AIMS Mathematics, 11(11): 38895–38932.
DOI: 10.3934/math.20261500
Received: 20 August 2026
Revised: 15 October 2026
Accepted: 26 October 2026
Published: 20 November 2026

https://www.aimspress.com/journal/Math

Research article

Modeling uncertainties in the tuberculosis-brucellosis coinfection with environmental transmission and Lévy noise

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Abstract: Existing mathematical studies of tuberculosis–brucellosis co-infection have provided valuable insights into disease transmission and control; however, many existing models do not simultaneously account for environmental transmission and sudden, extreme fluctuations that may affect the disease dynamics. In particular, the role of environmental contamination and rare stochastic events in shaping co-infections' persistence and outbreak patterns remains insufficiently explored. To address these limitations, we develop a stochastic model to study the dynamics of tuberculosis-brucellosis coinfection in the animal population. The model incorporates environmental load, which shows indirect transmission, as well as Lévy noise to represent sudden changes, such as increases or outbreaks. The total population is divided into different classes, including susceptible, infected with both diseases, recovered, and environmental compartments with no coinfection. The model is analyzed to determine the impact of stochastic fluctuations and environmental contamination on the disease dynamics, including the probability of outbreaks and potential disease persistence in human and animal populations with their corresponding $\mathcal{R}_0^s$ for different situations. Numerical simulations are obtained that illustrate how environmental load and rare extreme events influence infection peaks, disease prevalence, and the effectiveness of control strategies. The proposed study provides a stochastic approach for understanding the combined effects of environmental contamination and random fluctuations.

Keywords: stochastic tuberculosis-brucellosis coinfection model; environmental load; Lévy noise; livestock epidemiology; disease dynamics; extinction and persistence; basic reproductive number
Mathematics Subject Classification: 60H10, 92D30, 60G51, 60J27

1. Introduction

Brucellosis is an infectious disease caused by *Brucella* bacteria and mainly affects animals but it can spread to humans so it is considered a zoon-otic disease. Similarly tuberculosis is a contagious bacterial disease of animals. It can spread among animals through close contact and respiratory secretions. The scientific understanding of tuberculosis advanced substantially following Robert Koch's identification of the tubercle bacillus in 1882 [1], while the modern history of brucellosis began with David Bruce's isolation of the causative organism of Malta fever in 1887. It was initially reported in the United Kingdom in the 19th century [2, 3].

Tuberculosis and brucellosis remain significant public health concerns worldwide, particularly in developing and low-resource regions where socioeconomic, environmental, and healthcare challenges facilitate the diseases' persistence. Tuberculosis, caused primarily by *Mycobacterium tuberculosis* is one of the important sources of death from transmissible infections, affecting millions of individuals yearly despite significant advances in diagnosis, treatment, and prevention [4]. Similarly, brucellosis is a widespread disease caused by the *brucella* virus that can spread from animals to animals and from animals to humans through direct contact with infected animals and use of unpasteurized dairy products, although transmission from humans to animals is rare [5–7]. The disease imposes considerable health and economic burdens, particularly in regions where livestock farming constitutes a major component of rural livelihoods [8]. The coexistence of tuberculosis and brucellosis in many endemic regions raises important epidemiological concerns regarding coinfection dynamics [9]. Co-infected individuals may experience more severe disease progression and complications than those infected with a single pathogen, emphasizing the need for a comprehensive understanding of the interaction between these infectious diseases [10]. In 2008, evidence of simultaneous infection with bovine tuberculosis and brucellosis in slaughtered cattle in Ibadan, Nigeria was reported, demonstrating that these two infections can occur concurrently within the same cattle population [11]. Cowie et al. identified both the shared and disease-specific risk factors for bovine tuberculosis and brucellosis in cattle, reporting that intra- and inter-herd cattle contact was associated with both diseases, while wildlife exposure was specifically associated with tuberculosis and cattle movement between farms associated with brucellosis [12]. Gorsich et al. combined longitudinal data from African buffalo with a mathematical model to investigate the effects of bovine tuberculosis and brucellosis coinfection at both the individual and population scales [13]. Lokika et al. developed a mathematical coinfection model for bovine tuberculosis and brucellosis that incorporated seasonal transmission and investigated the effectiveness of culling infected animals as a control strategy [14]. Kumpai and Modnak developed a deterministic mathematical model for bovine tuberculosis and brucellosis coinfection and investigated its reproduction numbers, equilibrium properties, and optimal cost-effective control strategies [15]. Mathematical modeling is the best tool to study the dynamics of infectious diseases and also evaluate possible control policies. Historically, deterministic

compartmental models have been used to study the dynamics of infectious diseases, equilibrium points, and the basic reproduction number [16–19]. Both diseases exhibit chronic characteristics and share several risk factors, including poverty, occupational exposure, improper sanitation systems, and very poor access to healthcare services [20]. Clinical manifestations of brucellosis often overlap with those of tuberculosis, making differential diagnosis challenging and potentially resulting in delayed treatment and increased transmission [21]. In reality, disease transmission is influenced by numerous sources of uncertainty, including climatic variability, demographic changes, migration, behavioral adaptations, and public health interventions. Such uncertainties may significantly alter the transmission rates and disease outcomes [22, 23].

Although previous studies have made important contributions to the mathematical understanding of tuberculosis-brucellosis coinfection, most existing coinfection models are formulated within a deterministic framework. However, deterministic models generally assume that the epidemiological parameters remain fixed and therefore do not explicitly account for random environmental fluctuations, demographic variability, or sudden epidemiological disturbances. To address these limitations, stochastic epidemic models have emerged as a very accurate structure for describing transmissible infection dynamics under uncertainty. Stochastic epidemic models have demonstrated that environmental noise can substantially alter the disease persistence and extinction thresholds compared with their deterministic counterparts. Most stochastic epidemic models use Brownian motion to represent continuous environmental fluctuations. However, many epidemiological systems are also subject to sudden and unpredictable events such as natural disasters, class displacement, livestock movements, abrupt policy changes, and environmental contamination incidents [24, 25]. These events may induce discontinuous changes in disease dynamics that cannot be adequately represented by Gaussian noise alone. Lévy processes provide a natural mathematical framework for incorporating jump-type perturbations into epidemic models [26–28].

Stochastic differential equations incorporate random perturbations into epidemiological processes and allow researchers to investigate diseases' persistence, extinction, and long-term behavior in fluctuating environments [29, 30]. The inclusion of Lévy noise has attracted considerable attention in recent years because it enables the modeling of rare but impaction events and provides a more realistic representation of complex epidemiological systems [31–33]. Environmental transmission may therefore sustain disease circulation and complicate eradication efforts. Mathematical models incorporating environmental compartments have demonstrated the significant influence of environmental reservoirs on disease persistence and outbreak dynamics [34, 35]. In particular, mathematical models have been extensively used to investigate tuberculosis-related dynamics, reinfection mechanisms, treatment strategies, and coinfection phenomena [36–39]. However, deterministic models assume that epidemiological processes evolve under fixed environmental conditions and therefore fail to capture the random fluctuations that naturally occur in real-world disease systems [40]. Another important feature of infectious disease transmission is the role of environmental reservoirs.

Environmental contamination can facilitate indirect disease transmission and contribute to pathogens' persistence even when direct transmission pathways are reduced. This mechanism is particularly relevant for brucellosis, where *Brucella* species can survive in soil, water, animal waste, and contaminated farm environments for extended periods [41]. The integration of coinfection dynamics, environmental transmission, and stochastic perturbations represents an important

advancement in infectious disease modeling. Coinfection systems are inherently complex because pathogen interactions, host responses, and environmental factors collectively influence the diseases outcomes. Environmental reservoirs may enhance diseases' persistence, while stochastic fluctuations and sudden jump disturbances may alter outbreak trajectories and extinction probabilities. Consequently, mathematical frameworks that simultaneously account for these factors are essential for accurately characterizing disease dynamics and informing effective public health interventions [42–44].

The motivation in this study is to better understand the joint spread of tuberculosis and brucellosis under realistic environmental uncertainties. This study develops a stochastic coinfection model incorporating environmental transmission and Lévy noise. The model examines the effects of random fluctuations, environmental contamination, vaccination, and sudden disturbances on the disease dynamics. By analyzing solutions' existence, disease persistence, and extinction conditions. This study provides theoretical insights that can support more effective disease monitoring and control strategies while advancing the theory of stochastic epidemic modeling. We extend Model (2.1) to a hybrid stochastic (susceptible–infected–recovered–vaccinated) model for tuberculosis–brucellosis coinfection dynamics with multi type noise and environmental load. This approach improves upon existing models by using multiple techniques of stochasticity and environmental transmission pathways, offering a more realistic demonstration of disease dynamics in livestock. We analyze the model to explore the extinction and persistence conditions, basic reproduction numbers, and the environmental impact on the diseases' behavior, supported by sensitivity and numerical analysis.

The aim of this study are outlined below.

- To develop a biologically realistic stochastic (susceptible–infected–recovered–vaccinated) tuberculosis-Brucellosis infection disease model with different noise and environmental impact.
- To study the model validation, we will use different types of theoretical results in the terms of positivity, sensitivity, and stability by using stochastic techniques, including Itô's formula, stopping time arguments, and suitable Lyapunov functions.
- We will analyze the transmission of the disease when one or both diseases are present or die out from the population by using the condition of $\mathcal{R}_0$,
- To determine the long-term limits of the infected classes and investigate their variability under stochastic effects,
- To verify the theoretical results, we will use some numerical examples and apply the Milstein scheme for simulations.

