AIMS Mathematics, 11(11): 39075–39101.
DOI: 10.3934/math.20261505
Received: 08 September 2026
Revised: 24 October 2026
Accepted: 05 November 2026
Published: 23 November 2026

https://www.aimspress.com/journal/Math

Research article

The three largest Euler Sombor indices of bicyclic graphs with a fixed matching number

Wei Fang[1,*] **and Wangxin Huang**[2]

[1] Foundational Courses Department, Nanjing University of Industry Technology, Nanjing 210023, China

[2] School of Aeronautical Engineering, Nanjing University of Industry Technology, Nanjing 210023, China

* **Correspondence:** Email: fangwei@niit.edu.cn; Tel: 18856668600.

Abstract: For a graph G, the Euler Sombor index is defined by

$$EUS(G) = \sum_{uv \in E(G)} \sqrt{d(u)^2 + d(v)^2 + d(u)d(v)}.$$

Let $\mathcal{B}_{n,\mu}$ be the class of bicyclic graphs of order $n \geq 7$ with the matching number $2 \leq \mu \leq \lfloor n/2 \rfloor$. We determined the second and third largest values of the Euler Sombor index on $\mathcal{B}_{n,\mu}$ and proved that each extremal value is attained by a unique graph. The proof was organized by a stratification according to the maximum degree. We identified the leading graphs in the first lower strata, excluded the next finite strata by explicit comparison certificates, and controlled the remaining lower-degree tail by a global first Zagreb index estimate. For $\mu = 2$, the second and third extremal graphs are Π_n and Q_n; for $\mu = 3$, they are $T_{n,3}$ and R_n; and for $\mu \geq 4$, they are $W_{n,\mu}$ and $T_{n,\mu}$, respectively (all of these families are defined in Section 3). In the last regime, the gap between the second and third largest values equals $\sqrt{13} + 2\sqrt{3} - \sqrt{19} - \sqrt{7}$, and hence is independent of both n and μ. For $\mu \geq 3$, combining these results with the known maximum yields the complete top-three ordering. For $\mu = 2$, we additionally proved that $A_{n,2}$ is the unique maximizer. Consequently, the three largest Euler Sombor indices, together with their unique extremal graphs, were completely determined for every admissible pair (n, μ).

Keywords: Euler Sombor index; bicyclic graph; matching number; extremal graph; topological index; degree-based index
Mathematics Subject Classification: 05C09, 05C35, 05C70, 05C92

1. Introduction

All graphs in this paper are finite, simple, and connected. For a graph G, we write $V(G)$ and $E(G)$ for its vertex and edge sets, $n = |V(G)|$, $m = |E(G)|$, $d(u)$ for the degree of u, and $\Delta = \Delta(G)$ for the maximum degree. A vertex of degree Δ is called dominant. We use $N_G(u)$ and $N_G[u]$ for the open and closed neighborhoods of u. A vertex of degree one is pendant; the incident edge is a pendant edge. A support vertex is a vertex adjacent to a pendant vertex. A pendant path at v is a path starting at v whose internal vertices have a degree of two and whose terminal vertex is pendant.

A matching is a set of pairwise non-adjacent edges, and its maximum cardinality is the matching number $\mu = \mu(G)$. Relative to a matching M, vertices incident with M are called M-saturated and the others M-unsaturated. A maximum matching leaves exactly $n - 2\mu$ vertices unsaturated, so $2\mu \leq n$. We write P_t for the path on t vertices, C_t for the cycle on $t \geq 3$ vertices, K_t for the complete graph on t vertices, and $K_{1,t}$ for the star with t leaves. Thus P_3 has three vertices and two edges, whereas K_2 consists of one edge and its two endpoints.

A connected graph with $m = n + c - 1$ is c-cyclic; bicyclic graphs are the case $c = 2$. The integer $c = m - n + 1$ is the cyclomatic number, that is, the dimension of the cycle space of a connected graph. The base of a bicyclic graph is obtained by repeatedly deleting pendant vertices. It is either a theta graph $\Theta(p, q, r)$, formed by three internally disjoint paths joining the same two end vertices, where p, q, r are their lengths measured in edges (at most one length is one in a simple graph), or a dumbbell, formed by two cycles joined by a path or sharing one vertex. The dumbbell consisting of two triangles with one common vertex is the bowtie B^*; the common vertex is its center.

The Euler Sombor index is the degree-based invariant

$$EUS(G) = \sum_{uv \in E(G)} \sqrt{d(u)^2 + d(v)^2 + d(u)d(v)}. \tag{1.1}$$

It belongs to the Sombor family of degree-based indices [1,2] and was systematically studied by Tang, Li, and Deng [3]. The chemical applicability of Sombor indices was studied by Redžepović [4]; for the Euler Sombor index, molecular-tree bounds and applications were investigated by Zhang et al. [5]. In a chemical graph, a matching is a set of bonds no two of which share an atom; thus fixing μ imposes a natural bond-disjointness constraint while allowing branching and the two cycles to vary. Accordingly, $\mathcal{B}_{n,\mu}$ models two-cycle molecular skeletons with a prescribed maximum number of pairwise non-incident bonds, and the extremal rankings below identify the degree-connectivity patterns with the largest Euler Sombor index under this constraint.

Recent studies have derived Euler Sombor bounds for trees and chemical graphs [6], characterized extremal unicyclic graphs [7], and determined extremal values for unicyclic and bicyclic graphs without a matching constraint [8]. Other work has established extremal results for tricyclic graphs [9,10], with related results for maximal outerplanar graphs [11]. Further studies have considered trees with prescribed parameters [12] and discussed open problems for the index [13]. For bicyclic graphs with a prescribed matching number, Ali et al. [14] determined the first rank for $\mu \geq 3$ as a specialization of their bond incident degree index theorem. We write $\mathcal{B}_{n,\mu}$ for the class of bicyclic graphs of order n and matching number μ.

Theorem 1.1 (Known first-rank theorem; Euler–Sombor specialization of Ali et al. [14])**.** *Let $n \geq 7$, and $3 \leq \mu \leq \lfloor n/2 \rfloor$. The maximum Euler Sombor index over $\mathcal{B}_{n,\mu}$ is attained uniquely by the graph $A_{n,\mu}$*

defined in Definition 3.1(1).

Remark 1.2. A bond incident degree (BID) weight is a symmetric function $\varphi : \mathbb{N}_{>0}^2 \to \mathbb{R}$ assigning weight $\varphi(d(u), d(v))$ to an edge uv; the associated BID index is $\sum_{uv\in E(G)} \varphi(d(u), d(v))$. Ali et al. formulated their result for fixed-size bicyclic graphs and a general BID weight. Since a bicyclic graph has $m = n + 1$ and the choice $\varphi(x, y) = \sqrt{x^2 + y^2 + xy}$ gives EUS, Theorem 1.1 is precisely the $\mu \geq 3$ Euler–Sombor specialization of their result. We do not use their conditional $\mu = 2$ statement as an input; the first rank for $\mu = 2$ is verified directly in the proof of Theorem 5.1.

The problem addressed here is the next two ranks. Related parameter-constrained extremal problems have been studied for trees with prescribed parameters [12], for trees and unicyclic graphs with a prescribed diameter [15], and for unicyclic graphs with a prescribed diameter [16]. We determine, for every admissible n and μ, the second and third largest EUS values on $\mathcal{B}_{n,\mu}$ and prove uniqueness of the corresponding graphs. Together with the known first rank for $\mu \geq 3$ and the direct $\mu = 2$ comparison proved below, this yields the complete top-three ordering stated in Theorem 5.1. A notable feature is that, for $\mu \geq 4$, the gap between the second and third values is the constant

$$\sqrt{13} + 2\sqrt{3} - \sqrt{19} - \sqrt{7} = 0.065002636\ldots, \tag{1.2}$$

independent of both n and μ.

Section 2 collects the analytic facts used later, Section 3 defines the extremal graphs, Section 4 develops the maximum-degree stratification, and Section 5 proves the main result.

2. Preliminaries

For $x, y > 0$, set

$$f(x, y) = \sqrt{x^2 + y^2 + xy}, \tag{2.1}$$

so that $EUS(G) = \sum_{uv\in E(G)} f(d(u), d(v))$ by (1.1). The function f is symmetric and positively homogeneous of degree one, and $f(x, x) = \sqrt{3}\, x$. Table 1 lists the values needed below. Decimal approximations are included only to indicate the scale of selected gaps; every strict comparison used in the proof is established analytically or by exact rational certificates.

Table 1. Selected values of $f(x, y)$.

$f(1,1) = \sqrt{3} = 1.732050808$	$f(1,2) = \sqrt{7} = 2.645751311$
$f(1,3) = \sqrt{13} = 3.605551275$	$f(1,4) = \sqrt{21} = 4.582575695$
$f(2,2) = 2\sqrt{3} = 3.464101615$	$f(2,3) = \sqrt{19} = 4.358898944$
$f(2,4) = 2\sqrt{7} = 5.291502622$	$f(3,3) = 3\sqrt{3} = 5.196152423$
$f(3,4) = \sqrt{37} = 6.082762530$	$f(4,4) = 4\sqrt{3} = 6.928203230$

We shall use the following elementary properties.

Lemma 2.1. *For $x, y > 0$,*

$$\frac{\partial f}{\partial x}(x, y) = \frac{2x + y}{2f(x, y)} > 0, \tag{2.2}$$

so f is strictly increasing in each variable. Moreover,

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{3y^2}{4f(x,y)^3} > 0, \qquad \frac{\partial^2 f}{\partial x\,\partial y}(x,y) = -\frac{3xy}{4f(x,y)^3} < 0, \tag{2.3}$$

so f is strictly convex in each variable and its mixed second derivative is negative.

Proof. Differentiating (2.1) gives (2.2) and the numerator is positive. A second differentiation gives $\partial^2 f/\partial x^2 = (2f^2 - (2x+y)^2/2)/(2f^3)$, and $2f^2 - (2x+y)^2/2 = \frac{3}{2}y^2$, which is the first formula in (2.3). Similarly, $\partial^2 f/\partial x\partial y = (2f^2 \cdot 0 - (2x+y)(x+2y)/2 + f^2)/(2f^3)$, which, after simplification, equals $-3xy/(4f^3)$. □

For $t \geq 1$ and $c \geq 1$, set

$$g(t,c) = f(t+1,c) - f(t,c), \tag{2.4}$$

the gain in the contribution of an edge when the degree of one of its ends increases from t to $t+1$ while the other end keeps degree c.

