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Research article

A perimeter approach to best proximity point theory

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Abstract: In this paper, we improve an existing best proximity point result for perimeter proximal contractions by removing the continuity assumption. Additionally, we provide an example involving a discontinuous mapping to demonstrate the applicability of our result where the previous theorem cannot be applied. Furthermore, we introduce the notions of a non-self perimeter α-contraction and a perimeter weak P-property. Then, we establish a new best proximity point theorem based on these concepts. Several comparative examples are presented to illustrate the relationships between the newly introduced notions and their classical counterparts and to show that the obtained results are genuine extensions of existing ones.

Keywords: perimeter contraction; best proximity point; best periodic point; fixed point; proximal contraction
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1. Introduction and preliminaries

The fixed point theory is one of the central topics of nonlinear analyses and has numerous applications in the optimization theory, approximation theory, differential equations, and engineering sciences. One of the most fundamental results in this area is the Banach contraction principle [1], which guarantees the existence and uniqueness of fixed points for contraction mappings on complete metric spaces. Related developments include multivalued and weakly Picard mappings [2–4], Kannan-type, cyclic, and $\mathcal{R}$-contractions [5–7], and fixed point methods based on simulation functions [8–10]. Further connections to fractal operators, approximation theory, and fuzzy mappings are discussed in [11–13]. In particular, Petrov [14] introduced the following notion of perimeter contraction.

Definition 1 ([14]). *Let* (Π, δ) *be a metric space and* $F : \Pi \to \Pi$ *be a self-map. We say that* F *is a contracting perimeters of triangles if there exists a constant* $\alpha \in [0, 1)$ *for which*

$$\delta(F\zeta, F\varrho) + \delta(F\zeta, F\omega) + \delta(F\varrho, F\omega) \leq \alpha \left[\delta(\zeta, \varrho) + \delta(\zeta, \omega) + \delta(\varrho, \omega)\right]$$

whenever $\zeta, \varrho, \omega \in \Pi$ are pairwise distinct.

Petrov [14] proved the following fixed point theorem.

Theorem 1 ([14]). *Suppose that* (Π, δ) *is a complete metric space containing at least three elements. Let* $F : \Pi \to \Pi$ *satisfy the triangle-perimeter contractive condition. If* F *has no periodic point with a prime period of two, then there exists some* $\xi \in \Pi$ *such that* $F\xi = \xi$*. Furthermore,* F *can have no more than two fixed points.*

However, in many applications, the underlying mapping is non-self, that is, the mapping is defined from a subset M of a metric space into another subset R. In this case, fixed points may fail to exist whenever $M \cap R = \emptyset$. To overcome this difficulty, the concept of the best proximity point was introduced as an appropriate generalization of fixed points. A point $\zeta^* \in M$ is called a best proximity point of a mapping $F : M \to R$ if

$$\delta(\zeta^*, F\zeta^*) = \delta(M, R).$$

During the last few decades, several authors studied the existence and uniqueness results for best proximity points under various contractive conditions. In particular, proximal contraction mappings have been extensively investigated due to their strong geometric structure and their relationship with the classical fixed point theory.

In the best proximity point theory, there are generally two main approaches to obtain existence and uniqueness results. The first one is based on the P-property (or weak P-property) of the pair (M, R), which allows the transfer of the geometric structure of the sets into a contraction type condition. The second approach is based on proximal contractions where the contractive condition is directly imposed on points realizing the distance $\delta(M, R)$. Both approaches have been extensively studied and generalized. For foundational treatments of best proximity pairs and the P-property, see [15–17]. Developments involving cyclic, polynomial-type, and proximal coincidence contractions are discussed in [18–20]. Related applications to fractals and multivalued contractions appear in [21, 22]. Recall some definitions and theorems about the best proximity point theory.

Throughout this paper, (Π, δ) denotes a metric space, and M and R are nonempty subsets of Π. The distance between M and R is defined by the following:

$$\delta(M, R) = \inf\{\delta(\zeta, \varrho) : \zeta \in M,\ \varrho \in R\}.$$

Define the following subsets:

$$M_0(R) = \{\zeta \in M :\ \delta(\zeta, \varrho) = \delta(M, R) \text{ for some } \varrho \in R\},$$

and

$$R_0(M) = \{\varrho \in R :\ \delta(\zeta, \varrho) = \delta(M, R) \text{ for some } \zeta \in M\}.$$

Throughout the paper, we will denote $M_0(R)$ and $R_0(M)$ simply by M_0 and R_0, respectively.

Basha [23] introduced the notion of proximal contraction.

Definition 2 ([23]). *Let $F : M \to R$ be a mapping. Then, F is said to be a proximal contraction if there is $\alpha \in [0, 1)$ such that*

$$\left.\begin{array}{c} \delta(\varsigma_1, F\zeta_1) = \delta(M, R), \\ \delta(\varsigma_2, F\zeta_2) = \delta(M, R) \end{array}\right\} \Rightarrow \delta(\varsigma_1, \varsigma_2) \leq \alpha\delta(\zeta_1, \zeta_2)$$

for all $\varsigma_1, \varsigma_2, \zeta_1, \zeta_2 \in M$.

Theorem 2 ([23]). *Let M,R be closed sets in a complete metric space (Π, δ). Assume that $M_0 \neq \emptyset$ and $F : M \to R$ is a proximal contraction satisfying*

$$F(M_0) \subseteq R_0;$$

then, F admits a unique best proximity point.

Definition 3 ([24]). *The pair (M, R) is said to satisfy the P-property if*

$$\left.\begin{array}{c} \delta(\zeta_1, \varrho_1) = \delta(M, R), \\ \delta(\zeta_2, \varrho_2) = \delta(M, R) \end{array}\right\} \Rightarrow \delta(\zeta_1, \zeta_2) = \delta(\varrho_1, \varrho_2)$$

for all $\zeta_1, \zeta_2 \in M$ and $\varrho_1, \varrho_2 \in R$.

Theorem 3 ([24]). *Let (Π, δ) be a complete metric space and M,R be closed sets. Assume that $M_0 \neq \emptyset$, and the pair (M, R) satisfies the P -property. Let $F : M \to R$ be a non-self α-contraction, that is, there exists $\alpha \in [0, 1)$ such that*

$$\delta(F\zeta, F\varrho) \leq \alpha\delta(\zeta, \varrho)$$

for all $\zeta, \varrho \in M$. If

$$F(M_0) \subseteq R_0,$$

then F admits a unique best proximity point.