2. Stochastic model construction

The mechanism and study of the model are highlighted in Figure 1. In the dynamic system in (2.1), $\mathcal{S}(\mathtt{t})$ represents the susceptible animals population, $\mathcal{I}_1(t)$ is the brucellosis-infected animal class, $\mathcal{I}_2(t)$ is the tuberculosis co-infected animal class, $\mathcal{R}(t)$ represents recovered animal, and $(\mathcal{W}(\mathtt{t}))$ is the environmental load. Moreover, ϱ is the ratio of the brucellosis infected class, and $(1-\rho)$ represents the percentage of the tuberculosis infected class. The remaining parameters are presented the Table 1.

$$\begin{cases} \dot{\mathcal{S}} = \omega - \xi^* \mathcal{I}_1(t)\mathcal{S}(\mathtt{t}) - \beta^* \mathcal{S}(\mathtt{t})\mathcal{I}_2(t) - \gamma \mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t}) - (\psi + \phi)\mathcal{S}(\mathtt{t}), \\ \dot{\mathcal{I}}_1 = \varrho\Big(\xi^* \mathcal{I}_1(t)\mathcal{S}(t) + \beta^* \mathcal{S}(\mathtt{t})\mathcal{I}_2(t) + \gamma \mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t})\Big) - (\phi + \xi_1^*)\mathcal{I}_1(t), \\ \dot{\mathcal{I}}_2 = (1-\varrho)\Big(\xi^* \mathcal{I}_1(t)\mathcal{S}(\mathtt{t}) + \beta^* \mathcal{S}(\mathtt{t})\mathcal{I}_2(t) + \gamma \mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t})\Big) + \rho\xi_1^* \mathcal{I}_1(t) - (\delta_1^* + \xi_2^* + \phi)\mathcal{I}_2(t), \\ \dot{\mathcal{R}} = \xi_1^*(1-\rho)\mathcal{I}_1(\mathtt{t}) + \xi_2^* \mathcal{I}_2(t) + \psi \mathcal{S}(\mathtt{t}) - \phi \mathcal{R}(\mathtt{t}), \\ \dot{\mathcal{W}} = \lambda_1 \mathcal{I}_1(\mathtt{t}) + \lambda_2 \mathcal{I}_2(t) - \delta^* \mathcal{W}(\mathtt{t}). \end{cases} \tag{2.1}$$

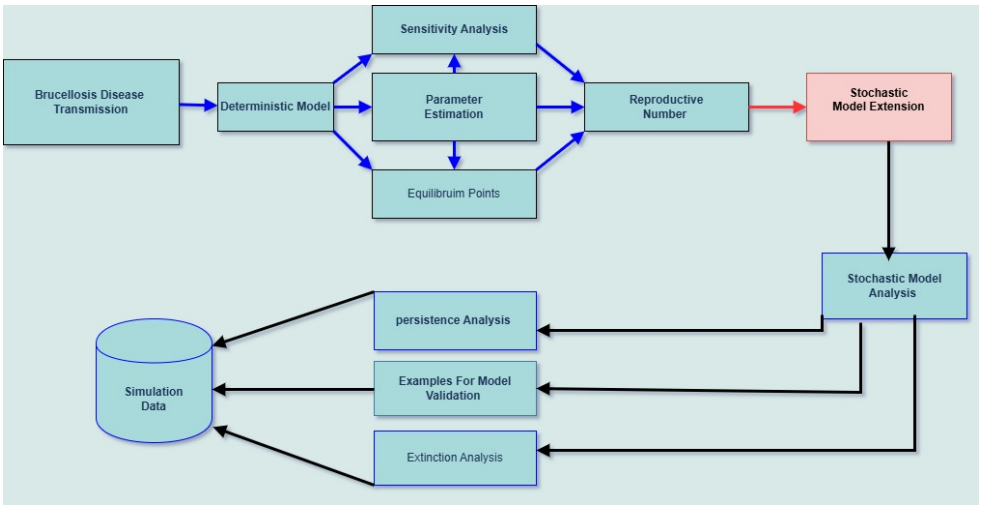


Figure 1. Schematic flowchart of the proposed methodology.

Table 1. Definitions of the model parameters and their meanings.

Notations	Meanings	Units
ω	Recruitment rate of newborn susceptible animals into the class	animals/year
ξ	Transmission rate of brucellosis infection I_1	year^{-1}
β^*	Transmission rate of tuberculosis infection I_2	year^{-1}
γ	Contribution rate to the environmental pathogen load $\mathcal{W}$	year^{-1}
ψ	Vaccination rate	year^{-1}
ϕ	Natural mortality rate per unit of time from each class	year^{-1}
ξ_1^*	$(1-\rho)$ is the recovery rate from brucellosis, and ρ is the progression rate from I_1 (tuberculosis) to brucellosis I_2	year^{-1}
δ_1^*	Indicates the disease-induced death rate associated with I_2 (brucellosis)	year^{-1}
ξ_2^*	Recovery rate from tuberculosis infection	year^{-1}
δ^*	Natural decay rate of the environmental pathogen load $\mathcal{W}$	year^{-1}
λ_1	Environmental transmission rate of brucellosis.	year^{-1}
λ_2	Environmental transmission rate of brucellosis.	year^{-1}

The equilibrium points for the deterministic model are given by $\mathbf{E}_0 = \left\{\frac{\omega}{\psi+\phi}, 0, 0, \frac{\phi\omega}{\phi(\psi+\phi)}, 0\right\}$, and its basic reproductive number can be calculated by using the next-generation matrix method of Diekmann, as given below with $\mathscr{F}(\mathscr{V})^{-1} = \left(\frac{\partial \mathscr{F}_i}{\partial x_i}\right)\left(\frac{\partial \mathscr{V}_i}{\partial x_i}\right)^{-1}$

$$\mathscr{F} = \begin{pmatrix} \varrho\left(\xi^*\mathcal{I}_1\mathcal{S} + \beta^*\mathcal{I}_2\mathcal{S} + \gamma\mathcal{S}\mathcal{W}\right) \\ (1-\varrho)\left(\xi^*\mathcal{I}_1\mathcal{S} + \beta^*\mathcal{I}_2\mathcal{S} + \gamma\mathcal{S}\mathcal{W}\right) \\ 0 \end{pmatrix}, \quad \mathscr{V} = \begin{pmatrix} (\phi+\xi_1^*)\mathcal{I}_1 \\ (\delta_1^*+\xi_2^*+\phi)\mathcal{I}_2 - \rho\xi_1^*\mathcal{I}_1 \\ \delta^*\mathcal{W} - \lambda_1\mathcal{I}_1 - \lambda_2\mathcal{I}_2 \end{pmatrix}.$$

By calculating the Jacobian of $\mathscr{F}$ and $\mathscr{V}$ as follows:

$$\mathscr{F} = \begin{pmatrix} \varrho\xi^*\mathcal{S}_0 & \varrho\beta^*\mathcal{S}_0 & \gamma\varrho\mathcal{S}_0 \\ (1-\varrho)\xi^*\mathcal{S}_0 & (1-\varrho)\beta^*\mathcal{S}_0 & \gamma(1-\varrho)\mathcal{S}_0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathscr{V}^{-1} = \begin{pmatrix} \frac{1}{\phi+\xi_1^*} & 0 & 0 \\ \frac{\rho\xi_1^*}{(\phi+\xi_1^*)(\delta_1^*+\xi_2^*+\phi)} & \frac{1}{\delta_1^*+\xi_2^*+\phi} & 0 \\ \frac{\lambda_1(\delta_1^*+\xi_2^*+\phi)+\lambda_2\rho\xi_1^*}{(\phi+\xi_1^*)(\delta_1^*+\xi_2^*+\phi)\delta^*} & \frac{\lambda_2}{(\delta_1^*+\xi_2^*+\phi)\delta^*} & \frac{1}{\delta^*} \end{pmatrix}.$$

$$\mathscr{F}\mathscr{V}^-1 = \begin{pmatrix} \varrho\xi^*\mathcal{S}_0 & \varrho\beta^*\mathcal{S}_0 & \gamma\varrho\mathcal{S}_0 \\ (1-\varrho)\xi^*\mathcal{S}_0 & (1-\varrho)\beta^*\mathcal{S}_0 & \gamma(1-\varrho)\mathcal{S}_0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\phi+\xi_1^*} & 0 & 0 \\ \frac{\rho\xi_1^*}{(\phi+\xi_1^*)(\delta_1^*+\xi_2^*+\phi)} & \frac{1}{\delta_1^*+\xi_2^*+\phi} & 0 \\ \frac{\lambda_1(\delta_1^*+\xi_2^*+\phi)+\lambda_2\rho\xi_1^*}{(\phi+\xi_1^*)(\delta_1^*+\xi_2^*+\phi)\delta^*} & \frac{\lambda_2}{(\delta_1^*+\xi_2^*+\phi)\delta^*} & \frac{1}{\delta^*} \end{pmatrix} =$$

$$\begin{pmatrix} \frac{\varrho\xi^*\mathcal{S}_0}{\phi+\xi_1^*} + \frac{\varrho\beta^*\mathcal{S}_0\rho\xi_1^*}{(\phi+\xi_1^*)(\delta_1^*+\xi_2^*+\phi)} + \frac{\gamma\varrho\mathcal{S}_0\left[\lambda_1(\delta_1^*+\xi_2^*+\phi)+\lambda_2\rho\xi_1^*\right]}{(\phi+\xi_1^*)(\delta_1^*+\xi_2^*+\phi)\delta^*} & \frac{\varrho\beta^*\mathcal{S}_0}{\delta_1^*+\xi_2^*+\phi} + \frac{\gamma\varrho\mathcal{S}_0\lambda_2}{(\delta_1^*+\xi_2^*+\phi)\delta^*} & \frac{\gamma\varrho\mathcal{S}_0}{\delta^*} \\ \frac{(1-\varrho)\xi^*\mathcal{S}_0}{\phi+\xi_1^*} + \frac{(1-\varrho)\beta^*\mathcal{S}_0\rho\xi_1^*}{(\phi+\xi_1^*)(\delta_1^*+\xi_2^*+\phi)} + \frac{\gamma(1-\varrho)\mathcal{S}_0\left[\lambda_1(\delta_1^*+\xi_2^*+\phi)+\lambda_2\rho\xi_1^*\right]}{(\phi+\xi_1^*)(\delta_1^*+\xi_2^*+\phi)\delta^*} & \frac{(1-\varrho)\beta^*\mathcal{S}_0}{\delta_1^*+\xi_2^*+\phi} + \frac{\gamma(1-\varrho)\mathcal{S}_0\lambda_2}{(\delta_1^*+\xi_2^*+\phi)\delta^*} & \frac{\gamma(1-\varrho)\mathcal{S}_0}{\delta^*} \\ 0 & 0 & 0 \end{pmatrix}.$$

