

Furthermore, we define the functions

$$\Phi_{1,R}(r) := 4\Big(2-\frac{\Phi'(r)}{|x|}\Big); \tag{4.17}$$

$$\Phi_{2,R} := -\frac{2(p_c-1)}{1+p_c}\Big(\Phi''+\Big(d-1+\frac{2b_c}{p_c-1}\Big)\frac{\Phi'}{|x|}-\frac{2d(p_c-1)+4b_c}{p_c-1}\Big). \tag{4.18}$$

We rewrite the second function above as follows:

$$\Phi_{2,R} = -\frac{2(p_c-1)}{1+p_c}\Big(\Phi''-2+\frac{2(d-1)+b}{2-b}\Big(\frac{\Phi'}{|x|}-2\Big)\Big)\geq 0, \tag{4.19}$$

where we have used (4.16). Moreover, the function $\Phi_{2,R}$ satisfies

$$\|\Phi_{2,R}\|_{L^\infty}\lesssim 1, \quad \|\nabla\Phi_{2,R}\|_{L^2}\lesssim \frac{1}{R}, \tag{4.20}$$

where the first estimate is immediate from the definition of the function, whereas the second follows as in [31, Eq (3.9)]. Finally, arguing as in [31, Eq (3.13)], it follows that for $0<\delta\ll 1$,

$$\delta\Phi_{2,R}^{\frac{2}{2-b_c}}-\Phi_{1,R}\leq 0 \quad \text{on} \quad B_R^c. \tag{4.21}$$

Note that [31, Eq (3.9)] and [31, Eq (3.13)] rely on an explicit computation in certain subdomains described by (4.15). To compute the terms involving second-order derivatives of the weight function $I_\Phi$, we recall the radial identity

$$\partial_j\partial_k = \left(\frac{\delta_{jk}}{r}-\frac{x_jx_k}{r^3}\right)\frac{d}{dr}+\frac{x_jx_k}{r^2}\frac{d^2}{dr^2}, \tag{4.22}$$

which will be used in conjunction with Proposition 2.2, to get

$$\begin{aligned} I_\Phi''(t) = {} & 4\int_{\mathbb{R}^d}\frac{\Phi'}{|x|}|\nabla u|^2\,dx+4\int_{\mathbb{R}^d}\left(\frac{\Phi''}{|x|^2}-\frac{\Phi'}{|x|^3}\right)|\nabla u\cdot x|^2\,dx-\int_{\mathbb{R}^d}\Delta^2\Phi|u|^2\,dx\\ & +\frac{2(p-1)}{1+p}\int_{\mathbb{R}^d}\left(\Phi''+(d-1)\frac{\Phi'}{|x|}\right)|u|^{1+p}\frac{dx}{|x|^b}+\frac{4b}{1+p}\int_{\mathbb{R}^d}\frac{\Phi'}{|x|}|u|^{1+p}\frac{dx}{|x|^b}\\ & -\frac{2(p_c-1)}{1+p_c}\int_{\mathbb{R}^d}\left(\Phi''+(d-1)\frac{\Phi'}{|x|}\right)|u|^{1+p_c}\frac{dx}{|x|^{b_c}}-\frac{4b_c}{1+p_c}\int_{\mathbb{R}^d}\frac{\Phi'}{|x|}|u|^{1+p_c}\frac{dx}{|x|^{b_c}}. \end{aligned} \tag{4.23}$$

We decompose the above integrals in the centered ball of radius $R>0$, denoted by $B_R$ and its complementary $B_R^c$. Using the next identities on $B_R$,

$$\Phi(r)=r^2, \quad \Phi'(r)=2r, \quad \Phi''(r)=2, \quad \forall\, r\in B_R, \tag{4.24}$$

we write the first part of $I''_\Phi$ as follows:

$$\begin{aligned}\left(I''_\Phi(t)\right)_1 &= 8\int_{B_R}|\nabla u|^2\,dx+\frac{4d(p-1)}{1+p}\int_{B_R}|u|^{1+p}\frac{dx}{|x|^b}+\frac{8b}{1+p}\int_{B_R}|u|^{1+p}\frac{dx}{|x|^b}\\&\quad-\frac{4d(p_c-1)}{1+p_c}\int_{B_R}|u|^{1+p_c}\frac{dx}{|x|^{b_c}}-\frac{8b_c}{1+p_c}\int_{B_R}|u|^{1+p_c}\frac{dx}{|x|^{b_c}}\\&=8\Big(\int_{B_R}|\nabla u|^2\,dx+\frac{d(p-1)}{2(1+p)}\int_{B_R}|u|^{1+p}\frac{dx}{|x|^b}+\frac{b}{1+p}\int_{B_R}|u|^{1+p}\frac{dx}{|x|^b}\\&\quad-\frac{d(p_c-1)}{2(1+p_c)}\int_{B_R}|u|^{1+p_c}\frac{dx}{|x|^{b_c}}-\frac{b_c}{1+p_c}\int_{B_R}|u|^{1+p_c}\frac{dx}{|x|^{b_c}}\Big).\end{aligned}\tag{4.25}$$

Using

$$p_c=1+\frac{2(2-b_c)}{d},$$

we rewrite (4.25) :

$$\begin{aligned}\left(I''_\Phi(t)\right)_1 &= 8\Big(\int_{\mathbb{R}^d}|\nabla u|^2\,dx+\frac{d(p-1)+2b}{2(1+p)}\int_{\mathbb{R}^d}|u|^{1+p}\frac{dx}{|x|^b}-\frac{2}{1+p_c}\int_{\mathbb{R}^d}|u|^{1+p_c}\frac{dx}{|x|^{b_c}}\Big)\\&\quad-8\Big(\int_{B_R^c}|\nabla u|^2\,dx+\frac{d(p-1)+2b}{2(1+p)}\int_{B_R^c}|u|^{1+p}\frac{dx}{|x|^b}-\frac{2}{1+p_c}\int_{B_R^c}|u|^{1+p_c}\frac{dx}{|x|^{b_c}}\Big).\end{aligned}\tag{4.26}$$

The second part of $I''_\Phi$ reads

$$\begin{aligned}\left(I''_\Phi(t)\right)_2 &= 4\int_{B_R^c}\frac{\Phi'}{|x|}|\nabla u|^2\,dx+4\int_{B_R^c}\left(\frac{\Phi''}{|x|^2}-\frac{\Phi'}{|x|^3}\right)|\nabla u\cdot x|^2\,dx-\int_{B_R^c}\Delta^2\Phi|u|^2\,dx\\&\quad+\frac{2(p-1)}{1+p}\int_{B_R^c}\left(\Phi''+(d-1)\frac{\Phi'}{|x|}\right)|u|^{1+p}\frac{dx}{|x|^b}+\frac{4b}{1+p}\int_{B_R^c}\frac{\Phi'}{|x|}|u|^{1+p}\frac{dx}{|x|^b}\\&\quad-\frac{2(p_c-1)}{1+p_c}\int_{B_R^c}\left(\Phi''+(d-1)\frac{\Phi'}{|x|}\right)|u|^{1+p_c}\frac{dx}{|x|^{b_c}}-\frac{4b_c}{1+p_c}\int_{B_R^c}\frac{\Phi'}{|x|}|u|^{1+p_c}\frac{dx}{|x|^{b_c}}.\end{aligned}\tag{4.27}$$

Hence, by (4.23), (4.26) and (4.27), it follows that

$$I''_\Phi(t)=4K(u(t))+I_1+I_2+I_3+I_4,\tag{4.28}$$

where the operator $K$ is given by

$$K(u):=2\Big(\int_{\mathbb{R}^d}|\nabla u|^2\,dx+\frac{d(p-1)+2b}{2(1+p)}\int_{\mathbb{R}^d}|u|^{1+p}\frac{dx}{|x|^b}-\frac{2}{1+p_c}\int_{\mathbb{R}^d}|u|^{1+p_c}\frac{dx}{|x|^{b_c}}\Big)\tag{4.29}$$

and

$$\begin{aligned}I_1&:=-\int_{\mathbb{R}^d}\Delta^2\Phi|u|^2\,dx,\\I_2&:=4\int_{\mathbb{R}^d}\left(\frac{\Phi'}{|x|}-2\right)|\nabla u|^2\,dx+4\int_{\mathbb{R}^d}\left(\frac{\Phi''}{|x|^2}-\frac{\Phi'}{|x|^3}\right)|\nabla u\cdot x|^2\,dx.\end{aligned}$$