Definition 4 ([25]). *The pair (M, R) is said to satisfy the generalized weak P-property if*

$$\left.\begin{array}{c} \delta(\zeta_1, \varrho_1) = \delta(M, R), \\ \delta(\zeta_2, \varrho_2) = \delta(M, R) \end{array}\right\} \Rightarrow \delta(\zeta_1, \zeta_2) \leq \delta(\varrho_1, \varrho_2)$$

for all $\zeta_1, \zeta_2 \in M$ with $\zeta_1 \neq \zeta_2$ and for all $\varrho_1, \varrho_2 \in R$.

Definition 5 ([26]). *Let $F : M \to R$ be a mapping and $\zeta \in M$. Then, the set of iterative sequences*

$$O_F(\zeta) = \{\{\zeta_s\} \subseteq M : \zeta_0 = \zeta \text{ and } \delta(\zeta_{s+1}, F\zeta_s) = \delta(M, R) \text{ for all } s \in \mathbb{N}\}$$

is called the orbit of ζ.

It can be seen that if we take $M = R = \Pi$ in Definition 5, then we have

$$O_F(\zeta) = \{\{F^s\zeta\} : s \in \mathbb{N}\}.$$

Definition 6 ([26]). *Let $F : M \to R$ be a mapping. The set M is called F-best orbitally complete whenever the following property holds: for any $\zeta \in M$ and any sequence*

$$\{\zeta_s\} \in O_F(\zeta),$$

if a subsequence $\{\zeta_{s_i}\}$ is Cauchy, then there exists some $\zeta^* \in M_0$ such that

$$\lim_{i\to\infty} \zeta_{s_i} = \zeta^*.$$

The requirement in Definition 6 concerns every initial point $\zeta \in M$, every sequence in $O_F(\zeta)$, and every Cauchy subsequence of that sequence.

Definition 7 ([26]). *Let $F : M \to R$ be a mapping. A point $\zeta \in M$ is said to be a best periodic point of F if there exists a sequence $\{\zeta_s\} \in O_F(\zeta)$ such that*

$$\delta(\zeta, F\zeta_m) = \delta(M, R)$$

for some $m \in \mathbb{N}$.

Here, the prime period is the smallest positive number of proximal steps required to return to the initial point. In particular, a point $\zeta \in M$ is a best periodic point of a prime period of two precisely when

$$\delta(\zeta, F\zeta) > \delta(M, R),$$

and there exists $\varrho \in M \setminus \{\zeta\}$ such that

$$\delta(\varrho, F\zeta) = \delta(\zeta, F\varrho) = \delta(M, R).$$

In the particular case of a self-mapping $F : \Pi \to \Pi$, a point $\zeta \in \Pi$ is said to be periodic if one of its positive iterates returns to the point itself; equivalently,

$$F^m(\zeta) = \zeta$$

for some $m \in \mathbb{N}$. Therefore, if we take $M = R = \Pi$ in Definition 7, it turns into the definition of periodic point of F.

Theorem 5 provides a continuity-free counterpart of Theorem 4. Both results concern perimeter proximal contractions satisfying $|M_0| \geq 3$ and $F(M_0) \subseteq R_0$. Theorem 4 assumes completeness of the ambient metric space, closedness of M and R, continuity of F, and Condition Λ. In Theorem 5, we instead use F-best orbital completeness of M and the absence of best periodic points of a prime period of two. Under these hypotheses, we obtain the same existence conclusion and the same upper bound of two best proximity points. Example 2 illustrates the applicability of Theorem 5 to a discontinuous mapping for which Theorem 4 cannot be applied.

In Theorem 6, the perimeter proximal contraction assumption is replaced by the combination of the perimeter weak P-property and the non-self perimeter α-contraction condition, while the other hypotheses of Theorem 5 are retained. The generalized weak P-property implies the perimeter weak P-property, and every non-self α-contraction is a non-self perimeter α-contraction. Examples 3 and 4 show that the converse implications fail.

2. Main results

Let us begin this section with the following definition.

Definition 8 ([27]). *Let $F : M \to R$ be a mapping. Then, we say that F is a perimeter proximal contraction mapping, if there exists a constant $\alpha \in [0, 1)$ such that*

$$\begin{aligned}\delta(\varsigma_1, F\zeta_1) &= \delta(M, R)\\ \delta(\varsigma_2, F\zeta_2) &= \delta(M, R)\\ \delta(\varsigma_3, F\zeta_3) &= \delta(M, R)\end{aligned}$$

implies

$$\delta(\varsigma_1, \varsigma_2) + \delta(\varsigma_1, \varsigma_3) + \delta(\varsigma_2, \varsigma_3) \leq \alpha(\delta(\zeta_1, \zeta_2) + \delta(\zeta_1, \zeta_3) + \delta(\zeta_2, \zeta_3))$$

for all three pairwise distinct points $\zeta_1, \zeta_2, \zeta_3 \in M$ and for all $\varsigma_1, \varsigma_2, \varsigma_3 \in M$.

In Definition 8, the distinctness requirement only applies to $\zeta_1,\ \zeta_2,\ \zeta_3$; the points $\varsigma_1,\ \varsigma_2,\ \varsigma_3$ may coincide.

Proposition 1. *Every proximal contraction is a perimeter proximal contraction mapping.*

Proof. Let $\zeta_1, \zeta_2, \zeta_3 \in M$ be pairwise distinct points and $\varsigma_1, \varsigma_2, \varsigma_3 \in M$ be points satisfying

$$\begin{aligned}\delta(\varsigma_1, F\zeta_1) &= \delta(M, R),\\ \delta(\varsigma_2, F\zeta_2) &= \delta(M, R),\\ \delta(\varsigma_3, F\zeta_3) &= \delta(M, R).\end{aligned}$$

Since F is a proximal contraction mapping, there exists α in $[0, 1)$ satisfying

$$\delta(\varsigma_i, \varsigma_j) \leq \alpha\delta(\zeta_i, \zeta_j)$$

for all $i, j = 1, 2, 3$. Adding the left and the right sides of the last inequalities, we get the following:

$$\delta(\varsigma_1, \varsigma_2) + \delta(\varsigma_1, \varsigma_3) + \delta(\varsigma_2, \varsigma_3) \leq \alpha(\delta(\zeta_1, \zeta_2) + \delta(\zeta_1, \zeta_3) + \delta(\zeta_2, \zeta_3)).$$

Then, the proof is done. □

The converse of Proposition 1 may not be true. The following example shows this fact.