By using the equilibrium point at $\mathcal{S}_0 = \frac{\omega}{\psi+\phi}$ for the necessary calculation, with $i \in [1,2]$ being the basic reproduction number, Model (2.1) can be written as $\mathcal{R}_0^d = \rho(\mathscr{F}\mathscr{V}^{-1})$.

$$\mathcal{R}_i^d = \frac{\omega\,\xi^*\{\beta^*(\rho\xi_1^* + \phi + \xi_1^* + \delta_1^* + \xi_2^* + \phi)\}}{(\psi+\xi_1^*)(\phi+\psi)(\delta_1^*+\xi_2^*+\phi)} + \frac{\gamma\omega}{\delta^*(\phi+\psi)}\left\{\frac{\lambda_1\varrho}{\phi+\xi_1^*} + \frac{\lambda_2(1-\rho)}{\delta_1^*+\xi_2^*+\phi}\right\}. \tag{2.2}$$

Figure 2 show the disease dynamics.

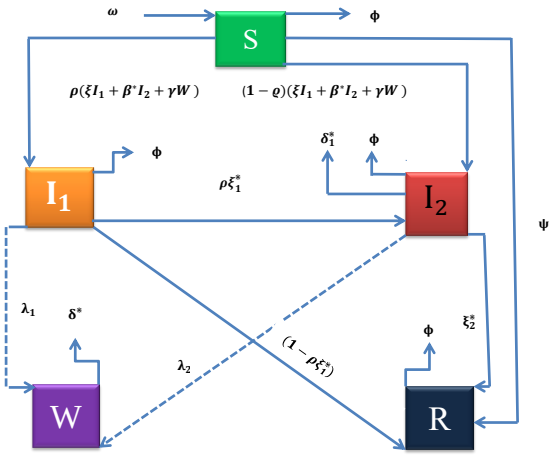


Figure 2. A schematic representation of the disease transmission dynamics.

$$
\begin{cases}
\dot{\mathcal{S}} = \Big[\omega_{r(\mathtt{t})} - \xi^*_{r(\mathtt{t})}\mathcal{I}_1(t)\mathcal{S}(\mathtt{t}) - \beta^*_{r(\mathtt{t})}\mathcal{I}_2(t)\mathcal{S}(\mathtt{t}) - \gamma_{r(\mathtt{t})}\mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t}) - (\psi_{r(\mathtt{t})} + \phi_{r(\mathtt{t})})\mathcal{S}(\mathtt{t})\Big]\mathtt{dt} \\
\qquad + \bar{\sigma}_{1,r(\mathtt{t})}\mathcal{S}(\mathtt{t})\, d\mathscr{W}_1(\mathtt{t}) + \int_Z K_{1,r(\mathtt{t})}\mathcal{S}_{(\mathtt{t}^-)}(\tilde{z})\, \widetilde{Q}_c(\mathtt{dt}, \mathtt{d}\tilde{z}), \\
\dot{\mathcal{I}}_1 = \Big[\varrho\Big(\xi^*_{r(\mathtt{t})}\mathcal{I}_1(t)\mathcal{S}(\mathtt{t}) + \beta^*_{r(\mathtt{t})}\mathcal{I}_2(t)\mathcal{S}(\mathtt{t}) + \gamma_{r(\mathtt{t})}\mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t})\Big) - (\phi_{r(t)} + \xi^*_{1,r(t)})\mathcal{I}_1(t)\Big]\mathtt{dt} \\
\qquad + \bar{\sigma}_{2,r(\mathtt{t})}\mathcal{I}_1(t)\, d\mathscr{W}_2(\mathtt{t}) + \int_Z K_{2,r(\mathtt{t})}\mathcal{I}_{1(\mathtt{t}^-)}(\tilde{z})\, \widetilde{Q}_c(\mathtt{dt}, \mathtt{d}\tilde{z}), \\
\dot{\mathcal{I}}_2 = \Big[(1-\varrho_{r(\mathtt{t})})\Big(\xi^*_{r(\mathtt{t})}\mathcal{I}_1(t)\mathcal{S}(\mathtt{t}) + \beta^*_{r(\mathtt{t})}\mathcal{I}_2(t)\mathcal{S}(\mathtt{t}) + \gamma_{r(\mathtt{t})}\mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t})\Big) + \rho\,\xi^*_{1,r(\mathtt{t})}\mathcal{I}_1(t) \\
\quad - (\delta^*_{1,r(\mathtt{t})} + \xi^*_{2,r(\mathtt{t})} + \phi_{r(\mathtt{t})})\mathcal{I}_2(t)\Big]\mathtt{dt} + \bar{\sigma}_{3,r(\mathtt{t})}\mathcal{I}_2(t)\, d\mathscr{W}_3(\mathtt{t}) + \int_Z K_{3,r(\mathtt{t})}\mathcal{I}_{2(\mathtt{t}^-)}(\tilde{z})\, \widetilde{Q}_c(\mathtt{dt}, \mathtt{d}\tilde{z}), \\
\dot{\mathcal{R}} = \Big[(1-\rho)\xi_{1,r(\mathtt{t})}\mathcal{I}_1(t) + \xi^*_{2,r(\mathtt{t})}\mathcal{I}_2(t) + \nu_{r(\mathtt{t})}\mathcal{S}(\mathtt{t}) - \phi_{r(\mathtt{t})}\mathcal{R}(t)\Big]\mathtt{dt} + \bar{\sigma}_{4,r(\mathtt{t})}\mathcal{R}(t)\, d\mathscr{W}_4(\mathtt{t}) \\
\qquad + \int_Z K_{4,r(\mathtt{t})}\mathcal{R}_{(\mathtt{t}^-)}(\tilde{z})\, \widetilde{Q}_c(\mathtt{dt}, \mathtt{d}\tilde{z}), \\
\dot{\mathcal{W}} = \Big[\lambda_{1,r(\mathtt{t})}\mathcal{I}_1(t) + \lambda_{2,r(\mathtt{t})}\mathcal{I}_2(t) - \delta^*_{r(\mathtt{t})}\mathcal{W}(\mathtt{t})\Big]\mathtt{dt} + \bar{\sigma}_{5,r(\mathtt{t})}\mathcal{W}(\mathtt{t})\, d\mathscr{W}_5(\mathtt{t}) + \int_Z K_{5,r(\mathtt{t})}\mathcal{W}(\mathtt{t}^-)(\tilde{z})\, \widetilde{Q}_c(\mathtt{dt}, \mathtt{d}\tilde{z}).
\end{cases}
\tag{2.3}
$$

In [45], the (susceptible–infected–recovered–susceptible) epidemic model was examined and extended by incorporating different types of noise to account for the disease dynamics. These results were subsequently extended to bounded-state Markov chain settings. Now, in this work, the author incorporates different jump terms and environmental fluctuations; the deterministic tuberculosis-brucellosis model is reformulated as a system of stochastic differential equations. In Model (2.4) the parameter $\bar{\sigma}_i$ represents the intensity of the stochastic disturbance. The stochastic processes the term $\mathscr{W}_i(\mathtt{t})\quad \forall i \in \{1...5\}$ represents are the stranded Brownian motion with $\mathscr{W}_i(0) = 0$, because the system has infinitely many small variations that cannot be measured separately. The term $\mathscr{K}_i$ denotes the jumps measured in the time t. $\tilde{Q}_{\mathtt{c}}(\mathtt{dt}, \mathtt{dx}) = Q(\mathtt{dt}, \mathtt{dx}) - \nu(\mathtt{dt}, \mathtt{dx})\,\mathtt{dt}, \mathtt{dt}$. In this expression, the $\tilde{Q}_c$ compensated Poisson measure, while $\nu(.)$ is the intensity measure, $dt\ \tilde{z}$ represents the magnitude or size of each jump, while the intensity measure represents the time. In addition,

$\mathcal{S}(\mathtt{t}^-)$, $\mathcal{I}_1(t^-)$, $\mathcal{I}_2(t^-)$, $\mathcal{R}(t^-)$ and $\mathcal{W}(\mathtt{t}^-)$ are the left limits of the $\mathcal{S}(\mathtt{t})$, $\mathcal{I}_1(t)$, $\mathcal{I}_2(t)$, $\mathcal{R}(t)$, and $\mathcal{W}(\mathtt{t})$. This model can be applied to study and predict the dynamics of tuberculosis and brucellosis coinfection in livestock populations, mostly when environmental contamination contributes to the diseases' spread and its control strategies such as vaccination, sanitation, and livestock management.