Lemma 2.2. *For all $t \geq 1$ and $c \geq 1$, we have $\frac{1}{2} < g(t,c) < 1$. Moreover, $g(t,c)$ is strictly increasing in t and strictly decreasing in c, and*

$$\lim_{t\to\infty} g(t,c) = 1, \qquad g(t,1) > g(t,2) > g(t,3) > \cdots. \tag{2.5}$$

Proof. By the mean value theorem, $g(t,c) = \partial f/\partial x(\xi,c)$ for some $\xi \in (t,t+1)$, and by (2.2), this equals $(2\xi + c)/(2f(\xi,c))$. Since $4f(\xi,c)^2 - (2\xi+c)^2 = 3c^2 > 0$, we get $\partial f/\partial x < 1$, and since $(2\xi+c)^2 - f(\xi,c)^2 = 3\xi^2 + 3\xi c > 0$, we get $\partial f/\partial x > \frac{1}{2}$. Monotonicity in t is the convexity (2.3) in the first variable, and monotonicity in c is the negativity of the mixed derivative in (2.3). Finally, $f(t+1,c) - f(t,c) \to 1$ as $t \to \infty$ because $f(t,c) = t\sqrt{1 + c/t + c^2/t^2} = t + c/2 + O(1/t)$. □

3. The extremal graphs

Throughout this section, n and μ are integers with $n \geq 7$, and $2 \leq \mu \leq \lfloor n/2 \rfloor$, and we write

$$\begin{aligned} k &= n - 2\mu + 1, \qquad D^* = \min\{n-1,\ n-\mu+2\}, \\ D_1 &= D^* - 1, \qquad D = n - \mu + 1. \end{aligned} \tag{3.1}$$

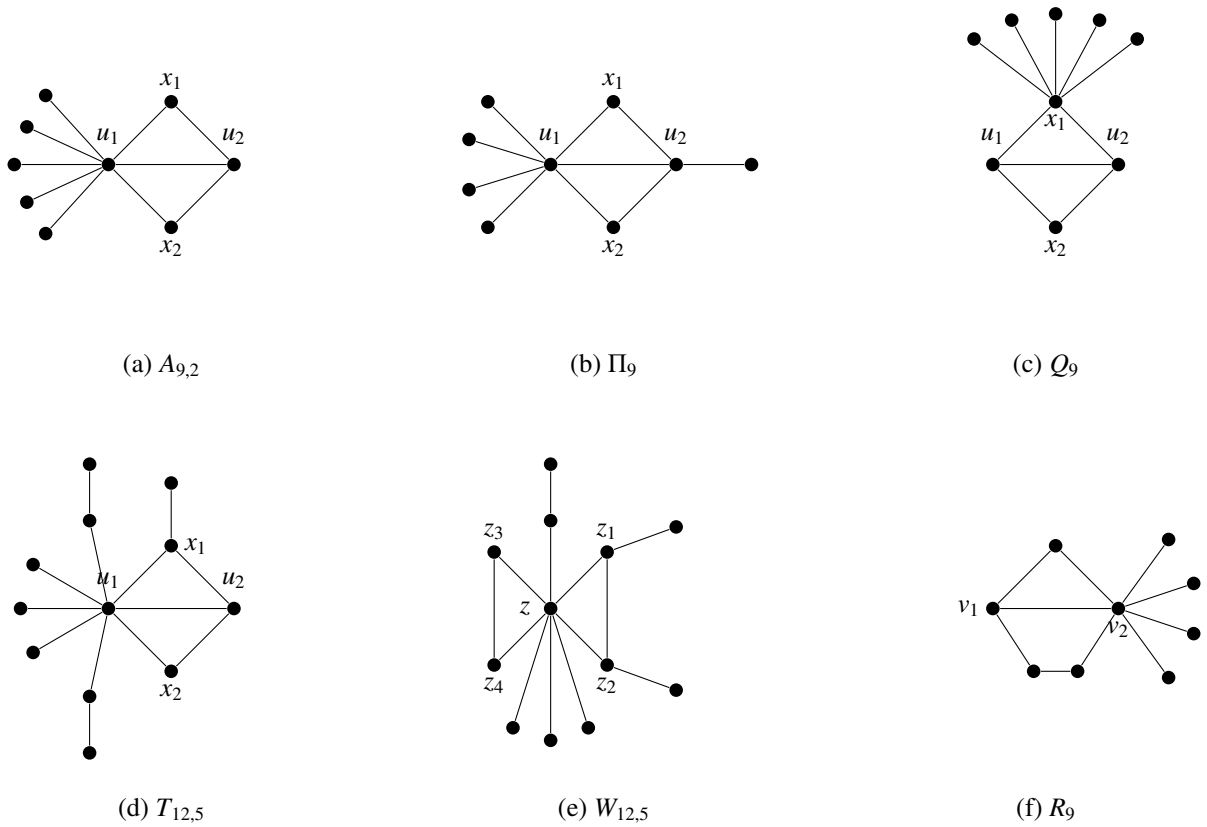
Thus $k \geq 1$ is the number of M-unsaturated vertices plus one and D^* is the largest degree available in $\mathcal{B}_{n,\mu}$. The symbol D_1 denotes the next maximum-degree stratum. For $\mu \geq 3$, we have $D_1 = D = n-\mu+1$, whereas for $\mu = 2$, one has $D^* = n-1$, $D_1 = n-2$, and $D = n-1$.

Definition 3.1. The notation $\Theta(1,2,2)$ means that the three paths between its two branch vertices have one, two, and two edges, respectively; similarly, $\Theta(1,2,3)$ uses lengths one, two, and three. Let $\Theta = \Theta(1,2,2)$ have vertices of degree three, u_1, u_2, and vertices of degree two, x_1, x_2. Let $\Theta' = \Theta(1,2,3)$ have degree three, v_1, v_2, and let B^* be the bowtie with center z, with its triangles being zz_1z_2z and zz_3z_4z. In all constructions below, $k = n - 2\mu + 1$ is the number of pendant edges attached at the specified dominant vertex.

(1) $A_{n,2}$ is obtained from Θ by attaching $n-4$ pendant edges at u_1; for $\mu \geq 3$, $A_{n,\mu}$ is obtained from B^* by attaching at z exactly $\mu - 3$ pendant paths of length two and k pendant edges.

(2) Π_n is obtained from Θ by attaching $n-5$ pendant edges at u_1 and one pendant edge at u_2.
(3) For $\mu \geq 3$, $T_{n,\mu}$ is obtained from Θ by attaching at u_1 exactly $\mu-3$ pendant paths of length two and k pendant edges, as well as one pendant edge at x_1.
(4) For $\mu \geq 4$, $W_{n,\mu}$ is obtained from B^* by attaching at z exactly $\mu-4$ pendant paths of length two and k pendant edges, as well as one pendant edge at each of z_1 and z_2.
(5) Q_n is obtained from Θ by attaching all $n-4$ pendant edges at x_1.
(6) R_n is obtained from Θ' by attaching $n-5$ pendant edges at v_2.

The representative graphs from Definition 3.1 are shown in Figure 1.



(a) $A_{9,2}$ (b) Π_9 (c) Q_9

(d) $T_{12,5}$ (e) $W_{12,5}$ (f) R_9

Figure 1. Representative extremal graphs from Definition 3.1. Each dot is a vertex and each solid segment is an edge. Panels (a)–(c) have $n = 9$, $\mu = 2$; panels (d)–(e) have $n = 12$, $\mu = 5$, and $k = 3$; panel (f) has $n = 9$, $\mu = 3$. The labeled core vertices are the attachment vertices used in the general constructions.

Lemma 3.2. *Each graph of Definition 3.1 is bicyclic of order n; the matching numbers are* $\mu(A_{n,\mu}) = \mu(T_{n,\mu}) = \mu(W_{n,\mu}) = \mu$, $\mu(\Pi_n) = \mu(Q_n) = 2$, *and* $\mu(R_n) = 3$. *The maximum degrees are* $\Delta(A_{n,\mu}) = D^*$, $\Delta(\Pi_n) = \Delta(Q_n) = D_1$, *and* $\Delta(R_n) = \Delta(T_{n,\mu}) = \Delta(W_{n,\mu}) = D$ *in their respective regimes. Their indices are*

$$EUS(A_{n,2}) = (n-4)f(n-1,1) + 2f(n-1,2) + f(n-1,3) + 2f(2,3), \tag{3.2}$$

$$EUS(A_{n,\mu}) = (\mu-3)f(1,2) + k\,f(D^*,1) + (\mu+1)f(D^*,2) + 2f(2,2), \tag{3.3}$$

$$EUS(\Pi_n) = (n-5)f(n-2,1) + 2f(n-2,2) + f(n-2,4) + f(1,4) + 2f(2,4), \tag{3.4}$$

$$EUS(T_{n,\mu}) = (\mu-3)f(1,2) + k\,f(D,1) + (\mu-2)f(D,2) + 2f(D,3) + f(1,3) + f(2,3) + f(3,3), \tag{3.5}$$

$$EUS(W_{n,\mu}) = (\mu-4)f(1,2) + k\,f(D,1) + (\mu-2)f(D,2) + 2f(D,3) + 2f(1,3) + f(2,2) + f(3,3), \tag{3.6}$$

$$EUS(Q_n) = (n-4)f(n-2,1) + 2f(n-2,3) + 2f(2,3) + f(3,3), \tag{3.7}$$

$$EUS(R_n) = (n-5)f(n-2,1) + 2f(n-2,2) + f(n-2,3) + f(2,2) + 2f(2,3). \tag{3.8}$$

Proof. Each graph is obtained from a theta graph or from the bowtie by attaching trees, so it is connected with $m = n+1$ and hence bicyclic; the vertex count is immediate from the construction. We verify the matching numbers and then read off the edge partitions.

For $A_{n,\mu}$ with $\mu \geq 3$, a maximum matching is obtained by taking the outer edge z_1z_2 of one triangle, the outer edge z_3z_4 of the other, the terminal edge of each of the $\mu-3$ pendant paths of length two, as well as one pendant edge at z; these $2+(\mu-3)+1 = \mu$ edges are pairwise non-adjacent. No larger matching exists: deleting z leaves a graph of matching number $\mu-1$, so $\mu(A_{n,\mu}) \leq 1+(\mu-1) = \mu$.

For $W_{n,\mu}$, take the terminal edges of the $\mu-4$ pendant paths, the two pendant edges at z_1, z_2, the outer edge z_3z_4, as well as one pendant edge at z. These μ edges are independent. After deleting z, the component on z_1, z_2, and their pendant neighbors has matching number two, the edge z_3z_4 contributes one, and the $\mu-4$ pendant-path components contribute $\mu-4$; hence $\mu(W_{n,\mu}) \leq 1+(\mu-1) = \mu$.

For $T_{n,\mu}$, take the terminal edges of the $\mu-3$ pendant paths, the pendant edge at x_1, the edge u_2x_2, and one pendant edge at u_1. They form a matching of size μ. Deleting u_1 leaves the $\mu-3$ terminal K_2 components together with a P_4 of matching number two, so $\mu(T_{n,\mu}) \leq 1+(\mu-1) = \mu$.

For $A_{n,2}$, deleting u_1 leaves a graph of matching number one, so every matching has a size of at most two; two independent edges are visible. For Π_n, the two vertices u_1, u_2 form a vertex cover, hence $\mu(\Pi_n) \leq 2$, and again two independent edges are visible. For Q_n, if a matching contains an edge incident with x_1, deleting x_1 leaves the triangle on u_1, u_2, x_2, whose matching number is one; if no edge incident with x_1 is used, the matching lies in that triangle. Thus $\mu(Q_n) \leq 2$, while a pendant edge at x_1 together with u_1u_2 gives a matching of size two. For R_n, a cover of size three is exhibited by $\{v_1, v_2\}$ together with the middle vertex of the path of length three, and three independent edges are visible.

The maximum degrees are as stated because, in $A_{n,\mu}$, the center z has a degree of $4+(\mu-3)+k = n-\mu+2$, while in $A_{n,2}$, the vertex u_1 has a degree of $3+(n-4) = n-1$. For Π_n and Q_n, the dominant degree is $n-2 = D_1$. For R_n, $T_{n,\mu}$, and $W_{n,\mu}$ (which occur only for $\mu \geq 3$), the dominant degree is $D = n-\mu+1 = D_1$.

