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Research article

Second-kind proximal contractions in complete fuzzy metric spaces with a stylized illustration from measles dynamics

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Abstract: We develop a common two-function framework for best proximity points of non-self mappings in complete non-Archimedean fuzzy metric spaces. In the spirit of Proinov, the contractive condition is written with a pair of auxiliary functions, and this single condition is used for four schemes, namely the fuzzy proximal contraction of the second kind and its interpolative Reich–Rus–Ćirić, Kannan, and Hardy–Rogers forms. For each scheme, two best proximity point theorems are proven under different conditions on the auxiliary functions. The Banach type and the Gregori–Sapena type fuzzy proximal contractions are obtained as special cases, and an example shows that this inclusion is strict. Worked examples illustrate the results. As a stylized illustration, the two branches of the period-two cycle of the logistic map, a caricature of biennial measles outbreaks, are treated as disjoint sets, and the best proximity point gives the parameter value at which the oscillation is mildest.

Keywords: best proximity point; fuzzy metric space; interpolative contraction; proximal contraction; Reich–Rus–Ćirić contraction; Kannan contraction; Hardy–Rogers contraction
Mathematics Subject Classification: 47H10, 54H25, 54A40

1. Introduction

The fixed point theory is a basic tool of nonlinear analysis for proving the existence, uniqueness, and approximation of solutions. The strength of such results depends on how the closeness of points is measured and on the contractive condition imposed on the mapping. Schweizer and Sklar [1] introduced statistical metric spaces, in which the distance between two points is described by a probability distribution. Zadeh [2] introduced fuzzy sets as a framework for uncertainty and partial membership. Kramosil and Michálek [3] combined these ideas and defined fuzzy metric spaces, and

George and Veeramani [4] modified this definition into the form that is now most widely used and studied its topological properties.

The fixed point theory in fuzzy metric spaces (FMS) developed quickly after these works. Grabiec [5] obtained some of the first fixed point theorems in this setting, and Gregori and Sapena [6] introduced contractive conditions adapted to fuzzy metric structures. In these works, convergence is expressed by fuzzy closeness tending to one instead of distances tending to zero, and this point of view has been used in many later generalizations.

Most classical fixed point theorems concern self mappings. If $\Phi : \mathcal{A}^o \longrightarrow \mathcal{B}^o$ and $\mathcal{A}^o \cap \mathcal{B}^o = \varnothing$, then the equation $\Phi\sigma^o = \sigma^o$ has no solution, no matter how strong the contraction is. In this case, one looks for a best proximity point, that is, a point whose image is as close to it as the two sets allow. Suzuki, Kikkawa, and Vetro [7] proved existence results of this kind in metric spaces with the property UC, a geometric condition on pairs of sets that extends uniform convexity to metric spaces. Basha [8] extended Banach type contractions to non-self mappings and developed a theory of optimal approximate solutions, which was strengthened in [9] to a global optimal approximation. Espínola, Kosuru, and Veeramani [10] studied the geometric side of the theory through the Pythagorean property. In this way, the best proximity point theory extends the fixed point theory to situations in which an exact fixed point cannot exist.

Interpolative contractions form another line of research. Instead of a single distance term, they combine several terms with suitable exponents in one contractive inequality. Karapınar [11] introduced the interpolative Kannan contraction. Karapınar and Agarwal [12] combined interpolation with simulation functions and obtained interpolative Rus–Reich–Ćirić contractions. Karapınar et al. [13] extended the idea to Hardy–Rogers type contractions, and Karapınar et al. [14] studied Perov interpolative contractions of Suzuki type.

Proinov [15] replaced the contraction constant by two auxiliary functions, which gives more flexible conditions for convergence. This approach has been extended to fuzzy metric spaces. Roldán López de Hierro et al. [16] proved Proinov type fixed point results in non-Archimedean fuzzy metric spaces, and Zhou et al. [17] gave a further approach in the same setting.

Interpolation and fuzzy contractive conditions have also been combined. Ishtiaq et al. [18] studied orthogonal fuzzy interpolative contractions and applied them to Volterra type integral equations and fractional differential equations. Hammad, Aydi, and Park [19] obtained fixed point results in fuzzy cone metric spaces and applied them to systems of Volterra integral equations. In a related direction, Alizadeh, Baleanu, and Rezapour [20] used the Caputo–Fabrizio fractional derivative to study the transient response of a parallel resistor–capacitor–inductor (RCL) circuit.

Altun and Taşdemir [21] introduced interpolative proximal contractions and proved best proximity point theorems for non-self mappings. Javed et al. [22] generalized these contractions to non-Archimedean fuzzy metric spaces, and Javed et al. [23] obtained best proximity point theorems for generalized fuzzy interpolative proximal contractions in complete non-Archimedean fuzzy metric spaces.

A unified two-function treatment of fuzzy proximal contractions (FPC) of the second kind and of their interpolative Reich–Rus–Ćirić, Kannan, and Hardy–Rogers forms is still missing. These contractions combine fuzzy closeness terms in different ways, and treating them by separate inequalities hides their common convergence mechanism. Moreover, when auxiliary functions are added to a contractive condition, its relation to the classical condition must be made precise. Otherwise,

a condition that only looks more general may in fact be more restrictive.

In this paper, we fix a complete non-Archimedean fuzzy metric space $(\mathcal{X}^o, \mathcal{H}^o, *)$, non-empty subsets $\mathcal{A}^o, \mathcal{B}^o \subseteq \mathcal{X}^o$, and a non-self mapping $\Phi : \mathcal{A}^o \longrightarrow \mathcal{B}^o$, and we write the contractive conditions with two auxiliary functions $\mathscr{P}$ and $\mathscr{Q}$ satisfying

$$\mathscr{Q}(t) > \mathscr{P}(t), \qquad t \in (0, 1).$$

To the best of our knowledge, Proinov type conditions [16, 17] and interpolative proximal contractions [21–23] have not been combined before for proximal contractions of the second kind, in which the contractive condition is imposed on the images of the points. Here, a single two-function condition is formulated for the fuzzy proximal contraction of the second kind and for its interpolative Reich–Rus–Ćirić, Kannan, and Hardy–Rogers forms, and two existence theorems are proven for each class under different hypotheses on the auxiliary functions. The Banach type and the Gregori–Sapena type fuzzy proximal contractions are shown to be particular cases of this condition, and an example is given in which no Banach type constant exists. Additionally, it is shown that the closedness of $\mathcal{A}^o$ cannot be removed.

We also give a dynamical illustration based on the logistic map

$$i_{l+1} = r\, i_l(1 - i_l).$$

May [24] showed that this simple map has rich dynamics, including period-doubling bifurcations and a route to chaos, and Earn et al. [25] explained the transitions between regular and irregular measles epidemics. Fixed point methods have also been used for equations arising in population dynamics with an effect on infectious diseases [26]. We regard the low and high prevalence branches of a period-two regime as two disjoint sets. The map between them gives a concrete example of fuzzy proximity and of a best proximity point. This construction is an illustration of the framework and not a substitute for the existence theory.

The paper is organized as follows: Section 2 recalls fuzzy metric spaces, proximity sets, approximate compactness, and best proximity points; Section 3 introduces the $(\mathscr{P}, \mathscr{Q})$-FPC of the second kind and its interpolative Reich–Rus–Ćirić, Kannan, and Hardy–Rogers forms, proves the corresponding best proximity point theorems and gives examples and special cases; Section 4 presents the logistic map illustration; and Section 5 contains the conclusions and some open problems.

2. Preliminaries

Before developing the main constructions, we fix the background notions, notation, and auxiliary facts needed later. Throughout, $\mathbb{N}$ stands for the positive integers and $\mathbb{R}^+$ for the positive real numbers.

Definition 2.1 ([1]). *A binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is called a continuous t-norm if $\forall\, \sigma^o, \nu, \xi, \zeta \in [0, 1]$, the following conditions are satisfied:*
(1) $\sigma^o * \nu = \nu * \sigma^o$;
(2) $(\sigma^o * \nu) * \xi = \sigma^o * (\nu * \xi)$;
(3) $\sigma^o * 1 = \sigma^o$;
(4) $\sigma^o * \nu \leq \xi * \zeta$ *whenever* $\sigma^o \leq \xi$ *and* $\nu \leq \zeta$;
and $*$ is continuous on $[0, 1] \times [0, 1]$.

Fuzzy metrics go back to Kramosil and Michálek [3]. George and Veeramani [4] later modified the definition, and we use their version throughout.

Definition 2.2 ([4]). *Let $\mathcal{X}^o$ be a non-empty set, and let $*$ be a continuous t-norm. A mapping $\mathcal{H}^o : \mathcal{X}^o \times \mathcal{X}^o \times (0,\infty) \longrightarrow (0,1]$ is called a fuzzy metric on $\mathcal{X}^o$ if $\forall\, \sigma^o, \nu^o, \xi \in \mathcal{X}^o$, and $\omega^o, \tau > 0$, the following conditions hold:*
(1) $\mathcal{H}^o(\sigma^o, \nu^o, \omega^o) > 0$;
(2) $\mathcal{H}^o(\sigma^o, \nu^o, \omega^o) = 1 \iff \sigma^o = \nu^o$;
(3) $\mathcal{H}^o(\sigma^o, \nu^o, \omega^o) = \mathcal{H}^o(\nu^o, \sigma^o, \omega^o)$;
(4) $\mathcal{H}^o(\sigma^o, \xi, \omega^o + \tau) \geq \mathcal{H}^o(\sigma^o, \nu^o, \omega^o) * \mathcal{H}^o(\nu^o, \xi, \tau)$; *and*
(5) *the mapping $\mathcal{H}^o(\sigma^o, \nu^o, \cdot) : (0,\infty) \longrightarrow (0,1]$ is continuous.*
The triple $(\mathcal{X}^o, \mathcal{H}^o, *)$ is called a fuzzy metric space.

Here, $\mathcal{H}^o(\sigma^o, \nu^o, \omega^o)$ measures how close σ^o and ν^o are at the parameter value ω^o. The larger this value, the nearer the two points are considered to be.

Convergence, Cauchy sequences, and completeness below are all understood in the usual fuzzy-metric sense.

Definition 2.3 ([4, 6]). *Let $(\mathcal{X}^o, \mathcal{H}^o, *)$ be an FMS and $\{\sigma_l^o\}$ be a sequence in $\mathcal{X}^o$. The sequence $\{\sigma_l^o\}$ converges to $\sigma^o \in \mathcal{X}^o$ if*

$$\lim_{l\to\infty} \mathcal{H}^o(\sigma_l^o, \sigma^o, \omega^o) = 1, \quad \forall\, \omega^o > 0.$$

Definition 2.4 ([4, 6]). *Let $(\mathcal{X}^o, \mathcal{H}^o, *)$ be an FMS. A sequence $\{\sigma_l^o\}$ in $\mathcal{X}^o$ is called a Cauchy sequence if $\forall\, \varepsilon \in (0,1)$ and $\forall\, \omega^o > 0$, $\exists\, l_0 \in \mathbb{N}$ such that*

$$\mathcal{H}^o(\sigma_l^o, \sigma_m^o, \omega^o) > 1 - \varepsilon, \quad \forall\, l, m \geq l_0.$$

The FMS $(\mathcal{X}^o, \mathcal{H}^o, *)$ is called complete if every Cauchy sequence in $\mathcal{X}^o$ converges to an element of $\mathcal{X}^o$.

Fix two non-empty subsets $\mathcal{A}^o$ and $\mathcal{B}^o$ of a fuzzy metric space $(\mathcal{X}^o, \mathcal{H}^o, *)$, and fix $\omega^o > 0$. We define

$$\mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o) = \sup\{\mathcal{H}^o(\sigma^o, \nu^o, \omega^o) : \sigma^o \in \mathcal{A}^o,\ \nu^o \in \mathcal{B}^o\}. \tag{2.1}$$

For a point $\sigma^o \in \mathcal{X}^o$, its fuzzy proximity to the set $\mathcal{B}^o$ is

$$\mathcal{H}^o(\sigma^o, \mathcal{B}^o, \omega^o) = \sup_{\nu^o \in \mathcal{B}^o} \mathcal{H}^o(\sigma^o, \nu^o, \omega^o). \tag{2.2}$$

The pair $(\mathcal{A}^o, \mathcal{B}^o)$ also determines two further sets,

$$\mathcal{A}_0^o(\omega^o) = \{\sigma^o \in \mathcal{A}^o : \exists\, \nu^o \in \mathcal{B}^o,\ \mathcal{H}^o(\sigma^o, \nu^o, \omega^o) = \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o)\}, \tag{2.3}$$

$$\mathcal{B}_0^o(\omega^o) = \{\nu^o \in \mathcal{B}^o : \exists\, \sigma^o \in \mathcal{A}^o,\ \mathcal{H}^o(\sigma^o, \nu^o, \omega^o) = \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o)\}. \tag{2.4}$$

Because $\mathcal{H}^o$ measures closeness rather than distance, $\mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o)$ records the greatest proximity the two sets can achieve. Therefore, taking a supremum in (2.1) plays the role that an infimum of distances plays in the classical metric theory of best proximity points.

Figure 1 sketches this geometrically for a pair of disjoint sets.

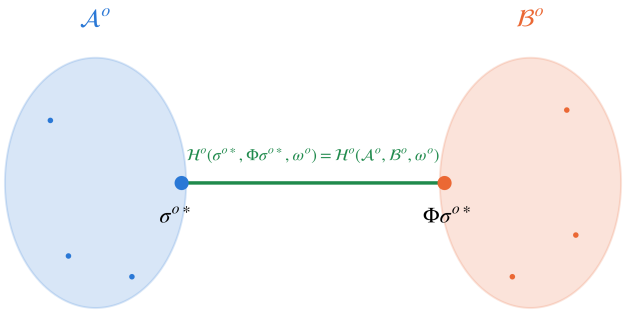


Figure 1. A best proximity point $\sigma^{o*} \in \mathcal{A}^o$ and its image $\Phi\sigma^{o*} \in \mathcal{B}^o$, which attain the optimal proximity of the two sets.

Additionally, our existence arguments rely on the following compactness requirement.

Definition 2.5 ([21, 22]). *Let $(\mathcal{X}^o, \mathcal{H}^o, *)$ be an FMS, and let $\mathcal{A}^o, \mathcal{B}^o$ be non-empty subsets of $\mathcal{X}^o$. The set $\mathcal{B}^o$ is said to be approximately compact (AMC) with respect to $\mathcal{A}^o$ if every sequence $\{v_l^o\} \subseteq \mathcal{B}^o$ that satisfies*

$$\mathcal{H}^o(\sigma^o, v_l^o, \omega^o) \longrightarrow \mathcal{H}^o(\sigma^o, \mathcal{B}^o, \omega^o)$$

for some $\sigma^o \in \mathcal{A}^o$ possesses a convergent subsequence in $\mathcal{B}^o$.

In the theorems below, this notion is used with the roles of the two sets interchanged. Thus, $\mathcal{A}^o$ is AMC with respect to $\mathcal{B}^o$ if every sequence $\{\sigma_l^o\} \subseteq \mathcal{A}^o$ with $\mathcal{H}^o(\eta, \sigma_l^o, \omega^o) \to \mathcal{H}^o(\eta, \mathcal{A}^o, \omega^o)$ for some $\eta \in \mathcal{B}^o$ has a convergent subsequence in $\mathcal{A}^o$.

In other words, if a sequence in $\mathcal{B}^o$ approaches the optimal fuzzy proximity from a point of $\mathcal{A}^o$, then the approximate compactness gives a convergent subsequence. This is used later to obtain a limit of the proximal iteration.

Now, we can state the central notion of the paper.

Definition 2.6 ([22]). *Let $(\mathcal{X}^o, \mathcal{H}^o, *)$ be an FMS, and let $\mathcal{A}^o, \mathcal{B}^o$ be non-empty subsets of $\mathcal{X}^o$. For a mapping*

$$\Phi : \mathcal{A}^o \longrightarrow \mathcal{B}^o,$$

an element $\sigma^{o*} \in \mathcal{A}^o$ is called a best proximity point of Φ if

$$\mathcal{H}^o(\sigma^{o*}, \Phi\sigma^{o*}, \omega^o) = \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o). \tag{2.5}$$

In words, a best proximity point of Φ has an image under Φ that is as fuzzy-close to it as the geometry of $\mathcal{A}^o$ and $\mathcal{B}^o$ allows. If $\mathcal{A}^o = \mathcal{B}^o$, then

$$\mathcal{H}^o(\mathcal{A}^o, \mathcal{A}^o, \omega^o) = 1.$$

Thus, (2.5) collapses to

$$\mathcal{H}^o(\sigma^{o*}, \Phi\sigma^{o*}, \omega^o) = 1,$$

forcing

$$\Phi\sigma^{o*} = \sigma^{o*}.$$

In other words, once the two sets coincide, a best proximity point is simply an ordinary fixed point.