Meanwhile, the Markov chain $r(t)$ randomly transitions among the states in $\mathbb{S}$, and we consider the epidemic model under a fixed period $l \in \mathbb{S}$. Specifically, by assuming that $r(t) = l$, the stochastic epidemic model corresponding to the regime l is obtained by

$$\begin{cases} \dot{\mathcal{S}} = \Big[\omega_l - \xi_l^* \mathcal{I}_2(t)\mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t}) - \beta_l^* \mathcal{I}_2(t)\mathcal{S}(\mathtt{t}) - \gamma_l \mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t}) - (\psi_l + \phi_l)\mathcal{S}(\mathtt{t})\Big]\mathtt{dt} \\ \qquad + \bar{\sigma}_{1,l}\mathcal{S}(\mathtt{t})\,\mathtt{d}\mathscr{W}_1(\mathtt{t}) + \int_Z \mathscr{K}_{1,l}\mathcal{S}_{(\mathtt{t}^-)}(\tilde{z})\,\widetilde{Q}_c(\mathtt{dtd}\tilde{z}), \\ \dot{\mathcal{I}}_1 = \Big[\varrho(\beta_l^* \mathcal{I}_2(t)\mathcal{S}(\mathtt{t}) + \xi_l^* \mathcal{I}_2(t)\mathcal{S}(\mathtt{t}) + \gamma_l \mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t})) - (\phi + \xi_{1,l}^*)\mathcal{I}_2(t)\Big]\mathtt{dt} \\ \qquad + \bar{\sigma}_{2,l}\mathcal{I}_2(t)\,\mathtt{d}\mathscr{W}_2(\mathtt{t}) + \int_Z \mathscr{K}_{2,l}\mathcal{I}_{1(\mathtt{t}^-)}(\tilde{z})\,\widetilde{Q}_c(\mathtt{dtd}\tilde{z}), \\ \dot{\mathcal{I}}_2 = \Big[(1-\varrho)(\xi_l^* \mathcal{I}_2(t)\mathcal{S}(\mathtt{t}) + \beta_l^* \mathcal{I}_2(t)\mathcal{S}(\mathtt{t}) + \gamma_l \mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t})) + \rho\xi_{1,l}^* \mathcal{I}_2(t) \\ \qquad - (\delta_{1,l}^* + \xi_{2,l}^* + \phi_l)\Big]\mathtt{dt} + \bar{\sigma}_{3,l}\mathcal{I}_2(t)\,\mathtt{d}\mathscr{W}_3(\mathtt{t}) + \int_Z \mathscr{K}_{3,l}\mathcal{I}_{2(\mathtt{t}^-)}(\tilde{z})\,\widetilde{Q}_c(\mathtt{dtd}\tilde{z}), \\ \dot{\mathcal{R}} = \Big[(1-\rho)\xi_{1,l}^* \mathcal{I}_2(t) + \xi_2^* \mathcal{I}_2(t) + \psi_l \mathcal{S}(\mathtt{t}) - \psi\mathcal{R}(t)\Big]\mathtt{dt} + \bar{\sigma}_{4,l}\mathcal{R}(t)\,\mathtt{d}\mathscr{W}_4(\mathtt{t}) + \int_Z \mathscr{K}_{4,l}\mathcal{R}_{(\mathtt{t}^-)}(\tilde{z})\,\widetilde{Q}_c(\mathtt{dtd}\tilde{z}), \\ \dot{\mathcal{W}} = \Big[\lambda_{1,l}\mathcal{I}_2(t) + \lambda_{2,l}\mathcal{I}_2(t) - \delta^*\mathcal{W}(\mathtt{t})\Big]\mathtt{dt} + \bar{\sigma}_{5,l}\mathcal{W}(\mathtt{t})\,\mathtt{d}\mathscr{W}_5(\mathtt{t}) + \int_Z \mathscr{K}_{5,l}\mathcal{W}_{(\mathtt{t}^-)}(\tilde{z})\,\widetilde{Q}_c(\mathtt{dtd}\tilde{z}). \end{cases} \tag{2.4}$$

2.1. Analytical study model

This section investigates the well-posedness and dynamic properties of the constructed stochastic mathematical epidemiological model (2.4). Initially, we show the positive existence of a positive global solution that remains bounded for all $\mathtt{t} \geq 0$. Next, we analyze the diseases' extinction and persistence in the population, using the condition $\mathcal{R}_0$ and examining the asymptotic behavior of the system.

2.2. Basic notions and applications

This subsection contains some basic definitions and results, which are very important for the better understanding of the next section. Initially ,we define the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_\mathtt{t}\}_{\mathtt{t}\geq 0}, \mathbf{P})$, which obeys the constraint that ($\mathcal{F}_0$ belongs to $\mathbf{P}$ null sets. There is a generator square matrix $\vartheta = (\theta_{pq})_{m\times m}$ for the Markov chain $r(r)$, which can be represented in the following way:

$$\mathbf{P}_{pq}(\Delta) = \Pr\{r(\mathtt{t}+\Delta) = q \mid r(\mathtt{t}) = p\} \begin{cases} \theta_{pq}\Delta + o(\Delta), & p \neq q, \\ 1 + \theta_{pp}\Delta + o(\Delta), & p = q, \end{cases}$$

where $o(\Delta) > 0$, and $\theta_{pq} > 0$ is the transition rate from state p to state q, and $p, q \in \mathbb{S}$. Furthermore, the unique stationary distribution is defined as $\Theta = (\Theta_1, \Theta_2, \ldots, \Theta_m)$. This distribution behaves like the Markov chain $r(t)$ which satisfies the condition of $\Theta\vartheta = 0$ and $\sum_{p=1\ q\neq p}^{m} \Theta_{pq} = 1, \quad p \in \mathbb{S}$. Suppose that $Z(t) \in \mathscr{R}$ is the trajectory of the given stochastic differential equations

$$\mathtt{d}x(t) = \mathbf{P}_1(x(t), \mathtt{t})\,dt + \mathbf{P}_2(y(t)), \mathtt{t})\,\mathtt{d}W(\mathtt{t}) + \int_Y \mathbf{P}_3(x(\mathtt{t}^-), \mathtt{t}, y)\,\widetilde{Q}(\mathtt{dt}, \mathtt{d}y), \tag{2.5}$$

where $\mathbf{P}_1 : \mathscr{R}^n \times \mathscr{R}^n \times \mathscr{R}_+ \times \mathscr{M} \to \mathscr{R}^n$, $\mathbf{P}_2 : \mathscr{R}^n \times \mathscr{R}^n \times \mathscr{R}_+ \times \mathscr{M} \to \mathscr{R}^{n\times d}$, and $\mathbf{P}_3 : \mathscr{R}^n \times \mathscr{R}^n \times \mathscr{M} \times \mathscr{R}_+ \times X \to \mathscr{R}^n$ are the measurable functions, and $x(\mathtt{t})$ is the limit of $x(\mathtt{t}^-)$. For $\mathtt{t} \in [0, \mathscr{T}]$, $W(.)$ is the D-dimensional Brownian motion and $\widetilde{Q}(dt, dy)$ is the 1-dimensional Poisson process. For the given function $\mathscr{G} \in (\mathbf{C}^{1,2}(\mathscr{R}^n \times \mathscr{R}_+; \mathscr{R}_+)$, the operator $\bar{\mathscr{L}}$ for the stochastic process in the proposed model (2.4) can be defined as:

$$\bar{\mathscr{L}}\mathscr{G}(x,\mathtt{t}) = \mathscr{G}_{\mathtt{t}}(x,\mathtt{t},l) + \sum_{j=1}^{m} \Theta_{lj}\mathscr{G}(x,\mathtt{t},j) + \mathscr{G}_x(x,\mathtt{t},l)\mathbf{P}_1(x,\mathtt{t},l) + \frac{1}{2}\mathbf{P}_2^T(x,\mathtt{t},l)\,\mathscr{G}_{xx}(x,\mathtt{t},l)\,\mathbf{P}_2(x,\mathtt{t},l)$$
$$+ \int_X \Big[\mathscr{G}(x + \mathbf{P}_3(x,\mathtt{t},l),\mathtt{t},l) - \mathscr{G}(x,\mathtt{t},l) - \mathscr{G}_x(x,\mathtt{t},l)\mathbf{P}_3(x,\mathtt{t},l)\Big]\nu(dx).$$

$$\mathscr{G}_{\mathtt{t}} = \frac{\partial \mathscr{G}}{\partial \mathtt{t}}, \qquad \mathscr{G}_x = \left(\frac{\partial \mathscr{G}}{\partial x_1}, \frac{\partial \mathscr{G}}{\partial x_2}, \ldots, \frac{\partial \mathscr{G}}{\partial x_n}\right), \mathscr{G}_{xx} = \left(\frac{\partial^2 \mathscr{G}}{\partial x_i \partial x_j}\right)_{n\times n}, \quad 1 \le i, j \le n.$$

Itô's formula, which explains (2.1) in [46] can be written with jump terms as follows:

$$\mathscr{G}(x,\mathtt{t}) = \bar{\mathscr{L}}\mathscr{G}(x,\mathtt{t},l) + \mathscr{G}_x(x,\mathtt{t},l)\mathbf{P}_1(x,\mathtt{t},l)W_{(1)} + \int_X \Big[\mathscr{G}(x + \mathbf{P}_3(x,\mathtt{t},l),\mathtt{t},l) - \mathscr{G}(x,\mathtt{t},l)\Big]\widetilde{Q}_c\nu(\mathtt{dt},\mathtt{d}x).$$