It remains to list the edge partitions, writing $m_{i,j}$ for the number of edges joining a vertex of degree i to a vertex of degree j. In $A_{n,2}$, the vertex u_1 has a degree of $n-1$, u_2 has a degree of three, x_1, x_2 each have degree two, and the $n-4$ pendant vertices each have degree one, so $m_{1,n-1} = n-4$, $m_{2,n-1} = 2$, $m_{3,n-1} = 1$, and $m_{2,3} = 2$, which is (3.2). In $A_{n,\mu}$, with $\mu \geq 3$, the center has a degree of D^*, the four vertices $z_1, \ldots, z_4$ and the $\mu-3$ inner vertices of the pendant paths each have degrees of two, and the remaining vertices are pendant; hence $m_{1,D^*} = k$, $m_{2,D^*} = 4+(\mu-3) = \mu+1$, $m_{2,2} = 2$ from the two outer triangle edges and $m_{1,2} = \mu-3$ from the ends of the pendant paths, which is (3.3). The remaining five partitions are obtained in the same way and are recorded in Table 2. □

Here $k = n-2\mu+1$; for $\mu \geq 3$, $D^* = n-\mu+2$ and $D = n-\mu+1$. Entries not listed in the table are zero.

Table 2. Edge partitions of the extremal graphs.

Graph	Partition	Regime
$A_{n,2}$	$m_{1,n-1} = n-4$, $m_{2,n-1} = 2$, $m_{3,n-1} = 1$, $m_{2,3} = 2$	$\mu = 2$
$A_{n,\mu}$	$m_{1,D^*} = k$, $m_{2,D^*} = \mu+1$, $m_{1,2} = \mu-3$, $m_{2,2} = 2$	$\mu \geq 3$
Π_n	$m_{1,n-2} = n-5$, $m_{2,n-2} = 2$, $m_{4,n-2} = 1$, $m_{1,4} = 1$, $m_{2,4} = 2$	$\mu = 2$
$T_{n,\mu}$	$m_{1,D} = k$, $m_{2,D} = \mu-2$, $m_{3,D} = 2$, $m_{1,2} = \mu-3$, $m_{1,3} = 1$, $m_{2,3} = 1$, $m_{3,3} = 1$	$\mu \geq 3$
$W_{n,\mu}$	$m_{1,D} = k$, $m_{2,D} = \mu-2$, $m_{3,D} = 2$, $m_{1,2} = \mu-4$, $m_{1,3} = 2$, $m_{2,2} = 1$, $m_{3,3} = 1$	$\mu \geq 4$
Q_n	$m_{1,n-2} = n-4$, $m_{3,n-2} = 2$, $m_{2,3} = 2$, $m_{3,3} = 1$	$\mu = 2$
R_n	$m_{1,D} = n-5$, $m_{2,D} = 2$, $m_{3,D} = 1$, $m_{2,2} = 1$, $m_{2,3} = 2$	$\mu = 3$

4. Stratification by maximum degree

The whole argument rests on the following bound, which for trees and unicyclic graphs is classical and which we prove here in the form valid for every cyclomatic number $c = |E(G)| - |V(G)| + 1$ (defined in the Introduction).

Lemma 4.1. *Let G be a connected c-cyclic graph of order n with matching number μ. Then*

$$\Delta(G) \leq n - \mu + c. \tag{4.1}$$

In particular, $\Delta(G) \leq n - \mu + 2$ for every bicyclic graph. For $\mu \geq 3$, this bicyclic bound is sharp, while for $\mu = 2$, the sharp bound is the simple-graph bound $\Delta \leq n - 1$.

Proof. Let v be a dominant vertex, $\Delta = d(v)$, and let M be a maximum matching, $|M| = \mu$. Set $T = V(G) \setminus N_G[v]$, so $|T| = n - 1 - \Delta$. We classify the edges of M according to their position relative to $N_G[v]$. At most one edge of M is incident with v. Let s be the number of edges of M with both ends in $N_G(v)$, let j be the number with one end in $N_G(v)$ and one end in T, and let i be the number with both ends in T; every edge of M falls into exactly one of these four classes, so

$$\mu \leq 1 + s + j + i. \tag{4.2}$$

Each edge counted by s joins two neighbors of v and therefore lies on a triangle through v; these triangles are edge-disjoint because the edges counted by s are pairwise non-adjacent, so they contribute s independent cycles to the cycle space of G, where $s \leq c$. The edges counted by j and i cover $j + 2i$ distinct vertices of T, so $j + 2i \leq |T| = n - 1 - \Delta$, and, in particular, $j + i \leq n - 1 - \Delta$. Substituting into (4.2) gives $\mu \leq 1 + c + n - 1 - \Delta$, which is (4.1). For $c = 2$ and $\mu \geq 3$, sharpness is witnessed by $A_{n,\mu}$ of Definition 3.1, whose center has a degree of $n - \mu + 2$ by Lemma 3.2. If $\mu = 2$, (4.1) only gives $\Delta \leq n$; simplicity improves this to $\Delta \leq n - 1$, attained by $A_{n,2}$. □

The case of equality in (4.1) is extremely rigid: it forces every inequality used in the proof to be tight, and this pins the graph down completely.

Lemma 4.2. *Let $n \geq 7$, $3 \leq \mu \leq \lfloor n/2 \rfloor$, and $G \in \mathcal{B}_{n,\mu}$. Then $\Delta(G) = n - \mu + 2$ if and only if $G \cong A_{n,\mu}$. For $\mu = 2$ and $n \geq 6$, one has $\Delta(G) = n - 1$ if and only if $G \cong A_{n,2}$.*

Proof. Sufficiency is Lemma 3.2. Conversely, suppose $\Delta = n-\mu+2$ with $\mu \geq 3$, and retain the notation of the proof of Lemma 4.1. Equality in (4.1) forces equality throughout, hence

$$\mu = 1 + s + j + i, \qquad s = 2, \qquad j + 2i = j + i = n - 1 - \Delta. \tag{4.3}$$

From $j + 2i = j + i$, we get $i = 0$, so no edge of M lies inside T, and $j = n - 1 - \Delta = \mu - 3$. Moreover, $s = 2$ means that M contains two non-adjacent edges z_1z_2 and z_3z_4 inside $N_G(v)$, and these produce two independent cycles, namely, the triangles vz_1z_2v and vz_3z_4v. Since G is bicyclic, its cycle space has a dimension of two, so these two triangles span it; consequently, the base of G is the bowtie with center v and G has no further cycle. In particular, $G[N_G(v)]$ consists exactly of the two edges z_1z_2 and z_3z_4, and every vertex of T lies in a tree attached to $N_G[v]$.

Finally, $i = 0$ and (4.3) give $j = |T| = \mu - 3$. Hence the j matching edges between $N_G(v)$ and T cover every vertex of T, and their endpoints in $N_G(v)$ are pairwise distinct. None of those endpoints can be one of z_1, z_2, z_3, z_4 because these four vertices are already M-saturated by the two edges counted by s.

We now use the fact that the two bowtie triangles already span the entire cycle space. Let $t, t' \in T$ have matched neighbors $a, a' \in N_G(v)$. If $tt' \in E(G)$, then $a \neq a'$ and $v - a - t - t' - a' - v$ is a cycle containing an edge outside the two bowtie triangles, so it is independent of them, which is a contradiction. Thus T is independent. Similarly, if some $t \in T$ had a second neighbor $b \in N_G(v) \setminus \{a\}$, then $v - a - t - b - v$ would be a third independent cycle. Therefore, every vertex of T has exactly its matched neighbor in $N_G(v)$ and no neighbor in T.

It follows that the vertices of T are attached to $\mu - 3$ distinct neighbors of v outside the bowtie, producing exactly $\mu - 3$ pendant paths of length two at v. Every remaining neighbor of v lies outside the base and outside these paths; it can have no further neighbor without creating an additional vertex in T. It is therefore pendant. Their number is $n - 5 - 2(\mu - 3) = n - 2\mu + 1 = k$. Thus G is precisely $A_{n,\mu}$.

For $\mu = 2$, the bound (4.1) reads $\Delta \leq n$, which is vacuous, and the relevant bound is $\Delta \leq n - 1$, valid in any graph of order n. If $\Delta = n - 1$, then v is adjacent to all other vertices, so $G = K_{1,n-1} + \{e_1, e_2\}$ for two edges inside $N_G(v)$. If e_1 and e_2 are non-adjacent, its four endpoints leave a further neighbor w of v because $n \geq 6$. Then $\{e_1, e_2, vw\}$ is a matching of size three, which is a contradiction. Hence e_1 and e_2 share a vertex, say, $e_1 = z_1z_2$ and $e_2 = z_2z_3$, and then the base of G is $\Theta(1, 2, 2)$ with vertices of degree three, v and z_2; this is $A_{n,2}$. □

By Lemma 4.2, the top maximum-degree stratum of $\mathcal{B}_{n,\mu}$ is a singleton. This gives a useful degree stratification, but a separate comparison is still required before an EUS runner-up can be placed on the next stratum.

Corollary 4.3. *Let $n \geq 7$, and $2 \leq \mu \leq \lfloor n/2 \rfloor$, and let $\mathcal{B}^j_{n,\mu} = \{G \in \mathcal{B}_{n,\mu} : \Delta(G) = D^* - j\}$ for $j \geq 0$. Then $\mathcal{B}^0_{n,\mu} = \{A_{n,\mu}\}$ and every graph other than $A_{n,\mu}$ belongs to a stratum with $j \geq 1$. Moreover, $\mathcal{B}^1_{n,\mu}$ is non-empty: it contains $T_{n,\mu}$ for $\mu \geq 3$ and Π_n for $\mu = 2$.*

Proof. This is immediate from Lemmas 4.1, 4.2, and 3.2. Notice that the statement is only a degree stratification: By itself, it does not assert that an EUS runner-up must lie in the first lower stratum. That separation is supplied by the complete comparison in Section 5. □

For $\mu \geq 3$, the first lower stratum has a particularly simple description once the dominant vertex is deleted. Throughout the paper, $\nu(H)$ denotes the matching number of an auxiliary graph H, while $\mu = \mu(G)$ is the fixed matching number of the original graph.

Lemma 4.4. *Let $n \geq 7$, $3 \leq \mu \leq \lfloor n/2 \rfloor$, and $G \in \mathcal{B}_{n,\mu}$ satisfy $\Delta(G) = D = n - \mu + 1$. Let v be a dominant vertex, set $H = G - v$, and call vertices of $N_G(v)$ roots of H. Then*

$$|E(H)| = \mu, \qquad \nu(H) \in \{\mu - 1, \mu\}.$$

Furthermore, exactly one of the following holds.

(a) $\nu(H) = \mu$. Then H is the disjoint union of μ copies of K_2 and $k - 2$ isolated vertices. Exactly two K_2 components have two roots; the other $\mu - 2$ have one root and one non-root, and every isolated vertex is a root.

(b) $\nu(H) = \mu - 1$ and the unique non-K_2 component is a P_3. If this path contains r roots, then $1 \leq r \leq 3$; among the remaining $\mu - 2$ copies of K_2, exactly $3 - r$ have two roots and $\mu + r - 5$ have one root. There are $k - 1$ isolated roots.

(c) $\nu(H) = \mu - 1$ and the unique non-K_2 component is a P_4. If this path contains r roots, then $1 \leq r \leq 3$; among the remaining $\mu - 3$ copies of K_2, exactly $3 - r$ have two roots and $\mu + r - 6$ have one root. There are k isolated roots.