We work throughout with the product t-norm,

$$a * b = ab, \qquad a, b \in [0, 1],$$

and, in the examples and in the estimate below, with a fuzzy metric induced from an ordinary metric d via

$$\mathcal{H}^o(\sigma^o, \nu^o, \omega^o) = e^{-d(\sigma^o, \nu^o)/\omega^o}, \qquad \sigma^o, \nu^o \in \mathcal{X}^o, \quad \omega^o > 0. \tag{2.6}$$

Under this induced fuzzy metric, the ordinary triangle inequality

$$d(\sigma^o, \xi) \leq d(\sigma^o, \nu^o) + d(\nu^o, \xi)$$

translates into

$$-\frac{d(\sigma^o, \xi)}{\omega^o} \geq -\frac{d(\sigma^o, \nu^o)}{\omega^o} - \frac{d(\nu^o, \xi)}{\omega^o}.$$

Hence,

$$\mathcal{H}^o(\sigma^o, \xi, \omega^o) \geq \mathcal{H}^o(\sigma^o, \nu^o, \omega^o) * \mathcal{H}^o(\nu^o, \xi, \omega^o), \qquad \sigma^o, \nu^o, \xi \in \mathcal{X}^o, \quad \omega^o > 0. \tag{2.7}$$

Inequality (2.7) is sharper than the general George–Veeramani triangle inequality, since here all three fuzzy-metric terms share the same parameter ω^o. This sharper form is what makes the cross term in the Hardy–Rogers type argument tractable, and it is used repeatedly in the proofs that follow. A fuzzy metric satisfying (2.7) is called non-Archimedean [16, 22].

Moreover, it is clear from (2.6) that

$$\mathcal{H}^o(\sigma^o, \nu^o, \omega^o) = e^{-d(\sigma^o, \nu^o)/\omega^o}$$

strictly decreases in the ordinary distance $d(\sigma^o, \nu^o)$, so that

$$\sup_{\substack{\sigma^o \in \mathcal{A}^o \\ \nu^o \in \mathcal{B}^o}} \mathcal{H}^o(\sigma^o, \nu^o, \omega^o) = \exp\left(-\frac{\inf\limits_{\substack{\sigma^o \in \mathcal{A}^o \\ \nu^o \in \mathcal{B}^o}} d(\sigma^o, \nu^o)}{\omega^o} \right). \tag{2.8}$$

Hence, maximizing the fuzzy proximity is equivalent to minimizing the corresponding ordinary distance. This links the fuzzy formulation of best proximity with its classical metric counterpart.

Under the standing assumptions of Section 3, the sets $\mathcal{A}_0^o(\omega^o)$ and $\mathcal{B}_0^o(\omega^o)$ do not depend on ω^o, and we simply write $\mathcal{A}_0^o$ and $\mathcal{B}_0^o$.

We write the two auxiliary functions used in the main results as

$$\mathscr{P}, \mathscr{Q} : (0, 1] \longrightarrow \mathbb{R},$$

and state the precise hypotheses on them alongside the relevant definitions and theorems in Section 3. The pair used in Example 3.2 is

$$\mathscr{P}(s) = s, \qquad \mathscr{Q}(s) = \sqrt{s}, \qquad s \in (0, 1], \tag{2.9}$$

which satisfies, for every $s \in (0, 1)$,

$$\mathscr{Q}(s) > \mathscr{P}(s).$$

Figure 2 displays the pair in (2.9) and the strict separation between the two auxiliary functions on $(0, 1)$.

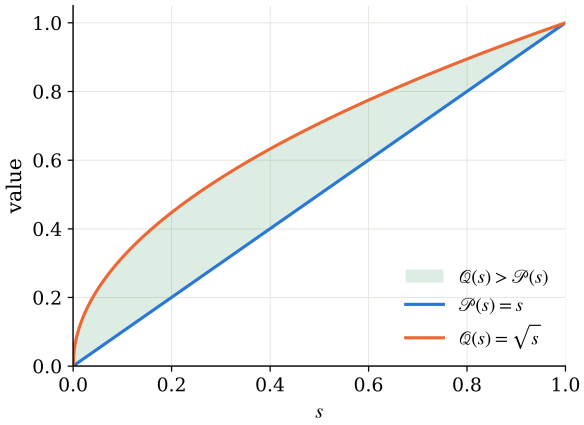


Figure 2. The auxiliary functions $\mathscr{P}(s) = s$ and $\mathscr{Q}(s) = \sqrt{s}$ of Example 3.2, with $\mathscr{Q} > \mathscr{P}$ on $(0, 1)$.

For the convenience of the reader, the main symbols used in the paper are collected in Table 1.

Table 1. List of symbols and their meanings.

Symbol	Meaning
$(\mathcal{X}^o, \mathcal{H}^o, *)$	fuzzy metric space with the product t-norm $*$
$\omega^o > 0$	parameter of the fuzzy metric
d	ordinary metric that induces $\mathcal{H}^o$ in (2.6)
$\mathcal{A}^o, \mathcal{B}^o$	non-empty subsets of $\mathcal{X}^o$
Φ	non-self mapping from $\mathcal{A}^o$ to $\mathcal{B}^o$
$\mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o)$	optimal fuzzy proximity of the two sets, see (2.1)
$\mathcal{H}^o(\sigma^o, \mathcal{B}^o, \omega^o)$	fuzzy proximity of a point to a set, see (2.2)
$\mathcal{A}_0^o, \mathcal{B}_0^o$	proximity sets, see (2.3) and (2.4)
σ^{o*}	best proximity point, see (2.5)
$\{\sigma_l^o\}$	proximal sequence, σ_{l+1}^o attains the optimal proximity to $\Phi\sigma_l^o$
$\mathscr{P}, \mathscr{Q}$	auxiliary functions from $(0, 1]$ to $\mathbb{R}$ with $\mathscr{Q} > \mathscr{P}$ on $(0, 1)$
FMS, FPC, AMC	fuzzy metric space, fuzzy proximal contraction, approximately compact

3. Main results

Throughout this section, we assume that $(\mathcal{X}^o, \mathcal{H}^o, *)$ is a non-Archimedean fuzzy metric space, that is, inequality (2.7) holds for all points and all $\omega^o > 0$. Additionally, we assume that the sets $\mathcal{A}_0^o$, $\mathcal{B}_0^o$, and the proximal relation $\mathcal{H}^o(\sigma^o, \eta, \omega^o) = \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o)$ do not depend on ω^o, and that the contractive conditions (3.1), (3.20), (3.26), and (3.32) hold for every $\omega^o > 0$. These assumptions are satisfied by the induced fuzzy metric (2.6) and by the stationary fuzzy metric $\mathcal{H}^o(\sigma^o, \nu^o, \omega^o) = e^{-d(\sigma^o, \nu^o)}$, and they are the setting of [16, 22].

The constructions below use several results from the literature, namely Karapınar's interpolative contractions [11], the interpolative Rus–Reich–Ćirić framework [12], the interpolative Hardy–Rogers contraction [13], the Proinov-style auxiliary-function approach [15–17], and the interpolative proximal-point framework [21, 22].

Because the $(\mathscr{P}, \mathscr{Q})$-FPC of the second kind and its interpolative Reich–Rus–Ćirić, Kannan, and Hardy–Rogers variants are the paper's principal constructions rather than background material, we introduce their definitions here rather than among the preliminaries.

3.1. Fuzzy proximal contractions of the second kind

Definition 3.1. *Let $(\mathcal{X}^o, \mathcal{H}^o, *)$ be a complete non-Archimedean FMS, and let $\mathcal{A}^o$ and $\mathcal{B}^o$ be non-empty subsets of $\mathcal{X}^o$. A mapping $\Phi : \mathcal{A}^o \to \mathcal{B}^o$ is said to be a $(\mathscr{P}, \mathscr{Q})$-FPC of the second kind if*

$$\left.\begin{array}{l} \mathcal{H}^o\left(\sigma_1^o, \Phi v_1, \omega^o\right) = \mathcal{H}^o\left(\mathcal{A}^o, \mathcal{B}^o, \omega^o\right), \\ \mathcal{H}^o\left(\sigma_2^o, \Phi v_2, \omega^o\right) = \mathcal{H}^o\left(\mathcal{A}^o, \mathcal{B}^o, \omega^o\right) \end{array}\right\} \Longrightarrow \mathscr{P}\left(\mathcal{H}^o\left(\Phi\sigma_1^o, \Phi\sigma_2^o, \omega^o\right)\right) \geq \mathscr{Q}\left(\mathcal{H}^o\left(\Phi v_1, \Phi v_2, \omega^o\right)\right), \quad (3.1)$$

$\forall\, \sigma_1^o$, σ_2^o, v_1, $v_2 \in \mathcal{A}^o$ with $\sigma_1^o \neq \sigma_2^o$, where $\mathscr{P}, \mathscr{Q} : (0, 1] \to \mathbb{R}$ are two auxiliary functions satisfying $\mathscr{Q}(t) > \mathscr{P}(t)$ $\forall\, t \in (0, 1)$.

This definition provides a two-function formulation of the classical proximal contraction of the second kind introduced by Basha [27]. The auxiliary functions $\mathscr{P}$ and $\mathscr{Q}$ characterize the contractive behavior through their strict functional separation.

Example 3.2. *Let $\mathcal{X}^o = \mathbb{R}^2$ and define $\mathcal{H}^o : \mathcal{X}^o \times \mathcal{X}^o \times (0, \infty) \to (0, 1]$ by*

$$\mathcal{H}^o(\sigma^o, \nu^o, \omega^o) = e^{-\frac{\mathfrak{d}(\sigma^o, \nu^o)}{\omega^o}},$$

where

$$\mathfrak{d}((\mathfrak{x}, \mathfrak{y}), (\mathfrak{a}, \mathfrak{b})) = |\mathfrak{x} - \mathfrak{a}| + |\mathfrak{y} - \mathfrak{b}|, \qquad (\mathfrak{x}, \mathfrak{y}), (\mathfrak{a}, \mathfrak{b}) \in \mathcal{X}^o.$$

Then, $(\mathcal{X}^o, \mathcal{H}^o, *)$ is an FMS. Let

$$\mathcal{A}^o = \{(0, \mathfrak{t}^o) : \mathfrak{t}^o > 0\}, \qquad \mathcal{B}^o = \{(1, \mathfrak{t}^o) : \mathfrak{t}^o > 0\},$$

so that $\mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o) = e^{-\frac{1}{\omega^o}}$. Take $\mathscr{P}(s) = s$ and $\mathscr{Q}(s) = \sqrt{s}$, and define $\Phi : \mathcal{A}^o \to \mathcal{B}^o$ by

$$\Phi(0, \mathfrak{t}^o) = \left(1, \frac{\mathfrak{t}^o}{3}\right).$$

Let $\sigma_1^o = (0, \mathfrak{a}_1)$ and $\sigma_2^o = (0, \mathfrak{a}_2)$, where $\mathfrak{a}_1 \neq \mathfrak{a}_2$. The proximal conditions

$$\mathcal{H}^o(\sigma_i^o, \Phi\nu_i^o, \omega^o) = \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o) = e^{-\frac{1}{\omega^o}}, \qquad i = 1, 2,$$

imply

$$\nu_i^o = (0, 3\mathfrak{a}_i), \qquad \Phi\nu_i^o = (1, \mathfrak{a}_i).$$

Moreover,

$$\Phi\sigma_i^o = \left(1, \frac{\mathfrak{a}_i}{3}\right),$$

and hence

$$\mathfrak{d}(\Phi\sigma_1^o, \Phi\sigma_2^o) = \frac{1}{3}|\mathfrak{a}_1 - \mathfrak{a}_2|,$$

while

$$\mathfrak{d}(\Phi\nu_1^o, \Phi\nu_2^o) = |\mathfrak{a}_1 - \mathfrak{a}_2|.$$

Therefore,

$$\mathscr{P}(\mathcal{H}^o(\Phi\sigma_1^o, \Phi\sigma_2^o, \omega^o)) = e^{-\frac{|\mathfrak{a}_1-\mathfrak{a}_2|}{3\omega^o}},$$

and

$$\mathscr{Q}(\mathcal{H}^o(\Phi\nu_1^o, \Phi\nu_2^o, \omega^o)) = e^{-\frac{|\mathfrak{a}_1-\mathfrak{a}_2|}{2\omega^o}}.$$

Since $\frac{1}{3} < \frac{1}{2}$*, we obtain the following:*

$$\mathscr{P}(\mathcal{H}^o(\Phi\sigma_1^o, \Phi\sigma_2^o, \omega^o)) \geq \mathscr{Q}(\mathcal{H}^o(\Phi\nu_1^o, \Phi\nu_2^o, \omega^o)), \qquad \mathfrak{a}_1 \neq \mathfrak{a}_2,\ \omega^o > 0. \tag{3.2}$$

Hence, Φ *is a* $(\mathscr{P}, \mathscr{Q})$*-FPC of the second kind on* $\mathcal{A}^o$*.*

The following lemmas are used in the proofs of the main results.

Lemma 3.3. *Let* $(\mathcal{X}^o, \mathcal{H}^o, *)$ *be an FMS, and let* $\{\sigma_l^o\} \subseteq \mathcal{X}^o$ *be a sequence such that*

$$\lim_{l\to\infty} \mathcal{H}^o(\Phi\sigma_l^o, \Phi\sigma_{l+1}^o, \omega^o) = 1, \qquad \forall\, \omega^o > 0.$$

If the sequence $\{\Phi\sigma_l^o\}$ *is not Cauchy, then* $\exists$ *subsequences* $\{\Phi\sigma_{l_k}^o\}$ *and* $\{\Phi\sigma_{p_k}^o\}$*, together with some* $\varepsilon \in (0, 1)$*, such that*

$$\lim_{k\to\infty} \mathcal{H}^o(\Phi\sigma_{l_k+1}^o, \Phi\sigma_{p_k+1}^o, \omega^o) = 1 - \varepsilon, \tag{3.3}$$

and

$$\begin{aligned} \lim_{k\to\infty} \mathcal{H}^o(\Phi\sigma_{l_k}^o, \Phi\sigma_{p_k}^o, \omega^o) &= \lim_{k\to\infty} \mathcal{H}^o(\Phi\sigma_{l_k+1}^o, \Phi\sigma_{p_k}^o, \omega^o) \\ &= \lim_{k\to\infty} \mathcal{H}^o(\Phi\sigma_{l_k}^o, \Phi\sigma_{p_k+1}^o, \omega^o) = 1 - \varepsilon. \end{aligned} \tag{3.4}$$

Proof. Since $\{\Phi\sigma_l^o\}$ is not Cauchy, there exist $\varepsilon \in (0, 1)$ and $\omega^o > 0$ such that for every $k \in \mathbb{N}$, there are integers $l_k > p_k \geq k$ with $\mathcal{H}^o(\Phi\sigma_{l_k}^o, \Phi\sigma_{p_k}^o, \omega^o) \leq 1 - \varepsilon$. We choose l_k to be the smallest such integer. Write $a_{i,j} = \mathcal{H}^o(\Phi\sigma_i^o, \Phi\sigma_j^o, \omega^o)$. Then, $a_{l_k-1,p_k} > 1 - \varepsilon$, and by (2.7),

$$1 - \varepsilon \geq a_{l_k,p_k} \geq a_{l_k,l_k-1}\, a_{l_k-1,p_k} > a_{l_k,l_k-1}(1 - \varepsilon).$$

Since $a_{l_k,l_k-1} \to 1$, we obtain $a_{l_k,p_k} \to 1 - \varepsilon$. Again, by (2.7),

$$a_{l_k+1,p_k} \geq a_{l_k+1,l_k}\, a_{l_k,p_k}, \qquad a_{l_k,p_k} \geq a_{l_k,l_k+1}\, a_{l_k+1,p_k},$$

and since $a_{l_k,l_k+1} \to 1$, these two inequalities give $a_{l_k+1,p_k} \to 1 - \varepsilon$. In the same way, with the second index shifted from p_k to $p_k + 1$, we obtain $a_{l_k,p_k+1} \to 1 - \varepsilon$ and $a_{l_k+1,p_k+1} \to 1 - \varepsilon$, which proves (3.3) and the remaining limits. □

Lemma 3.4. *Let* $\mathscr{P} : (0, 1] \to \mathbb{R}$ *be a non-decreasing function. Then, the following conditions are equivalent:*
(i) $\inf_{t>\varepsilon} \mathscr{P}(t) > -\infty$, $\forall\, \varepsilon \in (0, 1)$*;*

(ii) $\liminf_{t\to\varepsilon^+} \mathscr{P}(t) > -\infty$, $\forall\, \varepsilon \in (0,1)$; *and*
(iii) *For every sequence* $\{t_l\} \subset (0,1]$, *if* $\lim_{l\to\infty} \mathscr{P}(t_l) = -\infty$, *then* $\lim_{l\to\infty} t_l = 0$.

Lemma 3.5. *Let* $(\mathcal{X}^o, \mathcal{H}^o, *)$ *be an FMS, let* $\mathcal{A}^o, \mathcal{B}^o$ *be non-empty subsets of* $\mathcal{X}^o$, *and* $\Phi : \mathcal{A}^o \longrightarrow \mathcal{B}^o$ *satisfy the contractive condition* (3.1). *Suppose that* $\{\sigma_l^o\} \subseteq \mathcal{A}^o$ *satisfies the proximal iterative relation*

$$\mathcal{H}^o\left(\sigma_{l+1}^o, \Phi\sigma_l^o, \omega^o\right) = \mathcal{H}^o\left(\mathcal{A}^o, \mathcal{B}^o, \omega^o\right), \qquad l \geq 0, \tag{3.5}$$

$\forall\, \omega^o > 0$, *and assume that*

$$\lim_{l\to\infty} \mathcal{H}^o\left(\Phi\sigma_l^o, \Phi\sigma_{l+1}^o, \omega^o\right) = 1, \qquad \omega^o > 0. \tag{3.6}$$

Let $\mathscr{P}, \mathscr{Q} : (0,1] \to \mathbb{R}$, *where* $\mathscr{P}$ *is a continuous and non-decreasing function, and suppose that*

$$\liminf_{t\to s} \mathscr{Q}(t) > \mathscr{P}(s), \qquad s \in (0,1). \tag{3.7}$$

Then, the sequence $\{\Phi\sigma_l^o\}$ *is Cauchy in* $\mathcal{X}^o$.