Example 1. *Consider a function* $\delta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ *defined as follows:*

$$\delta(\zeta, \varrho) = \begin{cases} 0, & \zeta = \varrho;\\ \max\{\zeta, \varrho\}, & \textit{otherwise.}\end{cases}$$

Then, $([0, \infty), \delta)$ is a metric space. Let $M = [0, 1]$ and $R = [2, 3]$. Then, we get $\delta(M, R) = 2$, $M = M_0$, and $R_0 = \{2\}$. Now, we define a mapping $F : M \to R$ as follows:

$$F\zeta = \begin{cases} 2, & \zeta = 1;\\ \frac{\zeta+4}{2}, & \text{otherwise.}\end{cases}$$

For any $\varsigma, \zeta \in M$, we have $\varsigma \leq 1$ and $F\zeta \geq 2$, and hence,

$$\delta(\varsigma, F\zeta) = F\zeta.$$

Therefore,

$$\delta(\varsigma, F\zeta) = \delta(M, R) \iff F\zeta = 2 \iff \zeta \in \{0, 1\}.$$

Consequently, there are no three pairwise distinct points $\zeta_1, \zeta_2, \zeta_3 \in M$ for which points $\varsigma_1, \varsigma_2, \varsigma_3 \in M$ satisfy

$$\delta(\varsigma_i, F\zeta_i) = \delta(M, R), \qquad i = 1, 2, 3.$$

Thus, the implication in Definition 8 holds vacuously for any $\alpha \in [0, 1)$, and F is a perimeter proximal contraction. On the other hand, if we choose $\varsigma_1 = \zeta_1 = 0$, $\varsigma_2 = \zeta_2 = 1$ in M, then we have the following:

$$\delta(\varsigma_1, F\zeta_1) = \delta(M, R) = \delta(\varsigma_2, F\zeta_2).$$

Since $\delta(\varsigma_1, \varsigma_2) = \delta(\zeta_1, \zeta_2)$, the mapping is not a proximal contraction mapping.

Now, we recall the following condition.

Condition Λ [27]: Let M, R be two subsets of a metric space (Π, δ) and $F : M \to R$ be a mapping. Then, F is said to satisfy Condition Λ if there does not exist any pair of points (ζ, ϱ) with $\zeta \neq \varrho$ in M such that

$$\delta(\zeta, F\varrho) = \delta(\varrho, F\zeta) = \delta(M, R).$$

In Definition 7, the case $m = 1$ gives $\delta(\zeta, F\zeta_1) = \delta(M, R)$. Since $\{\zeta_s\} \in O_F(\zeta)$, we also have $\delta(\zeta_1, F\zeta) = \delta(M, R)$. Thus, when $\zeta_1 \neq \zeta$, these two equalities are excluded by Condition Λ. The following result, established in [27], will be used in the sequel.

Theorem 4 ([27])**.** *Suppose that (Π, δ) is a complete metric space and M, R are closed sets. Let $|M_0| \geq 3$ and $F : M \to R$ be a continuous perimeter proximal contraction such that $F(M_0) \subseteq R_0$. Then, if F satisfies Condition Λ ,then F possesses a best proximity point. Moreover, the best proximity points of F are at most two in number.*

Now we state our first main result. Observe that Theorem 4 was proven under the continuity assumption on F. Here, we obtain the same conclusion without assuming that F is continuous.

Theorem 5. *Let $F : M \to R$ be a mapping. Assume that $M_0 \neq \emptyset$, $|M_0| \geq 3$, and M is F-best orbitally complete. Suppose that F is a perimeter proximal contraction mapping satisfying $F(M_0) \subseteq R_0$. If F does not possess best periodic point of prime period 2, then F has a best proximity point in M. The number of best proximity points is at most two.*

Proof. Let $\zeta_0 \in M_0$ be arbitrary. If $\delta(\zeta_0, F\zeta_0) = \delta(M, R)$, then ζ_0 is a best proximity point of F, and the existence part of the proof is complete. Therefore, we may assume that

$$\delta(\zeta_0, F\zeta_0) > \delta(M, R). \tag{2.1}$$

Since $F\zeta_0 \in F(M_0) \subseteq R_0$, there exists $\zeta_1 \in M_0$ such that

$$\delta(\zeta_1, F\zeta_0) = \delta(M, R). \tag{2.2}$$

It follows from (2.1) and (2.2) that $\zeta_0 \neq \zeta_1$. If $\delta(\zeta_1, F\zeta_1) = \delta(M, R)$, then ζ_1 is a best proximity point of F, and the existence part of the proof is complete. Therefore, we may assume that

$$\delta(\zeta_1, F\zeta_1) > \delta(M, R). \tag{2.3}$$

Since $F\zeta_1 \in F(M_0) \subseteq R_0$, there exists $\zeta_2 \in M_0$ such that

$$\delta(\zeta_2, F\zeta_1) = \delta(M, R). \tag{2.4}$$

It follows from (2.3) and (2.4) that $\zeta_1 \neq \zeta_2$. If $\delta(\zeta_2, F\zeta_2) = \delta(M, R)$, then ζ_2 is a best proximity point of F, and the existence part of the proof is complete. Therefore, we may assume that

$$\delta(\zeta_2, F\zeta_2) > \delta(M, R). \tag{2.5}$$

Since $F\zeta_2 \in F(M_0) \subseteq R_0$, there exists $\zeta_3 \in M_0$ such that

$$\delta(\zeta_3, F\zeta_2) = \delta(M, R). \tag{2.6}$$

It follows from (2.5) and (2.6) that $\zeta_2 \neq \zeta_3$. On the other hand, (2.1) shows that ζ_0 is not a best proximity point of F. If $\delta(\zeta_0, F\zeta_1) = \delta(M, R)$, then this equality, together with (2.2) and $\zeta_0 \neq \zeta_1$, would imply that ζ_0 is a best periodic point of a prime period of two. This contradicts the hypothesis. Therefore,

$$\delta(\zeta_0, F\zeta_1) > \delta(M, R). \tag{2.7}$$

Combining (2.7) with (2.4), we obtain $\zeta_0 \neq \zeta_2$. Thus, the points ζ_0, ζ_1, and ζ_2 are pairwise distinct. Applying the perimeter proximal contraction condition to these points and the corresponding points ζ_1, ζ_2, and ζ_3 gives the following:

$$\delta(\zeta_1, \zeta_2) + \delta(\zeta_1, \zeta_3) + \delta(\zeta_2, \zeta_3) \leq \alpha\big[\delta(\zeta_0, \zeta_1) + \delta(\zeta_0, \zeta_2) + \delta(\zeta_1, \zeta_2)\big].$$

