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Review

Spectral density of the stretched exponential function: A comprehensive review on the derivations of its integral and series solutions (Pollard's formula) and fundamental properties

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Abstract: The stretched exponential function, a prominent completely monotonic function for characterizing anomalous relaxation and transport phenomena in complex, nonlinear systems across physics and materials science, can be mathematically represented as a continuous linear combination of exponential decay functions with non-negative weights, thereby giving rise to an associated spectral density $H(\tau,\beta)$. This article provides a comprehensive, rigorous, and accessible review of the step-by-step derivations of the key formulas and fundamental properties of $H(\tau,\beta)$. Specifically, it demonstrates how a continuous linear combination of exponential decay functions with non-negative weights converges to the stretched exponential function, thereby revealing the origin as well as the mathematical and physical significance of $H(\tau,\beta)$. It then demonstrates the derivation of the exact analytical solution for $H(\tau,\beta)$, widely known as Pollard's integral formula, along with its asymptotic series representation. In addition, the theoretical absolute convergence and the numerical convergence behavior of this asymptotic series are demonstrated. Finally, it demonstrates two fundamental mathematical and physical properties of $H(\tau,\beta)$, namely its non-negativity and normalization condition. Mastering these transparent derivations empowers researchers to establish modified Pollard-type representations for other completely monotonic functions, gain profound insights into the underlying mathematical and physical foundations, and select the optimal approach among the integral and series forms to facilitate efficient computations in practical applications.

Keywords: stretched exponential function; spectral density; distribution function; relaxation spectrum; completely monotonic function; Pollard's integral formula; anomalous relaxation

Mathematics Subject Classification: 26A48, 44A10, 33E20

1. Introduction

The stretched exponential function (also known as the Kohlrausch-Williams-Watts function [1–3]), $\phi(t) = e^{-(\frac{t}{\tau_0})^\beta}$, is a prominent completely monotonic function [4–6] defined on the domain $t \in [0, \infty)$, characterized by the parameters $\tau_0 \in (0, \infty)$ and $\beta \in (0,1]$. It serves as a prominent mathematical framework for characterizing anomalous relaxation and transport phenomena in complex, nonlinear systems across physics and materials science [7–9]. In contrast to Debye relaxation that governs systems near thermodynamic equilibrium through uniform kinetic processes [10–12], the stretched exponential behavior (representing non-Debye relaxation) serves as an intrinsic signature of complex nonlinear dynamics, structural disorder and heterogeneity, and many-body interactions [13–15]. Mathematically, Debye relaxation is described by first-order linear differential equations, exhibiting a single-scale, simple exponential decay with a single characteristic relaxation time [16–18]. In contrast, the stretched exponential behavior arises from a multi-scale superposition of simple exponential decays, involving a continuous spectrum of relaxation times [19–21]; capturing this phenomenon typically requires the use of differential equations containing nonlinear terms with respect to time [22–24]. Consequently, the stretched exponential function serves as a mathematical bridge between macroscopic, multi-scale, nonlinear dynamics and underlying microscopic, single-scale, linear dynamical processes.

The stretched exponential function can be mathematically represented as a continuous linear combination of exponential decay functions with non-negative weights [25–27]. This relationship is formulated through the following integral [28]:

$$e^{-(\frac{t}{\tau_0})^\beta} = \int_0^\infty H(\tau, \beta) e^{-\frac{t}{\tau}} d\tau, \tag{1}$$

where $H(\tau, \beta)$ denotes the spectral density or distribution function (commonly referred to as the relaxation spectrum or relaxation time distribution in physics) associated with the stretched exponential function; it represents the weight distribution of τ for a given β. Equation (1) implies that the stretched exponential function is inherently the Laplace transform of $H(\tau, \beta)$ after a specific variable transformation.

Through the application of analytic continuation and contour integration techniques in complex function theory, Pollard [29] obtained an exact analytical solution for $H(\tau, \beta)$, widely known as Pollard's integral formula:

$$H(\tau, \beta) = \frac{\tau_0}{\pi \tau^2} \int_0^\infty e^{-\frac{\tau_0 x}{\tau}} e^{-x^\beta \cos(\pi\beta)} \sin[x^\beta \sin(\pi\beta)]\, dx. \tag{2}$$

In Eq (2), the prefactor $\frac{\tau_0}{\pi\tau^2}$ ensures that $H(\tau, \beta)$ maintains correct dimensionality across different temporal scales while satisfying the normalization condition. In the integrand, the first term $e^{-\frac{\tau_0 x}{\tau}}$ represents the Laplace transform kernel, acting as an exponential decay weight. The combination of $e^{-x^\beta \cos(\pi\beta)}$ and $\sin[x^\beta \sin(\pi\beta)]$ constitutes the characteristic function of the Lévy stable distribution, physically describing non-Debye relaxation; these two terms ensure the absolute convergence of the improper integral for $\beta \in (0,1)$. It is crucial to explicitly distinguish the mathematical nature of $H(\tau, \beta)$ based on the value of β. For $\beta \in (0,1)$, $H(\tau, \beta)$ represents a

smooth, continuous spectral density. However, in the limiting case where $\beta = 1$, the stretched exponential function reduces to a simple exponential decay function; consequently, the associated spectral density transitions from a continuous function to a discrete Dirac delta measure concentrated entirely at the single characteristic relaxation time $\tau = \tau_0$, given by $H(\tau, 1) = \delta(\tau - \tau_0)$.

The integral representation of Pollard's formula provides an exact definition of the spectral density $H(\tau, \beta)$. However, the numerical evaluation of $H(\tau, \beta)$ using this integral form often poses severe challenges [11,19,20]. Since the integrand involves exponentially decaying and sine functions, it is highly oscillatory. This behavior leads to severe cancellation of positive and negative contributions across the integration domain, resulting in significant errors in conventional numerical integration methods due to insufficient sampling. To circumvent the computational difficulties, direct numerical integration of Pollard's integral formula is frequently avoided. Instead, researchers often rely on its series representation in the asymptotic limit as $\tau \to 0$ [11,19,20]. Theoretically, this infinite series is absolutely convergent to Pollard's integral formula for all $\tau \in (0, \infty)$. However, it must be truncated to a finite number of terms in practical numerical computations, inevitably introducing truncation errors.

The objective of this article is to provide a comprehensive, rigorous, and accessible review of the derivation of the key formulas and fundamental properties of the spectral density $H(\tau, \beta)$ associated with the stretched exponential function. First, in Section 2, we begin by demonstrating Eq (1), during which its non-negativity property is also established. In Section 3, we derive the exact analytical solution for $H(\tau, \beta)$, namely Pollard's integral formula. In Section 4, we derive the series representation of Pollard's integral formula in the asymptotic limit as $\tau \to 0$, demonstrate its theoretical absolute convergence, and investigate its numerical convergence behavior. In Section 5, we demonstrate another essential property of $H(\tau, \beta)$, namely its normalization condition. This is followed by the discussion section.

2. Derivation of Eq (1) and demonstration of the non-negativity of $H(\tau, \beta)$

Consider the linear combination of N exponential decay functions defined on $[0, \infty)$ with non-negative weights:

$$g(t) = g_1 e^{-\frac{t}{\tau_1}} + g_2 e^{-\frac{t}{\tau_2}} + \cdots + g_N e^{-\frac{t}{\tau_N}} = \sum_{i=1}^{N} g_i e^{-\frac{t}{\tau_i}}, \tag{3}$$

where $N \in \mathbb{N}$ denotes the order of the series, while $g_i \in [0, \infty)$ and $\tau_i \in (0, \infty)$ represent the weight and characteristic relaxation time of the i-th term, respectively. Such a formulation is widely known as the Prony series. By definition, the discrete point weight g_i is obtained by discretizing the continuous spectral density $H(\tau)$ (representing the spectral weight per unit interval of τ) within the i-th interval $\Delta\tau_i$ at $\tau = \tau_i$. Consequently, g_i is defined as the product of $H(\tau_i)$ and $\Delta\tau_i$, expressed as $g_i = H(\tau_i)\Delta\tau_i$. Substituting $g_i = H(\tau_i)\Delta\tau_i$ into Eq (3) yields

$$g(t) = \sum_{i=1}^{N} H(\tau_i) e^{-\frac{t}{\tau_i}} \Delta\tau_i. \tag{4}$$

Equation (4) can be interpreted as a discrete Riemann sum approximation of the continuous function $H(\tau)e^{-\frac{t}{\tau}}$ over a finite partition of the bounded interval $\tau \in [0, M]$ into subintervals of widths $\Delta\tau_i$. By taking the limit as the norm of the partition tends to zero (i.e., $\max(\Delta\tau_i) \to 0$ as $N \to \infty$), this

discrete sum converges to the definite Riemann integral over $[0, M]$. Subsequently, by taking the limit of the upper bound of the interval $M \to \infty$, we arrive at the following improper Riemann integral representation:

$$\int_0^{\infty} H(\tau) e^{-\frac{t}{\tau}} d\tau = \lim_{M\to\infty} (\lim_{N\to\infty} \sum_{i=1}^{N} H(\tau_i) e^{-\frac{t}{\tau_i}} \Delta\tau_i). \tag{5}$$

To rigorously establish the existence of a unique, non-negative, continuous spectral density associated with the stretched exponential function, we invoke the following classical theorem.