Furthermore, we compute

$$\begin{aligned} I_3 &:= \frac{2(p-1)}{1+p}\int_{\mathbb{R}^d}\left(\Phi'' + \Big(d-1+\frac{2b}{p-1}\Big)\frac{\Phi'}{|x|} - \frac{2d(p-1)+4b}{p-1}\right)|u|^{1+p}\,\frac{dx}{|x|^b} \\ &= \frac{2(p-1)}{1+p}\int_{\mathbb{R}^d}\left(\Phi'' - 2 + \Big(d-1+\frac{2b}{p-1}\Big)\Big(\frac{\Phi'}{|x|}-2\Big)\right)dx \\ &\le 0, \end{aligned} \tag{4.30}$$

where we used (4.15) in the last line. Moreover,

$$I_4 := -\frac{2(p_c-1)}{1+p_c}\int_{\mathbb{R}^d}\left(\Phi'' + \Big(d-1+\frac{2b_c}{p_c-1}\Big)\frac{\Phi'}{|x|} - \frac{2d(p_c-1)+4b_c}{p_c-1}\right)|u|^{1+p_c}\,\frac{dx}{|x|^{b_c}}.$$

We now focus on estimating the terms $I_1$, $I_2$, $I_3$, and $I_4$, which will be handled successively. First, by the mass conservation, it follows that

$$|I_1| \le \frac{C}{R^2}. \tag{4.31}$$

By (4.16), we have

$$\begin{aligned} I_2 &= 4\bigg(\int_{\mathbb{R}^d}\Big(\frac{\Phi'}{|x|}-2\Big)|\nabla u|^2\,dx + \int_{\mathbb{R}^d}\Big(\Phi'' - \frac{\Phi'}{|x|}\Big)\frac{|\nabla u\cdot x|^2}{|x|^2}\,dx\bigg) \\ &\le 4\int_{\mathbb{R}^d}\Big(\frac{\Phi'}{|x|}-2\Big)|\nabla u|^2\,dx. \end{aligned} \tag{4.32}$$

Now, we observe that

$$\int_{|x|\le R}\left(\Phi'' + \Big(d-1+\frac{2b}{p-1}\Big)\frac{\Phi'}{|x|} - \frac{2d(p-1)+4b}{p-1}\right)|u|^{1+p}\,\frac{dx}{|x|^b} = 0,$$

where we used the fact that $\Phi'(r) = 2r$, and $\Phi''(r) = 2$ for any $0 \le r \le R$. Let us write

$$\begin{aligned} I_4 &= \frac{2(p_c-1)}{1+p_c}\int_{|x|>R}\left(\Phi'' + \Big(d-1+\frac{2b_c}{p_c-1}\Big)\frac{\Phi'}{|x|} - \frac{2d(p_c-1)+4b_c}{p_c-1}\right)|u|^{1+p_c}|x|^{-b_c}\,dx \\ &= -\int_{|x|>R}\Phi_{2,R}|u|^{1+p_c}\,\frac{dx}{|x|^{b_c}}. \end{aligned} \tag{4.33}$$

To simplify the exposition, we start our attention to the case $d \neq 2$. Applying (2.6) together with Young's inequality, we obtain for $\delta > 0$