Proof. Assume, to the contrary, that $\{\Phi\sigma_l^o\}$ is not a Cauchy sequence. By Lemma 3.3, there exist $\varepsilon \in (0,1)$ and two subsequences, $\{\sigma_{l_k}^o\}$ and $\{\sigma_{p_k}^o\}$, such that

$$\lim_{k\to\infty} \mathcal{H}^o\left(\Phi\sigma_{l_k+1}^o, \Phi\sigma_{p_k+1}^o, \omega^o\right) = 1 - \varepsilon, \tag{3.8}$$

and

$$\lim_{k\to\infty} \mathcal{H}^o\left(\Phi\sigma_{l_k}^o, \Phi\sigma_{p_k}^o, \omega^o\right) = 1 - \varepsilon. \tag{3.9}$$

By the proximal iterative relation (3.5), we have the following:

$$\mathcal{H}^o\left(\sigma_{l_k+1}^o, \Phi\sigma_{l_k}^o, \omega^o\right) = \mathcal{H}^o\left(\mathcal{A}^o, \mathcal{B}^o, \omega^o\right),$$

$$\mathcal{H}^o\left(\sigma_{p_k+1}^o, \Phi\sigma_{p_k}^o, \omega^o\right) = \mathcal{H}^o\left(\mathcal{A}^o, \mathcal{B}^o, \omega^o\right).$$

For a sufficiently large k, the points $\sigma_{l_k+1}^o$ and $\sigma_{p_k+1}^o$ are distinct. Otherwise,

$$\mathcal{H}^o\left(\Phi\sigma_{l_k+1}^o, \Phi\sigma_{p_k+1}^o, \omega^o\right) = 1,$$

which contradicts (3.8) because $1-\varepsilon < 1$. Hence, applying the contractive condition (3.1) to $\sigma_1^o = \sigma_{l_k+1}^o$, $\sigma_2^o = \sigma_{p_k+1}^o$, $v_1 = \sigma_{l_k}^o$, and $v_2 = \sigma_{p_k}^o$, gives the following:

$$\mathscr{P}\left(\mathcal{H}^o\left(\Phi\sigma_{l_k+1}^o, \Phi\sigma_{p_k+1}^o, \omega^o\right)\right) \geq \mathscr{Q}\left(\mathcal{H}^o\left(\Phi\sigma_{l_k}^o, \Phi\sigma_{p_k}^o, \omega^o\right)\right). \tag{3.10}$$

Set

$$u_k = \mathcal{H}^o\left(\Phi\sigma_{l_k+1}^o, \Phi\sigma_{p_k+1}^o, \omega^o\right),$$

$$w_k = \mathcal{H}^o\left(\Phi\sigma_{l_k}^o, \Phi\sigma_{p_k}^o, \omega^o\right).$$

Then, by (3.8) and (3.9), $u_k \to 1-\varepsilon$, $w_k \to 1-\varepsilon$. Therefore, (3.10) becomes

$$\mathscr{P}(u_k) \geq \mathscr{Q}(w_k).$$

Taking the limit superior on the left-hand side and the limit inferior on the right-hand side yields the following:

$$\limsup_{k\to\infty} \mathscr{P}(u_k) \geq \liminf_{k\to\infty} \mathscr{Q}(w_k).$$

Since $\mathscr{P}$ is continuous,

$$\lim_{k\to\infty} \mathscr{P}(u_k) = \mathscr{P}(1-\varepsilon),$$

while

$$\liminf_{k\to\infty} \mathscr{Q}(w_k) \geq \liminf_{t\to(1-\varepsilon)} \mathscr{Q}(t).$$

Consequently,

$$\mathscr{P}(1-\varepsilon) \geq \liminf_{t\to(1-\varepsilon)} \mathscr{Q}(t),$$

which contradicts (3.7) with $s = 1-\varepsilon$. Thus, the assumption that $\{\Phi\sigma_l^o\}$ is not Cauchy is impossible. Hence, $\{\Phi\sigma_l^o\}$ is a Cauchy sequence in $\mathcal{X}^o$. □

Theorem 3.6. *Let $(\mathcal{X}^o, \mathcal{H}^o, *)$ be a complete non-Archimedean fuzzy metric space, and let $\mathcal{A}^o$ and $\mathcal{B}^o$ be non-empty closed subsets of $\mathcal{X}^o$ such that $\mathcal{A}^o$ is AMC with respect to $\mathcal{B}^o$. Let $\Phi : \mathcal{A}^o \to \mathcal{B}^o$ be a $(\mathscr{P}, \mathscr{Q})$-FPC of the second kind. Suppose that the following conditions hold:*
(i) *$\mathscr{P}$ is continuous and non-decreasing and $\liminf_{t\to s} \mathscr{Q}(t) > \mathscr{P}(s)$, for every $s \in (0,1)$;*
(ii) *$\mathcal{A}_0^o$ is a non-empty subset of $\mathcal{A}^o$ such that $\Phi\left(\mathcal{A}_0^o\right) \subseteq \mathcal{B}_0^o$; and*
(iii) *Φ is continuous.*

Then, Φ admits a best proximity point.

Proof. Let $\sigma_0^o \in \mathcal{A}_0^o$ be arbitrary. Since $\Phi(\sigma_0^o) \in \Phi(\mathcal{A}_0^o) \subseteq \mathcal{B}_0^o$, there exists $\sigma_1^o \in \mathcal{A}_0^o$ such that

$$\mathcal{H}^o(\sigma_1^o, \Phi\sigma_0^o, \omega^o) = \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o).$$

Proceeding inductively, we obtain a sequence $\{\sigma_l^o\} \subseteq \mathcal{A}_0^o$ which satisfies the following:

$$\mathcal{H}^o(\sigma_{l+1}^o, \Phi\sigma_l^o, \omega^o) = \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o), \qquad l \geq 0. \tag{3.11}$$

If $\sigma_{l+1}^o = \sigma_l^o$ for some l, then (3.11) gives

$$\mathcal{H}^o(\sigma_l^o, \Phi\sigma_l^o, \omega^o) = \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o),$$

and hence σ_l^o is a best proximity point of Φ. Therefore, assume that $\sigma_{l+1}^o \neq \sigma_l^o$ for every l. Using (3.11) at two consecutive indices and applying (3.1), we obtain the following:

$$\mathscr{P}\left(\mathcal{H}^o(\Phi\sigma_l^o, \Phi\sigma_{l+1}^o, \omega^o)\right) \geq \mathscr{Q}\left(\mathcal{H}^o(\Phi\sigma_{l-1}^o, \Phi\sigma_l^o, \omega^o)\right). \tag{3.12}$$

Set $D_l = \mathcal{H}^o(\Phi\sigma_l^o, \Phi\sigma_{l+1}^o, \omega^o)$. If $D_l = 1$ for some l, then $\Phi\sigma_l^o = \Phi\sigma_{l+1}^o$, and consequently

$$\mathcal{H}^o(\sigma_{l+1}^o, \Phi\sigma_{l+1}^o, \omega^o) = \mathcal{H}^o(\sigma_{l+1}^o, \Phi\sigma_l^o, \omega^o) = \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o).$$

Thus, σ_{l+1}^o is a best proximity point. Therefore, we may assume that $0 < D_l < 1$ for every l. Since $\mathscr{Q}(t) > \mathscr{P}(t)$ on $(0,1)$, Eq (3.12) implies the following:

$$\mathscr{P}(D_l) \geq \mathscr{Q}(D_{l-1}) > \mathscr{P}(D_{l-1}).$$

As $\mathscr{P}$ is non-decreasing, we obtain $D_l > D_{l-1}$. Hence, $\{D_l\}$ is increasing and bounded above by 1, and therefore

$$D_l \longrightarrow E$$

for some $E \in (0, 1]$. Suppose that $E < 1$. Since $D_l \to E$, we have

$$\lim_{l\to\infty} \mathscr{P}(D_l) = \mathscr{P}(E^-).$$

Consequently, from (3.12),

$$\mathscr{P}(E^-) \geq \liminf_{l\to\infty} \mathscr{Q}(D_{l-1}) \geq \liminf_{t\to E^-} \mathscr{Q}(t),$$

which contradicts Hypothesis (i). Hence, $E = 1$, and therefore

$$\lim_{l\to\infty} \mathcal{H}^o(\Phi\sigma_l^o, \Phi\sigma_{l+1}^o, \omega^o) = 1. \tag{3.13}$$

By Lemma 3.5, the sequence $\{\Phi\sigma_l^o\}$ is Cauchy. Since $\mathcal{X}^o$ is complete and $\mathcal{B}^o$ is closed, there exists $\eta^* \in \mathcal{B}^o$ such that

$$\Phi\sigma_l^o \longrightarrow \eta^*. \tag{3.14}$$

Using (3.11) together with inequality (2.7), we obtain the following:

$$\mathcal{H}^o(\sigma_{l+1}^o, \eta^*, \omega^o) \geq \mathcal{H}^o(\sigma_{l+1}^o, \Phi\sigma_l^o, \omega^o) * \mathcal{H}^o(\Phi\sigma_l^o, \eta^*, \omega^o).$$

Thus,

$$\mathcal{H}^o(\sigma_{l+1}^o, \eta^*, \omega^o) \geq \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o) * \mathcal{H}^o(\Phi\sigma_l^o, \eta^*, \omega^o).$$

Since $\sigma_{l+1}^o \in \mathcal{A}^o$ and $\eta^* \in \mathcal{B}^o$,

$$\mathcal{H}^o(\sigma_{l+1}^o, \eta^*, \omega^o) \leq \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o).$$

Therefore, taking $l \to \infty$ gives the following:

$$\mathcal{H}^o(\sigma_{l+1}^o, \eta^*, \omega^o) \longrightarrow \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o). \tag{3.15}$$

Additionally, since

$$\mathcal{H}^o(\sigma_{l+1}^o, \eta^*, \omega^o) \leq \mathcal{H}^o(\eta^*, \mathcal{A}^o, \omega^o) \leq \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o),$$

we have $\mathcal{H}^o(\eta^*, \sigma_{l+1}^o, \omega^o) \to \mathcal{H}^o(\eta^*, \mathcal{A}^o, \omega^o)$.

By approximate compactness of $\mathcal{A}^o$ with respect to $\mathcal{B}^o$, there exists a subsequence $\{\sigma_{l_k+1}^o\}$ converging to some $\sigma^{o*} \in \mathcal{A}^o$. By (2.7),

$$\mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o) \geq \mathcal{H}^o(\sigma^{o*}, \eta^*, \omega^o) \geq \mathcal{H}^o(\sigma^{o*}, \sigma_{l_k+1}^o, \omega^o) * \mathcal{H}^o(\sigma_{l_k+1}^o, \eta^*, \omega^o),$$

and by (3.15), the right-hand side tends to $\mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o)$. This yields the following:

$$\mathcal{H}^o(\sigma^{o*}, \eta^*, \omega^o) = \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o). \tag{3.16}$$

Since Φ is continuous, $\Phi\sigma_{l_k+1}^o \to \Phi\sigma^{o*}$. Additionally, by (3.14), $\Phi\sigma_{l_k+1}^o \to \eta^*$. Limits in a fuzzy metric space are unique [4], hence $\eta^* = \Phi\sigma^{o*}$, and (3.16) becomes

$$\mathcal{H}^o(\sigma^{o*}, \Phi\sigma^{o*}, \omega^o) = \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o).$$

Therefore, σ^{o*} is a best proximity point of Φ. □

Theorem 3.7. *Let $(\mathcal{X}^o, \mathcal{H}^o, *)$ be a complete non-Archimedean fuzzy metric space, and let $\mathcal{A}^o, \mathcal{B}^o$ be non-empty closed subsets of $\mathcal{X}^o$ such that $\mathcal{A}^o$ is AMC with respect to $\mathcal{B}^o$. Let $\Phi : \mathcal{A}^o \to \mathcal{B}^o$ be a $(\mathscr{P}, \mathscr{Q})$-FPC of the second kind. Suppose that the following hold:*
(i) *$\mathscr{P}$ is continuous and non-decreasing, and for every sequence $\{t_l\} \subset (0, 1]$, if $\{\mathscr{P}(t_l)\}$ and $\{\mathscr{Q}(t_l)\}$ are convergent and $\lim_{l\to\infty} \mathscr{P}(t_l) = \lim_{l\to\infty} \mathscr{Q}(t_l)$, then $\lim_{l\to\infty} t_l = 1$;*
(ii) *$\mathcal{A}_0^o$ is a non-empty subset of $\mathcal{A}^o$ such that $\Phi\left(\mathcal{A}_0^o\right) \subseteq \mathcal{B}_0^o$; and*
(iii) *Φ is continuous.*

Then, Φ admits a best proximity point.

Proof. Keeping the notation of Theorem 3.6, set $D_l = \mathcal{H}^o(\Phi\sigma_l^o, \Phi\sigma_{l+1}^o, \omega^o)$. Then,

$$\mathscr{P}(D_l) \geq \mathscr{Q}(D_{l-1}) > \mathscr{P}(D_{l-1}). \tag{3.17}$$

Since $\mathscr{P}$ is non-decreasing, (3.17) implies $D_l > D_{l-1}$. Hence, $\{D_l\}$ is increasing and, since $D_l \leq 1$, it converges to some $E \in (0, 1]$. Moreover, $\{\mathscr{P}(D_l)\}$ is increasing and $\mathscr{P}(D_l) \leq \mathscr{P}(1)$. Therefore, it is convergent. Suppose $\mathscr{P}(D_l) \to L$. Then, $\mathscr{P}(D_{l-1}) \to L$, and from (3.17),

$$\mathscr{P}(D_{l-1}) < \mathscr{Q}(D_{l-1}) \leq \mathscr{P}(D_l).$$

Thus, by the squeeze theorem,

$$\lim_{l\to\infty} \mathscr{Q}(D_{l-1}) = \lim_{l\to\infty} \mathscr{P}(D_{l-1}) = L.$$

Hypothesis (i), applied to $\{D_{l-1}\}$, now yields

$$\lim_{l\to\infty} D_l = 1,$$

that is,

$$\lim_{l\to\infty} \mathcal{H}^o(\Phi\sigma_l^o, \Phi\sigma_{l+1}^o, \omega^o) = 1. \tag{3.18}$$

It remains to verify the Cauchy property. Suppose that $\{\Phi\sigma_l^o\}$ is not Cauchy. By Lemma 3.3, there exist subsequences and some $\varepsilon \in (0, 1)$ such that

$$u_k = \mathcal{H}^o(\Phi\sigma_{l_k+1}^o, \Phi\sigma_{p_k+1}^o, \omega^o) \to 1 - \varepsilon$$

and

$$w_k = \mathcal{H}^o(\Phi\sigma_{l_k}^o, \Phi\sigma_{p_k}^o, \omega^o) \to 1 - \varepsilon.$$

As in the proof of Lemma 3.5, the points $\sigma_{l_k+1}^o$ and $\sigma_{p_k+1}^o$ are distinct for a large k, and the proximal contractive condition gives

$$\mathscr{P}(u_k) \geq \mathscr{Q}(w_k) > \mathscr{P}(w_k).$$

Since $\mathscr{P}$ is continuous and $u_k, w_k \to 1 - \varepsilon$, both $\mathscr{P}(u_k)$ and $\mathscr{P}(w_k)$ converge to $\mathscr{P}(1 - \varepsilon)$. Hence,

$$\lim_{k\to\infty} \mathscr{Q}(w_k) = \lim_{k\to\infty} \mathscr{P}(w_k).$$

By Hypothesis (i), $w_k \to 1$, which contradicts $w_k \to 1 - \varepsilon < 1$. Therefore, $\{\Phi\sigma_l^o\}$ is Cauchy. The remainder of the argument is identical to the corresponding final part of the proof of Theorem 3.6. Consequently, there exists $\sigma^{o*} \in \mathcal{A}^o$ such that

$$\mathcal{H}^o\left(\sigma^{o*}, \Phi(\sigma^{o*}), \omega^o\right) = \mathcal{H}^o\left(\mathcal{A}^o, \mathcal{B}^o, \omega^o\right).$$

Hence, σ^{o*} is a best proximity point of Φ. □

Theorems 3.6 and 3.7 only differ in Hypothesis (i). In Theorem 3.6, it is a local condition. It requires $\mathscr{Q}$ to stay strictly above $\mathscr{P}$ near every point $s \in (0,1)$, and it is used through Lemma 3.5. In Theorem 3.7, it is a sequential condition. It requires that $\mathscr{P}$ and $\mathscr{Q}$ have no common limit along a sequence that does not tend to 1, and the proof uses a squeeze argument instead of Lemma 3.5. Since $\mathscr{P}$ is continuous and $\mathscr{Q} > \mathscr{P}$ on $(0,1)$, Hypothesis (i) of Theorem 3.7 implies Hypothesis (i) of Theorem 3.6. Indeed, since $\mathscr{Q} > \mathscr{P}$ and $\mathscr{P}$ is continuous, $\liminf_{t\to s} \mathscr{Q}(t) \geq \mathscr{P}(s)$. If equality held for some $s \in (0,1)$, then there would be a sequence $t_l \to s$ with $\mathscr{Q}(t_l) \to \mathscr{P}(s)$ and $\mathscr{P}(t_l) \to \mathscr{P}(s)$, which would contradict Hypothesis (i) of Theorem 3.7. The converse does not hold. For $\mathscr{P}(t) = t$ and $\mathscr{Q}(t) = \sqrt{t}$, Hypothesis (i) of Theorem 3.6 holds, but $\mathscr{P}(t_l)$ and $\mathscr{Q}(t_l)$ both tend to 0 when $t_l \to 0$. Thus, Theorem 3.6 has the weaker hypothesis, while the sequential form in Theorem 3.7 is often easier to check, as in Corollaries 3.26 and 3.27. The same relation holds between Theorems 3.10 and 3.11, 3.14 and 3.15, and 3.18 and 3.19.