Only cases for which the displayed numbers are non-negative occur. In particular, up to the reversal of the distinguished path, the first lower stratum is described by finitely many rooted P_3 and P_4 patterns, together with case (a).

Proof. Since G is bicyclic and $d(v) = D = n - \mu + 1$,

$$|E(H)| = (n + 1) - D = \mu.$$

Deleting one vertex lowers the matching number by at most one, hence $\mu - 1 \leq \nu(H) \leq \mu$.

If $\nu(H) = \mu$, a matching of size μ uses every one of the μ edges of H, so those edges are pairwise disjoint. Thus H is a matching together with isolated vertices.

Suppose next that $\nu(H) = \mu - 1$, and let F be a maximum matching of H. Exactly one edge e of H is not in F. Every component not containing e is therefore either an edge of F or an isolated vertex. The edge e cannot join two F-unsaturated vertices because then $F \cup \{e\}$ would be a matching of size μ. Since the graph is simple, e also cannot have both ends in the same edge of F. Hence e either joins one endpoint of an edge of F to an F-unsaturated vertex, producing a P_3, or joins endpoints of two distinct edges of F, producing a P_4. This proves the unrooted parts of (b) and (c).

Every component of H contains at least one root because G is connected. All three forms found above are forests. If a tree component C of H contains $r(C)$ roots, then adjoining v and the $r(C)$ incident edges creates exactly $r(C) - 1$ independent cycles. Since G is bicyclic,

$$\sum_{C} (r(C) - 1) = 2. \tag{4.4}$$

Thus a K_2 component has either one root or two roots, and no K_2 has zero roots. In case (a), Eq (4.4) therefore forces exactly two two-root K_2 components; the rest have one root. The number of isolated vertices is $n - 1 - 2\mu = k - 2$.

In case (b), let r be the number of roots on the P_3 and let a be the number of two-root K_2 components. Equation (4.4) gives $(r-1)+a=2$, hence $a=3-r$. The remaining $(\mu-2)-a=\mu+r-5$ edge components have one root, and the number of isolated vertices is $n-1-[3+2(\mu-2)]=k-1$. The proof of (c) is identical: There are $\mu-3$ K_2 components, so $a=3-r$ of them have two roots, $\mu+r-6$ have one root, and the number of isolated vertices is $n-1-[4+2(\mu-3)]=k$. □

Remark 4.5. The rooted P_4 case is essential. For example, the graph on $\{v,a,b,c,d,e,f,p\}$ with edge set

$$\{va, vb, vc, vd, vp, bc, ae, ef, fd\}$$

has $n=8$, $\mu=4$, and $d(v)=5=n-\mu+1$. Here $G-v$ consists of the rooted path $aefd$ together with the edge bc and an isolated root p. Thus vertices outside $N_G[v]$ need not be pendant even though the whole first lower stratum still admits the finite description of Lemma 4.4.

The second lower stratum also admits a finite description, although its exceptional core is slightly larger. We use the following notation. Let T_5 be the tree obtained from $K_{1,3}$ by subdividing one edge once, and let T_6 be the tree obtained from $K_{1,3}$ by subdividing two distinct edges once. The paw is a triangle with one pendant edge.

Lemma 4.6. *Let $n\geq 7$, $3\leq\mu\leq\lfloor n/2\rfloor$, $G\in\mathcal{B}_{n,\mu}$ satisfy $\Delta(G)=n-\mu$. Let v be a dominant vertex, set $H=G-v$, and call the vertices of $N_G(v)$ roots. Then*

$$|E(H)|=\mu+1, \qquad \nu(H)\in\{\mu-1,\mu\}.$$

For a nontrivial connected component C of H, set $c(C)=|E(C)|-\nu(C)$. Every component with $c(C)=0$ is a copy of K_2, and the exceptional components are as follows.

(1) If $\nu(H)=\mu$, there is exactly one exceptional component, and it is P_3 or P_4.
(2) If $\nu(H)=\mu-1$, then either there are exactly two exceptional components, each isomorphic to P_3 or P_4, or there is exactly one exceptional component isomorphic to one of

$$C_3,\ K_{1,3},\ paw,\ C_4,\ T_5,\ P_5,\ T_6,\ P_6.$$

All remaining components are copies of K_2 or isolated vertices. Every component contains at least one root, and if $r(C)$ is the number of roots in a component and $\beta(C)=|E(C)|-|V(C)|+1$ is its cyclomatic number, then

$$\sum_C(\beta(C)+r(C)-1)=2. \tag{4.5}$$

Consequently, the second lower stratum is a finite rooted-core problem: The only unbounded data are the numbers of one-root K_2 components and isolated roots.

Proof. Since $d(v)=n-\mu$ and G has $n+1$ edges, $|E(H)|=\mu+1$. Deleting one vertex lowers the matching number by at most one, so $\mu-1\leq\nu(H)\leq\mu$. Hence

$$\sum_C c(C)=|E(H)|-\nu(H)\in\{1,2\},$$

where isolated vertices contribute zero. If a nontrivial connected component has $c(C)=0$, all its edges belong to one matching; connectedness then forces $C\cong K_2$.

It remains to classify connected components with $c(C) = 1$ or 2. If such a component has e edges and t vertices, connectedness gives $t \le e + 1$ and therefore

$$\nu(C) \le \left\lfloor \frac{t}{2} \right\rfloor \le \left\lfloor \frac{e+1}{2} \right\rfloor, \qquad c(C) \ge e - \left\lfloor \frac{e+1}{2} \right\rfloor = \left\lfloor \frac{e}{2} \right\rfloor.$$

Thus $c(C) \le 2$ implies $e \le 5$. Inspecting the connected simple graphs with at most five edges gives exactly

$c(C)$	Trees	Unicyclic graphs
1	P_3, P_4	none
2	$K_{1,3}, T_5, P_5, T_6, P_6$	C_3, paw, C_4.

Since the total cost is one or two, the asserted alternatives follow. Finally, every component of H contains a root because G is connected. Adding v to a component C increases its cyclomatic number by exactly $r(C)-1$; summing over the components of H and using the fact that G is bicyclic gives (4.5). □

5. The main result

Theorem 5.1. *Let $n \ge 7$, and $2 \le \mu \le \lfloor n/2 \rfloor$. The largest, second largest, and third-largest values of EUS over $\mathcal{B}_{n,\mu}$ are attained by exactly one graph each, according to Table 3.*

Table 3. The three largest Euler Sombor indices and their unique extremal graphs in $\mathcal{B}_{n,\mu}$.

Rank	$\mu = 2$	$\mu = 3$	$\mu \ge 4$
1st	$A_{n,2}$, (3.2)	$A_{n,3}$, (3.3)	$A_{n,\mu}$, (3.3)
2nd	Π_n, (3.4)	$T_{n,3}$, (3.5)	$W_{n,\mu}$, (3.6)
3rd	Q_n, (3.7)	R_n, (3.8)	$T_{n,\mu}$, (3.5)

For $\mu \ge 3$, the first row is Theorem 1.1; for $\mu = 2$, it is verified in the proof below. The new assertions are the second and third rows.

We first determine the two largest values on the first lower maximum-degree stratum.

Theorem 5.2. *Let $n \ge 7$, and $2 \le \mu \le \lfloor n/2 \rfloor$. On the stratum $\mathcal{B}^1_{n,\mu}$, the largest and second-largest EUS values are attained uniquely as follows:*

	$\mu = 2$	$\mu = 3$	$\mu \ge 4$
largest on $\mathcal{B}^1_{n,\mu}$	Π_n	$T_{n,3}$	$W_{n,\mu}$
second on $\mathcal{B}^1_{n,\mu}$	Q_n	R_n	$T_{n,\mu}$.

Proof. We begin with $\mu \ge 3$. Fix a dominant vertex v and set $H = G - v$. By Lemma 4.4, all possibilities are obtained from case (a), or from a rooted P_3 or rooted P_4. Write the vertices of the distinguished path in order as $y_1, \dots, y_\ell$ and record the root set $R \subseteq \{1, \dots, \ell\}$, up to reversal. Set

$$a_D = f(D,2) - f(D,1) = g(1,D), \qquad b_D = f(D,3) - f(D,2) = g(2,D).$$

Lemma 2.2 gives

$$0 < a_D < b_D < 1, \qquad g(1,2) > g(1,3) > a_D \quad (D > 3). \tag{5.1}$$

Assume first that $\mu \geq 4$. The rooted pattern P_4 with $R = \{2, 3\}$ is exactly $W_{n,\mu}$, while the pattern with $R = \{1, 2, 3\}$ is $T_{n,\mu}$. A useful intermediate value is obtained from P_4 with $R = \{1, 3\}$; when $\mu \geq 5$, the one-root pattern $R = \{2\}$ has the same edge partition. Denote this common value by U. A direct edge count gives

$$U = kf(D, 1) + (\mu - 4)f(1, 2) + (\mu - 1)f(D, 2) + f(D, 3) + f(1, 3) + 2f(2, 2) + f(2, 3).$$

The comparison between $T_{n,\mu}$ and U is

$$EUS(T_{n,\mu}) - U = b_D + f(1, 2) + f(3, 3) - 2f(2, 2) = b_D + \sqrt{7} - \sqrt{3} > 0. \tag{5.2}$$

Moreover, $EUS(W_{n,\mu}) - EUS(T_{n,\mu})$ is exactly the positive constant (1.2).

It remains to check that every other rooted pattern lies below U. Up to reversal and the equality of edge partitions, the remaining possibilities are the following seven classes; subtracting their indices from U gives, respectively,

$$\begin{aligned}
&(b_D - a_D) + (f(1, 3) - f(1, 2)) + (f(2, 3) - f(2, 2)),\\
&g(1, 3) - a_D,\\
&g(1, 2) - a_D,\\
&[g(1, 2) - a_D] + [g(1, 2) - g(1, 3)],\\
&b_D + f(1, 3) + f(2, 3) - 2f(2, 2),\\
&f(1, 3) + f(2, 2) - f(1, 2) - f(2, 3),\\
&(b_D - 2a_D) + f(1, 3) + f(2, 3) - 2f(1, 2).
\end{aligned}$$

They correspond in order to the rooted patterns $P_3 : \{1\}$ or $\{1, 3\}$, $P_3 : \{2\}$, $P_3 : \{1, 2\}$, $P_3 : \{1, 2, 3\}$, $P_4 : \{1\}$ or $\{1, 4\}$, $P_4 : \{1, 2\}$ or $\{1, 2, 4\}$, and case (a) of Lemma 4.4. The first four expressions are positive by (5.1) and the strict monotonicity properties of g. The fifth is positive because $f(1, 3) + f(2, 3) - 2f(2, 2) = \sqrt{13} + \sqrt{19} - 4\sqrt{3} > 0$; indeed, after squaring, this is equivalent to $\sqrt{247} > 8$. The sixth is (1.2); and for the last one, $b_D - 2a_D > -1$ while

$$f(1, 3) + f(2, 3) - 2f(1, 2) = \sqrt{13} + \sqrt{19} - 2\sqrt{7} > 1,$$

where the latter follows, for example, from $\sqrt{13} > 18/5$, $\sqrt{19} > 87/20$, and $\sqrt{7} < 8/3$. Thus every remaining pattern is strictly below U. For clarity, when $\mu = 4$ and the distinguished P_4 has two roots, the root sets up to reversal are exactly $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, and $\{2, 3\}$. The last is $W_{n,4}$, the second has a value of U, and the first and third are strictly below U by the sixth and fifth comparisons above. A one-root P_4 is infeasible at $\mu = 4$ because it would require $\mu + |R| - 6 = -1$ one-root K_2 components. This explicitly covers the boundary case $\mu = 4$, $|R| = 2$ and proves the $\mu \geq 4$ part.