$$\begin{aligned} I_4 &\lesssim \frac{1}{R^{b_c}}\int_{|x|>R}\Phi_{2,R}|u(t)|^{1+p_c}\,dx \\ &\lesssim \frac{1}{R^{b_c}}\left(\Big\|\nabla(\Phi_{2,R}^{\frac{1}{2-b_c}})u(t)\Big\|_{L^2} + \Big\|\Phi_{2,R}^{\frac{1}{2-b_c}}\nabla u(t)\Big\|_{L^2}\right)^{2-b_c}\|u_0\|_{L^2}^{\frac{4-2b_c+db_c}{d}} \\ &\lesssim \delta\Big\|\nabla(\Phi_{2,R}^{\frac{1}{2-b_c}})u(t)\Big\|_{L^2}^2 + \Big\|\Phi_{2,R}^{\frac{1}{2-b_c}}\nabla u(t)\Big\|_{L^2}^2 + \frac{\|u_0\|_{L^2}^{\frac{2(4-2b_c+db_c)}{b_c d}}}{\delta^{\frac{2-b_c}{b_c}}R^2}. \end{aligned} \tag{4.34}$$

Therefore, by (4.34) via (4.20), yields

$$I_4 \lesssim \delta \int_{|x|>R} \Phi_{2,R}^{\frac{2}{2-b_c}} |\nabla u|^2 \, dx + \frac{C(\delta, \|u_0\|_{L^2(\mathbb{R}^d)})}{R^2}. \tag{4.35}$$

Collecting (4.28) and (4.30)–(4.32), it follows that

$$I_\Phi''(t) \leq 4K(u) + \frac{C}{R^2} + C \int_{|x|>R} \Big(\delta \Phi_{2,R}^{\frac{2}{2-b_c}} - \Phi_{1,R}\Big) |\nabla u|^2 \, dx + \frac{C(\delta, \|u_0\|^2_{L^2(\mathbb{R}^d)})}{R^2}. \tag{4.36}$$

Hence, using the identity

$$K(u) = 2\Big(E(u) - \frac{d}{1+p}(p - 1 - \frac{2(2-b)}{d}) \int_{\mathbb{R}^d} |u(t,x)|^{1+p} \, \frac{dx}{|x|^b}\Big) \leq 2E(u), \tag{4.37}$$

via (4.36) and (4.21), we pick $0 < \delta \ll 1$ and get

$$I_\Phi''(t) \leq 8E(u_0) + \frac{C}{R^2}. \tag{4.38}$$

Moreover, because $E(u) = E(u_0) < 0$, we pick

$$R > \sqrt{\frac{C}{6|E(u_0)|}},$$

such that

$$I_\Phi''(t) \leq 8E(u_0) + \frac{C}{R^2} \leq 2E(u_0). \tag{4.39}$$

At this point, integrating twice (4.39), we then have the desired conclusion $T^{\max} < \infty$. Now, by (2.7), we write the following for $d = 2$:

$$\begin{aligned}
I_4 &= -\int_{|x|>R} \Phi_{2,R} \, |u|^{1+p_c} \, \frac{dx}{|x|^{b_c}} \\
&\lesssim \frac{1}{R^{b_c}} \Bigg( \int_{\mathbb{R}^d} \Big| \Phi_{2,R}^{\frac{1}{2-b_c/2}} u \Big|^2 \, dx + \int_{\mathbb{R}^d} \Big| \nabla \Big( \Phi_{2,R}^{\frac{1}{2-b_c/2}} \Big) u \Big|^2 \, dx + \int_{\mathbb{R}^d} \Big| \Phi_{2,R}^{\frac{1}{2-b_c/2}} \nabla u \Big|^2 \, dx \Bigg)^{1-b_c/4} \Bigg( \int_{\mathbb{R}^d} |u|^2 \, dx \Bigg)^{1-b_c/4} \\
&\lesssim \Bigg( \int_{\mathbb{R}^d} \delta \Big| \nabla \Big( \Phi_{2,R}^{\frac{1}{2-b_c/2}} \Big) u \Big|^2 \, dx + \delta \int_{\mathbb{R}^d} \Big| \Phi_{2,R}^{\frac{1}{2-b_c/2}} \nabla u \Big|^2 \, dx \Bigg)^{1-b_c/4} \frac{1}{R^{b_c}} \Bigg( \frac{1}{\delta} \int_{\mathbb{R}^d} |u|^2 \, dx \Bigg)^{1-b_c/4} \\
&\quad + \frac{1}{R^{b_c}} \Bigg( \int_{\mathbb{R}^d} |u_0(x)|^2 \, dx \Bigg)^{4-b}.
\end{aligned} \tag{4.40}$$