Assumption 2.1. *For every $n \in \mathscr{N}$ and each $\tilde{z} \in \tilde{Z}$, for some constant $\bar{\mathscr{D}}_n \ge 0$, for which, $\bar{\mathscr{V}}, \bar{\bar{\mathscr{V}}} \in \bar{\mathscr{R}}^n$, with $\max\{|\bar{\mathscr{V}}|, |\bar{\bar{\mathscr{V}}}|\} \le n$, the inequality below holds for $\forall i, j \in \mathscr{N}^5$ and $l \in \bar{\mathscr{M}}$:*

$$\int_{\tilde{\mathcal{Z}}} \left|\bar{\mathbf{K}}_i(\bar{\mathscr{V}},\tilde{z},l) - \bar{\mathbf{K}}_j(\bar{\bar{\mathscr{V}}},\tilde{z},l)\right|^2 \nu(d\tilde{z}) \le \bar{\mathscr{D}}_n\,|\bar{\mathscr{V}} - \bar{\bar{\mathscr{V}}}|^2,$$

where

$$\bar{\mathbf{K}}_1(\bar{\mathscr{V}},\tilde{z},l) = \varrho_{1,l}(\tilde{z})\,\mathcal{S}_{(\mathrm{t})},\ \bar{\mathbf{K}}_2(\bar{\mathscr{V}},\tilde{z},l) = \varrho_{2,l}(\tilde{z})\,\mathcal{I}_{1(\mathrm{t})},\ \bar{\mathbf{K}}_3(\bar{\mathscr{V}},\tilde{z},l) = \varrho_{3,l}(\tilde{z})\,\mathcal{I}_{2(\mathrm{t})},\ \bar{\mathbf{K}}_4(\bar{\mathscr{V}},\tilde{z},l)$$
$$= \varrho_{4,l}(\tilde{z})\,\mathcal{R}_{(\mathrm{t})}, \bar{\mathbf{K}}_5(\bar{\mathscr{V}},\tilde{z},l) = \varrho_{5,l}(\tilde{z})\,\mathcal{W}_{(\mathrm{t})}.$$

Assumption 2.2. *Let $y \in \bar{\mathscr{C}}([\mathtt{0}, +\infty) \times \Omega, (\mathtt{0}, +\infty))$. Let $\bar{\bar{\mathbf{K}}}_1$ and $\bar{\mathbf{K}}_{\mathtt{t}}$ be the two constants as with $\bar{\bar{\mathbf{K}}}_1$, $\bar{\mathbf{K}}_2 > 0$ such that*

$$\ln y(t) \ge \bar{\bar{\mathbf{K}}}_{1(t)} - \bar{\bar{\mathbf{K}}}_{2(t)} \int_0^1 y(s)ds + \mathcal{Y}(t), \forall t \ge 0 \quad \textit{where } \mathcal{Y}(t) \in \bar{\mathscr{C}}([0,+\infty)\times\Omega, (0,+\infty)), \textit{ and } \lim_{t\to\infty} \frac{\bar{\bar{\mathbf{K}}}_{1(\mathrm{t})}}{t} = 0,$$

then

$$\liminf_{t\to\infty} \langle y(\mathrm{t})\rangle \ge \frac{\bar{\mathbf{K}}_1}{\bar{\mathbf{K}}_2}.$$

Assumption 2.3. *Let $i \in \{1, 2\}$. For every $\tilde{z} \in \tilde{Z}$ and all $l \in S$, we assume that $1 + \varrho_{i,l}(\tilde{z}) > 0$, and there is a constant $\bar{C} > 0$ such that*

$$\int_{\tilde{Z}} \{\ln(\varrho_{i,l}(\tilde{z}) + 1)\}^2 \nu(d\tilde{z}) < \bar{C}.$$

2.2.1. Global solution of the model

Before proving the results, we define the region Ω in which the solution remains and changes, and $\Omega = \mho^* \times S$, where $\mho^*$ denotes the positive domain where

$$\Gamma^* = \left\{ Y(t) = (\mathcal{S}, \mathcal{I}_1, \mathcal{I}_2, \mathcal{W}, \mathcal{R}) \in \mathscr{R}_+^5 \ : \ \mathcal{S} + \sum_{i=1}^{2} \mathcal{I}_i + \mathcal{W} + \mathcal{R} \leq \frac{\breve{\omega}}{\hat{\phi}} \right\}. \tag{2.6}$$

Lemma 2.4. *The specified set* Γ^**, defined in* (2.6) *is almost surely positive invariant for the stochastic model* (2.4) *with jumps.*

Proof. Suppose that the initial values of the equation system Model (2.4) is $Y(0) \in \Gamma^*$, a positive constant n_0 can be chosen such that every component of $Y(t)$ in the system lies in the interval $\left[\frac{1}{n_0}, \mathtt{n}\right]$, for any positive integral number $n \geq n_0$). Accordingly, we define the stopping time $\hat{\bar{\tau}}$ as

$$\hat{\bar{\tau}}_{\mathtt{n}} = \inf \left\{ t \in [0, T_e] : Y(t) \in \mho^* \text{ and } Y(t) \notin \left(\frac{1}{n}, n\right)^5 \right\}.$$

We need to show that $\mathbf{P}\{\left(\hat{\bar{\tau}} = \infty\right)\} = 1$. It is sufficient to prove that $\mathbf{P}\{\left(\hat{\bar{\tau}} < \mathtt{t}\right)\} = \mathtt{0} \ \forall, \mathtt{t} > 0$, for any finite $n \quad \mathbf{P}(\bar{\tau} < \mathtt{t} >) \leq \mathbf{P}(\dot{\bar{\tau}}_{\mathtt{n}} < \mathtt{t} >), \forall\, \mathtt{t} > 0$. Therefore, $\lim_{\mathtt{n}\to\infty} \mathbf{P}(\hat{\bar{\tau}}_{\mathtt{n}} < \mathtt{t}) \to 0 \quad \text{as } \mathtt{n} \to \infty$. To achieve this we define the mapping $\bar{\mathscr{G}}_1 : \mathscr{R}_+^5 \longrightarrow \mathscr{R}_+$, which implies

$$\bar{\mathscr{G}}_1(Y(\mathtt{t})) = \frac{1}{\mathcal{S}(\mathtt{t})} + \frac{1}{\mathcal{I}_1(\mathtt{t})} + \frac{1}{\mathcal{I}_2(\mathtt{t})} + \frac{1}{\mathcal{R}(t)} + \frac{1}{\mathcal{W}(\mathtt{t})}.$$

In $\mathtt{s} \in [0, \min\{t, \bar{\tau}_{\mathtt{n}}\}]$, by using Itô's formula, we get:

$$\begin{aligned} d\bar{\mathscr{G}}_1(Y(s)) = {} & \bar{\mathscr{L}}\mathscr{G}_{1(s)})\, ds - \Big(\frac{\bar{\sigma}_{1,l}}{\mathcal{S}}\, d\mathscr{W}_1 + \frac{\bar{\sigma}_{2,l}}{\mathcal{I}_1}\, d\mathscr{W}_2 + \frac{\bar{\sigma}_{3,l}}{\mathcal{I}_2}\, d\mathscr{W}_3 + \frac{\bar{\sigma}_{4,l}}{\mathcal{W}}\, d\mathscr{W}_4 + \frac{\bar{\sigma}_{5,l}}{\mathcal{R}}\, d\mathscr{W}_5\Big) \\ & - \int_{\tilde{Z}} \Big[\frac{\mathscr{K}_{1,l}(\tilde{z})(\mathscr{K}_{1,l}(\tilde{z}) + 1)}{\mathcal{S}} + \frac{\mathscr{K}_{2,l}(\tilde{z})(\mathscr{K}_{2,l}(\tilde{z}) + 1)}{\mathcal{I}_1} + \frac{\mathscr{K}_{3,l}(\tilde{z})(\mathscr{K}_{3,l}(\tilde{z}) + 1)}{\mathcal{I}_2} \\ & + \frac{\mathscr{K}_{4,l}(\tilde{z})(\mathscr{K}_{4,l}(\tilde{z})) + 1}{\mathcal{W}} + \frac{\mathscr{K}_{5,l}(\tilde{z})(\mathscr{K}_{5,l}(\tilde{z}) + 1)}{\mathcal{R}} \Big] \tilde{Q}_c(ds, d\tilde{z}). \end{aligned} \tag{2.7}$$