If $\delta(\zeta_3, F\zeta_3) = \delta(M, R)$, then ζ_3 is a best proximity point of F, and the existence part of the proof is complete. Therefore, we may assume that

$$\delta(\zeta_3, F\zeta_3) > \delta(M, R). \tag{2.8}$$

Since $F\zeta_3 \in F(M_0) \subseteq R_0$, there exists $\zeta_4 \in M_0$ such that

$$\delta(\zeta_4, F\zeta_3) = \delta(M, R). \tag{2.9}$$

It follows from (2.8) and (2.9) that $\zeta_3 \neq \zeta_4$. On the other hand, (2.3) shows that ζ_1 is not a best proximity point of F. If $\delta(\zeta_1, F\zeta_2) = \delta(M, R)$, then this equality, together with (2.4) and $\zeta_1 \neq \zeta_2$, would imply that ζ_1 is a best periodic point of a prime period of two. This contradicts the hypothesis. Therefore,

$$\delta(\zeta_1, F\zeta_2) > \delta(M, R). \tag{2.10}$$

Combining (2.10) with (2.6), we obtain $\zeta_1 \neq \zeta_3$. Thus, the points ζ_1, ζ_2, and ζ_3 are pairwise distinct. Applying the perimeter proximal contraction condition to these points and the corresponding points ζ_2, ζ_3, and ζ_4 gives the following:

$$\delta(\zeta_2, \zeta_3) + \delta(\zeta_2, \zeta_4) + \delta(\zeta_3, \zeta_4) \leq \alpha\big[\delta(\zeta_1, \zeta_2) + \delta(\zeta_1, \zeta_3) + \delta(\zeta_2, \zeta_3)\big].$$

Continuing in this manner, if a best proximity point is obtained at any stage, then the existence part of the proof is complete. Otherwise, by the preceding arguments, we construct a sequence $\{\zeta_s\}$ in M_0 satisfying

$$\delta(\zeta_{s+1}, F\zeta_s) = \delta(M, R) \tag{2.11}$$

such that, for every $s \in \mathbb{N}_0$, $\delta(\zeta_s, F\zeta_s) > \delta(M,R)$ and the points ζ_s, ζ_{s+1}, and ζ_{s+2} are pairwise distinct. Hence, the perimeter proximal contraction condition gives

$$\left\{ \begin{array}{c} \delta(\zeta_{s+1},\zeta_{s+2}) + \delta(\zeta_{s+1},\zeta_{s+3}) \\ +\delta(\zeta_{s+2},\zeta_{s+3}) \end{array} \right\} \leq \alpha \left\{ \begin{array}{c} \delta(\zeta_s,\zeta_{s+1}) + \delta(\zeta_s,\zeta_{s+2}) \\ +\delta(\zeta_{s+1},\zeta_{s+2}) \end{array} \right\} \tag{2.12}$$

for all $s \in \mathbb{N}_0$ where ζ_s, ζ_{s+1} and ζ_{s+2} are pairwise distinct. For each $s \in \mathbb{N}_0$, define

$$P_s = \delta(\zeta_s,\zeta_{s+1}) + \delta(\zeta_s,\zeta_{s+2}) + \delta(\zeta_{s+1},\zeta_{s+2}), \tag{2.13}$$

and let $r = P_0$. By (2.12), we have

$$P_{s+1} \leq \alpha P_s$$

for all $s \in \mathbb{N}_0$. Thus, by induction,

$$\delta(\zeta_s,\zeta_{s+1}) \leq P_s \leq \alpha^s r \tag{2.14}$$

for all $s \in \mathbb{N}$. Consequently, for all $s, p \in \mathbb{N}$, the triangle inequality gives the following:

$$\begin{aligned} \delta(\zeta_s,\zeta_{s+p}) &\leq \sum_{j=0}^{p-1} \delta(\zeta_{s+j},\zeta_{s+j+1}) \\ &\leq r\sum_{j=0}^{p-1} \alpha^{s+j} \\ &= \frac{r\alpha^s(1-\alpha^p)}{1-\alpha} \\ &\leq \frac{r\alpha^s}{1-\alpha}. \end{aligned} \tag{2.15}$$

Since $0 \leq \alpha < 1$, the final bound tends to zero as $s \to \infty$, independently of p. Therefore, $\{\zeta_s\}$ is a Cauchy sequence in M. Since M is F-best orbitally complete, there exists a point $\zeta^* \in M_0$ such that $\zeta_s \to \zeta^*$. Since $\zeta^* \in M_0$ and $F\zeta^* \in F(M_0) \subseteq R_0$, there exists $\omega \in M_0$ such that

$$\delta(\omega, F\zeta^*) = \delta(M,R). \tag{2.16}$$

If $\delta(\zeta^*, F\zeta^*) = \delta(M,R)$, then the existence part of the proof is complete. Therefore, we may assume that

$$\delta(\zeta^*, F\zeta^*) > \delta(M,R).$$

For the sequence $\{\zeta_s\}$ constructed above, define the following:

$$U = \{s \in \mathbb{N}_0 : \zeta_s = \zeta^*\}.$$

Suppose that U is infinite. Then, we can choose $s_k \in U$ such that $s_k \to \infty$ as $k \to \infty$. Since $\zeta_{s_k} = \zeta^*$ and

$$\delta(\zeta_{s_k+1}, F\zeta_{s_k}) = \delta(M,R),$$

we obtain

$$\begin{aligned} \delta(M,R) &< \delta(\zeta^*, F\zeta^*) \\ &= \delta(\zeta_{s_k}, F\zeta_{s_k}) \\ &\leq \delta(\zeta_{s_k},\zeta_{s_k+1}) + \delta(\zeta_{s_k+1}, F\zeta_{s_k}) \\ &= \delta(\zeta_{s_k},\zeta_{s_k+1}) + \delta(M,R). \end{aligned}$$