Lemma 1. (Bernstein's theorem on completely monotonic functions) [30]
A real-valued function $f(t)$ *defined on* $[0, \infty)$ *is completely monotonic on* $(0, \infty)$ *if and only if it can be represented as the Laplace-Stieltjes transform of a unique, non-negative Borel measure* μ *on* $[0, \infty)$ *such that*

$$f(t) = \int_0^{\infty} e^{-\frac{t}{\tau}} d\mu(\tau) \tag{6}$$

for all $t \in (0, \infty)$.

By invoking Bernstein's theorem to the stretched exponential function $e^{-(\frac{t}{\tau_0})^{\beta}}$, its complete monotonicity on $(0, \infty)$ guarantees the existence of a unique, non-negative Borel measure μ on $[0, \infty)$ satisfying the representation in Eq (6). It is crucial to note that Bernstein's theorem guarantees only the existence of a generic measure, which could theoretically include discrete or singular components. The assertion that the measure associated with the stretched exponential function possesses a unique, non-negative, continuous spectral density requires an additional assumption: This measure is absolutely continuous with respect to the Lebesgue measure (the validity of this assumption will be justified in the final paragraph of this section). Under this assumption, the Radon-Nikodym theorem [31] guarantees the existence of a non-negative integrable spectral density $H(\tau)$ such that $d\mu(\tau) = H(\tau)d\tau$. Consequently, the stretched exponential function can be rigorously equated to the integral in Eq (5):

$$e^{-(\frac{t}{\tau_0})^{\beta}} = \int_0^{\infty} H(\tau) e^{-\frac{t}{\tau}} d\tau. \tag{7}$$

This is equivalent to substituting $f(t) = e^{-(\frac{t}{\tau_0})^{\beta}}$ and $d\mu(\tau) = H(\tau)d\tau$ into Eq (6). Since $H(\tau)$ is inherently dependent on β, Eq (7) can be rewritten as

$$e^{-(\frac{t}{\tau_0})^{\beta}} = \int_0^{\infty} H(\tau, \beta) e^{-\frac{t}{\tau}} d\tau. \tag{8}$$

This completes the derivation of Eq (1), which simultaneously demonstrates that the corresponding continuous spectral density $H(\tau, \beta)$ associated with the stretched exponential function is non-negative. Since Pollard's integral formula (which will be derived in the following section) is the exact analytical solution for $H(\tau, \beta)$, it must be non-negative as well. To be physically meaningful, the spectral density $H(\tau, \beta)$ (and consequently its exact analytical solution, namely Pollard's integral formula) must satisfy the non-negativity condition $H(\tau, \beta) \geq 0$ for all $\tau \in (0, \infty)$ and $\beta \in (0,1)$, thereby guaranteeing that the continuous superposition of exponential decay functions constitutes a thermodynamically admissible monotonic decay behavior [32–34].

It is crucial to note that in the preceding demonstration, we assume a priori that the measure

associated with the stretched exponential function is absolutely continuous, from which Eq (8) is subsequently derived. Indeed, applying the inverse Laplace-Stieltjes transform to the stretched exponential function with $\beta \in (0,1)$ guarantees the existence of a unique, well-defined, and absolutely continuous spectral density, namely Pollard's integral formula, which will be rigorously derived in the subsequent section. Furthermore, mathematically, the stretched exponential function is a superposition of infinitely many exponential decay functions with a continuous distribution of characteristic relaxation times; therefore, its corresponding spectral density must be absolutely continuous.

3. Derivation of Pollard's integral formula

In this section, we rigorously derive Pollard's integral formula, which represents the exact analytical solution for $H(\tau,\beta)$. To begin, we consider Eq (8):

$$e^{-(\frac{t}{\tau_0})^{\beta}} = \int_0^{\infty} H(\tau,\beta) e^{-\frac{t}{\tau}} d\tau. \tag{9}$$

To cast Eq (9) into the standard Laplace transform framework, we define the transform variable as the relaxation rate $s = \frac{1}{\tau}$ (which implies that $d\tau = -\frac{1}{s^2} ds$). Under this coordinate transformation, the domain $\tau \in (0,\infty)$ corresponds to $s \in (\infty, 0)$. To preserve the total spectral intensity under this coordinate transformation (ensuring that the integrated intensity within any interval remains invariant under this coordinate transformation), the time-domain spectral density $H(\tau,\beta)$ and its Laplace-domain counterpart $\rho(s)$ must satisfy $H(\tau,\beta)|d\tau| = \rho(s)|ds|$. Combining the relations $H(\tau,\beta)|d\tau| = \rho(s)|ds|$ and $d\tau = -\frac{1}{s^2} ds$ yields $H(\tau,\beta) = \frac{1}{\tau^2}\rho(\frac{1}{\tau})$. Substituting $H(\tau,\beta) = \frac{1}{\tau^2}\rho(\frac{1}{\tau})$ and the aforementioned relations associated with the coordinate transformation into Eq (9), and accounting for the reversal of integration limits, we obtain

$$e^{-(\frac{t}{\tau_0})^{\beta}} = \int_0^{\infty} \rho(s) e^{-st} ds = \mathcal{L}\{\rho(s)\}. \tag{10}$$

Equation (10) indicates that the stretched exponential function is essentially the Laplace transform of $\rho(s)$. Consequently, deriving $\rho(s)$ is equivalent to evaluating the inverse Laplace transform of $e^{-(\frac{t}{\tau_0})^{\beta}}$. By applying the Bromwich inversion formula, $\rho(s)$ can be expressed as a line integral along a vertical path in the complex plane:

$$\rho(s) = \mathcal{L}^{-1}\{e^{-(\frac{t}{\tau_0})^{\beta}}\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} e^{-(\frac{t}{\tau_0})^{\beta}} dt, \tag{11}$$

where the real constant γ is chosen to lie to the right of all singularities of the integrand.

Since $\beta \in (0,1)$ in this context, the stretched exponential term $e^{-(\frac{t}{\tau_0})^{\beta}}$ within the integrand cannot be expressed in terms of elementary functions, rending a direct analytical integration of Eq (11) unfeasible. Instead, this integral must be evaluated in the complex plane by invoking Cauchy's integral theorem and contour integration. The presence of $\beta \in (0,1)$ introduces a branch point at the origin $t = 0$. To ensure a single-valued integrand, we define the principal branch by introducing a branch cut along the negative real axis, thereby restricting the argument of t to $\theta \in (-\pi, \pi]$. To evaluate the integral along the Bromwich path Γ_1 in Eq (11) analytically, we construct a closed keyhole contour Γ in the complex plane, which consists of six distinct segments (Figure 1): The Bromwich line Γ_1 (from $\gamma - iR$ to $\gamma + iR$), two large circular arcs Γ_R and Γ_R' of radius R, an

infinitesimal circular loop Γ_ϵ of radius ϵ centered at the origin, and two parallel paths running along the upper and lower edges of the branch cut denoted as Γ_+ and Γ_-, respectively.