This leads to the following:

$$\begin{aligned}
\bar{\mathscr{L}}\bar{\mathscr{G}}_1(Y) = &-\frac{1}{\mathcal{S}^2}\Big[\omega_l - \xi_l^*\mathcal{I}_1\mathcal{S} - \beta_l^*\mathcal{I}_2\mathcal{S} - \gamma_l\mathcal{S}\mathcal{W} - (\psi_l + \phi_l)\mathcal{S}\Big] \\
&- \frac{1}{\mathcal{I}_1^2}\Big[\varrho(\xi_l\mathcal{I}_1\mathcal{S} + \beta_l^*\mathcal{I}_2\mathcal{S} + \gamma_l\mathcal{S}\mathcal{W}) - (\phi_l + \xi_{1,l}^*)\mathcal{I}_1\Big] \\
&- \frac{1}{\mathcal{I}_2^2}\Big[(1-\varrho)(\xi_l\mathcal{I}_1\mathcal{S} + \beta_l^*\mathcal{I}_2\mathcal{S} + \gamma_l\mathcal{S}\mathcal{W}) + \rho\gamma_{1,l}\mathcal{I}_1 - (\delta_1^* + \xi_{2,l}^* + \psi)\mathcal{I}_2\Big] \\
&- \frac{1}{\mathcal{R}^2}\Big[(1-\rho)\xi_{1,l}^*\mathcal{I}_1 + \xi_2^*\mathcal{I}_2 + \psi_l\mathcal{S} - \psi\mathcal{R}\Big] \\
&- \frac{1}{\mathcal{W}^2}\Big[\lambda_{1,l}\mathcal{I}_1 + \lambda_{2,l}\mathcal{I}_2 - \delta^*\mathcal{W}\Big] + \frac{\bar{\sigma}_{1,l}^2}{\mathcal{S}} + \frac{\bar{\sigma}_{2,l}^2}{\mathcal{I}_1} + \frac{\bar{\sigma}_{3,l}^2}{\mathcal{I}_2} \\
&+ \frac{\bar{\sigma}_{4,l}^2}{\mathcal{R}} + \frac{\bar{\sigma}_{5,l}^2}{\mathcal{W}} + \int_{\tilde{z}} \frac{\mathscr{K}_{1,l}^2(\tilde{z})}{(\mathscr{K}_{1,l}(\tilde{z})+1)\,\mathcal{S}}\,\nu(d\tilde{z}) + \int_{\tilde{z}} \frac{\mathscr{K}_{2,l}^2(\tilde{z})}{(\mathscr{K}_{2,l}(\tilde{z})+1)\,\mathcal{I}_1}\,\nu(d\tilde{z}) \\
&+ \int_{\tilde{z}} \frac{\mathscr{K}_{3,l}^2(\tilde{z})}{(\mathscr{K}_{3,l}(\tilde{z})+1)\,\mathcal{I}_2}\,\nu(d\tilde{z}) \\
&+ \int_{\tilde{z}} \frac{\mathscr{K}_{4,l}^2(\tilde{z})}{(\mathscr{K}_{4,l}(\tilde{z})+1)\,\mathcal{R}}\,\nu(d\tilde{z}) + \int_{\tilde{z}} \frac{\mathscr{K}_{5,l}^2(\tilde{z})}{(\mathscr{K}_{5,l}(\tilde{z})+1)\,\mathcal{W}}\,\nu(d\tilde{z}).
\end{aligned} \tag{2.8}$$

On the basis of (2.8), we can derive the following:

$$\begin{aligned}
\bar{\mathscr{L}}\bar{\mathscr{G}}_1(Y) = &-\frac{1}{\mathcal{S}^2}\Big[\omega_l - \xi_l^*\mathcal{I}_1\mathcal{S} - \beta_l^*\mathcal{I}_2\mathcal{S} - \gamma_l\mathcal{S}\mathcal{W} - (\psi_l + \phi_l)\mathcal{S}\Big] + \frac{\bar{\sigma}_{1,l}^2}{\mathcal{S}} + \int_{\tilde{z}} \frac{\mathscr{K}_{1,l}^2(\tilde{z})}{(\mathscr{K}_{1,l}(\tilde{z})+1)\,\mathcal{S}}\,\nu(d\tilde{z}) \\
&- \frac{1}{\mathcal{I}_1^2}\Big[\varrho(\xi_l^*\mathcal{I}_1\mathcal{S} + \beta_l^*\mathcal{I}_2\mathcal{S} + \gamma_l\mathcal{S}\mathcal{W}) - (\psi + \xi_{1,l}^*)\mathcal{I}_1\Big] + \frac{\bar{\sigma}_{2,l}^2}{\mathcal{I}_1} + \int_{\tilde{z}} \frac{\mathscr{K}_{2,l}^2(\tilde{z})}{(\mathscr{K}_{2,l}(\tilde{z})+1)\,\mathcal{I}_1}\,\nu(d\tilde{z}) \\
&- \frac{1}{\mathcal{I}_2^2}\Big[(1-\varrho)(\xi_l\mathcal{I}_1\mathcal{S} + \beta_l^*\mathcal{I}_2\mathcal{S} + \gamma_l\mathcal{S}\mathcal{W}) + \rho\xi_{1,l}^*\mathcal{I}_1 - (\delta_1^* + \xi_2^* + \psi)\mathcal{I}_2\Big] \\
&+ \frac{\bar{\sigma}_{3,l}^2}{\mathcal{I}_2} + \int_{\tilde{z}} \frac{\mathscr{K}_{3,l}^2(\tilde{z})}{(\mathscr{K}_{3,l}(\tilde{z})+1)\,\mathcal{I}_2}\,\nu(d\tilde{z}) \\
&- \frac{1}{\mathcal{R}^2}\Big[(1-\rho)\xi_{1,l}^*\mathcal{I}_1 + \xi_2^*\mathcal{I}_2 + \psi_l\mathcal{S} - \psi\mathcal{R}\Big] + \frac{\bar{\sigma}_{4,l}^2}{\mathcal{R}} + \int_{\tilde{z}} \frac{\mathscr{K}_{4,l}^2(\tilde{z})}{(\mathscr{K}_{4,l}(\tilde{z})+1)\,\mathcal{R}}\,\nu(d\tilde{z}) \\
&- \frac{1}{\mathcal{W}^2}\Big[\lambda_{1,l}\mathcal{I}_1 + \lambda_{2,l}\mathcal{I}_2 - \delta^*\mathcal{W} + \frac{\bar{\sigma}_{5,l}^2}{\mathcal{W}} + \int_{\tilde{z}} \frac{\mathscr{K}_{5,l}^2(\tilde{z})}{(\mathscr{K}_{5,l}(\tilde{z})+1)\,\mathcal{W}}\,\nu(d\tilde{z})\Big].
\end{aligned} \tag{2.9}$$

The Inequality (2.8) leads to the following:

$$\begin{aligned}
\bar{\mathscr{L}}\bar{\mathscr{G}}_1(Y) \le &\frac{1}{\mathcal{S}}\{\xi_l^*\mathcal{I}_1 + \beta_l^*\mathcal{I}_2 + \gamma_l\mathcal{W} + \psi_l + \phi_l\} + \frac{\bar{\sigma}_{1,l}^2}{\mathcal{S}} + \int_{\tilde{z}} \frac{\mathscr{K}_{1,l}^2(\tilde{z})}{(1+\mathscr{K}_{1,l}(\tilde{z}))\,\mathcal{S}}\,\nu(d\tilde{z}) + \frac{1}{\mathcal{I}_1}\{\phi_l + \xi_{1,l}^*\} + \frac{\bar{\sigma}_{2,l}^2}{\mathcal{I}_1} \\
&+ \int_{\tilde{z}} \frac{\mathscr{K}_{2,l}^2(\tilde{z})}{\mathcal{I}_1(1+\mathscr{K}_{2,l}(\tilde{z}))}\,\nu(d\tilde{z}) + \frac{1}{\mathcal{I}_2}\{\delta_1^* + \xi_{2,l}^* + \phi_l\}\frac{\bar{\sigma}_{3,l}^2}{\mathcal{I}_2} + \int_{\tilde{z}} \frac{\mathscr{K}_{3,l}^2(\tilde{z})}{\mathcal{I}_2(1+\mathscr{K}_{3,l}(\tilde{z}))}\,\nu(d\tilde{z}) + \frac{1}{\mathcal{R}}\phi_l + \frac{\bar{\sigma}_{4,l}^2}{\mathcal{R}} \\
&+ \int_{\tilde{z}} \frac{\mathscr{K}_{4,l}^2(\tilde{z})}{\mathcal{R}(1+\mathscr{K}_{4,l}(\tilde{z}))}\,\nu(d\tilde{z}) + \frac{1}{\mathcal{W}}\delta_1^* + \frac{\bar{\sigma}_{5,l}^2}{\mathcal{W}} + \int_{\tilde{z}} \frac{\mathscr{K}_{5,l}^2(\tilde{z})}{\mathcal{W}(1+\mathscr{K}_{5,l}(\tilde{z}))}\,\nu(d\tilde{z}).
\end{aligned} \tag{2.10}$$

Assume that

$$\begin{aligned}\bar{\mathscr{L}}\bar{\mathscr{G}}_1(Y) \leq\ & \frac{1}{\mathcal{S}}\Big\{\xi_l^*\mathcal{I}_1+\beta^*\mathcal{I}_2+\varrho_l\mathcal{W}+\psi_l+\phi_l+\bar{\sigma}_{1,l}^2+\int_{\tilde{z}}\frac{\mathscr{K}_{1,l}^2(\tilde{z})}{\mathscr{K}_{1,l}(\tilde{z})+1}\,\nu(d\tilde{z})\Big\}+\frac{1}{\mathcal{I}_1}\Big\{\phi_l+\xi_{1,l}+\bar{\sigma}_{2,l}^2\\ &+\int_{\tilde{z}}\frac{\mathscr{K}_{2,l}^2(\tilde{z})}{\mathscr{K}_{2,l}(\tilde{z})+1}\,\nu(d\tilde{z})\Big\}+\frac{1}{\mathcal{I}_2}\Big\{\delta_1^*+\xi_{2,l}+\phi_l+\frac{\bar{\sigma}_{3,l}^2}{\mathcal{I}_2}+\int_{\tilde{z}}\frac{\mathscr{K}_{3,l}^2(\tilde{z})}{\mathscr{K}_{3,l}(\tilde{z})+1}\,\nu(d\tilde{z})\Big\}\\ &+\frac{1}{\mathcal{R}}\Big\{\phi_l+\bar{\sigma}_{4,l}^2+\int_{\tilde{z}}\frac{\mathscr{K}_{4,l}^2(\tilde{z})}{\mathscr{K}_{4,l}(\tilde{z})+1}\,\nu(d\tilde{z})\Big\}+\frac{1}{\mathcal{W}}\Big\{\delta_1^*+\frac{\bar{\sigma}_{5,l}^2}{\mathcal{W}}+\int_{\tilde{z}}\frac{\mathscr{K}_{5,l}^2(\tilde{z})}{(\mathscr{K}_{5,l}(\tilde{z})+1)\mathcal{W}}\,\nu(d\tilde{z})\Big\}.\end{aligned}\tag{2.11}$$