Hence, by (4.40) via Young's estimate, it follows that

$$I_4 \lesssim \delta \int_{|x|>R} \Phi_{2,R}^{\frac{2}{2-b_c/2}} |\nabla u|^2 \, dx + \frac{\Big( \int_{\mathbb{R}^d} |u_0(x)|^2 \, dx \Big)^{2-\frac{b}{2}}}{R^{b_c}} + \frac{\delta \int_{\mathbb{R}^d} |u_0(x)|^2 \, dx}{R^2} + \frac{\Big( \int_{\mathbb{R}^d} |u_0(x)|^2 \, dx \Big)^{\frac{4}{b_c}-1}}{\delta^{\frac{4}{b_c}-1} R^4}. \tag{4.41}$$

Collecting (4.28), (4.30)–(4.32) and (4.41) similarly to (4.21) with $\frac{b_c}{2}$ rather than $b_c$, we pick $0 < \delta \ll 1$ and get

$$\begin{aligned} I''_{\Phi}(t) &\leq 4K(u) + \frac{C}{R^2} + C\int_{|x|>R} \left(\delta\Phi_{2,R}^{\frac{2}{2-\frac{b_c}{2}}} - \Phi_{1,R}\right)|\nabla u|^2\,dx \\ &\quad + C\left[\frac{\left(\int_{\mathbb{R}^d}|u_0(x)|^2\,dx\right)^{2-\frac{b}{2}}}{R^{b_c}} + \frac{\delta\int_{\mathbb{R}^d}|u_0(x)|^2\,dx}{R^2} + \frac{\left(\int_{\mathbb{R}^d}|u_0(x)|^2\,dx\right)^{\frac{4}{b_c}-1}}{\delta^{\frac{4}{b_c}-1}R^4}\right] \\ &\leq 8E(u_0) + C\left[\frac{1}{R^{b_c}} + \frac{1}{R^2} + \frac{1}{R^4}\right]. \end{aligned} \tag{4.42}$$

Hence, by (4.42) via the fact that $E(u_0) < 0$, we pick

$$R > \left(\frac{C}{6|E(u_0)|}\right)^{\frac{1}{b_c}}$$

such that

$$I''_{\Phi}(t) \leq 2E(u_0). \tag{4.43}$$

Integrating the differential inequality (4.43) twice in time, we deduce that the solution cannot exist beyond a finite time $T^{\max}$. This proves the claim. □

Having established the finite time blow-up claim (4.13), we now turn to the blow-up part of Theorem 1.1, for negative energy. For parameters $\alpha, \beta > 0$, we define the scaled function

$$u_{\alpha,\beta}(x) := \beta\alpha^{d/2}u(\alpha x). \tag{4.44}$$

A direct computation yields the following identities:

$$\int_{\mathbb{R}^d}|u_{\alpha,\beta}(x)|^2\,dx = \beta^2\int_{\mathbb{R}^d}|u(x)|^2\,dx, \tag{4.45}$$

$$\int_{\mathbb{R}^d}|\nabla u_{\alpha,\beta}(x)|^2\,dx = \alpha^2\beta^2\int_{\mathbb{R}^d}|\nabla u(x)|^2\,dx, \tag{4.46}$$

$$\int_{\mathbb{R}^d}|u_{\alpha,\beta}(x)|^{1+p}\,\frac{dx}{|x|^b} = \alpha^{\frac{d(p-1)+2b}{2}}\beta^{1+p}\int_{\mathbb{R}^d}|u(x)|^{1+p}\,\frac{dx}{|x|^b}. \tag{4.47}$$

Now, for a small parameter $0 < \delta \ll 1$, we choose the initial datum as a scaled version of the ground state:

$$u_0 := (\varphi_{b_c})_{\alpha,\beta}, \tag{4.48}$$

where the scaling parameter $\beta$ is selected so that the mass exceeds the ground-state level by a small amount:

$$\beta := \frac{\sqrt{\delta + \int_{\mathbb{R}^d}|\varphi_{b_c}(x)|^2\,dx}}{\sqrt{\int_{\mathbb{R}^d}|\varphi_{b_c}(x)|^2\,dx}}. \tag{4.49}$$