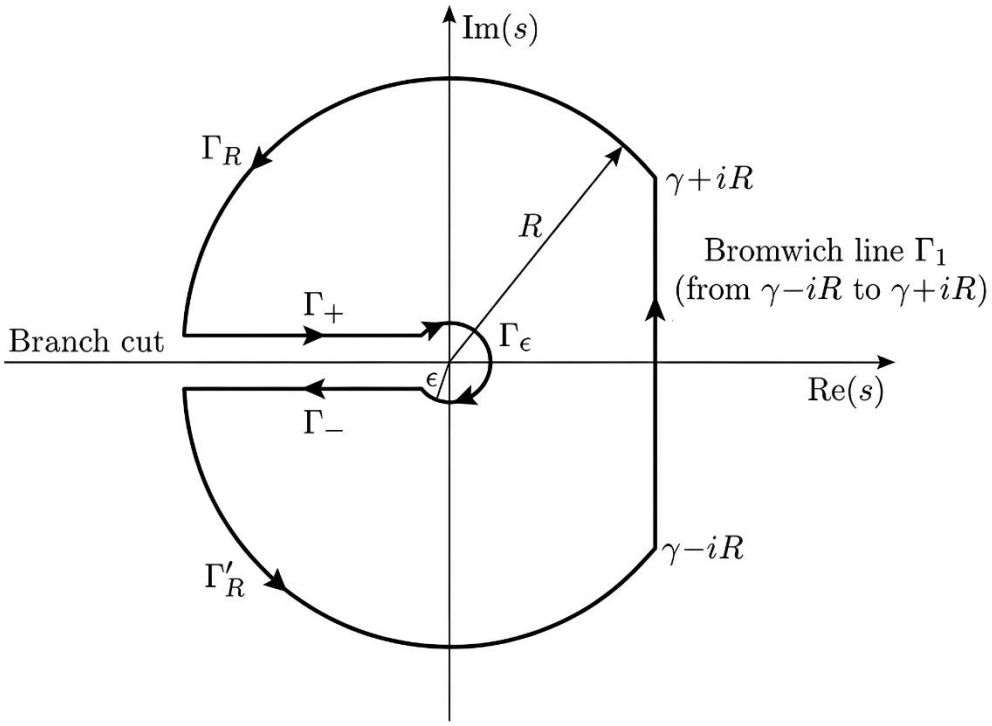


Figure 1. Illustration of the closed keyhole contour Γ in the complex plane for evaluating Pollard's integral formula, which consists of six distinct segments: the Bromwich line Γ_1 (from $\gamma - iR$ to $\gamma + iR$), two large circular arcs Γ_R and Γ'_R of radius R, an infinitesimal circular loop Γ_ϵ of radius ϵ centered at the origin, and two parallel paths running along the upper and lower edges of the branch cut denoted as Γ_+ and Γ_-, respectively.

By applying Cauchy's integral theorem [35], since the integrand in Eq (11) is holomorphic within and on the closed contour Γ, its contour integral around Γ vanishes:

$$\frac{1}{2\pi i}\oint_\Gamma e^{st}e^{-(\frac{t}{\tau_0})^\beta}dt = \frac{1}{2\pi i}\left(\int_{\Gamma_1} + \int_{\Gamma_R} + \int_{\Gamma'_R} + \int_{\Gamma_\epsilon} + \int_{\Gamma_+} + \int_{\Gamma_-}\right)e^{st}e^{-(\frac{t}{\tau_0})^\beta}dt = 0. \tag{12}$$

In Eq (12), the integral along the Bromwich path Γ_1 defines $\rho(s)$ according to Eq (11), namely $\frac{1}{2\pi i}\int_{\Gamma_1} e^{st}e^{-(\frac{t}{\tau_0})^\beta}dt = \rho(s)$. By virtue of Jordan's lemma [36], as $R \to \infty$, the integral contributions along the large circular arcs Γ_R and Γ'_R vanish in the left half of the complex plane. Similarly, as $\epsilon \to 0$, the contribution along the small circular loop Γ_ϵ around the origin also vanishes because the integrand remains bounded at the origin. Taking these asymptotic limits simultaneously, Eq (12) reduces to

$$\rho(s) + \frac{1}{2\pi i}\left(\int_{\Gamma_+} + \int_{\Gamma_-}\right)e^{st}e^{-(\frac{t}{\tau_0})^\beta}dt = 0. \tag{13}$$

Consequently, $\rho(s)$ is entirely balanced by the integrals along the branch cut:

$$\rho(s) = -\frac{1}{2\pi i}\left(\int_{\Gamma_+} + \int_{\Gamma_-}\right) e^{st} e^{-(\frac{t}{\tau_0})^{\beta}} dt. \tag{14}$$

To evaluate these branch-cut integrals, we parameterize the paths by introducing a real variable $u \in (0, \infty)$. On the upper cut edge Γ_+, the argument is fixed at $\theta = \pi$, implying $t = ue^{i\pi} = -u$ and $dt = -du$, with the integration path originating from $u = \infty$ and terminating at $u = 0$. Consequently, evaluating the integral along Γ_+ yields

$$\begin{aligned}\int_{\Gamma_+} e^{st} e^{-(\frac{t}{\tau_0})^{\beta}} dt &= \int_{\infty}^{0} e^{-su} e^{-(\frac{u}{\tau_0})^{\beta} e^{i\pi\beta}} (-du) = \int_{0}^{\infty} e^{-su} e^{-(\frac{u}{\tau_0})^{\beta}[\cos(\pi\beta) + i\sin(\pi\beta)]} du \\ &= \int_{0}^{\infty} e^{-su} e^{-(\frac{u}{\tau_0})^{\beta}\cos(\pi\beta)} e^{-i(\frac{u}{\tau_0})^{\beta}\sin(\pi\beta)} du.\end{aligned} \tag{15}$$

Conversely, on the lower cut edge Γ_-, the argument is fixed at $\theta = -\pi$, implying $t = ue^{-i\pi} = -u$ and $dt = -du$, with the integration path progressing in the opposite direction from $u = 0$ to $u = \infty$. Consequently, evaluating the integral along Γ_- yields

$$\begin{aligned}\int_{\Gamma_-} e^{st} e^{-(\frac{t}{\tau_0})^{\beta}} dt &= \int_{0}^{\infty} e^{-su} e^{-(\frac{u}{\tau_0})^{\beta} e^{-i\pi\beta}} (-du) = -\int_{0}^{\infty} e^{-su} e^{-(\frac{u}{\tau_0})^{\beta}[\cos(\pi\beta) - i\sin(\pi\beta)]} du \\ &= -\int_{0}^{\infty} e^{-su} e^{-(\frac{u}{\tau_0})^{\beta}\cos(\pi\beta)} e^{i(\frac{u}{\tau_0})^{\beta}\sin(\pi\beta)} du.\end{aligned} \tag{16}$$

Substituting Eqs (15) and (16) into Eq (14) yields

$$\rho(s) = -\frac{1}{2\pi i}\int_{0}^{\infty} e^{-su} e^{-(\frac{u}{\tau_0})^{\beta}\cos(\pi\beta)} [e^{-i(\frac{u}{\tau_0})^{\beta}\sin(\pi\beta)} - e^{i(\frac{u}{\tau_0})^{\beta}\sin(\pi\beta)}] du. \tag{17}$$

By utilizing the Euler identity $e^{-ia} - e^{ia} = -2i\sin(a)$, Eq (17) can be transformed from the complex form into the following real-valued form:

$$\rho(s) = \frac{1}{\pi}\int_{0}^{\infty} e^{-su} e^{-(\frac{u}{\tau_0})^{\beta}\cos(\pi\beta)} \sin[(\frac{u}{\tau_0})^{\beta}\sin(\pi\beta)]\, du. \tag{18}$$

This step demonstrates that, although analytic continuation and contour integration techniques in complex function theory are employed during the derivation, the resulting $\rho(s)$ is a purely real-valued function. This aligns with the physical requirement that any spectral density or distribution function must be real.