This implies

$$\begin{aligned}\bar{\mathscr{L}}\bar{\mathscr{G}}_1(Y) \leq\ & \frac{1}{\mathcal{S}}\Big\{\xi_l^*\frac{\check{\omega}}{\hat{\phi}}+\beta_l^*\frac{\check{\omega}}{\hat{\phi}}+\gamma_l\frac{\check{\omega}}{\hat{\phi}}+\psi_l+\phi_l+\bar{\sigma}_{1,l}^2+\int_{\tilde{z}}\frac{\mathscr{K}_{1,l}^2(\tilde{z})}{\mathscr{K}_{1,l}(\tilde{z})+1}\,\nu(d\tilde{z})\Big\}+\frac{1}{\mathcal{I}_1}\Big\{\phi_l+\xi_{1,l}^*+\bar{\sigma}_{2,l}^2\\ &+\int_{\tilde{z}}\frac{\mathscr{K}_{2,l}^2(\tilde{z})}{\mathscr{K}_{2,l}(\tilde{z})+1}\,\nu(d\tilde{z})\Big\}+\frac{1}{\mathcal{I}_2}\Big\{\delta_1^*+\xi_{2,l}^*+\phi_l+\frac{\bar{\sigma}_{3,l}^2}{\mathcal{I}_2}+\int_{\tilde{z}}\frac{\mathscr{K}_{3,l}^2(\tilde{z})}{\mathscr{K}_{3,l}(\tilde{z})+1}\,\nu(d\tilde{z})\Big\}\\ &+\frac{1}{\mathcal{R}}\Big\{\phi_l+\bar{\sigma}_{4,l}^2+\int_{\tilde{z}}\frac{\mathscr{K}_{4,l}^2(\tilde{z})}{\mathscr{K}_{4,l}(\tilde{z})+1}\,\nu(d\tilde{z})\Big\}+\frac{1}{\mathcal{W}}\Big\{\delta^*+\frac{\bar{\sigma}_{5,l}^2}{\mathcal{W}}+\int_{\tilde{z}}\frac{\mathscr{K}_{5,l}^2(\tilde{z})}{(\mathscr{K}_{5,l}(\tilde{z})+1)\mathcal{W}}\,\nu(d\tilde{z})\Big\}.\end{aligned}\tag{2.12}$$

By considering

$$\begin{aligned}\mathscr{B}=\max\Bigg\{&\frac{\check{\omega}}{\hat{\phi}}(\xi_l^*+\beta_l^*+\gamma_l)+\psi_l+\phi_l+\bar{\sigma}_{1,l}^2+\int_{\tilde{z}}\frac{\mathscr{K}_{1,l}^2(\tilde{z})}{\mathscr{K}_{1,l}(\tilde{z})+1}\,\nu(d\tilde{z}),\\ &\phi_l+\xi_{1,l}^*+\bar{\sigma}_{2,l}^2+\int_{\tilde{z}}\frac{\mathscr{K}_{2,l}^2(\tilde{z})}{\mathscr{K}_{2,l}(\tilde{z})+1}\,\nu(d\tilde{z}),\delta_1^*+\xi_{2,l}^*+\phi_l+\bar{\sigma}_{3,l}^2+\int_{\tilde{z}}\frac{\mathscr{K}_{3,l}^2(\tilde{z})}{\mathscr{K}_{3,l}(\tilde{z})+1}\,\nu(d\tilde{z}),\\ &\phi_l+\bar{\sigma}_{4,l}^2+\int_{\tilde{z}}\frac{\mathscr{K}_{4,l}^2(\tilde{z})}{\mathscr{K}_{4,l}(\tilde{z})+1}\,\nu(d\tilde{z}),\delta^*+\bar{\sigma}_{5,l}^2+\int_{\tilde{z}}\frac{\mathscr{K}_{5,l}^2(\tilde{z})}{\mathscr{K}_{5,l}(\tilde{z})+1}\,\nu(d\tilde{z})\Bigg\}.\end{aligned}\tag{2.13}$$

Rewriting Inequality (2.13), we get:

$$\bar{\mathscr{L}}\bar{\mathscr{G}}_1(Y)(s)\leq\mathscr{B}\,\bar{\mathscr{L}}\bar{\mathscr{G}}_1(Y)(s)).\tag{2.14}$$

By using the inequality (2.7), in (2.14), we get the following result:

$$\begin{aligned}\mathrm{d}\bar{\mathscr{G}}_1(Y(s))\leq\bar{\mathscr{L}}\bar{\mathscr{G}}_1(Y(s))\,\mathrm{d}s-\sum_{i=1}^{5}\frac{\bar{\sigma}_{i,l}}{i}\,\mathrm{d}\mathscr{W}_i-\int_{\widetilde{Z}}\Bigg(&\frac{\mathscr{K}_{1,l}(\widetilde{z})(1+\mathscr{K}_{1,l}(\widetilde{z}))}{\mathcal{S}}+\frac{\mathscr{K}_{2,l}(\widetilde{z})(1+\mathscr{K}_{2,l}(\widetilde{z}))}{\mathcal{I}_1}\\ &+\frac{\mathscr{K}_{3,l}(\widetilde{z})(1+\mathscr{K}_{3,l}(\widetilde{z}))}{\mathcal{I}_2}+\frac{\mathscr{K}_{4,l}(\widetilde{z})(1+\mathscr{K}_{4,l}(\widetilde{z}))}{\mathcal{W}}+\frac{\mathscr{K}_{5,l}(\widetilde{z})(1+\mathscr{K}_{5,l}(\widetilde{z}))}{\mathcal{R}}\Bigg)\widetilde{Q}_c(\mathrm{d}s,\mathrm{d}\widetilde{z}).\end{aligned}\tag{2.15}$$

Taking expectation on both sides of Inequality (2.13) and integrating over the interval $[0,s]$, where $s\in[0,\min\{t,\bar{\tau}_t\}]$, we obtain

$$\mathbb{E}\Big[\bar{\mathscr{G}}_1(Y(s))\Big]-\bar{\mathscr{G}}_1(Y(0))\leq\mathscr{B}\int_0^s\mathbb{E}\Big[\bar{\mathscr{G}}_1(Y(y))\Big]\,dy+\mathbb{E}[\mathscr{M}(s)]\,.$$

Since $\mathscr{M}(s)$ is a local martingale, it follows that $\mathbb{E}[\mathscr{M}(s)] = 0$. Hence we have,

$$\mathbb{E}\Big[\bar{\mathscr{G}}_1(Y(s))\Big] \leq \bar{\mathscr{G}}_1(Y(0)) + \mathscr{B}\int_0^s \mathbb{E}\Big[\bar{\mathscr{G}}_1(Y(y))\Big]\,dy.$$

By the help of Gronwall's inequality, we deduce that

$$\mathbb{E}\Big[\bar{\mathscr{G}}_1(Y(s))\Big] \leq \bar{\mathscr{G}}_1(Y(0))e^{\mathscr{B}s}, \qquad s \in [0,\, \min\{t, \bar{\tau}_n\}].$$

Restricting to the localized interval $[0,\, \mathtt{t} \wedge \bar{\tau}_{\mathtt{n}}]$, we obtain the inequality

$$\bar{\mathscr{E}}[\bar{\mathscr{G}}_1(Y(\min\{t, \bar{\tau}_n\}_{\mathtt{n}}))] \leq \bar{\mathscr{G}}_1(Y(0))\, e^{\mathscr{B}(\mathtt{t}\wedge\dot{\bar{\tau}}_{\mathtt{n}})} \leq \bar{\mathscr{G}}_1(Y(0))\, e^{\mathscr{B}\mathtt{t}}, \qquad \forall \mathtt{t} \geq 0. \tag{2.16}$$

Setting $\mathtt{s} = \mathtt{t} \wedge \dot{\bar{\tau}}_{\mathtt{n}} = \min\{t, \bar{\tau}_n\}$ in Inequality (2.16), where $\mathtt{t} \wedge \dot{\bar{\tau}}_{\mathtt{n}}$, and some values of $\bar{\mathscr{G}}_1$ are non-negative and satisfy $Y_i(\dot{\bar{\tau}}_{\mathtt{n}}) \leq \frac{1}{\mathtt{n}}$, we obtain the following result by combining Inequality (2.16) with Chebyshev's inequality.

$$\bar{\mathscr{E}}\{\bar{\mathscr{G}}_{\mathtt{t}}\{\{Y\{\min\{t, \bar{\tau}_n\}\}\} \geq \bar{\mathscr{E}}\{\bar{\mathscr{G}}_{\mathtt{t}}\{\{Y\{\bar{\tau}_{\mathtt{n}}\}\}\, \mathtt{t}_{\{\bar{\tau}_{\mathtt{n}}<1\}}\} \geq \mathtt{n}\,\mathbf{P}\{\bar{\tau}_{\mathtt{t}} < \mathtt{t}\}, \tag{2.17}$$

where $\mathbf{1}_{\{\dot{\bar{\tau}}_n<t\}}$ is the characteristic function of the event $\{\dot{\bar{\tau}}_n < t\}$.