Now let $\mu = 3$. The feasibility conditions in Lemma 4.4 leave only the two three-root P_4 patterns, the two two-root P_3 patterns, the one three-root P_3 pattern, and case (a). The P_4 root sets $\{1, 2, 3\}$ and $\{1, 2, 4\}$ give $T_{n,3}$ and R_n, respectively. Their difference is

$$EUS(T_{n,3}) - EUS(R_n) = b_D + \sqrt{13} + \sqrt{3} - \sqrt{19} > 0.$$

Subtracting each of the other four distinct edge partitions from $EUS(R_n)$ gives

$$g(1, 3) - a_D,$$

$$(b_D - a_D) + 2(f(2,3) - f(2,2)),$$
$$g(1,2) - a_D,$$
$$(b_D - 2a_D) + 2f(2,3) - f(2,2) - f(1,2),$$

which are all positive by (5.1). For the last expression, use $b_D - 2a_D > -1$ and $2f(2,3) - f(2,2) - f(1,2) = 2\sqrt{19} - 2\sqrt{3} - \sqrt{7} > 1$, using $\sqrt{19} > 87/20$, $\sqrt{3} < 7/4$, and $\sqrt{7} < 8/3$. This proves the $\mu = 3$ part.

It remains to treat $\mu = 2$. Here $D^* = n - 1$ and the first lower stratum has $\Delta = n - 2$. Let v be dominant and again set $H = G - v$. Then H has exactly three edges and there is exactly one non-root vertex. We claim that $\nu(H) = 1$. Indeed, if H contained two independent edges, its four endpoints would leave some neighbor of v uncovered because $d(v) = n - 2 \geq 5$; adding the edge from v to that uncovered neighbor would produce a matching of size three in G, which is a contradiction. A simple graph with three edges and a matching number of one is either a triangle or a $K_{1,3}$. Connectivity forces the unique non-root vertex to lie in that nontrivial component. The triangle gives Q_n. If the component is $K_{1,3}$, putting the non-root at a leaf gives Π_n, while putting it at the center gives a third graph, say, J_n, with

$$EUS(J_n) = (n-5)f(n-2,1) + 3f(n-2,2) + 3f(2,3).$$

Set $D = n-2$ and $c_D = f(D,4) - f(D,3) = g(3,D)$. Convexity, or Lemma 2.2, gives $0 < a_D < b_D < c_D$. Hence

$$EUS(\Pi_n) - EUS(Q_n) = (c_D - b_D + a_D) + f(1,4) + 2f(2,4) - 2f(2,3) - f(3,3) > 0,$$
$$EUS(Q_n) - EUS(J_n) = 2b_D - a_D + f(3,3) - f(2,3) > 0.$$

The fixed constant in the first line is positive; for instance, $\sqrt{21} > 9/2$, $\sqrt{7} > 13/5$, $\sqrt{19} < 22/5$, and $\sqrt{3} < 7/4$ give $\sqrt{21} + 4\sqrt{7} - 2\sqrt{19} - 3\sqrt{3} > 17/20 > 0$. Thus Π_n and Q_n are uniquely first and second on the stratum. □

The finite list in Lemma 4.6 also identifies natural benchmark graphs on the second lower stratum. For $\mu \geq 4$, let $X_{n,\mu}$ be obtained as follows. In the exceptional component T_6, root the vertex of degree three and the two vertices of degree two. Then, take $\mu-4$ further one-root copies of K_2 and $k = n-2\mu+1$ isolated roots, and join every root to a new vertex v. Then $d(v) = L = n - \mu$ and

$$\begin{aligned} EUS(X_{n,\mu}) = {} & kf(L,1) + (\mu-4)f(L,2) + 2f(L,3) + f(L,4) \\ & + (\mu-4)f(1,2) + 2f(1,3) + f(1,4) + 2f(3,4). \end{aligned} \tag{5.3}$$

For $\mu = 3$, let $X_{n,3}$ be obtained from the paw by rooting its three-valent vertex and one adjacent two-valent vertex, adding $n - 5$ isolated roots, and joining all roots to v. With $L = n - 3$,

$$\begin{aligned} EUS(X_{n,3}) = {} & (n-5)f(L,1) + f(L,3) + f(L,4) \\ & + f(1,4) + f(2,4) + f(3,4) + f(2,3). \end{aligned} \tag{5.4}$$

Both families occur in $\mathcal{B}^2_{n,\mu}$. In the finite rooted-core search, they are the persistent leaders of this stratum. The following comparison, however, is purely analytic and does not assume that leadership.

Proposition 5.3. *For every $n \geq 7$,*

$$EUS(X_{n,3}) < EUS(R_n).$$

For every $\mu \geq 4$ and $n \geq 2\mu$,

$$EUS(X_{n,\mu}) < EUS(T_{n,\mu}).$$

In fact, the two gaps are at least 0.315567283 . . . *and* 0.228607020 . . ., *respectively.*

Proof. For $\mu = 3$, set $L = n - 3 \geq 4$ and

$$h(L) = 2f(L + 1, 2) - f(L, 4).$$

By Lemma 2.2,

$$h(L + 1) - h(L) = 2g(L + 1, 2) - g(L, 4) > 0.$$

Subtracting (5.4) from (3.8) gives

$$\begin{aligned} EUS(R_n) - EUS(X_{n,3}) = (L - 2)g(L, 1) + g(L, 3) + h(L) \\ + f(2, 2) + f(2, 3) - f(1, 4) \\ - f(2, 4) - f(3, 4). \end{aligned}$$

The product in the first term is increasing with L. The same is true of $g(L, 3)$ and $h(L)$. Thus the right-hand side is smallest at $L = 4$. At that point, it equals

$$-3\sqrt{21} - 2\sqrt{37} - 2\sqrt{7} - 2\sqrt{3} + \sqrt{19} + 7 + 2\sqrt{31} + 2\sqrt{39}.$$

Using the evident three-decimal rational bounds obtained by squaring ($\sqrt{19} > 4.358$, $\sqrt{31} > 5.567$, $\sqrt{39} > 6.244$, and $\sqrt{21} < 4.583$, $\sqrt{37} < 6.083$, $\sqrt{7} < 2.646$, and $\sqrt{3} < 1.733$) gives the strict lower bound $0.307 > 0$.

Now let $\mu \geq 4$, set $L = n - \mu \geq \mu \geq 4$, and recall that $D = L + 1$ and $k = n - 2\mu + 1 \geq 1$. From (3.5) and (5.3),

$$\begin{aligned} EUS(T_{n,\mu}) - EUS(X_{n,\mu}) = kg(L, 1) + (\mu - 4)g(L, 2) + 2g(L, 3) + h(L) \\ + f(1, 2) - f(1, 3) + f(2, 3) + f(3, 3) - f(1, 4) - 2f(3, 4). \end{aligned}$$

Every variable term is nondecreasing when L, μ, or k increases. The minimum is therefore attained at $L = 4$, $\mu = 4$, $k = 1$, namely, at $n = 8, \mu = 4$. There it equals

$$-4\sqrt{37} - 2\sqrt{21} - \sqrt{13} - \sqrt{3} + \sqrt{7} + \sqrt{19} + \sqrt{31} + 2\sqrt{39} + 14.$$

The directed rational bounds $\sqrt{37} < 6.083$, $\sqrt{21} < 4.583$, $\sqrt{13} < 3.606$, $\sqrt{3} < 1.733$ and $\sqrt{7} > 2.645$, $\sqrt{19} > 4.358$, $\sqrt{31} > 5.567$, $\sqrt{39} > 6.244$ give a strict lower bound $0.221 > 0$. □

We now complete the finite comparison on the second lower stratum. For $L \geq 4$, set

$$a_i(L) = f(L, i + 1) - f(L, i) = g(i, L), \qquad i = 1, 2, 3.$$

Lemma 2.2 gives

$$\frac{1}{2} < a_1(L) < a_2(L) < a_3(L) < 1. \tag{5.5}$$

For a fixed rooted core, after the common terms are cancelled, the difference between its EUS value and the appropriate benchmark is of the form

$$q_1a_1(L) + q_2a_2(L) + q_3a_3(L) + C, \tag{5.6}$$

where q_i are integers and C is a fixed linear combination of the small values in Table 1. This follows because both graphs have the same dominant degree and therefore the coefficients of $f(L,1),\dots,f(L,4)$ sum to zero. The minimum of the variable part of (5.6) under (5.5) is attained at one of

$$(a_1(L),a_2(L),a_3(L)) \in \left\{\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right),\left(\tfrac{1}{2},\tfrac{1}{2},1\right),\left(\tfrac{1}{2},1,1\right),(1,1,1)\right\}.$$

Thus each rooted pattern has a finite exact certificate.

Theorem 5.4. *Let $n \geq 7$, and $3 \leq \mu \leq \lfloor n/2 \rfloor$.*

(1) If $\mu \geq 4$, the unique maximum of EUS on $\mathcal{B}^2_{n,\mu}$ is attained by $X_{n,\mu}$. Every other graph in that stratum satisfies

$$EUS(X_{n,\mu}) - EUS(G) > \delta_4,$$

where

$$\delta_4 = \frac{1}{2} + \sqrt{13} - 2\sqrt{3} - 2\sqrt{7} + \sqrt{37} = 1.432709568\dots.$$

(2) If $\mu = 3$, the maximum on $\mathcal{B}^2_{n,3}$ is attained by $X_{n,3}$. It is unique up to graph isomorphism for every $n \geq 7$. For $n = 7$, one further rooted T_5 configuration has the same value, but reconstructs a graph isomorphic to $X_{7,3}$.

Proof. We use Lemma 4.6. Up to automorphisms of the unrooted exceptional components, there are only finitely many root sets. For $\mu \geq 4$, explicit edge counts reduce every non-benchmark root orbit to (5.6). Minimizing over the four vertices of the order polytope above gives the category bounds in Table 4. The smallest one occurs for the alternative T_6 rooting and equals $\delta_4 > 0$. Hence $X_{n,\mu}$ is the unique leader of the stratum.

For $\mu = 3$, the same calculation gives the bounds in Table 5 for every exceptional-core category other than T_5. Among the T_5 rootings, only two need a sharper estimate. Their benchmark differences are, respectively,

$$a_1(4) - a_1(L), \qquad f(2,3) - f(1,3) - a_1(L).$$

Here $L = n - 3 \geq 4$. Since $a_1(L) = g(1,L)$ is decreasing in its second argument,

$$\begin{aligned} a_1(L) \leq a_1(4) &= f(4,2) - f(4,1) \\ &= 2\sqrt{7} - \sqrt{21} = 0.708926927\dots. \end{aligned}$$

The first difference is therefore nonnegative, with equality only at $L = 4$. The second is strictly positive because

$$f(2,3) - f(1,3) - a_1(4) = \sqrt{19} - \sqrt{13} - 2\sqrt{7} + \sqrt{21} > 0;$$

for example, directed rational bounds for the four radicals certify a positive margin exceeding 0.044. The remaining T_5 rootings have a certificate exceeding 1.686. At $L = 4$, the zero T_5 difference gives no second isomorphism class. Indeed, we write $X_{7,3}$ on $\{v,a,b,c,d,p,q\}$ with edges

$ab, bc, ca, ad, va, vb, vp, vq$. Deleting v gives the rooted paw benchmark; deleting the other degree-four vertex a gives the exceptional rooted T_5 together with an isolated root. Thus both configurations describe the same graph. This proves the assertion. The root-orbit edge counts and all sign decisions underlying the two tables are reproduced by `exact_certificates.py` in Supplementary Material S1, which uses directed rational intervals for every fixed radical and no floating-point sign test. □

The benchmark T_6 rooting is omitted. The displayed decimals are rounded downward and follow from (5.5) and Table 1.