To transform $\rho(s)$ back to the time-domain spectral density $H(\tau, \beta)$, we substitute $s = \frac{1}{\tau}$ into Eq (18) and apply the relation $H(\tau, \beta) = \frac{1}{\tau^2}\rho(\frac{1}{\tau})$, yielding

$$H(\tau, \beta) = \frac{1}{\pi\tau^2}\int_{0}^{\infty} e^{-\frac{u}{\tau}} e^{-(\frac{u}{\tau_0})^{\beta}\cos(\pi\beta)} \sin[(\frac{u}{\tau_0})^{\beta}\sin(\pi\beta)]\, du. \tag{19}$$

Finally, by performing the dimensionless variable substitution $u = \tau_0 x$ (which implies that $du = \tau_0 dx$), we arrive at Pollard's integral formula:

$$H(\tau,\beta)=\frac{\tau_0}{\pi\tau^2}\int_0^\infty e^{-\frac{\tau_0 x}{\tau}}e^{-x^\beta\cos(\pi\beta)}\sin[x^\beta\sin(\pi\beta)]\,dx. \tag{20}$$

This completes the derivation of Pollard's integral formula.

4. Series representation of Pollard's integral formula in the asymptotic limit as $\tau\to 0$

In this section, we derive the series representation of Pollard's integral formula in the asymptotic limit as $\tau\to 0$, demonstrate its theoretical absolute convergence, and investigate its numerical convergence behavior.

4.1. Mathematical derivation

To establish the series representation of Pollard's integral formula in the asymptotic limit as $\tau\to 0$, we begin by expanding the stretched exponential function $e^{-(\frac{t}{\tau_0})^\beta}$ into its corresponding Taylor series:

$$e^{-(\frac{t}{\tau_0})^\beta}=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(\frac{t}{\tau_0})^{\beta k}. \tag{21}$$

Next, we perform a term-by-term inverse Laplace transform of Eq (21). Since we restrict our interest to positive relaxation rates (i.e., $s>0$), the $k=0$ term can be reasonably excluded as it formally generates a Dirac delta function $\delta(s)$ concentrated at $s=0$. Consequently, the transformation yields

$$\rho(s)=\sum_{k=1}^{\infty}\frac{(-1)^k}{k!\,(\tau_0)^{\beta k}}\mathcal{L}^{-1}\{t^{\beta k}\}=\sum_{k=1}^{\infty}\frac{(-1)^k}{k!\,(\tau_0)^{\beta k}}\frac{s^{-(\beta k+1)}}{\Gamma(-\beta k)}. \tag{22}$$

In the derivation of Eq (22), we utilize the relation $\rho(s)=\mathcal{L}^{-1}\{e^{-(\frac{t}{\tau_0})^\beta}\}$ according to Eq (11), along with the inverse Laplace transform of the generalized power-law term t^α, given by $\mathcal{L}^{-1}\{t^\alpha\}=\frac{s^{-(\alpha+1)}}{\Gamma(-\alpha)}$, where Γ is the gamma function and α is set to βk. To further simplify Eq (22), we utilize Euler's reflection formula for the gamma function, which states that $\Gamma(a)\Gamma(1-a)=\frac{\pi}{\sin(\pi a)}$. Setting $a=-\beta k$, we obtain $\frac{1}{\Gamma(-\beta k)}=-\Gamma(1+\beta k)\frac{\sin(\pi\beta k)}{\pi}$. Substituting this relation into Eq (22) yields

$$\rho(s)=\frac{1}{\pi s}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\Gamma(1+\beta k)\sin(\pi\beta k)\,(\tau_0 s)^{-\beta k}. \tag{23}$$

Finally, mapping this Laplace-domain spectral density back to its time-domain counterpart through the relation $H(\tau,\beta)=\frac{1}{\tau^2}\rho(\frac{1}{\tau})$ yields

$$H(\tau,\beta)=\frac{1}{\pi\tau}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\Gamma(1+\beta k)\sin(\pi\beta k)\,(\frac{\tau}{\tau_0})^{\beta k}. \tag{24}$$

Equation (24) serves as the series representation of Pollard's integral formula in the asymptotic limit as $\tau\to 0$.

4.2. Demonstration of the theoretical absolute convergence

The series in Eq (24) is originally derived in the asymptotic limit as $\tau \to 0$; however, in fact, it is absolutely convergent for all $\tau \in (0, \infty)$ and $\beta \in (0,1)$. To rigorously demonstrate this fact, we apply a bounding technique and d'Alembert's ratio test, while leveraging the asymptotic behavior of the gamma function.

Let the general term of the series (excluding the prefactor $\frac{1}{\pi\tau}$) be denoted by a_k:

$$a_k = \frac{(-1)^{k-1}}{k!}\Gamma(1+\beta k)\sin(\pi\beta k)\, x^{\beta k}, \tag{25}$$

where $x = \frac{\tau}{\tau_0} > 0$. To demonstrate the absolute convergence of the series, we consider the series of absolute values $\sum_{k=1}^{\infty}|a_k|$. Utilizing the fundamental bounds $|(-1)^{k-1}| = 1$ and $|\sin(\pi\beta k)| \leq 1$, we majorize the absolute general term as follows:

$$|a_k| \leq \frac{\Gamma(1+\beta k)}{k!}x^{\beta k} = b_k. \tag{26}$$

By applying the comparison test to Eq (26), if the series $\sum_{k=1}^{\infty} b_k$ converges for all finite x, then the series $\sum_{k=1}^{\infty} a_k$ converges absolutely. The convergence of $\sum_{k=1}^{\infty} b_k$ can be determined through d'Alembert's ratio test, which requires investigating the limit of the ratio of successive terms:

$$\frac{b_{k+1}}{b_k} = \frac{\Gamma(1+\beta(k+1))}{\Gamma(1+\beta k)}\frac{1}{k+1}x^{\beta}. \tag{27}$$

To analyze the asymptotic behavior of the ratio $\frac{b_{k+1}}{b_k}$ as $k \to \infty$, we utilize the well-known asymptotic property of the gamma function derived from Stirling's approximation:

$$\frac{\Gamma(y+a)}{\Gamma(y)} \sim y^a \text{ as } y \to \infty. \tag{28}$$

The ratio of the gamma functions in Eq (27) can be evaluated by setting $y = 1+\beta k$ and $a = \beta$ in Eq (28). Hence, as $k \to \infty$, this ratio is asymptotically equivalent to

$$\frac{\Gamma(1+\beta(k+1))}{\Gamma(1+\beta k)} \sim (1+\beta k)^{\beta} \sim \beta^{\beta}k^{\beta}. \tag{29}$$

Substituting this asymptotic expression into Eq (27) and evaluating the limit as $k \to \infty$ yields

$$L = \lim_{k\to\infty}\frac{b_{k+1}}{b_k} = \lim_{k\to\infty}\frac{\beta^{\beta}k^{\beta}}{k+1}x^{\beta} = \beta^{\beta}x^{\beta}\lim_{k\to\infty}\frac{k^{\beta}}{k+1}. \tag{30}$$

The expression $\frac{k^{\beta}}{k+1}$ is asymptotically equivalent to $k^{\beta-1}$ as $k \to \infty$. Under the constraint $\beta \in (0,1)$, the exponent $\beta - 1$ is strictly negative. Consequently, $\lim_{k\to\infty}\frac{k^{\beta}}{k+1} \sim \lim_{k\to\infty} k^{\beta-1} = 0$, and the limit L evaluates to zero regardless of the magnitude of x:

$$L = \beta^{\beta}x^{\beta} \cdot 0 = 0. \tag{31}$$

According to d'Alembert's ratio test, since $L = 0 < 1$ for all finite $x > 0$, the series $\sum_{k=1}^{\infty} b_k$ converges for all such x. It is thus established that the series in Eq (24) is absolutely convergent.