Since $\{\zeta_s\}$ converges to ζ^*, we have the following:

$$\lim_{k\to\infty} \delta(\zeta_{s_k}, \zeta_{s_k+1}) = 0.$$

Passing to the limit yields

$$\delta(\zeta^*, F\zeta^*) = \delta(M, R),$$

which contradicts our assumption. Hence, U is finite. Define the following:

$$s_0 = \begin{cases} 0, & U = \varnothing; \\ \max U, & U \neq \varnothing. \end{cases}$$

For every $s > s_0$, the points $\zeta_s, \zeta_{s+1}, \zeta^*$ are pairwise distinct. By (2.11) and $\delta(\omega, F\zeta^*) = \delta(M, R)$, we can apply the perimeter proximal contraction condition to these points and the corresponding points $\zeta_{s+1}, \zeta_{s+2}, \omega$ to obtain the following:

$$\delta(\zeta_{s+1}, \zeta_{s+2}) + \delta(\zeta_{s+1}, \omega) + \delta(\zeta_{s+2}, \omega) \leq \alpha\big[\delta(\zeta_s, \zeta_{s+1}) + \delta(\zeta_s, \zeta^*) + \delta(\zeta_{s+1}, \zeta^*)\big]. \tag{2.17}$$

Since $\zeta_s \to \zeta^*$, every term on the right-hand side tends to zero, while the left-hand side tends to $2\delta(\zeta^*, \omega)$. Hence, $\delta(\zeta^*, \omega) = 0$, so $\zeta^* = \omega$. Consequently,

$$\delta(\zeta^*, F\zeta^*) = \delta(\omega, F\zeta^*) = \delta(M, R). \tag{2.18}$$

Thus, ζ^* is a best proximity point of F, completing the existence part of the proof. Assume that there exist at least three distinct ζ, ϱ, and $\omega \in M$ such that

$$\delta(\zeta, F\zeta) = \delta(\varrho, F\varrho) = \delta(\omega, F\omega) = \delta(M, R).$$

Hence, from the definition of perimeter proximal contraction, we have

$$\delta(\zeta, \varrho) + \delta(\zeta, \omega) + \delta(\varrho, \omega) \leq \alpha(\delta(\zeta, \varrho) + \delta(\zeta, \omega) + \delta(\varrho, \omega)),$$

which is a contradiction. □

In the following example, the mapping F is not continuous. Therefore, Theorem 4 is not applicable. Consequently, this example is important in illustrating that, although Theorem 4 cannot be used, all the assumptions of Theorem 5 are satisfied, and hence, Theorem 5 remains applicable.

Example 2. *Let $\mathbb{R}^2$ be endowed with the Euclidean metric. Consider the subsets $M = \{(\zeta, 0) : 0 \leq \zeta \leq 1\} \cup \{(1, 2)\}$ and $R = \{(0, 1), (1, 1), (5, 1)\}$. Hence, we get $\delta(M, R) = 1, M_0 = \{(0, 0), (1, 0), (1, 2)\}$ and $R_0 = \{(0, 1), (1, 1)\}$. Now, define the mapping $F : M \to R$ by $F(1, 2) = (0, 1)$ and*

$$F(\zeta, 0) = \begin{cases} (0, 1), & \zeta = 0; \\ (1, 1), & \zeta = 1; \\ (5, 1), & \textit{otherwise.} \end{cases}$$

For every $\zeta \in M$ and every sequence $\{\zeta_s\} \in O_F(\zeta)$, the orbit relation implies that $\zeta_s \in M_0$ for all $s \geq 1$. Since

$$M_0 = \{(0, 0), (1, 0), (1, 2)\}$$

is finite, every Cauchy subsequence of such an orbit is eventually constant and therefore converges to a point of M_0. Hence, M is F-best orbitally complete. It is clear that there does not exist a point with a prime period of two and we get $F(M_0) \subseteq R_0$. Now, we will show that F is a perimeter proximal contraction. Let ζ_1, ζ_2, and ζ_3 be three pairwise distinct points and ς_1, ς_2, and ς_3 be points in M satisfying the condition

$$\begin{aligned}\delta(\varsigma_1, F\zeta_1) &= \delta(M, R),\\ \delta(\varsigma_2, F\zeta_2) &= \delta(M, R),\\ \delta(\varsigma_3, F\zeta_3) &= \delta(M, R).\end{aligned}$$

Then, we have two cases. Let $\alpha = \frac{2\sqrt{5}}{3+\sqrt{5}} \in [0, 1)$.

Case 1: Let $\varsigma_1 = \zeta_1 = \varsigma_2 = (0, 0), \varsigma_3 = \zeta_3 = (1, 0), \zeta_2 = (1, 2)$. Hence, we have the following:

$$\begin{aligned}\delta(\varsigma_1, \varsigma_2) + \delta(\varsigma_2, \varsigma_3) + \delta(\varsigma_1, \varsigma_3) &= 0 + 1 + 1\\ &= 2\\ &< \frac{2\sqrt{5}}{3+\sqrt{5}}\{\sqrt{5} + 1 + 2\}\\ &= \alpha[\delta(\zeta_1, \zeta_2) + \delta(\zeta_2, \zeta_3) + \delta(\zeta_1, \zeta_3)].\end{aligned}$$

Case 2: Let $\varsigma_1 = \zeta_1 = \varsigma_2 = (0, 0), \varsigma_3 = \zeta_2 = (1, 2), \zeta_3 = (1, 0)$. Hence, we have the following:

$$\begin{aligned}\delta(\varsigma_1, \varsigma_2) + \delta(\varsigma_2, \varsigma_3) + \delta(\varsigma_1, \varsigma_3) &= 0 + \sqrt{5} + \sqrt{5}\\ &= 2\sqrt{5}\\ &\leq \frac{2\sqrt{5}}{3+\sqrt{5}}\{1 + 2 + \sqrt{5}\}\\ &= \alpha[\delta(\zeta_1, \zeta_2) + \delta(\zeta_2, \zeta_3) + \delta(\zeta_1, \zeta_3)].\end{aligned}$$