Thus, by (4.45) and (4.48), it follows that

$$\|u_0\|^2_{L^2(\mathbb{R}^d)} = \|\varphi_{b_c}\|^2_{L^2(\mathbb{R}^d)} + \delta. \tag{4.50}$$

Using the scaling relations (4.45)–(4.47), the energy of the initial data takes the form

$$\begin{aligned} E(u_0) &= \int_{\mathbb{R}^d} |\nabla u_0(x)|^2\,dx - \frac{2}{1+p_c}\int_{\mathbb{R}^d} |u_0(x)|^{1+p_c}\,\frac{dx}{|x|^{b_c}} + \frac{2}{1+p}\int_{\mathbb{R}^d} |u_0(x)|^{1+p}\,\frac{dx}{|x|^{b}} \\ &= \alpha^2\beta^2\int_{\mathbb{R}^d} |\nabla \varphi_{b_c}(x)|^2\,dx - \frac{2\alpha^2\beta^{1+p_c}}{1+p_c}\int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^{1+p_c}\,\frac{dx}{|x|^{b_c}} \\ &\quad + \frac{2\alpha^{\frac{d(p-1)+2b}{2}}\beta^{1+p}}{1+p}\int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^{1+p}\,\frac{dx}{|x|^{b}} \\ &= \alpha^2\beta^2\bigg(\int_{\mathbb{R}^d} |\nabla \varphi_{b_c}(x)|^2\,dx - \frac{2\beta^{p_c-1}}{1+p_c}\int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^{1+p_c}\,\frac{dx}{|x|^{b_c}} \\ &\quad + \frac{2\alpha^{\frac{d(p-1)+2b}{2}-2}\beta^{p-1}}{1+p}\int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^{1+p}\,\frac{dx}{|x|^{b}}\bigg). \end{aligned} \tag{4.51}$$

Substituting the Pohozaev identities (2.5) into (4.51), we obtain

$$E(u_0) = 2\alpha^2\beta^2\bigg(\frac{1}{p_c-1}\left(1-\beta^{p_c-1}\right)\int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^2\,dx + \frac{\alpha^{\frac{d(p-1)+2b}{2}-2}\beta^{p-1}}{1+p}\int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^{1+p}\,\frac{dx}{|x|^{b}}\bigg). \tag{4.52}$$

We now choose the parameter $\alpha$ sufficiently large so that

$$\alpha > \left(\frac{\frac{1}{p_c-1}\left(\beta^{p_c-1}-1\right)\displaystyle\int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^2\,dx}{\frac{\beta^{p-1}}{1+p}\displaystyle\int_{\mathbb{R}^d} |\varphi_{b_c}(x)|^{1+p}\,\frac{dx}{|x|^{b}}}\right)^{\frac{1}{2-\frac{d(p-1)+2b}{2}}}. \tag{4.53}$$

Finally, (4.53) via (4.52) implies that

$$E(u_0) < 0. \tag{4.54}$$

The proof is finished via (4.54) and the previous claim (4.13).

## 5. Numerical simulation

In this section, we set $p = 3$, $d = \lambda = 1$, and $b = b_c/2$, and we present some numerical simulations. Specifically, we plot the $L^2$-norm of the gradient of the solution to (INLS) with $u_0 \in \{\varphi_{b_c}, 2\varphi_{b_c}\}$, where $\varphi_{b_c}$ is the ground state solution to (1.6). We consider three cases: $b_c \in \{0, 1/4, 3/4\}$. For $u_0 = 2\varphi_{b_c}$, we observe finite-time blow-up, and the singularity of the inhomogeneous term appears to accelerate the blow-up speed. For instance, when $b_c = 3/4$, the blow-up is clearly faster than in the case $b_c = 1/4$; see Figure 1. On the other hand, for $u_0 = \varphi_{b_c}$, the solution exists globally in time; see Figure 2. This is consistent with Theorem 1.1.