4.3. Numerical convergence behavior

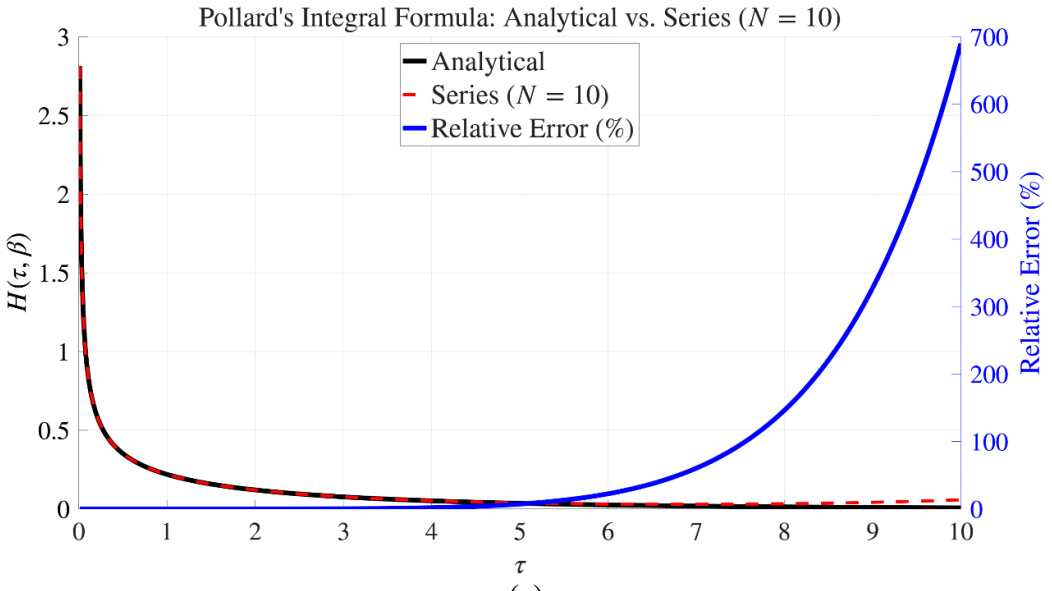
Theoretically, the series in Eq (24) is absolutely convergent for all $\tau \in (0, \infty)$, as demonstrated in Section 4.2. However, while it is inherently an infinite series, it must be truncated to a finite number of terms in practical numerical computations, inevitably introducing truncation errors. In the following, we investigate the numerical convergence behavior of the truncated form of the series in Eq (24) as the number of terms increases.

This investigation examines the specific case where $\beta = 0.5$, under which Pollard's integral formula yields an analytical solution:

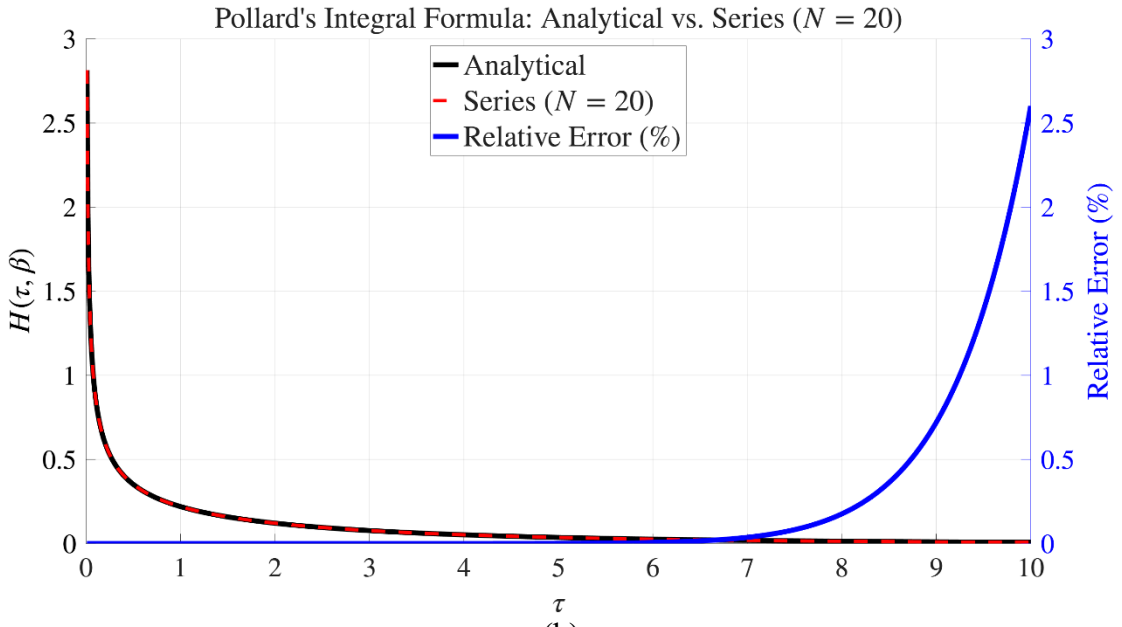
$$H(\tau, 0.5) = \frac{1}{2\sqrt{\pi\tau_0\tau}} e^{-\frac{\tau}{4\tau_0}}. \tag{32}$$

Utilizing this analytical solution as the gold standard, we compute the values of $H(\tau, 0.5)$ at $\tau_0 = 1$ across the range $\tau \in [0.01, 10]$. These values are then compared with the results obtained from the truncated form of the series in Eq (24). Figures 2a through 2c illustrate the results for the truncated series with 10, 20, and 30 terms, respectively. Interestingly, although the series in Eq (24) is originally derived in the asymptotic limit as $\tau \to 0$, it exhibits excellent convergence (i.e., excellent agreement with the analytical solution) even at τ values substantially larger than τ_0. The range of τ that ensures convergence expands as the number of terms increases. This perfectly aligns with the demonstration in Section 4.2 that the series in Eq (24) is absolutely convergent for all $\tau \in (0, \infty)$ rather than being restricted to the regime of extremely small τ, although it is originally derived in the asymptotic limit as $\tau \to 0$.

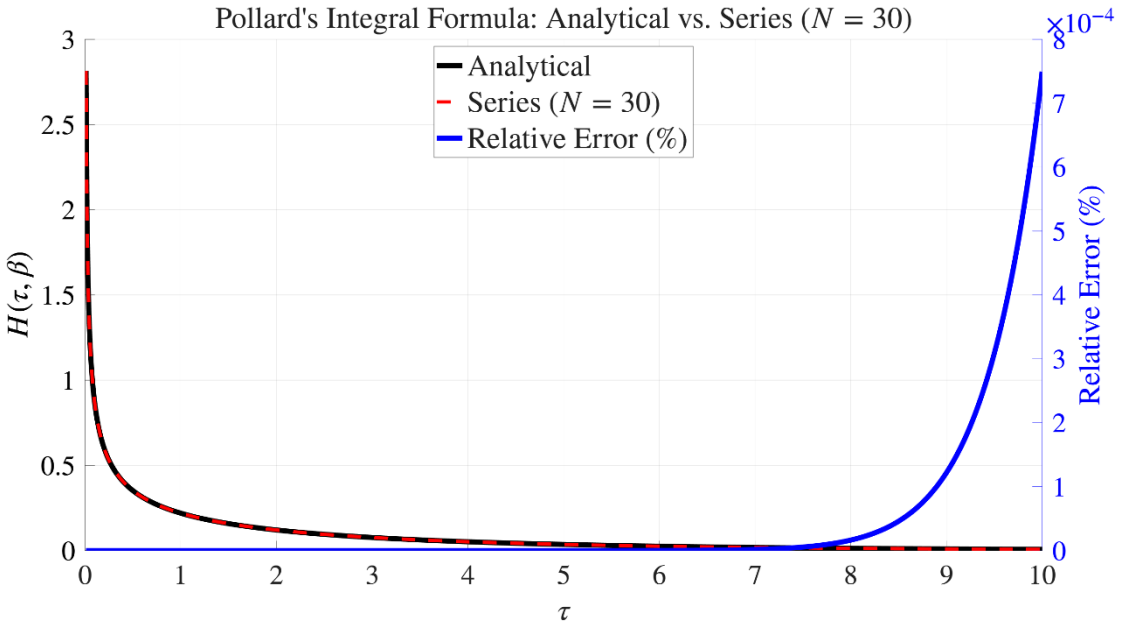
Figure 2d illustrates the quantification of the convergence range for each truncation order by illustrating the threshold value of τ at which the relative error between the analytical solution and the series exceeds 1% as the number of terms increases from 1 to 20. Consistent with expectations, the results demonstrate that increasing the number of terms shifts the 1% error threshold toward a larger value of τ. Notably, when the number of terms reaches 19, the error remains below 1% up to $\tau = 9.25$, demonstrating an exceptionally wide convergence range. Furthermore, as shown in Figure 2(c), when the series is extended to 30 terms, the error remains below 0.1% across the entire investigated τ range.