Table 4. Certificate bounds for $EUS(X_{n,\mu}) - EUS(G)$, $\mu \geq 4$.

Core	Lower bound	Core	Lower bound
T_6	1.432709	paw	1.497712
T_5	1.641636	$P_4 + P_4$	1.790444
$K_{1,3}$	3.283272	P_6	3.334150
$P_3 + P_4$	3.522495	P_5	3.710044
C_4	4.877856	C_3	5.253750
$P_3 + P_3$	5.254546	P_4	5.442095
P_3	7.174146		

Two special T_5 rootings are treated separately; the displayed decimals are rounded downward.

Table 5. Certificate bounds for $EUS(X_{n,3}) - EUS(G)$.

Core	Lower bound	Core	Lower bound
$K_{1,3}$	1.394984	paw	1.477130
T_5	1.686057	P_5	1.712332
C_4	3.380144	$P_3 + P_3$	3.386839
P_4	3.444383	C_3	3.756038
P_3	5.176434		

For matching number two, the second lower stratum is even more rigid. Set $L = n - 3$ and define, for $n \geq 8$,

$$EUS(Y_n) = (n-6)f(L,1) + 2f(L,2) + f(L,5) + 2f(1,5) + 2f(2,5), \tag{5.7}$$

and for $n \geq 7$, set

$$EUS(Z_n) = (n-6)f(L,1) + 3f(L,2) + f(1,4) + 3f(2,4). \tag{5.8}$$

The graphs are obtained from a $K_{1,4}$ component in $G - v$: For Y_n, the center and two leaves are roots, while for Z_n, three leaves are roots. All remaining vertices of $G - v$ are isolated roots.

Proposition 5.5. *On $\mathcal{B}^2_{n,2}$, the largest value is attained by Y_n for $n \geq 8$; for $n = 7$, the stratum consists only of the Z_7 pattern. In all cases,*

$$\max\{EUS(G) : G \in \mathcal{B}^2_{n,2}\} < EUS(Q_n).$$

Proof. Let v be dominant and $H = G - v$. Since $d(v) = n - 3$ and $|E(G)| = n + 1$, H has four edges, and exactly two vertices of H are non-roots. If $n \geq 8$ and $\nu(H) \geq 2$, a matching of two edges in H

saturates at most four roots, whereas there are $n-3 \geq 5$ roots; adjoining v to an unsaturated root gives a matching of size three in G, which is a contradiction. For $n = 7$, suppose instead that H has a two-edge matching. Every such matching must saturate all four roots; otherwise, an unsaturated root can be matched to v, producing a matching of size three in G. Hence the two matching edges are disjoint root–root edges. Since each of the two non-roots must lie in a component containing a root, H has an edge joining a root to a non-root. That edge is disjoint from one of the two root–root matching edges, and the resulting two-edge matching leaves a root unsaturated; adjoining the corresponding edge to v again gives a three-edge matching, which is a contradiction. Hence $\nu(H) = 1$ for every $n \geq 7$.

A simple graph with four edges and a matching number of one is $K_{1,4}$ together with isolated vertices: Four pairwise intersecting edges must have a common endpoint, since, otherwise, three form a triangle and a fourth simple edge cannot meet all three. Connectivity forces every isolated vertex of H to be a root, so the two non-roots lie in $K_{1,4}$. Up to isomorphism, they are either two leaves, giving Y_n, or the center and one leaf, giving Z_n. The first pattern is infeasible at $n = 7$ because its center would have a degree of five while $d(v) = 4$.

For $n \geq 8$,

$$\begin{aligned} EUS(Y_n) - EUS(Z_n) = {} & f(L,5) - f(L,2) + 2f(1,5) + 2f(2,5) \\ & - f(1,4) - 3f(2,4) > 0, \end{aligned}$$

because $f(L,5) - f(L,2) = g(2,L) + g(3,L) + g(4,L) > 3/2$, while the fixed remainder $2\sqrt{31} + 2\sqrt{39} - \sqrt{21} - 6\sqrt{7}$ is positive; for example, $\sqrt{31} > 11/2$, $\sqrt{39} > 31/5$, $\sqrt{21} < 23/5$, and $\sqrt{7} < 8/3$ already give a positive lower bound.

It remains to compare Y_n with Q_n. Set $F(L) = EUS(Q_{L+3}) - EUS(Y_{L+3})$ for $L \geq 5$. Direct subtraction gives

$$F(5) = -4\sqrt{39} - 4\sqrt{31} - 2\sqrt{3} + 2\sqrt{19} + 6\sqrt{7} + 4\sqrt{43} > \frac{12}{125} > 0,$$

where the displayed rational lower bound follows from directed three-decimal bounds for the radicals. Moreover,

$$\begin{aligned} F(L+1) - F(L) = {} & L\big(g(L+1,1) - g(L,1)\big) + 3g(L,1) \\ & + 2g(L+1,3) - 2g(L,2) - g(L,5). \end{aligned}$$

The first term is positive and $3g(L,1) > 3/2$. Also $g(L+1,3) > g(L,5)$ by both monotonicities in Lemma 2.2, whereas $g(L+1,3) > 1/2$ and $g(L,2) < 1$. Hence the second line is greater than $-3/2$, so $F(L+1) - F(L) > 0$. Thus $Y_n < Q_n$ for every $n \geq 8$. Finally, direct substitution in (3.7) and (5.8) gives

$$\begin{aligned} EUS(Q_7) - EUS(Z_7) = {} & -12\sqrt{7} - 2\sqrt{21} + 3\sqrt{3} + 2\sqrt{19} \\ & + 14 + 3\sqrt{31} > \frac{739}{200} > 0, \end{aligned}$$

again by directed three-decimal rational bounds. □

5.1. Finite certificates for the third through fifth lower strata

The next three strata share the same deletion structure, so we treat them uniformly. This removes the repeated core catalogues while keeping the finite part of the proof completely explicit and reproducible.

Lemma 5.6. *Let $3 \le j \le 5$, $n \ge 7$, $3 \le \mu \le \lfloor n/2 \rfloor$, and $G \in \mathcal{B}^j_{n,\mu}$. If v is a dominant vertex, $H = G - v$, and the vertices of $N_G(v)$ are called roots. Then,*

$$|E(H)| = \mu + j - 1, \qquad \nu(H) \in \{\mu - 1, \mu\}.$$

For each nontrivial component C of H, set $c(C) = |E(C)| - \nu(C)$. Components with $c(C) = 0$ are copies of K_2, and

$$\sum_C c(C) \in \{j - 1, j\}. \tag{5.9}$$

Every exceptional component C with $c(C) > 0$ has at most $2j + 1$ edges and at most $2j + 2$ vertices. Moreover, if $r(C)$ is the number of roots in C and $\beta(C) = |E(C)| - |V(C)| + 1$, then

$$\sum_C (\beta(C) + r(C) - 1) = 2. \tag{5.10}$$

Thus, after isolated roots and one-root K_2 components are removed, each of the strata $j = 3, 4, 5$ is a finite rooted-core problem.

Proof. For $\mu \ge 3$, we have $D^* = n - \mu + 2$ and hence $d(v) = n - \mu + 2 - j$. Since a bicyclic graph has $n + 1$ edges, $|E(H)| = \mu + j - 1$. Deleting one vertex lowers the matching number by at most one, so $\mu - 1 \le \nu(H) \le \mu$, which gives (5.9). A connected graph of cost zero is K_2.

If an exceptional connected component has e edges and t vertices, then $t \le e + 1$ and

$$c(C) = e - \nu(C) \ge e - \left\lfloor \frac{e + 1}{2} \right\rfloor = \left\lfloor \frac{e}{2} \right\rfloor.$$

Since $c(C) \le j$, it follows that $e \le 2j + 1$ and $t \le 2j + 2$. Every component of H contains a root, and adjoining v to a component C increases its cyclomatic number by $r(C) - 1$. Summing over all components and using the fact that G is bicyclic gives (5.10). □

Lemma 5.7 (Catalogue completeness)**.** *Under the hypotheses of Lemma 5.6, every exceptional rooted component that can occur for $j \in \{3, 4, 5\}$ is produced by the following finite procedure: Choose a non-isomorphic tree on at most $2j + 2$ vertices, add zero, one, or two non-edges, retain the resulting connected graph C when $1 \le c(C) \le j$ and $\beta(C) \le 2$, and then choose a non-empty root set $R \subseteq V(C)$ satisfying*

$$\beta(C) + |R| - 1 \le 2.$$

Quotienting this list by rooted graph isomorphism cannot remove any distinct EUS configuration.

Proof. Let C be an exceptional component of an actual graph in the stratum. By Lemma 5.6, $|V(C)| \le 2j + 2$. Since every component of $H = G - v$ contains a root, $r(C) \ge 1$; all summands in (5.10) are therefore non-negative, and hence $\beta(C) + r(C) - 1 \le 2$. In particular, $\beta(C) \le 2$ and $r(C) \le 3 - \beta(C)$. Choose a spanning tree T of C. The number of edges of C outside T is exactly $|E(C)| - |V(C)| + 1 = \beta(C) \le 2$, so C is obtained from T by adding zero, one, or two non-edges. Thus the stated generation contains every possible unrooted core, and enumerating all admissible non-empty root subsets contains every rooted core. Finally, rooted-isomorphic components have the same internal degree pairs and the same degrees on their roots after the edge to v is restored, so they make identical contributions to every term used in the certificate comparison. □

For the exact certificates, we use the quantitative form

$$g(t,c) > 1 - \frac{3c^2}{8t(t+1)} \qquad (t,c \geq 1). \tag{5.11}$$

Indeed,

$$1 - \frac{\partial f}{\partial x}(x,c) = \frac{3c^2}{2f(x,c)[2f(x,c)+2x+c]} < \frac{3c^2}{8x^2},$$

and integration over $[t, t+1]$ gives (5.11).

Theorem 5.8 (Finite lower-stratum certificates)**.** *Define*

$$\Phi_3(n,\mu) = \begin{cases} EUS(Q_n), & \mu = 2, \\ EUS(R_n), & \mu = 3, \\ EUS(T_{n,\mu}), & \mu \geq 4. \end{cases}$$

For every admissible n, μ and each $j \in \{3,4,5\}$,

$$G \in \mathcal{B}^j_{n,\mu} \quad \Longrightarrow \quad EUS(G) < \Phi_3(n,\mu).$$

If a parameter–stratum pair is empty, the implication is understood vacuously; the certificate programs retain only feasible non-empty rooted configurations. The exact certificates have the sizes and conservative uniform margins shown in Table 6. In that table, "cases" denotes distinct aggregate signatures retained by the supplied implementation after the exact matching and degree feasibility filters; the last column gives coarse rational lower bounds asserted by the program. The same signature convention is used for all three strata.