Example 3.8. *Let $\mathcal{X}^o = \mathbb{R}$ be endowed with the usual metric $d(\sigma^o, \nu^o) = |\sigma^o - \nu^o|$, and define*

$$\mathcal{H}^o(\sigma^o, \nu^o, \omega^o) = e^{-\frac{|\sigma^o - \nu^o|}{\omega^o}}, \qquad \omega^o > 0,$$

together with the product t-norm. Consider $\mathcal{A}^o = \{0, 10, 30, 100\}$, $\mathcal{B}^o = \{1, 11, 31, 101\}$. Since both sets are finite, they are closed and compact. Moreover,

$$\mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o) = e^{-\frac{1}{\omega^o}},$$

and $\mathcal{A}_0^o = \mathcal{A}^o$, $\mathcal{B}_0^o = \mathcal{B}^o$. Define $\Phi : \mathcal{A}^o \to \mathcal{B}^o$ by the following:

$$\Phi(0) = 1, \qquad \Phi(10) = 1, \qquad \Phi(30) = 11, \qquad \Phi(100) = 31.$$

Clearly, $\Phi(\mathcal{A}_0^o) = \{1, 11, 31\} \subseteq \mathcal{B}_0^o$. Let the auxiliary functions be as follows:

$$\mathscr{P}(t) = \ln t, \qquad \mathscr{Q}(t) = \frac{1}{2}\ln t, \qquad t \in (0, 1].$$

Then, $\mathscr{P}$ is continuous and non-decreasing, and $\mathscr{Q}(t) > \mathscr{P}(t)$, $t \in (0,1)$. Under (2.6)*, Condition* (3.1) *with these functions is equivalent to the following:*

$$|\Phi\sigma_1^o - \Phi\sigma_2^o| \leq \frac{1}{2}|\Phi v_1 - \Phi v_2|.$$

The proximal conditions give $\sigma_i^o = \Phi v_i - 1$, so $v \in \{0, 10\}$ gives $\sigma^o = 0$, $v = 30$ gives $\sigma^o = 10$, and $v = 100$ gives $\sigma^o = 30$. For the pairs $\{0, 10\}$, $\{0, 30\}$, and $\{10, 30\}$ of values of σ_1^o, σ_2^o, the left-hand side equals 0, 10, and 10, and the right-hand side equals 5, 15, and 10, respectively. For instance, for $v_1 = 10$ and $v_2 = 100$ we get $\sigma_1^o = 0$ and $\sigma_2^o = 30$. Then,

$$\mathscr{P}(\mathcal{H}^o(\Phi\sigma_1^o, \Phi\sigma_2^o, \omega^o)) = \ln e^{-10/\omega^o} = -10/\omega^o$$

and

$$\mathscr{Q}(\mathcal{H}^o(\Phi v_1, \Phi v_2, \omega^o)) = \frac{1}{2}\ln e^{-30/\omega^o} = -15/\omega^o,$$

and indeed $-10/\omega^o \geq -15/\omega^o$. Hence, Φ is a $(\mathscr{P}, \mathscr{Q})$-FPC of the second kind. Since $\mathcal{A}^o$ is finite, Φ is continuous and $\mathcal{A}^o$ is AMC with respect to $\mathcal{B}^o$. Thus, all the assumptions of Theorems 3.6 and 3.7 hold, and $\sigma^{o*} = 0$ is a best proximity point, since $|0 - \Phi(0)| = 1$.

Figure 3 displays every admissible case of Example 3.8, not a continuous surface. Under (2.6), the condition does not depend on ω^o, and each row is one admissible case. In every case, the circle lies on or inside the square, and equality only occurs for $\{\sigma_1^o, \sigma_2^o\} = \{10, 30\}$. The figure illustrates the verification given above and is not used as a proof.

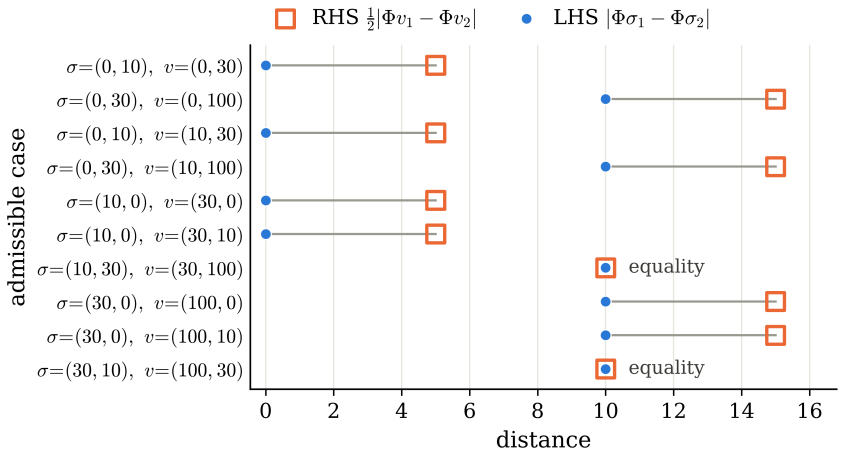


Figure 3. Example 3.8 with $\mathscr{P}(t) = \ln t$ and $\mathscr{Q}(t) = \frac{1}{2}\ln t$. Each row is an admissible case, and the horizontal axis shows LHS $= |\Phi\sigma_1^o - \Phi\sigma_2^o|$ and RHS $= \frac{1}{2}|\Phi v_1 - \Phi v_2|$. LHS $\leq$ RHS in all ten cases.

3.2. Interpolative Reich–Rus–Ćirić type proximal contractions

Definition 3.9. *Let $(\mathcal{X}^o, \mathcal{H}^o, *)$ be a complete non-Archimedean fuzzy metric space, and let $\mathcal{A}^o, \mathcal{B}^o$ be non-empty subsets of $\mathcal{X}^o$. A mapping $\Phi : \mathcal{A}^o \to \mathcal{B}^o$ is said to be a $(\mathscr{P}, \mathscr{Q})$-interpolative Reich–Rus–Ćirić type FPC of the second kind if $\exists\, \alpha, \beta \in (0,1)$ with $\alpha + \beta < 1$ such that*

$$\left.\begin{aligned} \mathcal{H}^o\left(\sigma_1^o, \Phi v_1, \omega^o\right) &= \mathcal{H}^o\left(\mathcal{A}^o, \mathcal{B}^o, \omega^o\right) \\ \mathcal{H}^o\left(\sigma_2^o, \Phi v_2, \omega^o\right) &= \mathcal{H}^o\left(\mathcal{A}^o, \mathcal{B}^o, \omega^o\right) \end{aligned}\right\} \tag{3.19}$$

$$\Longrightarrow \mathscr{P}\left(\mathcal{H}^o\left(\Phi\sigma_1^o, \Phi\sigma_2^o, \omega^o\right)\right) \geq \mathscr{Q}\left(\begin{array}{c}\left[\mathcal{H}^o\left(\Phi v_1, \Phi v_2, \omega^o\right)\right]^{\alpha}\left[\mathcal{H}^o\left(\Phi v_1, \Phi\sigma_1^o, \omega^o\right)\right]^{\beta} \\ \left[\mathcal{H}^o\left(\Phi v_2, \Phi\sigma_2^o, \omega^o\right)\right]^{1-\alpha-\beta}\end{array}\right), \tag{3.20}$$

$\forall\, \sigma_1^o, \sigma_2^o, v_1, v_2 \in \mathcal{A}^o$ that satisfy $\sigma_i^o \neq v_i$ for $i \in \{1,2\}$, where $\mathscr{P}, \mathscr{Q} : (0,1] \to \mathbb{R}$ are auxiliary functions such that $\mathscr{Q}(t) > \mathscr{P}(t)\ \forall\, t \in (0,1)$.

Theorem 3.10. *Let $(\mathcal{X}^o, \mathcal{H}^o, *)$ be a complete non-Archimedean fuzzy metric space, and let $\mathcal{A}^o$ and $\mathcal{B}^o$ be non-empty closed subsets of $\mathcal{X}^o$ such that $\mathcal{A}^o$ is AMC with respect to $\mathcal{B}^o$. Let $\Phi : \mathcal{A}^o \to \mathcal{B}^o$ be a $(\mathscr{P}, \mathscr{Q})$-interpolative Reich–Rus–Ćirić type FPC of the second kind. Suppose that the following hold:*
(i) *$\mathscr{P}$ is a non-decreasing function and $\liminf_{t\to\varepsilon^-} \mathscr{Q}(t) > \mathscr{P}(\varepsilon^-)$, $\forall\, \varepsilon \in (0,1)$;*
(ii) *$\mathcal{A}_0^o$ is a non-empty subset of $\mathcal{A}^o$ such that $\Phi\left(\mathcal{A}_0^o\right) \subseteq \mathcal{B}_0^o$; and*
(iii) *Φ is continuous.*

Then, Φ admits a best proximity point.

Proof. Mirroring the construction used for Theorem 3.6, build a sequence $\{\sigma_l^o\} \subseteq \mathcal{A}_0^o$ with

$$\mathcal{H}^o(\sigma_{l+1}^o, \Phi\sigma_l^o, \omega^o) = \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o), \qquad l \geq 0.$$

Should $\sigma_l^o = \sigma_{l+1}^o$ occur for some l, then σ_l^o is already a best proximity point of Φ, so we may assume $\sigma_l^o \neq \sigma_{l+1}^o$ for every l. Write $D_l = \mathcal{H}^o(\Phi\sigma_l^o, \Phi\sigma_{l+1}^o, \omega^o)$. Applying the contractive condition (3.20) to two consecutive terms of the sequence gives the following:

$$\mathscr{P}(D_l) \geq \mathscr{Q}\left(D_{l-1}^{\alpha+\beta} D_l^{1-\alpha-\beta}\right). \tag{3.21}$$

Since $\mathscr{Q}(t) > \mathscr{P}(t)$ for $t \in (0,1)$,

$$\mathscr{P}(D_l) > \mathscr{P}\left(D_{l-1}^{\alpha+\beta} D_l^{1-\alpha-\beta}\right).$$

As $\mathscr{P}$ is non-decreasing, it follows that

$$D_l > D_{l-1}^{\alpha+\beta} D_l^{1-\alpha-\beta},$$

and hence

$$D_l^{\alpha+\beta} > D_{l-1}^{\alpha+\beta}.$$

Since $\alpha + \beta > 0$, we obtain

$$D_l > D_{l-1}.$$

Thus, $\{D_l\}$ is increasing and bounded above by 1, so $D_l \to E$ for some $E \in (0,1]$. We claim that $E = 1$. Suppose that $E < 1$ and put

$$G_l = D_{l-1}^{\alpha+\beta} D_l^{1-\alpha-\beta}.$$

Since $D_{l-1} < D_l < E$, we have $G_l < D_l < E$ and $G_l \to E$. Therefore, taking the limit inferior in (3.21) yields

$$\mathscr{P}(E^-) \geq \liminf_{l\to\infty} \mathscr{Q}(G_l) \geq \liminf_{t\to E^-} \mathscr{Q}(t) > \mathscr{P}(E^-),$$

which is impossible. Hence, $E = 1$, and consequently

$$\lim_{l\to\infty} \mathcal{H}^o(\Phi\sigma_l^o, \Phi\sigma_{l+1}^o, \omega^o) = 1. \tag{3.22}$$

Next, we show that $\{\Phi\sigma_l^o\}$ is Cauchy. Suppose otherwise. By Lemma 3.3, there exist subsequences $\{\Phi\sigma_{l_k}^o\}$ and $\{\Phi\sigma_{p_k}^o\}$ and some $\varepsilon \in (0,1)$ such that, with $c = 1 - \varepsilon$,

$$\mathcal{H}^o(\Phi\sigma_{l_k+1}^o, \Phi\sigma_{p_k+1}^o, \omega^o) \to c$$

and

$$\mathcal{H}^o(\Phi\sigma_{l_k}^o, \Phi\sigma_{p_k}^o, \omega^o) \to c.$$

Using (3.20) with $\sigma_1^o = \sigma_{l_k+1}^o$, $\sigma_2^o = \sigma_{p_k+1}^o$, $v_1 = \sigma_{l_k}^o$, and $v_2 = \sigma_{p_k}^o$, we obtain

$$\mathscr{P}(R_k) \geq \mathscr{Q}(S_k),$$

where

$$R_k = \mathcal{H}^o(\Phi\sigma_{l_k+1}^o, \Phi\sigma_{p_k+1}^o, \omega^o)$$

and

$$S_k = \left[\mathcal{H}^o(\Phi\sigma_{l_k}^o, \Phi\sigma_{p_k}^o, \omega^o)\right]^\alpha \left[\mathcal{H}^o(\Phi\sigma_{l_k}^o, \Phi\sigma_{l_k+1}^o, \omega^o)\right]^\beta \left[\mathcal{H}^o(\Phi\sigma_{p_k}^o, \Phi\sigma_{p_k+1}^o, \omega^o)\right]^{1-\alpha-\beta}.$$

By (3.22) and Lemma 3.3,

$$R_k \to c, \qquad S_k \to c^\alpha.$$

Since $0 < c < 1$ and $0 < \alpha < 1$,

$$c < c^\alpha.$$

Hence, for a sufficiently large k, $R_k < S_k$. Since $\mathscr{P}$ is non-decreasing and $\mathscr{Q}(t) > \mathscr{P}(t)$, we obtain

$$\mathscr{P}(R_k) \leq \mathscr{P}(S_k) < \mathscr{Q}(S_k),$$

which contradicts $\mathscr{P}(R_k) \geq \mathscr{Q}(S_k)$. Therefore, $\{\Phi\sigma_l^o\}$ is a Cauchy sequence. The remaining argument is identical to the final part of the proof of Theorem 3.6. Hence, there exists $\sigma^{o*} \in \mathcal{A}^o$ such that

$$\mathcal{H}^o(\sigma^{o*}, \Phi\sigma^{o*}, \omega^o) = \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o).$$

Therefore, σ^{o*} is a best proximity point of Φ. □

Theorem 3.11. *Let $(\mathcal{X}^o, \mathcal{H}^o, *)$ be a complete non-Archimedean fuzzy metric space, and let $\mathcal{A}^o$ and $\mathcal{B}^o$ be non-empty closed subsets of $\mathcal{X}^o$ such that $\mathcal{A}^o$ is AMC with respect to $\mathcal{B}^o$. Let $\Phi : \mathcal{A}^o \to \mathcal{B}^o$ be a $(\mathscr{P}, \mathscr{Q})$-interpolative Reich–Rus–Ćirić type FPC of the second kind. Suppose that the following hold:*
(i) *$\mathscr{P}$ is non-decreasing and, for every sequence $\{t_l\} \subset (0, 1]$, if $\{\mathscr{P}(t_l)\}$ and $\{\mathscr{Q}(t_l)\}$ are convergent sequences satisfying $\lim_{l\to\infty} \mathscr{P}(t_l) = \lim_{l\to\infty} \mathscr{Q}(t_l)$, then $\lim_{l\to\infty} t_l = 1$;*
(ii) *$\mathcal{A}_0^o$ is a non-empty subset of $\mathcal{A}^o$ such that $\Phi\left(\mathcal{A}_0^o\right) \subseteq \mathcal{B}_0^o$; and*
(iii) *Φ is continuous.*

Then, Φ admits a best proximity point.

Proof. Retaining the notation of Theorem 3.10, put

$$D_l = \mathcal{H}^o(\Phi\sigma_l^o, \Phi\sigma_{l+1}^o, \omega^o).$$

Again, we have the following:

$$\mathscr{P}(D_l) \geq \mathscr{Q}\left(D_{l-1}^{\alpha+\beta} D_l^{1-\alpha-\beta}\right) > \mathscr{P}\left(D_{l-1}^{\alpha+\beta} D_l^{1-\alpha-\beta}\right). \tag{3.23}$$

Since $\mathscr{P}$ is non-decreasing, (3.23) implies $D_l > D_{l-1}$. Hence, $\{D_l\}$ is increasing and bounded above by 1. Put $G_l = D_{l-1}^{\alpha+\beta} D_l^{1-\alpha-\beta}$. Since $D_{l-1} < G_l < D_l$, we have the following:

$$\mathscr{P}(D_{l-1}) \leq \mathscr{P}(G_l) < \mathscr{Q}(G_l) \leq \mathscr{P}(D_l).$$

The increasing sequence $\{\mathscr{P}(D_l)\}$ is bounded above by $\mathscr{P}(1)$ and therefore converges, say to L. Hence, by the squeeze theorem,

$$\lim_{l\to\infty} \mathscr{P}(G_l) = \lim_{l\to\infty} \mathscr{Q}(G_l) = L.$$

Hypothesis (i) applied to $\{G_l\}$ yields $G_l \to 1$. Since $G_l \leq D_l \leq 1$, it follows that

$$\lim_{l\to\infty} \mathcal{H}^o(\Phi\sigma_l^o, \Phi\sigma_{l+1}^o, \omega^o) = 1. \tag{3.24}$$

It remains to verify that $\{\Phi\sigma_l^o\}$ is Cauchy. Otherwise, Lemma 3.3 provides subsequences for which, for some $c \in (0, 1)$, $R_k \to c$, $S_k \to c^\alpha$, where R_k denotes the left-hand fuzzy-metric term in (3.20) and S_k

is its interpolative right-hand argument. Since $0 < \alpha < 1$, we have $c < c^{\alpha}$, and hence $R_k < S_k$ for all sufficiently large k. Therefore,

$$\mathscr{P}(R_k) \leq \mathscr{P}(S_k) < \mathscr{Q}(S_k),$$

which contradicts the contractive inequality $\mathscr{P}(R_k) \geq \mathscr{Q}(S_k)$. Thus, $\{\Phi\sigma_l^o\}$ is Cauchy. The remaining argument is identical to the final part of the proof of Theorem 3.10. Consequently, there exists $\sigma^{o*} \in \mathcal{A}^o$ such that

$$\mathcal{H}^o\left(\sigma^{o*}, \Phi\sigma^{o*}, \omega^o\right) = \mathcal{H}^o\left(\mathcal{A}^o, \mathcal{B}^o, \omega^o\right).$$

Hence, σ^{o*} is a best proximity point of Φ. □

Example 3.12. *Let $\mathcal{X}^o = \mathbb{R}$ with the usual metric $d(\sigma^o, \nu^o) = |\sigma^o - \nu^o|$, and consider*

$$\mathcal{H}^o(\sigma^o, \nu^o, \omega^o) = e^{-|\sigma^o - \nu^o|/\omega^o}$$

for $\omega^o > 0$. Let

$$\mathcal{A}^o = \{0, 4, 20, 100, 500\}, \qquad \mathcal{B}^o = \{1, 5, 21, 101, 501\}.$$

Then, $\mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o) = e^{-1/\omega^o}$ and $\mathcal{A}_0^o = \mathcal{A}^o$, $\mathcal{B}_0^o = \mathcal{B}^o$.