Then, F is a perimeter proximal contraction. Hence, all the hypotheses of Theorem 5 are satisfied. Therefore, F has exactly two best proximity points, namely $(0, 0)$ and $(1, 0)$. Moreover, F is not continuous. Indeed, let $\{(\zeta_s, 0)\} \subset M$ be any sequence such that $\zeta_s \to 1$ and $\zeta_s \neq 1$ for all $s \in \mathbb{N}$. Then, $0 < \zeta_s < 1$ for all sufficiently large s. Therefore,

$$F(\zeta_s, 0) = (5, 1)$$

for all sufficiently large s, and hence,

$$\lim_{s\to\infty} F(\zeta_s, 0) = (5, 1).$$

On the other hand, $F(1, 0) = (1, 1)$. Therefore,

$$\lim_{s\to\infty} F(\zeta_s, 0) \neq F(1, 0),$$

which shows that F is not continuous at $(1, 0)$. On the other hand, if we take $\varsigma_1 = \zeta_1 = (0, 0), \varsigma_2 = \zeta_2 = (1, 0)$, then

$$\delta(\varsigma_1, \varsigma_2) = 1 = \delta(\zeta_1, \zeta_2).$$

Hence, F is not a proximal contraction.

Now, we present the following definition.

Definition 9. *Let* (Π, δ) *be a metric space. The pair* (M, R) *is said to satisfy the perimeter weak P-property if*

$$\begin{aligned}\delta(\zeta_1, \varrho_1) &= \delta(M, R)\\ \delta(\zeta_2, \varrho_2) &= \delta(M, R)\\ \delta(\zeta_3, \varrho_3) &= \delta(M, R)\end{aligned}$$

implies

$$\delta(\zeta_1, \zeta_2) + \delta(\zeta_1, \zeta_3) + \delta(\zeta_2, \zeta_3) \leq \delta(\varrho_1, \varrho_2) + \delta(\varrho_1, \varrho_3) + \delta(\varrho_2, \varrho_3) \tag{2.19}$$

for all distinct points $\zeta_1,\ \zeta_2,\ \zeta_3 \in M$ and for all $\varrho_1,\ \varrho_2,\ \varrho_3 \in R$.

The distinctness requirement only applies to the points ζ_1, ζ_2, and $\zeta_3 \in M$. The corresponding points $\varrho_1,\ \varrho_2,\ \varrho_3 \in R$ are not required to be pairwise distinct. It is clear that if the pair (M, R) has the weak P-property, then the pair has the perimeter weak P-property. The converse may not be true. The following example shows this fact.

Example 3. *Let* $(\mathbb{R}^2, \delta)$ *be the Euclidean metric space, and* $M = \{(-1, 0), (5, 2), (-5, 1)\}$, $R = \{(-3, 0), (5, 0), (-5, 3)\}$. *Then, we get* $\delta(M, R) = 2$. *Let* ζ_1, ζ_2, *and* ζ_3 *be distinct points* $\in M$ *and* ϱ_1, ϱ_2, *and* ϱ_3 *be points* $\in R$ *satisfying the equations*

$$\begin{aligned}\delta(\zeta_1, \varrho_1) &= \delta(M, R)\\ \delta(\zeta_2, \varrho_2) &= \delta(M, R)\\ \delta(\zeta_3, \varrho_3) &= \delta(M, R)\end{aligned}$$

Each point of M has a unique corresponding point of R at a distance of $\delta(M, R) = 2$. Thus, up to a simultaneous permutation of the indices, we have $\zeta_1 = (-1, 0), \zeta_2 = (5, 2), \zeta_3 = (-5, 1)$, and $\varrho_1 = (-3, 0), \varrho_2 = (5, 0), \varrho_3 = (-5, 3)$.

$$\begin{aligned}\delta(\zeta_1, \zeta_2) + \delta(\zeta_1, \zeta_3) + \delta(\zeta_2, \zeta_3) &= \sqrt{40} + \sqrt{17} + \sqrt{101}\\ &\leq 8 + \sqrt{13} + \sqrt{109}\\ &= \delta(\varrho_1, \varrho_2) + \delta(\varrho_1, \varrho_3) + \delta(\varrho_2, \varrho_3).\end{aligned}$$

Since both perimeter sums are invariant under simultaneous permutations of the indices, the inequality holds for every admissible triple. Therefore, the pair (M, R) has the perimeter weak P-property. On the other hand, it does not have the generalized weak P-property. If we take $\zeta_1 = (-1, 0), \zeta_2 = (-5, 1)$ and $\varrho_1 = (-3, 0), \varrho_2 = (-5, 3)$ satisfying

$$\begin{aligned}\delta(\zeta_1, \varrho_1) &= \delta(M, R),\\ \delta(\zeta_2, \varrho_2) &= \delta(M, R);\end{aligned}$$

then, we get

$$\delta(\zeta_1, \zeta_2) = \sqrt{17} > \sqrt{13} = \delta(\varrho_1, \varrho_2).$$

Definition 10. *Let (Π, δ) be a metric space and $F : M \to R$ be a mapping. Then, we say that F is a non-self perimeter α-contraction mapping, if there exists a constant $\alpha \in [0, 1)$ such that*

$$\left\{ \begin{array}{c} \delta(F\zeta_1, F\zeta_2) + \delta(F\zeta_1, F\zeta_3) \\ +\delta(F\zeta_2, F\zeta_3) \end{array} \right\} \leq \alpha \left\{ \begin{array}{c} \delta(\zeta_1, \zeta_2) + \delta(\zeta_1, \zeta_3) \\ +\delta(\zeta_2, \zeta_3) \end{array} \right\} \tag{2.20}$$

for all three pairwise distinct points ζ_1, ζ_2, and $\zeta_3 \in M$.

It is clear that every non-self α-contraction mapping is a non-self perimeter α-contraction mapping. The converse may not be true.