Table 6. Certificate summary for strata $j = 3, 4, 5$.

j	Regime	Cases	Certified $\Phi_3 - EUS$ margin
3	$\mu \geq 4$	1044	> 1.76
3	$\mu = 3$	91	> 2.38
3	$\mu = 2$	2	> 4.44
4	$\mu \geq 4$	5991	> 6.71
4	$\mu = 3$	130	> 9.34
4	$\mu = 2$	2	> 10.95
5	$\mu \geq 4$	29,476	> 14.99
5	$\mu = 3$	148	> 17.38
5	$\mu = 2$	2	> 21.33

Proof. We now specify the finite certificate used in the proof. The supplied exact scripts regenerate every catalogue and every number in Table 6. For $\mu \geq 3$, Lemmas 5.6 and 5.7 reduce the proof to a finite list of rooted connected cores. Exact isomorphism reduction gives

$$2,\ 8,\ 29,\ 138,\ 756$$

connected core types of costs $1, 2, 3, 4, 5$, respectively. Root subsets have size at most $3-\beta(C)$ by (5.10); they are reduced up to rooted isomorphism. For each $j = 3, 4, 5$, a lossless compression is used: A rooted component is represented by its cost, root excess, edge and vertex counts, matching number, the coefficients of its fixed internal-edge EUS sum in square-free radicals after the root edges to v are restored, the multiset of degrees occurring at roots, the maximum restored core degree, and a flag recording whether a root can be left unsaturated by a maximum matching of the component. These data determine every fixed EUS term, every contribution incident with v, and every feasibility count used below, so two components with the same signature are interchangeable in the inequality certificate. Component signatures are then combined to the total cost of $j-1$ or j and root excess two.

Set $L = \Delta(G)$ and $a_i(L) = f(L, i+1) - f(L, i)$. The dominant vertex of the comparison graph defining Φ_3 has a degree of $L + j - 1$. After the common isolated-root and one-root K_2 terms are cancelled, every surviving signature yields a lower bound of the form

$$(j-1)L + C + \sum_i q_i a_i(L) + \sum_r b_r \sum_{h=0}^{j-2} g(L+h, r), \tag{5.12}$$

where C is an integral linear combination of the square roots of integers. By Lemma 2.2,

$$\frac{1}{2} < a_1(L) < a_2(L) < \cdots < 1,$$

so the middle term in (5.12) is minimized at a step vertex of the closed-order polytope $\frac{1}{2} \le a_1 \le a_2 \le \cdots \le 1$. Every term in the last sum is bounded from below by (5.11). The coefficient vectors at the comparison graph are

$$(b_1, b_2, b_3) = \begin{cases} (k, \mu-2, 2), & \mu \ge 4, \\ (L+j-4, 2, 1), & \mu = 3, \\ (L+j-3, 0, 2), & \mu = 2. \end{cases}$$

For a fixed signature in the regime $\mu \ge 4$, this reduction to the smallest feasible parameters is itself monotone and does not rely on a numerical guess. Let e and $|V_{\text{core}}|$ be, respectively, the total numbers of edges and vertices in the selected exceptional cores, and let

$$\rho = \sum_C (\beta(C) + r(C) - 1)$$

be their total root excess. Since the full root excess is two, $a = 2 - \rho$ of the remaining K_2 components have two roots. Let s be the total number of remaining K_2 components and let I be the number of isolated roots. Edge and vertex counts give

$$s = \mu + j - 1 - e, \qquad I = k - 2j - |V_{\text{core}}| + 2e.$$

Thus $s - a$ of these K_2 components have one root. Substituting these counts shows that the coefficient vector (q_i) and the fixed radical term C in (5.12) are independent of k and μ: The k- and μ-terms cancel between the comparison graph and the reconstructed signature. The remaining parameter-dependent part is bounded below by

$$(j-1)L + kA_1(L) + (\mu-2)A_2(L) + 2A_3(L),$$

$$A_r(L) = \sum_{h=0}^{j-2} \left(1 - \frac{3r^2}{8(L+h)(L+h+1)}\right),$$

with $L = k + \mu + 1 - j$. On every feasible case, $A_1, A_2 > 0$ and each $A_r(L)$ is increasing in L. Hence increasing either k or μ cannot decrease the certified lower bound, so it suffices to check every coordinatewise minimal feasible pair. If the core-degree constraint requires $\mu + k$ to increase, the program checks each way to distribute that finite increase between μ and k; every feasible pair lies coordinatewise above one of these checked pairs. For $\mu = 3$, one has $k = L + j - 4$, and the same monotonicity gives the corresponding one-parameter reduction. For completeness, the matching filter can be imposed exactly. An isolated root can be left unsaturated. For a nontrivial rooted component C, a root x can be left unsaturated by a maximum matching precisely when $\nu(C - x) = \nu(C)$. Hence if $\nu(H) = \mu$, feasibility requires $I = 0$ and no such root in any exceptional component. If $\nu(H) = \mu - 1$, feasibility requires either $I > 0$ or such a root in an exceptional component. One-root and two-root K_2 components never supply such a root.

For $\mu = 2$, H has $j + 2$ edges, exactly j non-root vertices, and $\nu(H) \in \{1, 2\}$. It therefore has at most two nontrivial components, each with at most seven edges. The program generates them by the same spanning-tree procedure and combines their rooted signatures exhaustively. Writing V_0 for their total order gives $I = L + j - V_0$. When $\nu(H) = 2$, the exact filter forces $I = 0$ and no exposable root; when $\nu(H) = 1$, it requires $I > 0$ or an exposable root. The lower bound is increasing in the remaining free parameter L, so the smallest feasible L suffices. These exact filters account for the compact case counts in Table 6.

All fixed radicals in C are enclosed by directed rational intervals to 30 decimal places: Lower endpoints are used for positive coefficients and upper endpoints for negative coefficients. All subsequent sign arithmetic is rational; consequently no floating-point sign decision enters the proof. The margins displayed in Table 6 are deliberately coarse rational lower bounds hard-coded as assertions in the exact programs; longer decimal values printed by the scripts are diagnostic displays only. The program `exact_certificates.py`, with the command and environment documented in Supplementary Material S1, regenerates every catalogue, case count, and certificate. Its outputs include `summary.json` and the full rational certificate records in `certificates.json.gz`. The independent script `graph_checks.py` checks the displayed graph constructions and closed forms for $7 \le n \le 40$ and verifies the $n = 7$ isomorphism above. Isomorphism-invariant hashes are used only as a prefilter: a candidate is discarded as a duplicate only after an exact graph isomorphism test. This proves the claim for $j = 3, 4, 5$. □

Lemma 5.9 (Zagreb tail bound)**.** *Let G be a connected bicyclic graph of order n, matching number μ, and maximum degree Δ. Then*

$$EUS(G) < \sum_{x \in V(G)} d(x)^2 \le (n - \mu + 2)\Delta + 3n + \mu + 4. \tag{5.13}$$

Proof. For every edge xy,

$$f(d(x), d(y)) < d(x) + d(y),$$

because $(d(x) + d(y))^2 - f(d(x), d(y))^2 = d(x)d(y) > 0$. Summing over the edges gives

$$EUS(G) < \sum_{xy \in E(G)} (d(x) + d(y)) = \sum_{x \in V(G)} d(x)^2.$$

Let p be the number of pendant vertices and let S be the set of their distinct support vertices, that is, the vertices adjacent to at least one pendant vertex. Choosing one pendant edge at each support vertex

gives a matching, hence $|S| \le \mu$. In a maximum matching, at most one pendant vertex adjacent to each member of S can be saturated. Therefore, at least $p - |S|$ pendant vertices are unsaturated. Since a maximum matching leaves exactly $n - 2\mu$ vertices unsaturated,

$$p - |S| \le n - 2\mu, \qquad \text{and hence} \qquad p \le n - \mu.$$

For each vertex x of degree at least three, set $a_x = d(x) - 2$. Since G is bicyclic,

$$\sum_{x \in V(G)} (d(x) - 2) = 2,$$

so

$$\sum_{d(x) \ge 3} a_x = p + 2. \tag{5.14}$$

Writing $M_1(G) = \sum_x d(x)^2$ for the first Zagreb index and separating vertices of degree one, two, and at least three gives

$$M_1(G) = 4n + p + 8 + \sum_{d(x) \ge 3} a_x^2.$$

Since $a_x \le \Delta - 2$, Eq (5.14) yields

$$\sum a_x^2 \le (\Delta - 2)(p + 2).$$

Consequently,

$$M_1(G) \le 4n + 4 + 2\Delta + p(\Delta - 1) \le 4n + 4 + 2\Delta + (n - \mu)(\Delta - 1),$$

which is exactly the second inequality in (5.13). □

Theorem 5.10 (Uniform tail separation)**.** *Let $\Phi_3(n, \mu)$ be as defined in Theorem 5.8. If $G \in \mathcal{B}_{n,\mu}$ satisfies*

$$\Delta(G) \le D^* - 6,$$

then

$$EUS(G) < \Phi_3(n, \mu).$$

Proof. We use repeatedly the elementary strict bound

$$f(D, r) > D + \frac{r}{2} \qquad (D, r > 0), \tag{5.15}$$

because $f(D, r)^2 - (D + r/2)^2 = 3r^2/4$.

First, suppose $\mu \ge 3$ and set $D = n - \mu + 1$, so $D^* = D + 1$. From Lemma 5.9 and $\Delta \le D - 5$,

$$EUS(G) < D^2 - D + 4\mu - 4. \tag{5.16}$$

If $\mu = 3$, then $\Phi_3(n, 3) = EUS(R_n)$ and (3.8), together with (5.15), give

$$EUS(R_n) > D^2 + \frac{D}{2} + 2 + 2\sqrt{3} + 2\sqrt{19}.$$

Subtracting the right-hand side of (5.16) with $\mu = 3$ leaves

$$\frac{3D}{2} - 6 + 2\sqrt{3} + 2\sqrt{19} > 0.$$

Thus every sixth or lower stratum lies below R_n.

Now let $\mu \geq 4$. Since $n \geq 2\mu$, one has $D \geq \mu + 1$ and $k = n - 2\mu + 1 = D - \mu$. From (3.5) and (5.15),

$$EUS(T_{n,\mu}) > D^2 + \frac{D}{2} + \frac{\mu}{2} + 1 + (\mu - 3)\sqrt{7} + \sqrt{13} + \sqrt{19} + 3\sqrt{3}.$$

After subtracting (5.16), the difference is greater than

$$\frac{3D}{2} - \frac{7\mu}{2} + 5 + (\mu - 3)\sqrt{7} + \sqrt{13} + \sqrt{19} + 3\sqrt{3}.$$

Using $D \geq \mu + 1$, this is at least

$$\mu(\sqrt{7} - 2) + \frac{13}{2} - 3\sqrt{7} + \sqrt{13} + \sqrt{19} + 3\sqrt{3},$$

which is increasing in μ and is already positive at $\mu = 4$. Therefore, every graph with $\Delta \leq D^* - 6$ lies below $T_{n,\mu}$.

Finally, let $\mu = 2$ and set $D = n - 2$. Then $D^* = D + 1$ and $\Delta \leq D - 5 = n - 7$. Lemma 5.9 gives

$$EUS(G) < n(n - 7) + 3n + 6 = D^2 + 2.$$

On the other hand, (3.7) and (5.15) give

$$EUS(Q_n) > D^2 + \frac{D}{2} + 2 + 2\sqrt{19} + 3\sqrt{3} > D^2 + 2.$$

Hence the entire sixth-and-lower tail lies below Q_n as well. □

Proof of Theorem 5.1. By Corollary 4.3, every graph in $\mathcal{B}_{n,\mu}$ belongs to exactly one stratum $\mathcal{B}^j_{n,\mu}$ with an integer $j \geq 0$. Thus the proof is complete once the mutually disjoint cases

$$j = 0, \qquad j = 1, \qquad j = 2, \qquad j \in \{3, 4, 5\}, \qquad j \geq 6$$

have been covered (an empty stratum requires no separate argument).