Define $\Phi : \mathcal{A}^o \to \mathcal{B}^o$ by the following:

$$\Phi(0) = 1, \quad \Phi(4) = 1, \quad \Phi(20) = 5, \quad \Phi(100) = 21, \quad \Phi(500) = 101.$$

Take $\alpha = \frac{2}{5}$ and $\beta = \frac{1}{5}$, and define the following:

$$\mathscr{P}(t) = -(\ln t)^2, \qquad \mathscr{Q}(t) = -\frac{1}{2}(\ln t)^2, \qquad t \in (0, 1].$$

Clearly, $\mathscr{P}$ is non-decreasing and $\mathscr{Q}(t) > \mathscr{P}(t)$ for every $t \in (0, 1)$. Moreover,

$$\liminf_{t \to \varepsilon^-} \mathscr{Q}(t) > \mathscr{P}(\varepsilon^-), \qquad \varepsilon \in (0, 1),$$

and the equal-limit condition of Theorem 3.11 is also satisfied. Under (2.6)*, Condition* (3.20) *with these functions is equivalent to $|\Phi\sigma_1^o - \Phi\sigma_2^o| \leq M/\sqrt{2}$, where*

$$M = \alpha|\Phi\nu_1 - \Phi\nu_2| + \beta|\Phi\nu_1 - \Phi\sigma_1^o| + (1 - \alpha - \beta)|\Phi\nu_2 - \Phi\sigma_2^o|.$$

The admissible pairs (σ^o, ν) with $\sigma^o \neq \nu$ are $(0, 4)$*,* $(4, 20)$*,* $(20, 100)$*, and* $(100, 500)$*. A direct check of all sixteen combinations shows that the inequality holds. The closest case is $(\sigma_1^o, \nu_1) = (20, 100)$ and $(\sigma_2^o, \nu_2) = (0, 4)$, where the left-hand side equals* 4 *and the right-hand side is approximately* 7.92*. In this case, $\Phi\sigma_1^o = 5$, $\Phi\sigma_2^o = 1$, $\Phi\nu_1 = 21$, and $\Phi\nu_2 = 1$, so*

$$M = \frac{2}{5} \cdot 20 + \frac{1}{5} \cdot 16 + \frac{2}{5} \cdot 0 = 11.2.$$

Hence, the left-hand side of (3.20) *equals $-16/(\omega^o)^2$ and the right-hand side equals*

$$-\frac{1}{2}(11.2)^2/(\omega^o)^2 = -62.72/(\omega^o)^2.$$

Thus, Φ is a $(\mathscr{P}, \mathscr{Q})$-interpolative Reich–Rus–Ćirić type FPC of the second kind. Finally, $\sigma^{o*} = 0$ is a best proximity point of Φ, since $|0 - \Phi(0)| = 1$.

Figure 4 displays every admissible case of Example 3.12. Under (2.6), the condition does not depend on ω^o, and each row is one admissible case. The circle lies to the left of the square in every case with a positive bound, and the closest case is the one identified in Example 3.12.

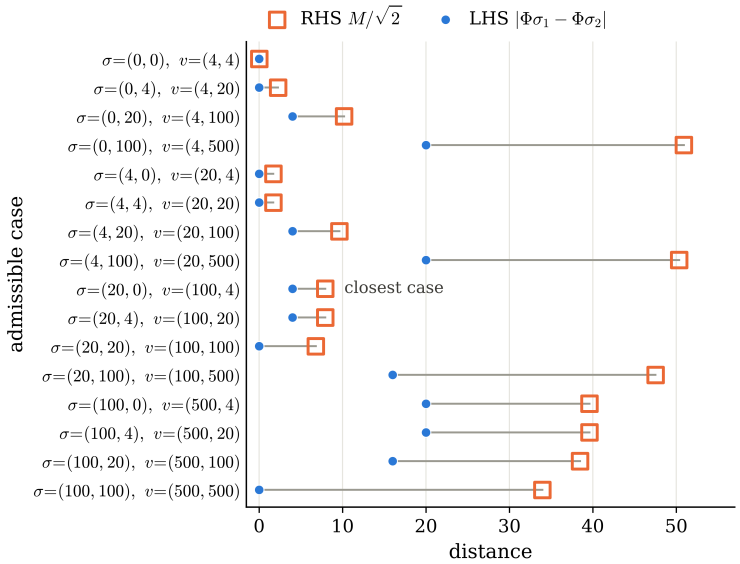


Figure 4. Example 3.12 with $\alpha = \frac{2}{5}, \beta = \frac{1}{5}$, $\mathscr{P}(t) = -(\ln t)^2$, and $\mathscr{Q}(t) = -\frac{1}{2}(\ln t)^2$. Each row is an admissible case, and the horizontal axis shows LHS $= |\Phi\sigma_1^o - \Phi\sigma_2^o|$ and RHS $= M/\sqrt{2}$. LHS $\leq$ RHS in all sixteen cases.

3.3. Interpolative Kannan type proximal contractions

Definition 3.13. *Let $(\mathcal{X}^o, \mathcal{H}^o, *)$ be a complete non-Archimedean fuzzy metric space, and let $\mathcal{A}^o$ and $\mathcal{B}^o$ be non-empty subsets of $\mathcal{X}^o$. A mapping $\Phi : \mathcal{A}^o \to \mathcal{B}^o$ is said to be a $(\mathscr{P}, \mathscr{Q})$-interpolative Kannan type fuzzy proximal contraction of the second kind if $\exists\, \alpha \in (0,1)$ such that*

$$\left.\begin{aligned} \mathcal{H}^o\left(\sigma_1^o, \Phi v_1, \omega^o\right) &= \mathcal{H}^o\left(\mathcal{A}^o, \mathcal{B}^o, \omega^o\right) \\ \mathcal{H}^o\left(\sigma_2^o, \Phi v_2, \omega^o\right) &= \mathcal{H}^o\left(\mathcal{A}^o, \mathcal{B}^o, \omega^o\right) \end{aligned}\right\} \Longrightarrow \mathscr{P}\left(\mathcal{H}^o\left(\Phi\sigma_1^o, \Phi\sigma_2^o, \omega^o\right)\right) \tag{3.25}$$

$$\geq \mathscr{Q}\left(\left[\mathcal{H}^o\left(\Phi v_1, \Phi\sigma_1^o, \omega^o\right)\right]^{\alpha}\left[\mathcal{H}^o\left(\Phi v_2, \Phi\sigma_2^o, \omega^o\right)\right]^{1-\alpha}\right). \tag{3.26}$$

This inequality holds $\forall\, \sigma_1^o, \sigma_2^o, v_1, v_2 \in \mathcal{A}^o$ satisfying $\sigma_i^o \neq v_i$ for $i \in \{1,2\}$, where $\mathscr{P}, \mathscr{Q} : (0,1] \to \mathbb{R}$ are auxiliary functions such that $\mathscr{Q}(t) > \mathscr{P}(t)$, $\forall\, t \in (0,1)$.

Theorem 3.14. *Let $(\mathcal{X}^o, \mathcal{H}^o, *)$ be a complete non-Archimedean fuzzy metric space, and let $\mathcal{A}^o$ and $\mathcal{B}^o$ be non-empty closed subsets of $\mathcal{X}^o$ such that $\mathcal{A}^o$ is AMC with respect to $\mathcal{B}^o$. Let $\Phi : \mathcal{A}^o \to \mathcal{B}^o$ be a $(\mathscr{P}, \mathscr{Q})$-interpolative Kannan type fuzzy proximal contraction of the second kind. Suppose that the following hold:*

(i) *$\mathscr{P}$ is non-decreasing and $\liminf_{t\to\varepsilon^-} \mathscr{Q}(t) > \mathscr{P}(\varepsilon^-)$, $\forall\, \varepsilon \in (0,1)$;*

(ii) *$\mathcal{A}_0^o$ is a non-empty subset of $\mathcal{A}^o$ such that $\Phi\left(\mathcal{A}_0^o\right) \subseteq \mathcal{B}_0^o$; and*

(iii) *Φ is continuous.*

Then, Φ admits a best proximity point.

Proof. Following the same construction as in Theorem 3.6, we obtain a sequence $\{\sigma_l^o\} \subseteq \mathcal{A}_0^o$ which satisfies the following:

$$\mathcal{H}^o(\sigma_{l+1}^o, \Phi\sigma_l^o, \omega^o) = \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o).$$

If $\sigma_l^o = \sigma_{l+1}^o$ for some l, then σ_l^o is at once a best proximity point, so we may take $\sigma_l^o \neq \sigma_{l+1}^o$ for every l. Put $D_l = \mathcal{H}^o(\Phi\sigma_l^o, \Phi\sigma_{l+1}^o, \omega^o)$. Applying (3.26) to two consecutive terms gives the following:

$$\mathscr{P}(D_l) \geq \mathscr{Q}\left(D_{l-1}^{\alpha} D_l^{1-\alpha}\right) > \mathscr{P}\left(D_{l-1}^{\alpha} D_l^{1-\alpha}\right). \tag{3.27}$$

Since $\mathscr{P}$ is non-decreasing, (3.27) implies $D_l > D_{l-1}^{\alpha} D_l^{1-\alpha}$. Therefore, $D_l^{\alpha} > D_{l-1}^{\alpha}$, and, since $\alpha > 0$, $D_l > D_{l-1}$. Thus, $\{D_l\}$ is increasing and bounded above by 1. Hence, $D_l \to E$ for some $E \in (0, 1]$.

Suppose that $E < 1$ and put

$$G_l = D_{l-1}^{\alpha} D_l^{1-\alpha}.$$

Since $D_{l-1} < D_l$, we have

$$D_{l-1} < G_l < D_l < E,$$

and consequently $G_l \to E$ from the left. Therefore, (3.27) yields

$$\mathscr{P}(E^-) \geq \liminf_{l\to\infty} \mathscr{Q}(G_l) \geq \liminf_{t\to E^-} \mathscr{Q}(t) > \mathscr{P}(E^-),$$

which is impossible. Hence, $E = 1$. Therefore,

$$\lim_{l\to\infty} \mathcal{H}^o(\Phi\sigma_l^o, \Phi\sigma_{l+1}^o, \omega^o) = 1. \tag{3.28}$$

Since $\omega^o > 0$ is arbitrary, the above convergence holds for every $\omega^o > 0$. Now, we show that $\{\Phi\sigma_l^o\}$ is Cauchy. Suppose otherwise. By Lemma 3.3, there exist subsequences $\{\Phi\sigma_{l_k}^o\}$ and $\{\Phi\sigma_{p_k}^o\}$ and some $\varepsilon \in (0, 1)$ such that

$$R_k = \mathcal{H}^o(\Phi\sigma_{l_k+1}^o, \Phi\sigma_{p_k+1}^o, \omega^o) \to c = 1 - \varepsilon < 1.$$

Applying (3.26) with $\sigma_1^o = \sigma_{l_k+1}^o$, $v_1 = \sigma_{l_k}^o$, $\sigma_2^o = \sigma_{p_k+1}^o$, and $v_2 = \sigma_{p_k}^o$, we obtain $\mathscr{P}(R_k) \geq \mathscr{Q}(S_k)$, where

$$S_k = \left[\mathcal{H}^o(\Phi\sigma_{l_k}^o, \Phi\sigma_{l_k+1}^o, \omega^o)\right]^{\alpha} \left[\mathcal{H}^o(\Phi\sigma_{p_k}^o, \Phi\sigma_{p_k+1}^o, \omega^o)\right]^{1-\alpha}.$$

By (3.28), $S_k \to 1$, whereas $R_k \to c < 1$. Hence, $R_k < S_k$ for all sufficiently large k. Since $\mathscr{P}$ is non-decreasing and $\mathscr{Q}(t) > \mathscr{P}(t)$ for $t \in (0, 1)$,

$$\mathscr{P}(R_k) \leq \mathscr{P}(S_k) < \mathscr{Q}(S_k),$$

which contradicts $\mathscr{P}(R_k) \geq \mathscr{Q}(S_k)$. Therefore, $\{\Phi\sigma_l^o\}$ is Cauchy. The remaining argument exactly follows the final part of the proof of Theorem 3.6. Indeed, completeness of $\mathcal{X}^o$, closedness of $\mathcal{B}^o$, and approximate compactness of $\mathcal{A}^o$ with respect to $\mathcal{B}^o$ yield a point $\sigma^{o*} \in \mathcal{A}^o$ which satisfies the following:

$$\mathcal{H}^o\left(\sigma^{o*}, \Phi\sigma^{o*}, \omega^o\right) = \mathcal{H}^o\left(\mathcal{A}^o, \mathcal{B}^o, \omega^o\right).$$

Hence, σ^{o*} is a best proximity point of Φ. □

Theorem 3.15. *Let $(\mathcal{X}^o, \mathcal{H}^o, *)$ be a complete non-Archimedean fuzzy metric space, and let $\mathcal{A}^o$ and $\mathcal{B}^o$ be non-empty closed subsets of $\mathcal{X}^o$ such that $\mathcal{A}^o$ is AMC with respect to $\mathcal{B}^o$. Let $\Phi : \mathcal{A}^o \to \mathcal{B}^o$ be a $(\mathscr{P}, \mathscr{Q})$-interpolative Kannan type fuzzy proximal contraction of the second kind. Suppose that the following hold:*
(i) *$\mathscr{P}$ is non-decreasing and, for every sequence $\{t_l\} \subset (0, 1]$, if $\{\mathscr{P}(t_l)\}$ and $\{\mathscr{Q}(t_l)\}$ are convergent sequences such that $\lim_{l\to\infty} \mathscr{P}(t_l) = \lim_{l\to\infty} \mathscr{Q}(t_l)$, then $\lim_{l\to\infty} t_l = 1$;*
(ii) *$\mathcal{A}_0^o$ is a non-empty subset of $\mathcal{A}^o$ such that $\Phi\left(\mathcal{A}_0^o\right) \subseteq \mathcal{B}_0^o$; and*
(iii) *Φ is continuous.*

Then, Φ admits a best proximity point.