(a)


Pollard's Integral Formula: Analytical vs. Series ($N = 20$)
Analytical
Series ($N = 20$)
Relative Error (%)
$H(\tau, \beta)$
Relative Error (%)
τ

(b)


Pollard's Integral Formula: Analytical vs. Series ($N = 30$)
Analytical
Series ($N = 30$)
Relative Error (%)
$\times 10^{-4}$
$H(\tau, \beta)$
Relative Error (%)
τ

(c)

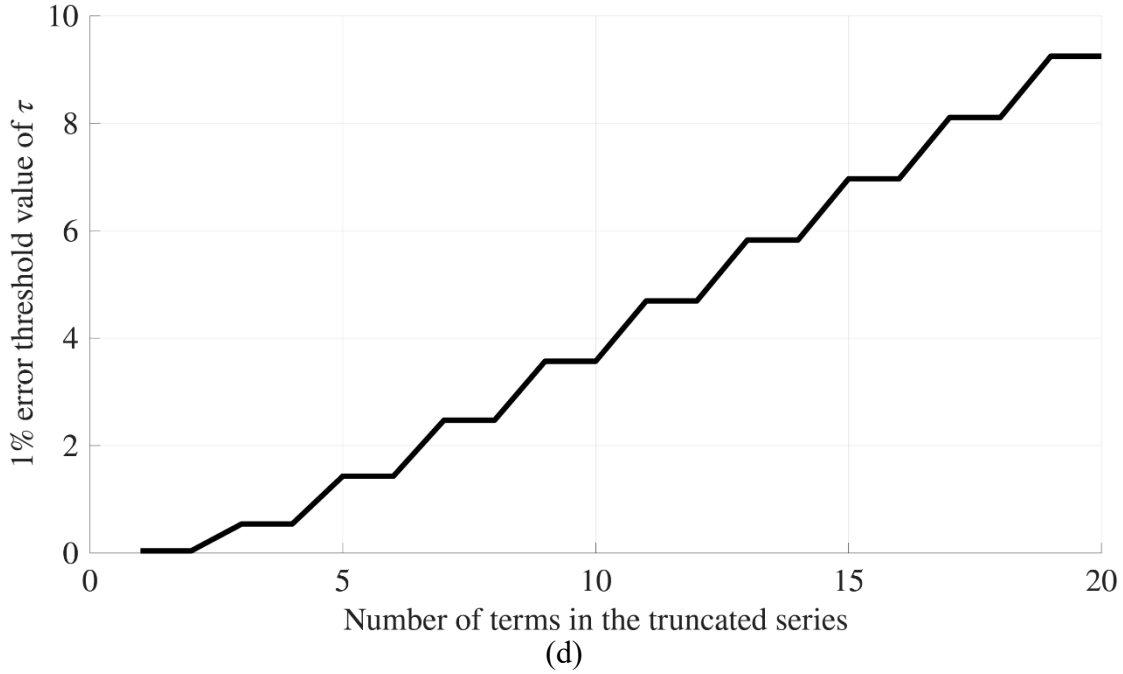


Figure 2. Numerical convergence behavior of the truncated form of the series in Eq (24) with an increasing number of terms. (a–c) Comparison between the analytical solution of Pollard's integral formula at $\beta = 0.5$ and the truncated series with (a) 10, (b) 20, and (c) 30 terms. (d) Quantification of the convergence range for each truncation order by illustrating the threshold value of τ at which the relative error between the analytical solution and the series exceeds 1% as the number of terms increases from 1 to 20.

5. Demonstration of the normalization condition of $H(\tau, \beta)$

To ensure the physical and statistical integrity of the continuous spectrum framework, the spectral density $H(\tau, \beta)$ (and consequently its exact analytical solution, namely Pollard's integral formula) must satisfy the normalization condition: $\int_0^\infty H(\tau, \beta)d\tau = 1$. Since $H(\tau, \beta)$ functions as a probability density in statistical physics, this condition represents the conservation of total probability (or total energy intensity). Mathematically, it ensures that the total weight of all relaxation times sums to unity, directly corresponding to the initial value of 1 for the stretched exponential function $e^{-(\frac{t}{\tau_0})^\beta}$ at $t = 0$. In this section, we demonstrate this normalization condition by utilizing a coordinate transformation and invoking the classical initial value theorem of the Laplace transform. To this end, the integral of the time-domain spectral density $H(\tau, \beta)$ is evaluated over its entire support $\tau \in (0, \infty)$:

$$I = \int_0^\infty H(\tau, \beta)d\tau. \tag{33}$$

Direct substitution of Pollard's integral formula into Eq (33) yields an integrand that is highly singular and divergent near the origin, due to the leading factor $\frac{\tau_0}{\pi\tau^2}$. Such an approach is therefore analytically intractable. To resolve this, we leverage the relation $H(\tau, \beta) = \frac{1}{\tau^2}\rho(\frac{1}{\tau})$ (where $s = \frac{1}{\tau}$, which implies that $d\tau = -\frac{1}{s^2}ds$) to facilitate the demonstration. Under this mapping, the domain $\tau \in (0, \infty)$ corresponds to $s \in (\infty, 0)$. Substituting these relations into Eq (33) and reserving the

limits of integration yields

$$I = \int_0^{\infty} H(\tau, \beta) d\tau = \int_{\infty}^{0} \rho(s)(-ds) = \int_0^{\infty} \rho(s) ds. \tag{34}$$

Equation (34) demonstrates that the total area under the time-domain spectral density curve $H(\tau, \beta)$ is mathematically identical to that under the Laplace-domain spectral density curve $\rho(s)$.

Recall from Eq (10) that the stretched exponential function $e^{-(\frac{t}{\tau_0})^{\beta}}$ is essentially the Laplace transform of the Laplace-domain spectral density $\rho(s)$:

$$e^{-(\frac{t}{\tau_0})^{\beta}} = \int_0^{\infty} \rho(s) e^{-st} ds. \tag{35}$$

The continuous limit of Eq (35) is now evaluated as the time variable t approaches zero from the right ($t \to 0^+$):

$$\lim_{t \to 0^+} e^{-(\frac{t}{\tau_0})^{\beta}} = \lim_{t \to 0^+} \int_0^{\infty} \rho(s) e^{-st} ds, \tag{36}$$

where the left-hand side of Eq (36) is

$$\lim_{t \to 0^+} e^{-(\frac{t}{\tau_0})^{\beta}} = e^0 = 1 \tag{37}$$

and the right-hand side of Eq (36) is

$$\lim_{t \to 0^+} \int_0^{\infty} \rho(s) e^{-st} ds = \int_0^{\infty} \rho(s) (\lim_{t \to 0^+} e^{-st}) ds = \int_0^{\infty} \rho(s) \cdot 1 \cdot ds = \int_0^{\infty} \rho(s) ds. \tag{38}$$

In the evaluation of Eq (38), the interchange of the limit and the integral is justified by Lebesgue's dominated convergence theorem [37], given that $\rho(s)$ has been rigorously established as non-negative (i.e., $\rho(s) \geq 0$) and absolutely integrable.

Subsequently, by substituting Eqs (37) and (38) into Eq (36), we obtain

$$\int_0^{\infty} \rho(s) ds = 1. \tag{39}$$

Finally, combining Eqs (33), (34), and (39) yields

$$\int_0^{\infty} H(\tau, \beta) d\tau = 1. \tag{40}$$

This completes the demonstration of the normalization condition of the spectral density $H(\tau, \beta)$ (and consequently its exact analytical solution, namely Pollard's integral formula).