Assume first that $\mu \geq 3$. The case $j = 0$ is the known unique maximizer $A_{n,\mu}$ by Theorem 1.1. Theorem 5.2 gives the two leaders of the $j = 1$ stratum. Theorem 5.4 together with Proposition 5.3 places the $j = 2$ stratum strictly below the stated third value. Theorem 5.8 excludes $j = 3, 4, 5$, and Theorem 5.10 excludes every stratum with $j \geq 6$. This proves the table for $\mu \geq 3$.

Now let $\mu = 2$. Lemma 4.2 identifies the top stratum as the singleton $\{A_{n,2}\}$, while Theorem 5.2 places Π_n and Q_n first and second on the $j = 1$ stratum. Proposition 5.5, Theorems 5.8 and 5.10 place every stratum with $j \geq 2$ strictly below Q_n. It remains to compare $A_{n,2}$ with Π_n. Set $L = n - 2 \geq 5$. From (3.2) and (3.4),

$$EUS(A_{n,2}) - EUS(\Pi_n) = (L - 3)g(L, 1) + f(L + 1, 1) + 2g(L, 2)$$

$$+ f(L+1,3) - f(L,4) + 2f(2,3) - f(1,4) - 2f(2,4).$$

By Lemma 2.2, $g(L,1) > \frac{1}{2}$, $2g(L,2) > 1$, and $f(L+1,1) > L+1$. Moreover,

$$f(L+1,3)^2 - f(L,4)^2 = L - 3 > 0,$$

so $f(L+1,3) > f(L,4)$. Consequently,

$$EUS(A_{n,2}) - EUS(\Pi_n) > \frac{3L+1}{2} + 2\sqrt{19} - \sqrt{21} - 4\sqrt{7}.$$

Since $L \geq 5$, the right-hand side is larger than $8 + 87/10 - 23/5 - 53/5 = 3/2 > 0$, using $\sqrt{19} > 87/20$, $\sqrt{21} < 23/5$, and $\sqrt{7} < 53/20$. Thus $A_{n,2}$ is the unique first-ranked graph, and the table follows for $\mu = 2$ as well. □

The closed forms of the first-lower leaders exhibit three distinct numerical regimes.

For $\mu \geq 4$, subtraction of (3.5) from (3.6) gives the identity

$$\begin{aligned} EUS(W_{n,\mu}) - EUS(T_{n,\mu}) &= f(1,3) + f(2,2) - f(1,2) - f(2,3) \\ &= \sqrt{13} + 2\sqrt{3} - \sqrt{7} - \sqrt{19}. \end{aligned} \tag{5.17}$$

This is positive; indeed $\sqrt{13} > 18/5$, $\sqrt{3} > 173/100$, $\sqrt{19} < 109/25$, and $\sqrt{7} < 53/20$ give the strict lower bound $1/20$. Hence it is the positive constant in (1.2).

For $\mu = 3$, one has $D = n - 2$, and direct subtraction of (3.8) from (3.5) gives

$$\begin{aligned} EUS(T_{n,3}) - EUS(R_n) &= f(D,3) - f(D,2) + f(1,3) + f(3,3) - f(2,2) - f(2,3) \\ &= f(D,3) - f(D,2) + \sqrt{13} + \sqrt{3} - \sqrt{19}. \end{aligned} \tag{5.18}$$

Since $f(D,3) - f(D,2) \to \frac{1}{2}$ as $D \to \infty$, the limiting gap is

$$\tfrac{1}{2} + \sqrt{3} + \sqrt{13} - \sqrt{19} = 1.478703139\ldots.$$

In particular, the often convenient bound "greater than 1.5" is not valid for all n; the value is already below 1.5 at $n = 89$.

For $\mu = 2$, subtraction of (3.7) from (3.4) yields

$$\begin{aligned} EUS(\Pi_n) - EUS(Q_n) = &- f(n-2,1) + 2f(n-2,2) + f(n-2,4) \\ &- 2f(n-2,3) + f(1,4) + 2f(2,4) \\ &- 2f(2,3) - f(3,3), \end{aligned} \tag{5.19}$$

whose limit is

$$\tfrac{1}{2} + \sqrt{21} + 4\sqrt{7} - 2\sqrt{19} - 3\sqrt{3} = 1.751630629\ldots.$$

Thus the bound "greater than 1.78 for all $n \geq 7$" is likewise false; the gap is below 1.78 beginning at $n = 66$.

Theorem 5.10 is deliberately based on the global first Zagreb index rather than on the false edgewise relaxation

$$\Delta f(\Delta,3) + (n+1-\Delta) f(3,3).$$

For example, the graph on $\{a, b, c, d, e, f, g, h\}$ with edge set

$$\{ab, ac, ad, ae, af, bc, bd, bg, bh\}$$

has $n = 8$, $\mu = 2$, and $\Delta = 5$, but its Euler Sombor index exceeds that relaxation. The Zagreb argument avoids this obstruction because it retains the whole degree sequence through $\sum d(v)^2$ and uses the matching number only to control the number of leaves.

6. Conclusions

Theorem 5.1 completes the top-three ordering of the Euler Sombor index on $\mathcal{B}_{n,\mu}$ for every admissible n and μ. The comparison between $W_{n,\mu}$ and $T_{n,\mu}$ suggests the following more-general question for bond incident degree indices.

Problem 6.1. Characterize the symmetric functions φ for which $W_{n,\mu}$ attains the second-largest value of

$$\sum_{uv \in E(G)} \varphi(d(u), d(v))$$

over $\mathcal{B}_{n,\mu}$ when $\mu \geq 4$, and $T_{n,\mu}$ does so when $\mu = 3$, assuming the corresponding global extremal classification.

For $\mu \geq 4$, the local comparison of these two graphs turns on the sign of

$$\varphi(1, 3) + \varphi(2, 2) - \varphi(1, 2) - \varphi(2, 3).$$

For the Euler Sombor index, this is the small positive constant (1.2); for other indices, it may vanish or change sign.

It is also natural to determine the fourth and later extremal values. The minimum and second minimum appear to require different methods because their extremal graphs lie in much larger low-degree strata.

Supplementary material

Supplementary Material S1 contains the complete verification source `exact_certificates.py`, the graph checks in `graph_checks.py`, dependency and execution instructions, the full run log, the regenerated summary tables, and the exact rational certificate records. The verification uses Python and NetworkX; integer square roots and rational arithmetic determine all signs. The finite comparisons in Theorems 5.4 and 5.8 can therefore be rerun without access to any unpublished script.

Author contributions

Wei Fang: Conceptualization, methodology, formal analysis, and writing–review and editing; Wangxin Huang: Validation and writing–original draft. All authors discussed the results, reviewed the manuscript, and approved the final version. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

OpenAI (ChatGPT) was used to assist with formatting the manuscript and with language-level editing of the abstract. The authors independently verified, revised, and finalized every mathematical statement, proof, figure, table, and numerical output, and take full responsibility for the content of the manuscript.

Acknowledgments

This research was funded by the Start-up Fund for New Talented Researchers of Nanjing University of Industry Technology (Grant No. YK23-12-02). The authors would like to thank the anonymous referees for their careful reading of this manuscript and their valuable suggestions, which have greatly improved the quality of the original work.

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. I. Gutman, Geometric approach to degree-based topological indices: Sombor indices, *MATCH Commun. Math. Comput. Chem.*, **86** (2021), 11–16.
2. I. Gutman, Relating Sombor and Euler indices, *Vojnoteh. Glas.*, **72** (2024), 1–12. https://doi.org/10.5937/vojtehg72-48818
3. Z. Tang, Y. Li, H. Deng, The Euler Sombor index of a graph, *Int. J. Quantum Chem.*, **124** (2024), e27387. https://doi.org/10.1002/qua.27387
4. I. Redžepović, Chemical applicability of Sombor indices, *J. Serb. Chem. Soc.*, **86** (2021), 445–457. https://doi.org/10.2298/JSC201215006R
5. Q. Zhang, F. Tchier, Z. Hussain, A. Aslam, X. Zuo, On the bounds for the Euler Sombor index of molecular trees and its applications, *J. Appl. Math. Comput.*, **71** (2025), 645–663. https://doi.org/10.1007/s12190-025-02506-z
6. B. Khanra, S. Das, Euler Sombor index of trees, unicyclic and chemical graphs, *MATCH Commun. Math. Comput. Chem.*, **94** (2025), 525–548. https://doi.org/10.46793/match.94-2.525K
7. A. P. Tache, R. M. Tache, I. Stroe, Extremal unicyclic graphs for the Euler Sombor index, *MATCH Commun. Math. Comput. Chem.*, **94** (2025), 561–578. https://doi.org/10.46793/match.94-2.561T
8. Z. Su, Z. Tang, Extremal unicyclic and bicyclic graphs of the Euler Sombor index, *AIMS Mathematics*, **10** (2025), 6338–6354. https://doi.org/10.3934/math.2025289
9. G. Ö. Kızılırmak, On Euler Sombor index of tricyclic graphs, *MATCH Commun. Math. Comput. Chem.*, **94** (2025), 247–262. https://doi.org/10.46793/match.94-1.247K
10. A. M. Albalahi, A. M. Alanazi, A. M. Alotaibi, A. E. Hamza, A. Ali, Optimizing the Euler-Sombor index of (molecular) tricyclic graphs, *MATCH Commun. Math. Comput. Chem.*, **94** (2025), 549–560. https://doi.org/10.46793/match.94-2.549A

11. Y. Hu, J. Fang, Y. Liu, Z. Lin, Bounds on the Euler Sombor index of maximal outerplanar graphs, *Electron. J. Math.*, **8** (2024), 39–47. https://doi.org/10.47443/ejm.2024.053
12. P. Das, S. Mondal, A. Pal, Extremal trees for Euler Sombor index with given parameters, *MATCH Commun. Math. Comput. Chem.*, **95** (2026), 777–811. https://doi.org/10.46793/match.95-3.14525
13. K. C. Das, J. Bera, Resolving open problems on the Euler Sombor index, *MATCH Commun. Math. Comput. Chem.*, **95** (2026), 901–918. https://doi.org/10.46793/match.95-3.16625
14. A. Ali, A. M. Albalahi, A. M. Alanazi, A. A. Bhatti, T. Alraqad, H. Saber, et al., On bond incident degree indices of fixed-size bicyclic graphs with given matching number, *Mathematics*, **12** (2024), 3806. https://doi.org/10.3390/math12233806
15. Z. Raza, S. Akhter, S. Mondal, M. Ghorbani, Maximal tree and unicyclic graph for Euler Sombor index with given diameter, *AKCE Int. J. Graphs Comb.*, **22** (2025), 289–304. https://doi.org/10.1080/09728600.2025.2517077
16. S. Sekar, S. Balachandran, S. Elumalai, H. Liu, Maximum Euler Sombor index of unicyclic graphs with given diameter, *MATCH Commun. Math. Comput. Chem.*, **95** (2026), 759–776. https://doi.org/10.46793/match.95-3.12225



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)