Proof. As in the proof of Theorem 3.14, write $D_l = \mathcal{H}^o(\Phi\sigma_l^o, \Phi\sigma_{l+1}^o, \omega^o)$. We again find that

$$\mathscr{P}(D_l) \geq \mathscr{Q}\left(D_{l-1}^{\alpha} D_l^{1-\alpha}\right) > \mathscr{P}\left(D_{l-1}^{\alpha} D_l^{1-\alpha}\right). \tag{3.29}$$

Since $\mathscr{P}$ is non-decreasing, (3.29) implies $D_l > D_{l-1}$. Hence, $\{D_l\}$ is increasing and bounded above by 1. Put $G_l = D_{l-1}^{\alpha} D_l^{1-\alpha}$. Since $D_{l-1} < D_l$, we have $D_{l-1} < G_l < D_l$. Consequently,

$$\mathscr{P}(D_{l-1}) \leq \mathscr{P}(G_l) < \mathscr{Q}(G_l) \leq \mathscr{P}(D_l).$$

The sequence $\{\mathscr{P}(D_l)\}$ is increasing and bounded above by $\mathscr{P}(1)$, and hence convergent. Therefore, by the squeeze theorem,

$$\lim_{l\to\infty} \mathscr{P}(G_l) = \lim_{l\to\infty} \mathscr{Q}(G_l).$$

By Hypothesis (i), $G_l \to 1$. Since $G_l < D_l \leq 1$, it follows that

$$\lim_{l\to\infty} \mathcal{H}^o(\Phi\sigma_l^o, \Phi\sigma_{l+1}^o, \omega^o) = 1. \tag{3.30}$$

Following the Cauchy argument established in the proof of Theorem 3.14, the sequence $\{\Phi\sigma_l^o\}$ is Cauchy. The remaining argument is identical to the final part of the proof of Theorem 3.14. Hence, there exists $\sigma^{o*} \in \mathcal{A}^o$ such that

$$\mathcal{H}^o\left(\sigma^{o*}, \Phi\sigma^{o*}, \omega^o\right) = \mathcal{H}^o\left(\mathcal{A}^o, \mathcal{B}^o, \omega^o\right).$$

Therefore, σ^{o*} is a best proximity point of Φ. □

Example 3.16. *Let $\mathcal{X}^o = \mathbb{R}$ with*

$$\mathcal{H}^o(\sigma^o, \nu^o, \omega^o) = e^{-\frac{|\sigma^o - \nu^o|}{\omega^o}}, \qquad \omega^o > 0,$$

and consider $\mathcal{A}^o = \{1, 2, 3, 4, 5\}$, $\mathcal{B}^o = \{4, 5, 6, 7, 8\}$. Then, $\mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o) = 1, \mathcal{A}_0^o = \mathcal{B}_0^o = \{4, 5\}$. Define $\Phi : \mathcal{A}^o \to \mathcal{B}^o$ by the following:

$$\Phi(1) = 4, \quad \Phi(2) = 4, \quad \Phi(3) = 5, \quad \Phi(4) = 4, \quad \Phi(5) = 4.$$

Clearly, $\Phi(\mathcal{A}_0^o) \subseteq \mathcal{B}_0^o$. Choose $\alpha = \frac{1}{3}$, $\mathscr{P}(t) = (\ln t)^3$, and $\mathscr{Q}(t) = \frac{1}{2}(\ln t)^3$ for $t \in (0, 1]$. Then, $\mathscr{P}$ is non-decreasing and $\mathscr{Q}(t) > \mathscr{P}(t)$ for every $t \in (0, 1)$. Moreover, the equal-limit condition of Theorem 3.15 is satisfied. Here, $\mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o) = 1$, so the proximal conditions mean $\sigma_i^o = \Phi v_i$. With

$\sigma_i^o \neq v_i$, *the admissible pairs* (σ^o, v) *are* $(4,1)$, $(4,2)$, $(5,3)$, *and* $(4,5)$. *Since* $\Phi(4) = \Phi(5) = 4$, *we have* $\mathcal{H}^o(\Phi\sigma_1^o, \Phi\sigma_2^o, \omega^o) = 1$ *and the left-hand side equals* $\mathscr{P}(1) = 0$. *The right-hand side is* $\mathscr{Q}$ *evaluated at a number in* $(0,1]$, *and* $\mathscr{Q} \leq 0$ *there. For instance, for* $v_1 = 1$ *and* $v_2 = 3$, *we have* $\sigma_1^o = 4$ *and* $\sigma_2^o = 5$. *Since* $\Phi(1) = \Phi(4) = 4$ *and* $|\Phi(3) - \Phi(5)| = 1$, *the right-hand side equals*

$$\mathscr{Q}(e^{-(1-\alpha)/\omega^o}) = \frac{1}{2}\Big(-\frac{2}{3\omega^o}\Big)^3 = -\frac{4}{27(\omega^o)^3},$$

which is negative, while the left-hand side equals 0. *Hence, for all sixteen admissible combinations,*

$$\mathscr{P}\left(\mathcal{H}^o(\Phi\sigma_1^o, \Phi\sigma_2^o, \omega^o)\right) \geq \mathscr{Q}\left(\left[\mathcal{H}^o(\Phi v_1, \Phi\sigma_1^o, \omega^o)\right]^{\alpha}\left[\mathcal{H}^o(\Phi v_2, \Phi\sigma_2^o, \omega^o)\right]^{1-\alpha}\right).$$

Hence, Φ *is a* $(\mathscr{P}, \mathscr{Q})$*-interpolative Kannan type fuzzy proximal contraction of the second kind. Finally,* $\sigma^{o*} = 4$ *is a best proximity point of* Φ. *Since* $\mathcal{A}^o \cap \mathcal{B}^o \neq \emptyset$ *in this example, the best proximity point is a fixed point of* Φ.

Figure 5 displays every admissible case of Example 3.16. Under (2.6), the condition does not depend on ω^o, and each row is one admissible case. It agrees with the analytic check given in the example.

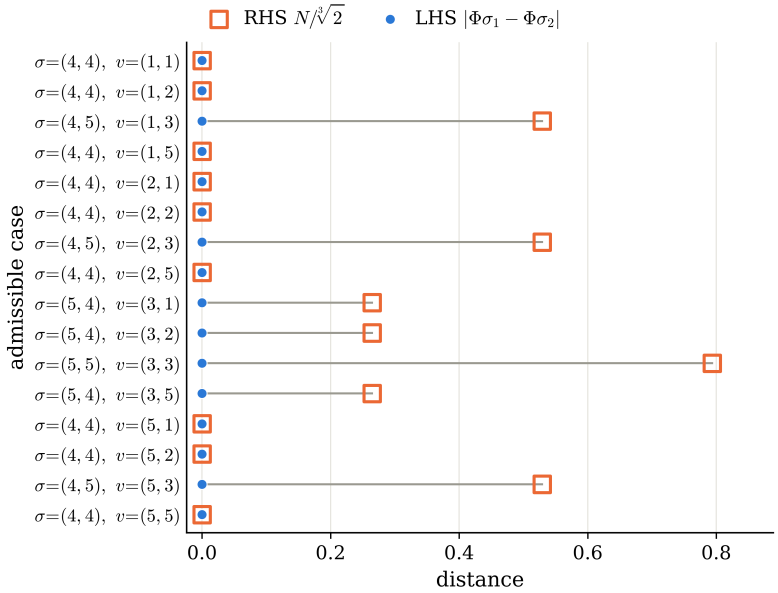


Figure 5. Example 3.16 with $\alpha = \frac{1}{3}$, $\mathscr{P}(t) = (\ln t)^3$, and $\mathscr{Q}(t) = \frac{1}{2}(\ln t)^3$. Each row is an admissible case, and the horizontal axis shows LHS $= |\Phi\sigma_1^o - \Phi\sigma_2^o| = 0$ and RHS $= N/\sqrt[3]{2}$, where $N = \alpha|\Phi v_1 - \Phi\sigma_1^o| + (1-\alpha)|\Phi v_2 - \Phi\sigma_2^o|$.

3.4. Interpolative Hardy–Rogers type proximal contractions

Definition 3.17. *Let* $(\mathcal{X}^o, \mathcal{H}^o, *)$ *be a complete non-Archimedean fuzzy metric space, and let* $\mathcal{A}^o$ *and* $\mathcal{B}^o$ *be non-empty subsets of* $\mathcal{X}^o$. *A mapping* $\Phi : \mathcal{A}^o \to \mathcal{B}^o$ *is called a* $(\mathscr{P}, \mathscr{Q})$*-interpolative Hardy–Rogers type FPC of the second kind if* $\exists\, \alpha, \beta, \gamma, \delta \in (0,1)$ *satisfying*

$$\alpha + \beta + \gamma + 2\delta \leq 1$$

such that

$$\left.\begin{array}{l}\mathcal{H}^o\left(\sigma_1^o, \Phi v_1, \omega^o\right) = \mathcal{H}^o\left(\mathcal{A}^o, \mathcal{B}^o, \omega^o\right) \\ \mathcal{H}^o\left(\sigma_2^o, \Phi v_2, \omega^o\right) = \mathcal{H}^o\left(\mathcal{A}^o, \mathcal{B}^o, \omega^o\right)\end{array}\right\} \Longrightarrow \mathscr{P}\left(\mathcal{H}^o\left(\Phi\sigma_1^o, \Phi\sigma_2^o, \omega^o\right)\right) \tag{3.31}$$

$$\geq \mathscr{Q}\left(\begin{array}{l}\left[\mathcal{H}^o\left(\Phi v_1, \Phi v_2, \omega^o\right)\right]^\alpha \left[\mathcal{H}^o\left(\Phi v_1, \Phi\sigma_1^o, \omega^o\right)\right]^\beta \\ \left[\mathcal{H}^o\left(\Phi v_2, \Phi\sigma_2^o, \omega^o\right)\right]^\gamma \left[\mathcal{H}^o\left(\Phi v_1, \Phi\sigma_2^o, \omega^o\right)\right]^\delta \\ \left[\mathcal{H}^o\left(\Phi v_2, \Phi\sigma_1^o, \omega^o\right)\right]^{1-\alpha-\beta-\gamma-\delta}\end{array}\right). \tag{3.32}$$

The above inequality holds $\forall\ \sigma_1^o, \sigma_2^o, v_1, v_2 \in \mathcal{A}^o$ *satisfying* $\sigma_i^o \neq v_i$, $i \in \{1, 2\}$, *where* $\mathscr{P}, \mathscr{Q} : (0, 1] \to \mathbb{R}$ *satisfy* $\mathscr{Q}(t) > \mathscr{P}(t)$, $t \in (0, 1)$.

Theorem 3.18. *Let* $(\mathcal{X}^o, \mathcal{H}^o, *)$ *be a complete non-Archimedean fuzzy metric space, and let* $\mathcal{A}^o$ *and* $\mathcal{B}^o$ *be non-empty closed subsets of* $\mathcal{X}^o$ *such that* $\mathcal{A}^o$ *is AMC with respect to* $\mathcal{B}^o$. *Let* $\Phi : \mathcal{A}^o \to \mathcal{B}^o$ *be a* $(\mathscr{P}, \mathscr{Q})$*-interpolative Hardy–Rogers type FPC of the second kind. Suppose that the following hold:*
(i) $\mathscr{P}$ *and* $\mathscr{Q}$ *are non-decreasing and* $\liminf_{t\to\varepsilon^-} \mathscr{Q}(t) > \mathscr{P}(\varepsilon^-)\ \forall\ \varepsilon \in (0, 1)$*;*
(ii) $\mathcal{A}_0^o$ *is a non-empty subset of* $\mathcal{A}^o$ *such that* $\Phi\left(\mathcal{A}_0^o\right) \subseteq \mathcal{B}_0^o$*; and*
(iii) Φ *is continuous.*

Then, Φ *admits a best proximity point.*

Proof. As in the proofs of Theorems 3.6–3.14, we build a sequence $\{\sigma_l^o\} \subseteq \mathcal{A}_0^o$ such that

$$\mathcal{H}^o(\sigma_{l+1}^o, \Phi\sigma_l^o, \omega^o) = \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o), \qquad l \geq 0. \tag{3.33}$$

If $\sigma_l^o = \sigma_{l+1}^o$ for some l, then (3.33) immediately shows that σ_l^o is a best proximity point. Hence, assume that $\sigma_l^o \neq \sigma_{l+1}^o$ for every l. Put

$$D_l = \mathcal{H}^o(\Phi\sigma_l^o, \Phi\sigma_{l+1}^o, \omega^o).$$

Applying the Hardy–Rogers contractive condition (3.32) with $\sigma_1^o = \sigma_l^o$, $\sigma_2^o = \sigma_{l+1}^o$, $v_1 = \sigma_{l-1}^o$, $v_2 = \sigma_l^o$, and using $\mathcal{H}^o(x, x, \omega^o) = 1$, we obtain the following:

$$\mathscr{P}(D_l) \geq \mathscr{Q}\left(D_{l-1}^{\alpha+\beta} D_l^\gamma \mathcal{H}^o(\Phi\sigma_{l-1}^o, \Phi\sigma_{l+1}^o, \omega^o)^\delta\right).$$

By (2.7),

$$\mathcal{H}^o(\Phi\sigma_{l-1}^o, \Phi\sigma_{l+1}^o, \omega^o) \geq D_{l-1} D_l.$$

Since $\mathscr{Q}$ is non-decreasing, it follows that

$$\mathscr{P}(D_l) \geq \mathscr{Q}\left(D_{l-1}^{\alpha+\beta+\delta} D_l^{\gamma+\delta}\right). \tag{3.34}$$

Set $p = \alpha+\beta+\delta$, $q = \gamma+\delta$, and $s = p+q = \alpha+\beta+\gamma+2\delta \leq 1$. Since $\mathscr{Q}(t) > \mathscr{P}(t)$ for $t \in (0, 1)$, (3.34) gives $\mathscr{P}(D_l) > \mathscr{P}(D_{l-1}^p D_l^q)$. As $\mathscr{P}$ is non-decreasing,

$$D_l > D_{l-1}^p D_l^q.$$

Suppose that $D_l \leq D_{l-1}$. Then,

$$D_{l-1}^p D_l^q \geq D_l^{p+q} = D_l^s \geq D_l,$$

because $0 < D_l \leq 1$ and $s \leq 1$. This is a contradiction. Consequently,

$$D_l > D_{l-1}$$

for every l. Thus, $\{D_l\}$ is increasing and bounded above by 1, and hence $D_l \to E$ for some $E \in (0, 1]$. We claim that $E = 1$. Suppose that $E < 1$. If $s < 1$, then letting $l \to \infty$ in $D_l > D_{l-1}^p D_l^q$ gives $E \geq E^s$. However, since $0 < E < 1$ and $s < 1$, $E^s > E$, which is impossible. It remains to consider $s = 1$. Put $G_l = D_{l-1}^p D_l^q$. Since $p, q > 0$, $p + q = 1$, and $D_{l-1} < D_l$, we have $D_{l-1} < G_l < D_l$, and hence $G_l \to E$ from the left. Therefore, from (3.34),

$$\mathscr{P}(E^-) \geq \liminf_{l\to\infty} \mathscr{Q}(G_l) \geq \liminf_{t\to E^-} \mathscr{Q}(t) > \mathscr{P}(E^-),$$

which is again impossible. Hence, $E = 1$. Consequently,

$$\lim_{l\to\infty} \mathcal{H}^o(\Phi\sigma_l^o, \Phi\sigma_{l+1}^o, \omega^o) = 1. \tag{3.35}$$

Now, we prove that $\{\Phi\sigma_l^o\}$ is Cauchy. Suppose, to the contrary, that it is not Cauchy. By Lemma 3.3, there exist subsequences indexed by l_k and p_k, and some $c \in (0, 1)$, such that

$$R_k = \mathcal{H}^o(\Phi\sigma_{l_k+1}^o, \Phi\sigma_{p_k+1}^o, \omega^o) \to c,$$

and the corresponding mixed fuzzy-metric terms also converge to c. Applying (3.32) with

$$\sigma_1^o = \sigma_{l_k+1}^o, \qquad \sigma_2^o = \sigma_{p_k+1}^o, \qquad v_1 = \sigma_{l_k}^o, \qquad v_2 = \sigma_{p_k}^o,$$

we obtain

$$\mathscr{P}(R_k) \geq \mathscr{Q}(S_k),$$

where

$$\begin{aligned} S_k = &[\mathcal{H}^o(\Phi\sigma_{l_k}^o, \Phi\sigma_{p_k}^o, \omega^o)]^\alpha [\mathcal{H}^o(\Phi\sigma_{l_k}^o, \Phi\sigma_{l_k+1}^o, \omega^o)]^\beta \\ &[\mathcal{H}^o(\Phi\sigma_{p_k}^o, \Phi\sigma_{p_k+1}^o, \omega^o)]^\gamma [\mathcal{H}^o(\Phi\sigma_{l_k}^o, \Phi\sigma_{p_k+1}^o, \omega^o)]^\delta \\ &[\mathcal{H}^o(\Phi\sigma_{p_k}^o, \Phi\sigma_{l_k+1}^o, \omega^o)]^{1-\alpha-\beta-\gamma-\delta}. \end{aligned}$$

Using (3.35) and Lemma 3.3, we obtain $S_k \to c^{\alpha+\delta+1-\alpha-\beta-\gamma-\delta} = c^{1-\beta-\gamma}$. Since $0 < 1 - \beta - \gamma < 1$ and $0 < c < 1$, we have $c < c^{1-\beta-\gamma}$. Hence, $R_k < S_k$ for all sufficiently large k. Since $\mathscr{P}$ is non-decreasing,

$$\mathscr{P}(R_k) \leq \mathscr{P}(S_k) < \mathscr{Q}(S_k),$$

which contradicts $\mathscr{P}(R_k) \geq \mathscr{Q}(S_k)$. Therefore, $\{\Phi\sigma_l^o\}$ is a Cauchy sequence. The remaining argument is the same as the common final part of the proofs of Theorems 3.6–3.14. Since $\mathcal{X}^o$ is complete, $\mathcal{B}^o$ is closed, and $\mathcal{A}^o$ is approximately compact with respect to $\mathcal{B}^o$, there exists $\sigma^{o*} \in \mathcal{A}^o$ such that

$$\mathcal{H}^o(\sigma^{o*}, \Phi\sigma^{o*}, \omega^o) = \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o).$$

Hence, σ^{o*} is a best proximity point of Φ. □

Theorem 3.19. *Let $(\mathcal{X}^o, \mathcal{H}^o, *)$ be a complete non-Archimedean fuzzy metric space, and let $\mathcal{A}^o$ and $\mathcal{B}^o$ be non-empty closed subsets of $\mathcal{X}^o$ such that $\mathcal{A}^o$ is AMC with respect to $\mathcal{B}^o$. Let $\Phi : \mathcal{A}^o \to \mathcal{B}^o$ be a $(\mathscr{P}, \mathscr{Q})$-interpolative Hardy–Rogers type FPC of the second kind. Suppose that the following hold:*

(i) $\mathscr{P}$ *and* $\mathscr{Q}$ *are non-decreasing and, for every sequence* $\{t_l\} \subset (0,1]$*, if* $\{\mathscr{P}(t_l)\}$ *and* $\{\mathscr{Q}(t_l)\}$ *are convergent sequences satisfying* $\lim_{l\to\infty} \mathscr{P}(t_l) = \lim_{l\to\infty} \mathscr{Q}(t_l)$*, then* $\lim_{l\to\infty} t_l = 1$*;*
(ii) $\mathcal{A}_0^o$ *is a non-empty subset of* $\mathcal{A}^o$ *such that* $\Phi\left(\mathcal{A}_0^o\right) \subseteq \mathcal{B}_0^o$*; and*
(iii) Φ *is continuous.*