Example 4. *Let $\Pi = \{a, b, c, p, q, r\}$, and define a function $\delta : \Pi \times \Pi \to \mathbb{R}$ as follows:*

(i) $\delta(\zeta, \zeta) = 0$ for all $\zeta \in \Pi$;
(ii) $\delta(\zeta, \varrho) = \delta(\varrho, \zeta)$ for all $\zeta, \varrho \in \Pi$;
(iii)

$$\delta(a, b) = \delta(p, r) = 1,$$
$$\delta(a, c) = \delta(b, c) = 100,$$
$$\delta(p, q) = \delta(q, r) = 50;$$

(iv) for all remaining pairs,

$$\delta(\zeta, \varrho) = 1000.$$

It is clear that (Π, δ) is a metric space. Let $M = \{a, b, c\}$ and $R = \{p, q, r\}$. Define a mapping $F : M \to R$ by

$$F(a) = p, \qquad F(b) = q, \qquad F(c) = r.$$

Hence, F is a nonself perimeter α-contraction. Indeed, for $\alpha = \frac{2}{3}$, we get the following:

$$\begin{aligned} \delta(Fa, Fb) + \delta(Fb, Fc) + \delta(Fa, Fc) &= \delta(p, q) + \delta(q, r) + \delta(p, r) \\ &= 50 + 50 + 1 \\ &\leq \frac{2}{3}(1 + 100 + 100) \\ &= \alpha(\delta(a, b) + \delta(a, c) + \delta(b, c)). \end{aligned}$$

On the other hand, F is not a non-self α-contraction. If we take $\zeta = a$ and $\varrho = b$, then we have

$$\begin{aligned} \delta(Fa, Fb) &= 50 \\ &\leq \alpha \cdot 1 \\ &= \alpha\delta(a, b), \end{aligned}$$

which is impossible for all $\alpha \in [0, 1)$.

Theorem 6. *Let $F : M \to R$ be a mapping on a metric space (Π, δ). Assume that $M_0 \neq \emptyset$, $|M_0| \geq 3$, and M is F-best orbitally complete and the pair (M, R) has the perimeter weak P-property. Suppose that F is a non-self perimeter α-contraction mapping satisfying $F(M_0) \subseteq R_0$. If F does not possess best periodic point of a prime period of two, then F has a best proximity point in M. The number of best proximity points is at most two.*

Proof. Let $\zeta_0 \in M_0$ be an arbitrary point. As in the proof of Theorem 5, we can construct a sequence $\{\zeta_s\}$ satisfying

$$\delta(\zeta_{s+1}, F\zeta_s) = \delta(M, R) \tag{2.21}$$

for every $s \in \mathbb{N}_0$, $\delta(\zeta_s, F\zeta_s) > \delta(M, R)$ and the points $\zeta_s, \zeta_{s+1}, \zeta_{s+2}$ are pairwise distinct. We apply the perimeter weak P-property to the pairwise distinct points $\zeta_{s+1}, \zeta_{s+2}, \zeta_{s+3} \in M$ and the corresponding points $F\zeta_s, F\zeta_{s+1}, F\zeta_{s+2} \in R$. By (2.21), these points satisfy the following:

$$\delta(\zeta_{s+i}, F\zeta_{s+i-1}) = \delta(M, R), \qquad i = 1, 2, 3.$$

Therefore,

$$\delta(\zeta_{s+1}, \zeta_{s+2}) + \delta(\zeta_{s+1}, \zeta_{s+3}) + \delta(\zeta_{s+2}, \zeta_{s+3}) \leq \delta(F\zeta_s, F\zeta_{s+1}) + \delta(F\zeta_s, F\zeta_{s+2}) + \delta(F\zeta_{s+1}, F\zeta_{s+2}) \tag{2.22}$$

for all $s \in \mathbb{N}_0$. Hence, from 2.22 and 2.20, for all $s \in \mathbb{N}$, we have the following:

$$\begin{aligned}
\delta(\zeta_{s+1}, \zeta_{s+2}) &\leq \delta(\zeta_{s+1}, \zeta_{s+2}) + \delta(\zeta_{s+1}, \zeta_{s+3}) + \delta(\zeta_{s+2}, \zeta_{s+3}) \\
&\leq \delta(F\zeta_s, F\zeta_{s+1}) + \delta(F\zeta_s, F\zeta_{s+2}) + \delta(F\zeta_{s+1}, F\zeta_{s+2}) \\
&\leq \alpha(\delta(\zeta_s, \zeta_{s+1}) + \delta(\zeta_s, \zeta_{s+2}) + \delta(\zeta_{s+1}, \zeta_{s+2})) \\
&\leq \alpha(\delta(F\zeta_{s-1}, F\zeta_s) + \delta(F\zeta_{s-1}, F\zeta_{s+1}) + \delta(F\zeta_s, F\zeta_{s+1})) \\
&\leq \alpha^2(\delta(\zeta_{s-1}, \zeta_s) + \delta(\zeta_{s-1}, \zeta_{s+1}) + \delta(\zeta_s, \zeta_{s+1})) \\
&\quad \vdots \\
&\leq \alpha^{s+1} r,
\end{aligned}$$

where $r = \delta(\zeta_0, \zeta_1) + \delta(\zeta_0, \zeta_2) + \delta(\zeta_1, \zeta_2)$. Hence, as in the proof of Theorem 5, we can show that the sequence $\{\zeta_s\}$ is a Cauchy sequence in M. Since M is F-best orbitally complete, there exists a point $\zeta^* \in M_0$ such that $\zeta_s \to \zeta^*$. Since $\zeta^* \in M_0$ and $F\zeta^* \in F(M_0) \subseteq R_0$, there exists $\omega \in M_0$ such that

$$\delta(\omega, F\zeta^*) = \delta(M, R). \tag{2.23}$$

If $\delta(\zeta^*, F\zeta^*) = \delta(M, R)$, then the existence part of the proof is complete. Therefore, we may assume that

$$\delta(\zeta^*, F\zeta^*) > \delta(M, R).$$

For the sequence $\{\zeta_s\}$ constructed above, define

$$U = \{s \in \mathbb{N}_0 : \zeta_s = \zeta^*\}.$$

It can be shown, as in the proof of Theorem 5, that U is finite. Then, we have $\zeta_s \neq \zeta^*$ for all $s > s_0$ where

$$s_0 = \begin{cases} 0, & U = \emptyset; \\ \max U, & \text{otherwise.} \end{cases}$$