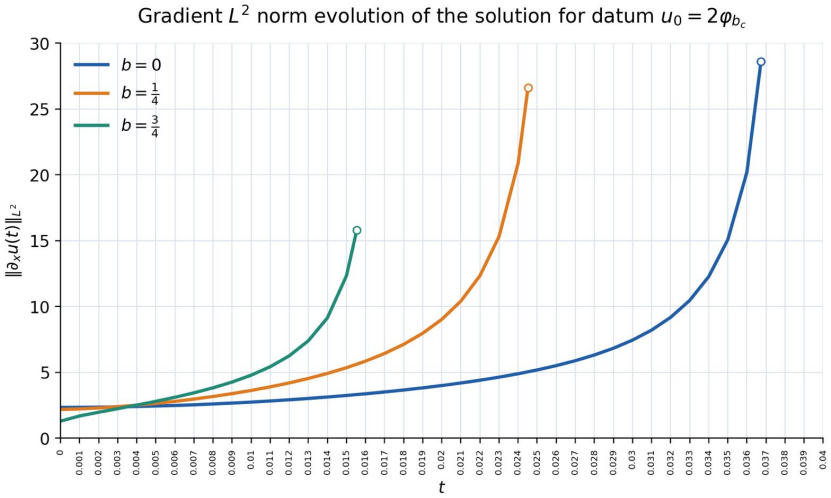


**Figure 1.** Blow-up.

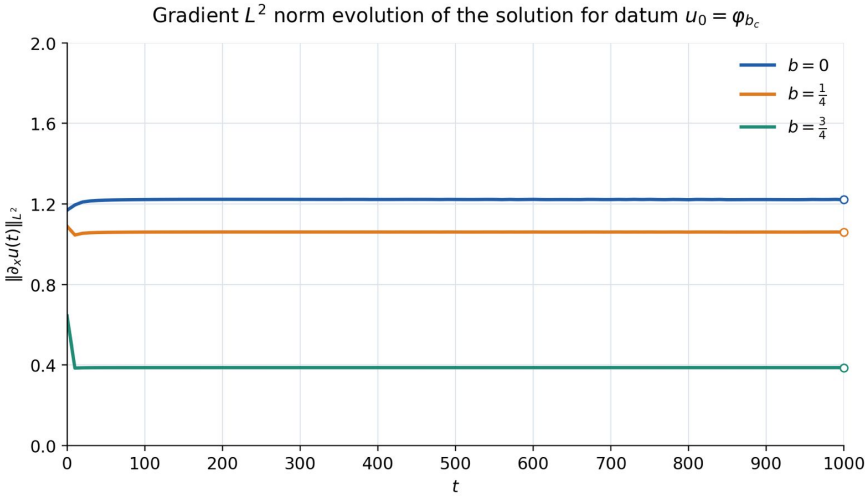


**Figure 2.** Global existence.

## 6. Conclusions

We have resolved the Cauchy problem for an inhomogeneous nonlinear Schrödinger equation with competing nonlinearities—a focusing term at the mass-critical level and a defocusing, mass-subcritical perturbation—and established a sharp mass threshold for unconditional global existence of energy solutions. Our analysis demonstrates that two structural features of the model drive the result. The spatially dependent coefficients break translation invariance, which allows us to eliminate entirely the radial symmetry hypotheses that are commonly indispensable in concentration–compactness based blow-up arguments. At the same time, the defocusing perturbation destroys the scaling symmetry of the pure critical equation, and this has a decisive dynamical consequence: minimal-mass blow-up solutions, a hallmark of the unperturbed critical problem, are ruled out. These results complement

those of [39] by settling the mass-subcritical perturbation regime, and they extend the results of [38] to the inhomogeneous setting while permitting nonradial initial data. Numerical simulations confirm the theoretical predictions. The methods developed here are robust and should apply directly to a broader class of subcritical perturbations and to the analysis of threshold dynamics.

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## Use of Generative-AI tools declaration

During the preparation of this work, the authors used free AI tool for language editing. The authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

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## Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest. No data-sets were generated or analyzed during the current study. This research received no external funding.

## Dedication

In loving memory of the second author's father, Hamdoun Saanouni (1948–2026).

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