6. Discussion

This article provides a comprehensive, rigorous, and accessible review of the step-by-step derivation of the key formulas and fundamental properties of the spectral density $H(\tau, \beta)$ associated with the stretched exponential function. Serving as a historical milestone, Pollard's integral formula resolved a long-standing challenge by rigorously establishing the existence and uniqueness of the exact analytical solution of this spectral density. It serves as a foundational cornerstone for modeling

anomalous relaxation and transport phenomena in complex, nonlinear systems across physics and materials science, while its series representation provides vital computational and practical tractability. Elucidating the derivation of Pollard's integral formula and its series representation provides profound mathematical, physical, pedagogical, and practical advantages. First, understanding the derivation of Pollard's integral formula clarifies the origin of each specific term within the formula, thereby gaining deep insights into the mathematical and physical characteristics governing non-Debye relaxation. Second, mastering this analytical framework empowers researchers to establish modified Pollard-type representations for other completely monotonic functions. Furthermore, this deep understanding may inspire researchers in selecting the optimal approach among the integral and series forms, thereby facilitating more efficient computations in practical applications.

It is important to emphasize that the series representation of Pollard's integral formula, namely Eq (24), is absolutely convergent for all $\tau \in (0, \infty)$, despite being originally derived in the asymptotic limit as $\tau \to 0$. This implies that, theoretically, the series can be utilized to compute values for any positive τ, rather than being restricted to the regime of extremely small τ. In practical computations, however, truncation is necessary for this infinite series, inevitably introducing truncation errors; as demonstrated in Section 4.3, a higher number of terms in the truncated series yields a wider convergence range. Building upon the preliminary investigation provided in this article, conducting a more detailed analysis of the numerical convergence behavior of Eq (24) remains an important and compelling direction for future research.

This review focuses on the exact analytical derivation of the spectral density for a noise-free stretched exponential function. However, it is essential to acknowledge the practical inverse problem encountered in experimental data analysis. Extracting spectra from experimental data typically requires solving a numerically ill-posed inverse Laplace transform, where direct inversion triggers severe, unphysical oscillations. To circumvent this and ensure physically meaningful spectra, advanced numerical regularization algorithms are widely employed. Prominent examples include the maximum entropy method [38], which imposes information-theoretic constraints to reconstruct the most probable distribution, and Tikhonov regularization [39], which introduces a penalty term to stabilize and smooth the extracted spectra. Indeed, Tikhonov regularization has gained substantial popularity as a robust tool for directly extracting continuous relaxation spectra from complex experimental data [40]. Recognizing these advanced numerical regularization algorithms bridges the gap between the analytical framework presented in this review and the practical demands of experimental data analysis.

It is essential to acknowledge the remarkable recent advances in nonlinear relaxation theory and to delineate how it distinctively differs from and complements the conventional spectral representation reviewed in this article. The continuous superposition of exponential decay functions (mathematically anchored by Pollard's integral formula) provides the analytical framework for identifying the specific distribution of relaxation times that yields the stretched exponential behavior. However, such a spectral representation does not uniquely dictate the underlying microscopic or mesoscopic mechanism, serving instead as a purely phenomenological representation. Conversely, a profound and modern dynamical description is provided by the Trachenko-Zaccone (TZ) equation [41,42], which addresses a fundamentally different yet vital question: How do underlying microscopic or mesoscopic dynamics natively generate non-Debye relaxation in the first place? In the TZ equation framework, non-Debye relaxation arises directly from a nonlinear first-order rate equation governed by dynamic feedback between interacting local relaxation events [43]. Crucially, this paradigm exposes the inherent limitations of the conventional spectral representation. As dictated by Bernstein's theorem, the spectral representation is mathematically viable only for completely

monotonic stretched exponential behavior ($0 < \beta < 1$), which ensures a unique, non-negative Borel measure. By contrast, compressed exponential behavior ($\beta > 1$) is not completely monotonic and therefore cannot be physically represented by a positive distribution of exponential decay functions. Remarkably, the TZ equation framework transcends this mathematical limitation, smoothly generating both stretched and compressed exponential behaviors by altering the nature of the nonlinear feedback mechanism [43]. This universal nonlinear dynamical framework has been shown to govern a diverse range of phenomena, spanning from condensed matter physics to broader classes of complex dynamical systems, such as population dynamics [44]. Clarifying this distinction demonstrates that the conventional spectral representation and the TZ equation framework are not competing paradigms, but rather complementary perspectives. Synthesizing the mathematical elegance of the former with the physical insights of the latter provides a unified and comprehensive framework for investigating complex nonlinear relaxation phenomena across the physical sciences.

Regarding future research directions, the systematic and rigorous mathematical framework presented in this review provides a critical foundation for future theoretical advancements. First, the entire mathematical framework detailed herein can be directly extended to other models governing non-Debye relaxation, such as the family of Mittag-Leffler functions and various modified stretched exponential functions. Deriving exact spectral density representations for such broader classes of completely monotonic functions remains a formidable but essential task. Second, while the mathematical framework of this review is anchored in the stretched exponential regime ($0 < \beta < 1$), future theoretical efforts will be required to develop a mathematical framework capable of characterizing the compressed exponential regime ($\beta > 1$), where non-negativity and complete monotonicity break down. Finally, from a computational perspective, integrating the series representation in Eq (24) with modern machine learning algorithms or advanced numerical regularization techniques (e.g., Tikhonov regularization) could offer a promising pathway to next-generation, noise-resilient hybrid inversion algorithms, thereby catalyzing future methodological innovations in the study of complex nonlinear relaxation phenomena.

7. Conclusions

In summary, a comprehensive, rigorous, and accessible derivation of the key formulas and fundamental properties of $H(\tau, \beta)$ associated with the stretched exponential function has been presented in this article. By demonstrating how a continuous linear combination of exponential decay functions with non-negative weights converges to the stretched exponential function, the origin as well as the mathematical and physical significance of $H(\tau, \beta)$ have been elucidated. Furthermore, the exact analytical solution for $H(\tau, \beta)$ (widely known as Pollard's integral formula) and its asymptotic series representation are systematically derived, accompanied by a verification of the theoretical absolute convergence and the numerical convergence behavior of the series. Two essential properties of $H(\tau, \beta)$, namely its non-negativity and normalization condition, are also formally established. These transparent derivations provide a solid foundation for comprehensive understanding and learning, deepening theoretical insights, and facilitating further theoretical and applied research.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

1. S. Yoshioka, Y. Aso, S. Kojima, Usefulness of the Kohlrausch-Williams-Watts stretched exponential function to describe protein aggregation in lyophilized formulations and the temperature dependence near the glass transition temperature, *Pharm. Res.*, **18** (2001), 256–260. https://doi.org/10.1023/A:1011082309058
2. D. Apitz, P. M. Johansen, Limitations of the stretched exponential function for describing dynamics in disordered solid materials, *J. Appl. Phys.*, **97** (2005). https://doi.org/10.1063/1.1852069
3. L. Lévy, H. Che, A. Weller, Using the stretched exponential function for automatic processing of time-domain induced polarization data and further interpretation, *Geophys. J. Int.*, **245** (2026), ggag010. https://doi.org/10.1093/gji/ggag010
4. M. Zhong, R. J. Loy, R. S. Anderssen, Approximating the Kohlrausch function by sums of exponentials, *ANZIAM J.*, **54** (2013), 306–323. https://doi.org/10.1017/S1446181113000229
5. E. C. de Oliveira, F. Mainardi, J. Vaz Jr, Fractional models of anomalous relaxation based on the Kilbas and Saigo function, *Meccanica*, **49** (2014), 2049–2060. https://doi.org/10.1007/s11012-014-9930-0
6. K. Górska, A. Horzela, A. Lattanzi, T. K. Pogány, On complete monotonicity of three parameter Mittag-Leffler function, *Appl. Anal. Discrete Math.*, **15** (2021), 118–128. https://doi.org/10.2298/AADM190226025G
7. J. P. Bouchaud, Anomalous relaxation in complex systems: From stretched to compressed exponentials, *Anomalous Transport-Found. Appl.*, 2008, 327–345. https://doi.org/10.1002/9783527622979.ch11
8. A. V. Milovanov, J. J. Rasmussen, K. Rypdal, Stretched-exponential decay functions from a self-consistent model of dielectric relaxation, *Phys. Lett. A*, **372** (2008), 2148–2154. https://doi.org/10.1016/j.physleta.2007.11.025
9. Z. W. Wu, W. Kob, W. H. Wang, L. Xu, Stretched and compressed exponentials in the relaxation dynamics of a metallic glass-forming melt, *Nat. Commun.*, **9** (2018), 5334. https://doi.org/10.1038/s41467-018-07759-w
10. M. F. Shlesinger, E. W. Montroll, On the Williams–Watts function of dielectric relaxation, *P. Natl. A. Sci.*, **81** (1984), 1280–1283. https://doi.org/10.1073/pnas.81.4.1280
11. R. M. Hill, L. A. Dissado, Debye and non-Debye relaxation, *J. Phys. C-Solid State Phys.*, **18** (1985), 3829–3836. https://doi.org/10.1088/0022-3719/18/19/021
12. A. K. Singh, Non-Debye to Debye spectral shift in solid solutions of orientationally disordered crystals, *Phys. Rev. E*, **109** (2024), 065003. https://doi.org/10.1103/PhysRevE.109.065003