Since $\delta(\zeta^*, F\zeta^*) > \delta(M, R)$ and $\delta(\omega, F\zeta^*) = \delta(M, R)$, we have $\omega \neq \zeta^*$. As $\zeta_s \to \zeta^*$, there exists $s_1 \geq s_0$ such that $\zeta_s \neq \omega$ whenever $s > s_1$. Thus, for every $s > s_1$, both triples $(\zeta_s, \zeta_{s+1}, \zeta^*)$ and $(\zeta_{s+1}, \zeta_{s+2}, \omega)$

consist of pairwise distinct points. Using the orbit relations and $\delta(\omega, F\zeta^*) = \delta(M,R)$, the perimeter weak P-property and the non-self perimeter contraction condition give the following:

$$\begin{aligned}&\delta(\zeta_{s+1},\zeta_{s+2}) + \delta(\zeta_{s+1},\omega) + \delta(\zeta_{s+2},\omega)\\ &\quad\le \delta(F\zeta_s, F\zeta_{s+1}) + \delta(F\zeta_s, F\zeta^*) + \delta(F\zeta_{s+1}, F\zeta^*)\\ &\quad\le \alpha\big[\delta(\zeta_s,\zeta_{s+1}) + \delta(\zeta_s,\zeta^*) + \delta(\zeta_{s+1},\zeta^*)\big].\end{aligned} \tag{2.24}$$

Since $\zeta_s \to \zeta^*$, the final bound tends to zero, while the left-hand side tends to $2\delta(\zeta^*,\omega)$. Hence, $\zeta^* = \omega$, and therefore

$$\delta(\zeta^*, F\zeta^*) = \delta(\omega, F\zeta^*) = \delta(M,R). \tag{2.25}$$

Thus, ζ^* is a best proximity point of F, completing the existence part of the proof. Assume that there exist at least three distinct ζ, ϱ, and $\omega \in M$ such that

$$\begin{aligned}\delta(\zeta, F\zeta) &= \delta(M,R),\\ \delta(\varrho, F\varrho) &= \delta(M,R),\\ \delta(\omega, F\omega) &= \delta(M,R).\end{aligned}$$

Hence, by using the perimeter weak P-property and from 2.20, we get

$$\begin{aligned}\left\{\begin{array}{c}\delta(\zeta,\varrho)+\delta(\zeta,\omega)\\ +\delta(\varrho,\omega)\end{array}\right\} &\le \left\{\begin{array}{c}\delta(F\zeta,F\varrho)+\delta(F\zeta,F\omega)\\ +\delta(F\varrho,F\omega)\end{array}\right\}\\ &\le \alpha\left\{\begin{array}{c}\delta(\zeta,\varrho)+\delta(\zeta,\omega)\\ +\delta(\varrho,\omega)\end{array}\right\},\end{aligned}$$

and we have

$$\delta(\zeta,\varrho) + \delta(\zeta,\omega) + \delta(\varrho,\omega) \le \alpha(\delta(\zeta,\varrho) + \delta(\zeta,\omega) + \delta(\varrho,\omega)),$$

which is a contradiction. □

Example 5. *Let $\mathbb{R}^2$ be endowed with the taxicab metric, and consider*

$$M = \{(0,0),(0,1),(0,2)\}, \qquad R = \{(10,0),(10,1),(10,2)\}.$$

Then, we have $\delta(M,R) = 10$, $M_0 = M$, and $R_0 = R$. The unique correspondence at a distance of $\delta(M,R)$ is $(0,i) \leftrightarrow (10,i)$. As this correspondence preserves pairwise distances, (M,R) has the perimeter weak P-property. Define $F : M \to R$ by the following:

$$F(0,0) = (10,2), \qquad F(0,1) = F(0,2) = (10,1).$$

Then, $F(M_0) \subseteq R_0$. Starting from any point $\zeta_0 \in M$, the proximal orbit reaches $(0,1)$ after at most two steps and remains constant thereafter. Consequently, M is F-best orbitally complete, and F has no best periodic point of a prime period of two. For any three pairwise distinct points ζ_1, ζ_2, and $\zeta_3 \in M$, we have the following:

$$\begin{aligned}&\delta(F\zeta_1, F\zeta_2) + \delta(F\zeta_1, F\zeta_3) + \delta(F\zeta_2, F\zeta_3)\\ &\quad= 1 + 1 + 0\\ &\quad= 2\\ &\quad= \frac{1}{2}(1 + 2 + 1)\\ &\quad= \frac{1}{2}\big[\delta(\zeta_1,\zeta_2) + \delta(\zeta_1,\zeta_3) + \delta(\zeta_2,\zeta_3)\big].\end{aligned}$$

Hence, F is a non-self perimeter α-contraction with $\alpha = \frac{1}{2}$. All the hypotheses of Theorem 6 are satisfied. Therefore, $(0, 1)$ is a best proximity point of F. On the other hand,

$$\delta(F(0,0),F(0,1)) = 1 = \delta((0,0),(0,1)),$$

so F is not a non-self contraction with any constant in $[0, 1)$.

3. Conclusions

In this paper, we have investigated best proximity point results for non-self mappings satisfying three-point perimeter-type contractive conditions. The main purpose was to extend the existing results in the literature while weakening some of their assumptions.

First, we revisited the class of perimeter proximal contraction mappings and an existing best proximity point theorem established under a continuity assumption. We obtained a corresponding best proximity point result without requiring the mapping F to be continuous. An example involving a discontinuous mapping was presented to illustrate the applicability of the obtained result in a situation where the previously known continuity-based theorem cannot be applied.

In the second part of the paper, we introduced the notion of the perimeter weak P-property. We showed that this property is weaker than the generalized weak P-property and provided an example demonstrating that the converse implication does not hold. Then, we introduced non-self perimeter α-contractions and established a best proximity point theorem under the perimeter weak P-property. The corresponding examples show that the newly introduced class of mappings properly contains the class of non-self α-contractions and that the new geometric condition allows the treatment of cases not covered by the stronger classical conditions.

Author contributions

Mustafa Aslantaş: Conceptualization, methodology, formal analysis, investigation, writing–original draft, writing–review & editing; Naisan Khalaf Mosah Mosah: Conceptualization, methodology, formal analysis, investigation; Hakan Sahin: Conceptualization, investigation, writing–original draft, writing–review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

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