Then, Φ *admits a best proximity point.*

Proof. Take $D_l = \mathcal{H}^o(\Phi\sigma_l^o, \Phi\sigma_{l+1}^o, \omega^o)$ as in the proof of Theorem 3.18, and put $p = \alpha+\beta+\delta$, $q = \gamma+\delta$, $s = p + q = \alpha + \beta + \gamma + 2\delta \leq 1$. As shown there,

$$\mathscr{P}(D_l) \geq \mathscr{Q}\left(D_{l-1}^p D_l^q\right) > \mathscr{P}\left(D_{l-1}^p D_l^q\right). \tag{3.36}$$

Since $\mathscr{P}$ is non-decreasing, (3.36) implies $D_l > D_{l-1}$. Hence, $\{D_l\}$ is increasing and bounded above by 1, so $D_l \to E$ for some $E \in (0,1]$. If $s < 1$ and $E < 1$, then passing to the limit in $D_l > D_{l-1}^p D_l^q$ gives $E \geq E^s > E$, which is a contradiction. Hence, $E = 1$. Now, suppose that $s = 1$ and define $G_l = D_{l-1}^p D_l^q$. Since $D_{l-1} < D_l$ and $p + q = 1$, $D_{l-1} < G_l < D_l$. Therefore,

$$\mathscr{P}(D_{l-1}) \leq \mathscr{P}(G_l) < \mathscr{Q}(G_l) \leq \mathscr{P}(D_l).$$

The sequence $\{\mathscr{P}(D_l)\}$ is increasing and bounded above by $\mathscr{P}(1)$, and hence convergent. By the squeeze theorem,

$$\lim_{l\to\infty} \mathscr{P}(G_l) = \lim_{l\to\infty} \mathscr{Q}(G_l).$$

Thus, Hypothesis (i) gives $G_l \to 1$. Since $G_l < D_l \leq 1$, we obtain the following:

$$\lim_{l\to\infty} \mathcal{H}^o(\Phi\sigma_l^o, \Phi\sigma_{l+1}^o, \omega^o) = 1. \tag{3.37}$$

Now, the Cauchy argument is exactly the same as in Theorem 3.18. Hence, $\{\Phi\sigma_l^o\}$ is Cauchy. The remaining completeness and approximate-compactness argument also follows from the final part of Theorem 3.18. Consequently, there exists $\sigma^{o*} \in \mathcal{A}^o$ such that

$$\mathcal{H}^o\left(\sigma^{o*}, \Phi\sigma^{o*}, \omega^o\right) = \mathcal{H}^o\left(\mathcal{A}^o, \mathcal{B}^o, \omega^o\right).$$

Therefore, σ^{o*} is a best proximity point of Φ. □

3.5. Corollaries

The following consequences are obtained from the preceding $(\mathscr{P}, \mathscr{Q})$-contractive framework by suppressing the auxiliary-function mechanism.

Corollary 3.20. *Under the hypotheses of Theorem 3.6, the mapping* $\Phi : \mathcal{A}^o \to \mathcal{B}^o$ *satisfies the classical FPC of the second kind, namely*

$$\left.\begin{aligned} \mathcal{H}^o(\sigma_1^o, \Phi v_1, \omega^o) &= \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o) \\ \mathcal{H}^o(\sigma_2^o, \Phi v_2, \omega^o) &= \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o) \end{aligned}\right\} \Longrightarrow \mathcal{H}^o(\Phi\sigma_1^o, \Phi\sigma_2^o, \omega^o) \geq \mathcal{H}^o(\Phi v_1, \Phi v_2, \omega^o).$$

Corollary 3.21. *Under the hypotheses of Theorem 3.10, the mapping $\Phi : \mathcal{A}^o \to \mathcal{B}^o$ satisfies the classical interpolative Reich–Rus–Ćirić type fuzzy proximal contraction of the second kind, namely*

$$\left.\begin{aligned}\mathcal{H}^o(\sigma_1^o, \Phi v_1, \omega^o) &= \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o)\\ \mathcal{H}^o(\sigma_2^o, \Phi v_2, \omega^o) &= \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o)\end{aligned}\right\}$$

$$\mathcal{H}^o(\Phi\sigma_1^o, \Phi\sigma_2^o, \omega^o) \geq \left[\mathcal{H}^o(\Phi v_1, \Phi v_2, \omega^o)\right]^{\alpha} \left[\mathcal{H}^o(\Phi v_1, \Phi\sigma_1^o, \omega^o)\right]^{\beta} \left[\mathcal{H}^o(\Phi v_2, \Phi\sigma_2^o, \omega^o)\right]^{1-\alpha-\beta},$$

where $\alpha, \beta \in (0, 1)$, and $\alpha + \beta < 1$.

Corollary 3.22. *Under the hypotheses of Theorem 3.14, the mapping $\Phi : \mathcal{A}^o \to \mathcal{B}^o$ satisfies the classical interpolative Kannan type FPC of the second kind, namely*

$$\left.\begin{aligned}\mathcal{H}^o(\sigma_1^o, \Phi v_1, \omega^o) &= \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o)\\ \mathcal{H}^o(\sigma_2^o, \Phi v_2, \omega^o) &= \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o)\end{aligned}\right\}$$

$$\mathcal{H}^o(\Phi\sigma_1^o, \Phi\sigma_2^o, \omega^o) \geq \left[\mathcal{H}^o(\Phi v_1, \Phi\sigma_1^o, \omega^o)\right]^{\alpha} \left[\mathcal{H}^o(\Phi v_2, \Phi\sigma_2^o, \omega^o)\right]^{1-\alpha},$$

where $\alpha \in (0, 1)$.

Corollary 3.23. *Under the hypotheses of Theorem 3.18, the mapping $\Phi : \mathcal{A}^o \to \mathcal{B}^o$ satisfies the classical interpolative Hardy–Rogers type FPC of the second kind, namely*

$$\left.\begin{aligned}\mathcal{H}^o(\sigma_1^o, \Phi v_1, \omega^o) &= \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o)\\ \mathcal{H}^o(\sigma_2^o, \Phi v_2, \omega^o) &= \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o)\end{aligned}\right\}$$

$$\begin{aligned}\mathcal{H}^o(\Phi\sigma_1^o, \Phi\sigma_2^o, \omega^o) \geq\ & \left[\mathcal{H}^o(\Phi v_1, \Phi v_2, \omega^o)\right]^{\alpha} \left[\mathcal{H}^o(\Phi v_1, \Phi\sigma_1^o, \omega^o)\right]^{\beta} \left[\mathcal{H}^o(\Phi v_2, \Phi\sigma_2^o, \omega^o)\right]^{\gamma}\\ & \left[\mathcal{H}^o(\Phi v_1, \Phi\sigma_2^o, \omega^o)\right]^{\delta} \left[\mathcal{H}^o(\Phi v_2, \Phi\sigma_1^o, \omega^o)\right]^{1-\alpha-\beta-\gamma-\delta},\end{aligned}$$

where $\alpha, \beta, \gamma, \delta \in (0, 1)$, and $\alpha + \beta + \gamma + 2\delta \leq 1$.

Remark 3.24. *Corollaries 3.20–3.23 show that the corresponding classical proximal contractive inequalities are recovered from the proposed $(\mathscr{P}, \mathscr{Q})$-framework once the auxiliary-function mechanism is suppressed. These include the basic proximal, Reich–Rus–Ćirić, Kannan, and Hardy–Rogers type contractions of the second kind. The comparison is summarized in Table 2.*

Table 2. Comparison of the contractive schemes with related results.

Contraction type	Fixed-point / Interpolative	Proximal	Fuzzy proximal	This work
FPC of the second kind	–	[27]	[28]	Theorems 3.6, 3.7
Reich–Rus–Ćirić interpolative	[12]	[21]	[22]	Theorems 3.10, 3.11
Kannan interpolative	[11]	[21]	[22]	Theorems 3.14, 3.15
Hardy–Rogers interpolative	[13]	[21]	[22]	Theorems 3.18, 3.19

Remark 3.25. *All proofs in this section follow the same three steps. The conditions on $\mathscr{P}$ and $\mathscr{Q}$ force $\mathcal{H}^o(\Phi\sigma_l^o, \Phi\sigma_{l+1}^o, \omega^o)$ to increase to* 1*; then, the sequence $\{\Phi\sigma_l^o\}$ is shown to be Cauchy, and completeness together with approximate compactness give the best proximity point. The two forms of Hypothesis (i) control the first step in different ways, which is why each class has two existence theorems. Lemma 3.5 is only used in Theorem 3.6. Theorems 3.10, 3.14, and 3.18 directly prove the Cauchy property from the monotonicity of $\mathscr{P}$, so their Hypothesis (i) does not require the continuity of $\mathscr{P}$. All eight theorems end with the argument given at the end of the proof of Theorem 3.6, which uses* (2.7) *and the continuity of* Φ*. In all proofs, we may assume $0 < D_l < 1$ for every l, since $D_l = 1$ for some l already gives a best proximity point, as shown in the proof of Theorem 3.6.*

The next two corollaries show that known contractions are obtained from the present framework by particular choices of $\mathscr{P}$ and $\mathscr{Q}$.

Corollary 3.26. *Let $(\mathcal{X}^o, \mathcal{H}^o, *)$, $\mathcal{A}^o$, $\mathcal{B}^o$, and $\mathcal{A}_0^o$ be as in Theorem 3.7, and let Φ be continuous. Suppose that there exists $k \in (0,1)$ such that*

$$\mathcal{H}^o(\Phi\sigma_1^o, \Phi\sigma_2^o, \omega^o) \geq \left[\mathcal{H}^o(\Phi v_1, \Phi v_2, \omega^o)\right]^k$$

whenever $\sigma_1^o \neq \sigma_2^o$ and

$$\mathcal{H}^o(\sigma_i^o, \Phi v_i, \omega^o) = \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o), \quad \text{for } i = 1, 2.$$

Then, Φ has a best proximity point.

Proof. Take $\mathscr{P}(t) = \ln t$ and $\mathscr{Q}(t) = k \ln t$. Then, $\mathscr{Q} > \mathscr{P}$ on $(0,1)$, and $\mathscr{P}(a) \geq \mathscr{Q}(b)$ is equivalent to $a \geq b^k$. The function $\mathscr{P}$ is continuous and increasing. If $\ln t_l \to L$ and $k \ln t_l \to L$, then $L = 0$ and $t_l \to 1$. The result follows from Theorem 3.7. □

For the induced fuzzy metric (2.6), the condition of Corollary 3.26 reads as follows:

$$d(\Phi\sigma_1^o, \Phi\sigma_2^o) \leq k\, d(\Phi v_1, \Phi v_2),$$

which is the proximal contraction of the second kind of Basha [27].

Corollary 3.27. *Under the assumptions of Corollary 3.26 on the space, the sets, and Φ, suppose that there exists $k \in (0,1)$ such that*

$$\frac{1}{\mathcal{H}^o(\Phi\sigma_1^o, \Phi\sigma_2^o, \omega^o)} - 1 \leq k\left(\frac{1}{\mathcal{H}^o(\Phi v_1, \Phi v_2, \omega^o)} - 1\right)$$

for the same points. Then, Φ has a best proximity point.

Proof. Take

$$\mathscr{P}(t) = 1 - \frac{1}{t} \quad \text{and} \quad \mathscr{Q}(t) = k(1 - \frac{1}{t}).$$

Since $\mathscr{P} < 0$ on $(0,1)$, we have $\mathscr{Q} > \mathscr{P}$ there, and $\mathscr{P}(a) \geq \mathscr{Q}(b)$ is the displayed inequality. If $\mathscr{P}(t_l)$ and $k\mathscr{P}(t_l)$ have the same limit, this limit is 0 and $t_l \to 1$. The result follows from Theorem 3.7. □

Remark 3.28. *The same choice $\mathscr{P}(t) = \ln t$ and $\mathscr{Q}(t) = k \ln t$ in Definitions 3.9, 3.13, and 3.17 leads to fuzzy versions of the interpolative proximal contractions of Altun and Taşdemir [21] with a contraction constant k. Their best proximity points follow from Theorems 3.11, 3.15, and 3.19.*

Remark 3.29. *The mapping of Example 3.2 satisfies* (3.1) *but has no best proximity point, since*

$$\mathfrak{d}((0, \mathfrak{t}^o), \Phi(0, \mathfrak{t}^o)) = 1 + \frac{2}{3}\mathfrak{t}^o > 1, \quad \text{for every } \mathfrak{t}^o > 0.$$

The proximal sequence $(0, \mathfrak{a}/3^l)$ converges to $(0,0) \notin \mathcal{A}^o$. Hence, the closedness of $\mathcal{A}^o$ cannot be dropped.

Examples 3.8, 3.12, and 3.16 use $\mathscr{Q} = c\,\mathscr{P}$, so under (2.6), they reduce to conditions with a fixed contraction constant. The following example shows that the framework also covers mappings for which no such constant exists.

Example 3.30. *Let $\mathcal{X}^o = \mathbb{R}^2$ with the Euclidean metric d and $\mathcal{H}^o(\sigma^o, \nu^o, \omega^o) = e^{-d(\sigma^o, \nu^o)}$ for all $\omega^o > 0$, with the product t-norm. By the triangle inequality for d, this is a complete non-Archimedean fuzzy metric space. Let*

$$\mathcal{A}^o = \{(x,0) : x \geq 0\}, \quad \mathcal{B}^o = \{(x,1) : x \geq 0\}, \quad \Phi(x,0) = \big(f(x), 1\big), \quad f(x) = \frac{x}{1+x},$$

and let $\mathscr{P}(t) = \ln t$ and

$$\mathscr{Q}(t) = \frac{\ln t}{1 - \ln t}$$

for $t \in (0,1]$.

The sets $\mathcal{A}^o$ and $\mathcal{B}^o$ are closed and $\mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o) = e^{-1}$. The points $(x,0)$ and $(y,1)$ only attain this value when $x = y$, so $\mathcal{A}^o_0 = \mathcal{A}^o$, $\mathcal{B}^o_0 = \mathcal{B}^o$, and $\Phi(\mathcal{A}^o_0) \subseteq \mathcal{B}^o_0$. If $\eta = (y,1)$ and

$$\mathcal{H}^o((x_l, 0), \eta, \omega^o) \to e^{-1},$$

then $x_l \to y$; hence, $\mathcal{A}^o$ is AMC with respect to $\mathcal{B}^o$. The mapping Φ is continuous.

The function $\mathscr{P}$ is continuous and increasing, and $\mathscr{Q}(t) > \mathscr{P}(t)$ for $t \in (0,1)$ because $\ln t < 0$ and $1 - \ln t > 1$. Since $\mathscr{Q}$ is continuous, Condition (3.7) *holds. If $\mathscr{P}(t_l)$ and $\mathscr{Q}(t_l)$ have the same finite limit L, then $L = L/(1-L)$, so $L = 0$ and $t_l \to 1$.*

Writing $v_i = (\bar{v}_i, 0)$, the proximal conditions give $\sigma^o_i = (u_i, 0)$ with $u_i = f(\bar{v}_i) \in [0,1)$. Put $D = |u_1 - u_2| > 0$. Since

$$f(u_1) - f(u_2) = \frac{u_1 - u_2}{(1+u_1)(1+u_2)}$$

and $(1+u_1)(1+u_2) \geq 1 + D$, we obtain

$$d(\Phi\sigma^o_1, \Phi\sigma^o_2) = \frac{D}{(1+u_1)(1+u_2)} \leq \frac{D}{1+D}.$$

As $d(\Phi v_1, \Phi v_2) = D$, this is Condition (3.1)*. Hence, all hypotheses of Theorems 3.6 and 3.7 hold. The best proximity point is $\sigma^{o*} = (0,0)$, since $d\big((x,0), \Phi(x,0)\big) = 1$ only for $x = 0$.*

For $\bar{v}_2 = 0$ and $\bar{v}_1 = \varepsilon/(1-\varepsilon)$ with $0 < \varepsilon < 1$, we have $d(\Phi v_1, \Phi v_2) = \varepsilon$ and $d(\Phi\sigma_1^o, \Phi\sigma_2^o) = \varepsilon/(1+\varepsilon)$. The ratio tends to 1 as $\varepsilon \to 0$, so no $k \in (0,1)$ satisfies the condition of Corollary 3.26. In the same way, since

$$(e^{\varepsilon/(1+\varepsilon)} - 1)/(e^{\varepsilon} - 1) \to 1,$$

no $k \in (0,1)$ satisfies the condition of Corollary 3.27. Accordingly, the proximal sequence $x_l = x_0/(1 + l\,x_0)$ converges to 0 at the rate $1/l$ and not geometrically.

Remark 3.31. *In Example 3.30, each point of $\mathcal{B}_0^o$ has a unique proximal point in $\mathcal{A}_0^o$. In such cases, the problem can also be reduced to a fixed point problem for a self mapping on $\Phi(\mathcal{A}_0^o)$, to which Proinov type results [16, 17] apply. The theorems of this section directly treat the proximal condition of the second kind and cover the four contractive structures by one argument.*

4. A dynamical illustration of the best-proximity concept

We illustrate the best proximity point concept of Definitions 2.6 and 3.1 in a setting outside the metric fixed point theory, namely a classical discrete-time population map with a period-doubling bifurcation. As explained in Remark 4.3 below, the contractive inequality verified in the examples is, in this particular construction, satisfied in a degenerate (vacuous) way, so the value of this section lies in showing how the abstract notions $\mathcal{A}^o$, $\mathcal{B}^o$, Φ, $\mathcal{A}_0^o$, and $\mathcal{B}_0^o$ and the best proximity point translate into a concrete question about a bifurcating dynamical system, rather than in exercising the full iterative machinery of Section 3.