13. J. C. Phillips, Stretched exponential relaxation in molecular and electronic glasses, *Rep. Prog. Phys.*, **59** (1996), 1133–1207. https://doi.org/10.1088/0034-4885/59/9/003
14. X. Xia, P. G. Wolynes, Microscopic theory of heterogeneity and nonexponential relaxations in supercooled liquids, *Phys. Rev. Lett.*, **86** (2001), 5526. https://doi.org/10.1103/PhysRevLett.86.5526
15. B. Sturman, E. Podivilov, M. Gorkunov, Origin of stretched exponential relaxation for hopping-transport models, *Phys. Rev. Lett.*, **91** (2003), 176602. https://doi.org/10.1103/PhysRevLett.91.176602
16. K. S. Cole, R. H. Cole, Dispersion and absorption in dielectrics I. Alternating current characteristics, *J. Chem. Phys.*, **9** (1941), 341–351. https://doi.org/10.1063/1.1750906
17. R. Böhmer, K. L. Ngai, C. A. Angell, D. J. Plazek, Nonexponential relaxations in strong and fragile glass formers, *J. Chem. Phys.*, **99** (1993), 4201–4209. https://doi.org/10.1063/1.466117
18. R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: A fractional dynamics approach, *Phys. Rep.*, **339** (2000), 1–77. https://doi.org/10.1016/S0370-1573(00)00070-3
19. C. P. Lindsey, G. D. Patterson, Detailed comparison of the Williams–Watts and Cole–Davidson functions, *J. Chem. Phys.*, **73** (1980), 3348–3357. https://doi.org/10.1063/1.440530
20. F. Alvarez, A. Alegra, J. Colmenero, Relationship between the time-domain Kohlrausch–Williams–Watts and frequency-domain Havriliak–Negami relaxation functions, *Phys. Rev. B*, **44** (1991), 7306. https://doi.org/10.1103/PhysRevB.44.7306
21. M. D. Ediger, Spatially heterogeneous dynamics in supercooled liquids, *Ann. Rev. Phys. Chem.*, **51** (2000), 99–128. https://doi.org/10.1146/annurev.physchem.51.1.99
22. F. Brouers, O. Sotolongo-Costa, Generalized fractal kinetics in complex systems (application to biophysics and biotechnology), *Physica A*, **368** (2006), 165–175. https://doi.org/10.1016/j.physa.2005.12.062
23. W. Chen, X. Hei, H. Sun, D. Hu, Stretched exponential stability of nonlinear Hausdorff dynamical systems, *Chaos Soliton. Fract.*, **109** (2018), 259–264. https://doi.org/10.1016/j.chaos.2018.03.002
24. A. Lukichev, Physical meaning of the stretched exponential Kohlrausch function, *Phys. Lett. A*, **383** (2019), 2983–2987. https://doi.org/10.1016/j.physleta.2019.06.029
25. J. Hristov, Response functions in linear viscoelastic constitutive equations and related fractional operators, *Math. Model. Nat. Pheno.*, **14** (2019), 305. https://doi.org/10.1051/mmnp/2018067
26. J. C. Mauro, Y. Z. Mauro, On the Prony series representation of stretched exponential relaxation, *Physica A*, **506** (2018), 75–87. https://doi.org/10.1016/j.physa.2018.04.047
27. D. C. Johnston, Stretched exponential relaxation arising from a continuous sum of exponential decays, *Phys. Rev.*, **74** (2006), 184430. https://doi.org/10.1103/PhysRevB.74.184430
28. C. Y. Lin, Interconnection between the stretched exponential function and the Prony series: A concise and rigorous revisit, *Mathematics*, **14** (2026), 2338. https://doi.org/10.3390/math14132338
29. H. Pollard, The representation of $e^{-x^{\lambda}}$ as a Laplace integral, *B. Am. Math. Soc.*, **52** (1946), 908–910. https://doi.org/10.1090/S0002-9904-1946-08672-3
30. S. Bernstein, Sur les fonctions absolument monotones, *Acta Math.*, **52** (1929), 1–66. https://doi.org/10.1007/BF02592679
31. M. Capiński, P. E. Kopp, *The Radon—Nikodym theorem*, In: Measure, Integral and Probability, London: Springer London, 2004, 187–240. https://doi.org/10.1007/978-1-4471-0645-6_7
32. A. Lion, Thermomechanically consistent formulations of the standard linear solid using fractional derivatives, *Arch. Mech.*, **53** (2001), 253–273.
33. E. C. De Oliveira, F. Mainardi, J. Vaz Jr, Models based on Mittag-Leffler functions for anomalous relaxation in dielectrics, *Eur. Phys. J.-Spec. Top.*, **193** (2011), 161–171. https://doi.org/10.1140/epjst/e2011-01388-0

34. F. Mainardi, R. Garrappa, On complete monotonicity of the Prabhakar function and non-Debye relaxation in dielectrics, *J. Comput. Phys.*, **293** (2015), 70–80. https://doi.org/10.1016/j.jcp.2014.08.006
35. B. Simon, Basic complex analysis, *Am. Math. Soc.*, 2015. https://doi.org/10.1090/simon/002.1
36. A. Vidras, A. Yger, Multidimensional residue theory and applications, *Am. Math. Soc.*, **275** (2023). https://doi.org/10.1090/surv/275
37. T. Tao, An introduction to measure theory, *Am. Math. Soc.*, **126** (2011).
38. A. K. Livesey, J. C. Brochon, Analyzing the distribution of decay constants in pulse-fluorimetry using the maximum entropy method, *Biophys. J.*, **52** (1987), 693–706. https://doi.org/10.1016/S0006-3495(87)83264-2
39. A. N. Tikhonov, Regularization of incorrectly posed problems, *Sov. Math. Dokl.*, **4** (1963), 1624–1627.
40. A. Arsenault, T. Imai, P. M. Singer, K. M. Suzuki, M. Fujita, Magnetic inhomogeneity in charge-ordered La 1.885 Sr 0.115 CuO 4 studied by NMR, *Phys. Rev. B*, **101** (2020), 184505. https://doi.org/10.1103/PhysRevB.101.184505
41. V. V. Ginzburg, O. V. Gendelman, A. Zaccone, Unifying physical framework for stretched-exponential, compressed-exponential, and logarithmic relaxation phenomena in glassy polymers, *Macromolecules*, **57** (2024), 2520–2529. https://doi.org/10.1021/acs.macromol.3c02480
42. A. Zaccone, V. V. Ginzburg, O. V. Gendelman, S. F. Mauro, J. C. Mauro, The Trachenko-Zaccone equation: Nonlinear relaxation from glasses to complex systems, *arXiv preprint*, 2026.
43. K. Trachenko, A. Zaccone, Slow stretched-exponential and fast compressed-exponential relaxation from local event dynamics, *J. Phys.-Condens. Mat.*, **33** (2021), 315101. https://doi.org/10.1088/1361-648X/ac04cd
44. A. Zaccone, K. Trachenko, Global population crisis scenarios predicted by a general nonlinear dynamical model, *Chaos Soliton. Fract.*, **209** (2026), 118542. https://doi.org/10.1016/j.chaos.2026.118542



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