4.1. The discrete map and its bifurcations

Consider the discrete-time quadratic map

$$i_{l+1} = \Phi_r(i_l) := r\,i_l(1 - i_l), \qquad i_l \in [0,1],\ r > 0, \tag{4.1}$$

that is, the logistic map, whose bifurcation structure is completely understood [24]. Earn, Rohani, Bolker, and Grenfell [25] explained the transitions between regular and irregular multi-year cycles observed in measles incidence data through changes in parameters such as birth and vaccination rates. In the same spirit, we adopt the following stylized epidemiological reading of (4.1). Here, i_l denotes the fraction of the population infected in generation l and r plays the role, in this stylized discrete-generation model, of the disease's basic reproduction number. We emphasize that this identification of r with the epidemiological reproduction number $\mathcal{R}_0$ is a modeling convention rather than a claim derived here from first principles. For the purposes of this section, r may equally well be read simply as the growth/transmission parameter of the abstract map (4.1).

For $0 < r \le 1$, the disease-free state $i = 0$ is the attractor. At $r = 1$, a transcritical bifurcation occurs, and for $r > 1$, the endemic equilibrium $i^*(r) = 1 - 1/r$ becomes the attractor. This endemic equilibrium, in turn, loses stability at $r = 3$, where a period-doubling bifurcation produces a stable two-cycle $\{p_1(r), p_2(r)\}$, $p_1(r) < p_2(r)$, given explicitly for $r \in (3, 1 + \sqrt{6}\,]$ by

$$p_{1,2}(r) = \frac{(r+1) \mp \sqrt{(r-3)(r+1)}}{2r}, \tag{4.2}$$

representing a disease that oscillates between a low-prevalence year and a high-prevalence year, the biennial pattern documented for measles before mass vaccination [25]. Figure 6 displays the full bifurcation diagram of Φ_r, including the period-doubling route to chaos as $r \to 4$, and Table 3 summarizes the successive dynamical regimes and their epidemiological reading.

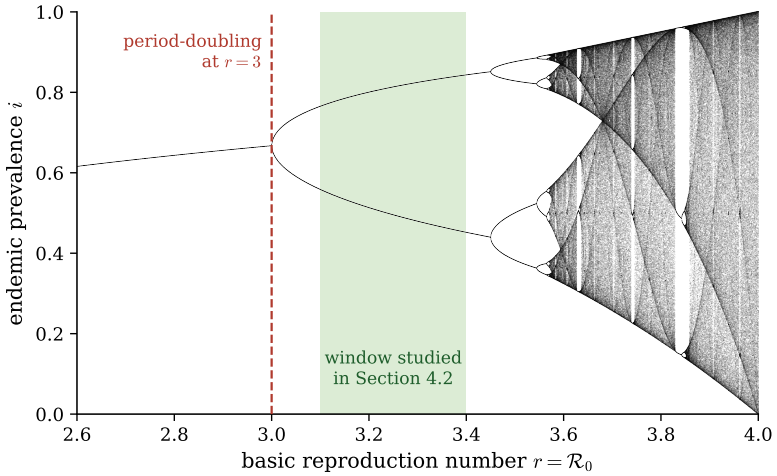


Figure 6. Bifurcation diagram of the logistic map $i_{l+1} = r\, i_l(1 - i_l)$ for $r \in [2.6, 4]$. The dashed line marks $r = 3$, and the shaded band is the window $[3.1, 3.4]$ of Section 4.2.

Table 3. Dynamical regimes of the logistic map (4.1) and their epidemiological meaning.

Range of r	Attractor	Epidemiological meaning
$0 < r \le 1$	$i = 0$	disease dies out
$1 < r < 3$	$i^*(r) = 1 - 1/r$	steady endemic level
$3 < r \le 1 + \sqrt{6}$	2-cycle $\{p_1(r), p_2(r)\}$	biennial epidemic
$1 + \sqrt{6} < r < r_4$	4-cycle	four-year cycle
r near 4	aperiodic	chaotic outbreaks

4.2. The best proximity point of the epidemic map

Now, we show how the sets $\mathcal{A}^o, \mathcal{B}^o$ and the mapping Φ of the present paper arise from this model, and how the resulting best proximity point carries a direct epidemiological meaning. Fix the parameter window $r \in [3.1, 3.4] \subset (3, 1 + \sqrt{6})$, work in $(\mathcal{X}^o, \mathcal{H}^o, *)$ with $\mathcal{X}^o = \mathbb{R}^2$, Euclidean distance d, and

$$\mathcal{H}^o(\sigma^o, \nu^o, \omega^o) = e^{-d(\sigma^o, \nu^o)/\omega^o}$$

as in (2.6), and set

$$\mathcal{A}^o = \{(r, p_1(r)) : r \in [3.1, 3.4]\}, \qquad \mathcal{B}^o = \{(r, p_2(r)) : r \in [3.1, 3.4]\},$$

the low- and high-prevalence branches of the two-cycle over this window. Define $\Phi : \mathcal{A}^o \to \mathcal{B}^o$ by $\Phi(r, p_1(r)) = (r, \Phi_r(p_1(r))) = (r, p_2(r))$. Thus, one epidemiological generation later, at the same value of r, the population moves from the low-prevalence phase to the high-prevalence phase of the cycle.

Proposition 4.1. *Writing out the two-cycle formula (4.2), the gap between the two branches is as follows:*

$$\begin{aligned} g(r) &:= p_2(r) - p_1(r) \\ &= \frac{(r+1) + \sqrt{(r-3)(r+1)}}{2r} - \frac{(r+1) - \sqrt{(r-3)(r+1)}}{2r} \\ &= \frac{\sqrt{(r-3)(r+1)}}{r}, \end{aligned}$$

and g is strictly increasing on $[3, 1+\sqrt{6}]$.

Proof. Write

$$g(r)^2 = (r-3)(r+1)/r^2 = 1 - 2/r - 3/r^2.$$

Then,

$$\frac{d}{dr}\left(g(r)^2\right) = 2/r^2 + 6/r^3 > 0$$

for every $r > 0$, so $g(r)^2$, and hence $g(r) \geq 0$ is strictly increasing on $(3, 1+\sqrt{6}]$. □

In particular, g is increasing on the window $[3.1, 3.4]$, from $g(3.1) = 0.2066$ to $g(3.4) = 0.3902$.

Proposition 4.2.

$$\mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o) = e^{-g(3.1)/\omega^o} = e^{-0.2066/\omega^o}, \quad \textit{for every } \omega^o > 0,$$

the sets $\mathcal{A}_0^o = \{(3.1, p_1(3.1))\}$ and $\mathcal{B}_0^o = \{(3.1, p_2(3.1))\}$ are singletons with $\Phi(\mathcal{A}_0^o) = \mathcal{B}_0^o$, and $\sigma^{o*} =$ (3.1, 0.5580) is the unique best proximity point of Φ.

Proof. By Proposition 4.1, g is minimized on $[3.1, 3.4]$ uniquely at the left endpoint $r = 3.1$. Since

$$\mathcal{H}^o((r, p_1(r)), (r, p_2(r)), \omega^o) = e^{-g(r)/\omega^o}$$

is a strictly decreasing function of $g(r)$, among pairs with the same r, the supremum defining $\mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o)$ is uniquely attained at $r = 3.1$. Since

$$p_{1,2}(r) = \frac{1}{2}(1 + \frac{1}{r}) \mp \frac{1}{2}g(r),$$

the function p_1 is decreasing on $[3.1, 3.4]$. Moreover,

$$p_2'(r) = \frac{1}{2}(g'(r) - \frac{1}{r^2}) > 0,$$

because

$$g'(r) = \frac{1 + 3/r}{r^2 g(r)} > \frac{1}{r^2}$$

when $0 < g(r) < 1$. Hence, for $r \neq r'$, $d((r, p_1(r)), (r', p_2(r'))) \geq p_2(r') - p_1(r) > g(3.1)$. This gives the stated value and singleton sets $\mathcal{A}_0^o, \mathcal{B}_0^o$. Hypothesis (*ii*) of Theorem 3.6 is met trivially, and

$$\mathcal{H}^o(\sigma^{o*}, \Phi\sigma^{o*}, \omega^o) = e^{-0.2066/\omega^o} = \mathcal{H}^o(\mathcal{A}^o, \mathcal{B}^o, \omega^o)$$

identifies σ^{o*} as the best proximity point. □

Remark 4.3 (Scope of the dynamical illustration)**.** *In the present construction, $\mathcal{A}_0^o$ and $\mathcal{B}_0^o$ are singletons. Consequently, the hypothesis of Definition 3.1 cannot be applied to two distinct points $\sigma_1^o, \sigma_2^o \in \mathcal{A}_0^o$ satisfying the prescribed optimal proximity conditions. Indeed, the only possible choice is $\sigma_1^o = \sigma_2^o = (3.1, p_1(3.1))$, which is excluded by the requirement $\sigma_1^o \neq \sigma_2^o$. Hence, the corresponding contractive condition is vacuously satisfied.*

Accordingly, the existence of σ^{o*} in Proposition 4.2 follows from the direct computation of the optimal proximity pair rather than from a nontrivial application of the iterative argument in Theorem 3.6. This feature results from the strict monotonicity of the gap between $\mathcal{A}^o$ and $\mathcal{B}^o$ with respect to the underlying parameter. On any non-degenerate compact parameter interval over which this gap remains strictly monotone, the extremal pair is unique, and hence $\mathcal{A}_0^o$ and $\mathcal{B}_0^o$ are singletons.

By contrast, Examples 3.2, 3.8, 3.12, and 3.16 provide non-vacuous illustrations of the corresponding contractive conditions through admissible distinct proximal configurations. Therefore, the purpose of the present dynamical construction is different. It provides a concrete interpretation of the abstract objects $\mathcal{A}^o$, $\mathcal{B}^o$, and Φ, and the best proximity point in terms of a computable bifurcating population model.

Figure 7 shows $\mathcal{A}^o$ and $\mathcal{B}^o$ as the two analytic branches of the two-cycle over $[3.1, 3.4]$, together with σ^{o*}, its image $\Phi(\sigma^{o*})$, and the segment of length 0.2066 joining them.

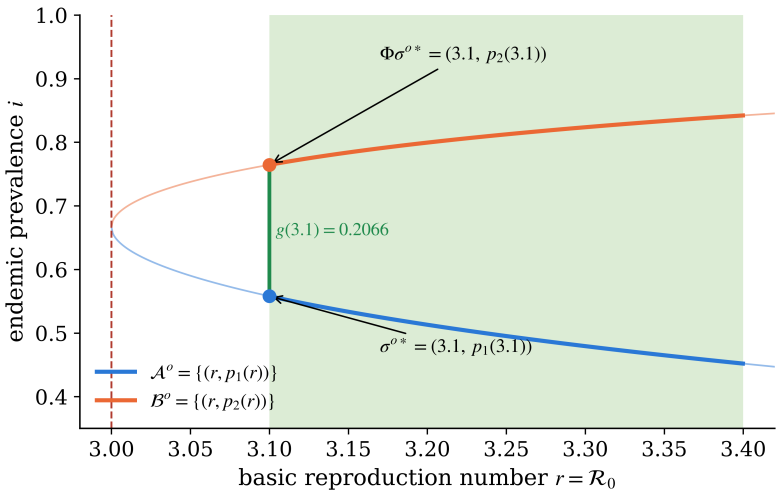


Figure 7. The best proximity point $\sigma^{o*} = (3.1, p_1(3.1))$ and its image $\Phi\sigma^{o*}$. The thick curves are $\mathcal{A}^o$ and $\mathcal{B}^o$ over the shaded window $[3.1, 3.4]$, and the segment between the two points has length $g(3.1) = 0.2066$.

Remark 4.4. *Epidemiologically, σ^{o*} identifies the value of the reproduction number, within the studied range, at which the low and high prevalence phases of a biennial epidemic cycle come closest to one another. Moving r away from the period-doubling threshold $r = 3$ widens the swing of the epidemic. As $r \downarrow 3$, the gap $g(r) \to 0$, and in the limit*

$$p_1(3) = p_2(3) = \frac{2}{3}.$$

Since

$$\Phi_3(\frac{2}{3}) = \frac{2}{3},$$

we have

$$\Phi(3, \frac{2}{3}) = (3, \frac{2}{3}),$$

and the best proximity equality reduces to the fixed point equation $\Phi\sigma^{o*} = \sigma^{o*}$.

Remark 4.5 (Sensitivity to the choice of window)**.** *The location and size of the best proximity point depend on the window chosen. Table 4 reports* σ^{o*} *and the gap* $g(r_{\text{left}})$ *for windows* $[r_{\text{left}}, 3.4]$ *with several choices of* $r_{\text{left}} \in (3, 3.4]$*. By Proposition 4.1, the best proximity point always sits at* $r = r_{\text{left}}$*, and the corresponding gap shrinks toward* 0 *as* $r_{\text{left}} \downarrow 3$*.*

Table 4. Location of the best proximity point and the gap g for the windows $[r_{\text{left}}, 3.4]$.

r_{left}	$p_1(r_{\text{left}})$	$p_2(r_{\text{left}})$	gap $g(r_{\text{left}})$
3.02	0.6186	0.7125	0.0939
3.05	0.5902	0.7377	0.1475
3.10	0.5580	0.7646	0.2066
3.15	0.5335	0.7840	0.2505
3.20	0.5130	0.7995	0.2864
3.30	0.4794	0.8236	0.3442

4.3. Discussion and limitations

The bifurcation structure of (4.1), including the two-cycle (4.2), its stability, and the bifurcation values, is classical [24] and is not obtained from the results of Section 3. The location and the uniqueness of σ^{o*} in Proposition 4.2 follow from the monotonicity of g.

The best proximity setting contributes the formulation of the problem. For $r \in [3.1, 3.4]$, the endemic equilibrium is unstable and the state alternates between the disjoint sets $\mathcal{A}^o$ and $\mathcal{B}^o$. Hence, Φ has no fixed point, and the closeness of the two phases is a best proximity question. The best proximity point maximizes $\mathcal{H}^o(\sigma^o, \Phi\sigma^o, \omega^o)$, which is equivalent to minimizing the amplitude $g(r)$ of the two-cycle. Thus, σ^{o*} marks the mildest biennial oscillation in the window. It is not an equilibrium of the model, since $\Phi\sigma^{o*} \neq \sigma^{o*}$. As $r \downarrow 3$, the best proximity point becomes the fixed point $(3, \frac{2}{3})$, as noted in Remark 4.4.

The space $\mathbb{R}^2$ with the induced fuzzy metric and the product t-norm is complete and non-Archimedean, the sets $\mathcal{A}^o$ and $\mathcal{B}^o$ are compact, and Φ is continuous. Since $\mathcal{A}_0^o$ is a singleton, the contractive condition (3.1) vacuously holds for $\mathscr{P}(t) = \ln t$ and $\mathscr{Q}(t) = \frac{1}{2}\ln t$, and Theorem 3.7 applies. Its conclusion coincides with Proposition 4.2. Therefore, the illustration does not use the contractive part of the theorems, and the logistic map is a simplified model of measles dynamics rather than a fitted epidemiological model. Quantitative conclusions about a specific disease require compartmental models calibrated to data, such as the models with vaccination, variants, imported cases, and optimal control developed for COVID-19 in [29–31]. The present construction is not of this type and is not claimed to be specific to measles. An application in which $\mathcal{A}_0^o$ contains more than one point is left for future work.

5. Conclusions

In this paper, we developed a two-function approach to the best proximity point theory in complete non-Archimedean fuzzy metric spaces. It covers the fuzzy proximal contraction of the second kind and its interpolative Reich–Rus–Ćirić, Kannan, and Hardy–Rogers forms. For each class, we proved two existence theorems under different hypotheses on the auxiliary functions. Although the four classes combine fuzzy closeness terms and interpolation parameters in different ways, their proofs follow the same convergence mechanism.

Moreover, we described the position of the framework with respect to the classical theory. Corollaries 3.20–3.23 recover the basic, Reich–Rus–Ćirić, Kannan, and Hardy–Rogers proximal inequalities of the second kind when the auxiliary functions are removed. Conversely, Corollaries 3.26 and 3.27 and Remark 3.28 show that the Banach type and the Gregori–Sapena type fuzzy proximal contractions and their interpolative forms are particular cases of the framework, and Example 3.30 shows that this inclusion is strict. Remark 3.29 shows that the closedness of $\mathscr{A}^o$ cannot be removed.

As an illustration, we considered the logistic map and treated the low and high prevalence branches of a period-two orbit as two disjoint sets. The optimal fuzzy proximity between them was explicitly computed. The bifurcation analysis itself is classical, and the best proximity point is obtained by direct computation. As explained in Remark 4.3, the proximity sets are singletons in this construction, so the example illustrates the framework but does not test the contractive condition.

Several directions remain open. One is the extension to Boyd–Wong and Matkowski type proximal contractions, where the step that excludes a limit smaller than 1 depends on a control function rather than on the separation $\mathscr{Q} > \mathscr{P}$. Another is the setting of partial and b-fuzzy metric spaces, where the fuzzy metric need not be continuous and the final limiting step of the proofs must be redone. A further question is whether all four contractive schemes follow from a single abstract principle. Uniqueness, stability, and constructive approximations of best proximity points in the proposed framework are also left for future work.

Author contributions

M. A. S, J. Z., C. L., and A. A. A.: Conceptualization, methodology, software, validation, formal analysis, writing—original draft; M. A. S, and C. L.: Visualization, writing—review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

No Generative-AI tools were used in this paper.

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Conflict of interest

The authors declare no conflicts of interest.

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