$$\bar{\mathscr{E}}\{\bar{\mathscr{G}}_1\{Y\{\{\min\{t, \bar{\tau}_n\}\}\} \geq \mathtt{n}\,\bar{\tau}_{\mathtt{n}} < \mathtt{t}. \tag{2.18}$$

From the above Inequality, we have

$$\bar{\mathscr{E}}\Big\{\bar{\mathscr{G}}_1\{Y\{\{\min\{t, \bar{\tau}_n\}\}\}\, .\mathbf{1}_{\{\bar{\tau}_{\mathtt{n}}<\mathtt{t}\}}\Big\} \geq \bar{\mathscr{E}}\{\bar{\mathscr{G}}_1\{Y\{\bar{\tau}_{\mathtt{n}}\}\}\mathbf{1}_{\{\bar{\tau}_{\mathtt{n}}<\mathtt{t}\}}\} = \mathtt{n}\,\mathbf{P}\{\bar{\tau}_{\mathtt{n}} < \mathtt{t}\}. \tag{2.19}$$

Thus, it can be written as

$$\bar{\mathscr{E}}\Big\{\bar{\mathscr{G}}_1\{Y\{\{\min\{t, \bar{\tau}_n\}\}\, .\mathbf{1}_{\{\bar{\tau}_{\mathtt{n}}<\mathtt{t}\}}\Big\} \geq \bar{\mathscr{E}}\{\bar{\mathscr{G}}_1\{Y\{\bar{\tau}_{\mathtt{n}}\}\}\mathbf{1}_{\{\bar{\tau}_{\mathtt{n}}<\mathtt{t}\}}\} \geq\ \mathbf{P}\{\bar{\tau}_{\mathtt{n}} < \mathtt{t}\}. \tag{2.20}$$

Utilizing Inequality (2.16) in (2.17), we get

$$\mathtt{n}\mathbf{P}\{\bar{\tau}_{\mathtt{n}} < \mathtt{t}\} \leq \bar{\mathscr{G}}_1\{Y\{0\}\}\, e^{\mathscr{B}\mathtt{t}}, \tag{2.21}$$

which can be expressed as

$$\mathbf{P}\{\bar{\tau}_{\mathtt{n}} < \mathtt{t}\} \leq \frac{1}{\mathtt{n}}\bar{\mathscr{G}}_1\{Y(0)\}\, e^{\mathscr{B}\mathtt{t}}. \tag{2.22}$$

As $n \to \infty$, we have $\lim_{n\to\infty} \sup \mathbf{P}\{\bar{\tau}_{\mathtt{n}} < \mathtt{t}\} = 0$. Consequently, the stopping times $\bar{\tau}_{\mathtt{n}}$ tend to infinity almost surely, which implies that the solution remains positive and bounded at $\forall \mathtt{t} \geq \mathtt{0}$with probability one. Hence, the proof is complete. □

Theorem 2.5. *Let us consider that Assumptions 2.1 and 2.2 hold true and let* $Y(\mathtt{0}) = \{\mathcal{S}(\mathtt{0}), \mathcal{I}_1(\mathtt{0}), \mathcal{I}_2(\mathtt{0}), \mathcal{R}(\mathtt{0}), \mathcal{W}(\mathtt{0})\} \in \mathbb{R}^5_+$ *be the initial values, and let* $Y(\mathtt{t}) = \{\mathcal{S}(\mathtt{t}), \mathcal{I}_1(t), \mathcal{I}_2(t), \mathcal{R}(t), \mathcal{W}(\mathtt{t})\}$ *be the solution of the stochastic system* (2.4) *for* $t \geq 0$. *Then the solution remains consistently in the positive region* $\mathbb{R}^5_+$ *almost surely; that is,* $Y(t) \in \mathbb{R}^5_+,\ \forall\, t \geq 0$.

Proof. Based on Assumptions 2.1 and 2.2, the coefficient of the stochastic model (2.4) will have the Lipschitz property, and for the initial conditions $Y(0), r(0)$, there is a corresponding solution $Y(t), r(t)$ in the positive domain $\mho^*$. Consequently, there is defined on $[0, \hat{\bar{\tau}}_e)$, where $\hat{\bar{\tau}}_e$ denotes the explosion time. It remains to show that $\hat{\bar{\tau}}_e = \infty$ almost surely. To this end, we construct a suitable Lyapunov function and show that the solution cannot exit the bounded domain Ω in finite time. Let $g_0 \geq 1$ be an integer such that $(Y(0), \tilde{r}(0)) \in \left[\frac{1}{g_0}, g_0\right]$. For each integer $b \geq g_0$, define the stopping time

$$\hat{\bar{\tau}}_{\mathtt{g}} = \inf\left\{\mathtt{t} \in [0, \hat{\bar{\tau}}_{\mathtt{t}}] : \mathcal{S}(\mathtt{t}) \notin \left(\frac{1}{\mathtt{g}}, \mathtt{g}\right), \mathcal{I}_1(t) \notin \left(\frac{1}{\mathtt{g}}, \mathtt{g}\right), \mathcal{I}_2(\mathtt{t}) \notin \left(\frac{1}{\mathtt{g}}, \mathtt{g}\right), \mathcal{R}(t) \notin \left(\frac{1}{\mathtt{g}}, \mathtt{g}\right), \mathcal{W}(\mathtt{t}) \notin \left(\frac{1}{\mathtt{g}}, \mathtt{g}\right)\right\},$$

where

$$\forall \hat{\bar{\tau}}_{g_0} \leq \hat{\bar{\tau}}_g \leq \hat{\bar{\tau}}_e, \text{and } g \geq g_0, \hat{\bar{\tau}}_\infty = \lim_{g\to\infty} \hat{\bar{\tau}}_g.$$

We have remained to show that $\hat{\bar{\tau}}_\infty = \infty$. For this, assuming the contrary path, that there exists is a finite constant $\bar{T} > 0$ and $0 < \varepsilon < 1$ subject to $\mathbf{P}\{\hat{\bar{\tau}}_\infty \leq \hat{T}\} \geq \varepsilon$. Under this assumption, we construct an appropriate Lyapunov $\mathscr{C}^2$ function, $\bar{\mathscr{G}}_1 : \Omega \to \mathscr{R}_+$, which is defined as

$$Y(t) = \mathcal{S}(\mathtt{t}) + \mathcal{R}(t) + \mathcal{W}(t) - 2 - \ln(\mathcal{S}(\mathtt{t})\mathcal{R}(t)\mathcal{W}(t)) + \sum_{i=1}^{2} \{I_{i(t)} - 1 - \ln(I_i)\}. \tag{2.23}$$

It is clear that the function is most positive for all values of Y. It is obvious that the expression $\mathbf{U} = \mathscr{U} - 1 - \ln \mathscr{U}$, for all $\mathtt{U}$. Now if we apply Ito's formula on $\bar{\mathscr{G}}_2$, we obtain the following result:

$$\begin{aligned}
\mathtt{d}\bar{\mathscr{G}}_2 = {} & \bar{\mathscr{L}}\bar{\mathscr{G}}_2\, dt + \bar{\sigma}_{1,l}(\mathcal{S} - 1)\, d\mathscr{W}_1 + \bar{\sigma}_{4,l}(\mathcal{R} - 1)\, d\mathscr{W}_4 + \bar{\sigma}_{5,l}(\mathcal{W} - 1)\, d\mathscr{W}_5 \\
& + 2\sum_{i=1}^{2} \bar{\sigma}_{i+1,l}(I_i - 1)\, \mathtt{d}W_{i+1} \\
& + \int_{\tilde{Z}} \Bigg\{ \mathscr{K}_{4,l}(\tilde{z})\,\mathcal{R} + \mathscr{K}_{1,l}(\tilde{z})\,\mathcal{S} + \mathscr{K}_{5,l}(\tilde{z})\,\mathcal{W} + \sum_{i=1}^{2} \mathscr{K}_{i+1,l}(\tilde{z})\, I_i \\
& \qquad - \ln\Big[(1 + \mathscr{K}_{1,l}(\tilde{z}))\,\mathcal{S}(1 + \mathscr{K}_{4,l}(\tilde{z}))\,\mathcal{R}(1 + \mathscr{K}_{5,l}(\tilde{z}))\,\mathcal{W}\Big] \\
& \qquad - \sum_{i=1}^{2} \ln(1 + \mathscr{K}_{i+1,l}(\tilde{z}))\, I_i \Bigg\} \tilde{Q}_{\mathtt{c}}(\mathtt{dt}, \mathtt{d}\tilde{z}),
\end{aligned} \tag{2.24}$$

in which

$$\begin{aligned}\bar{\mathscr{L}}\bar{\mathscr{G}}_2(Y) &= \left(1-\frac{1}{S}\right)\left[\omega_l - \xi_l^*\mathcal{I}_1\mathcal{S} - \beta_l^*\mathcal{I}_2\mathcal{S} - \gamma_l\mathcal{S}\mathcal{W} - (\psi_l+\phi_l)\mathcal{S}\right]\\ &-\left(1-\frac{1}{\mathcal{I}_1}\right)\left[\varrho(\xi_l\mathcal{I}_1\mathcal{S}+\beta_l^*\mathcal{I}_2\mathcal{S}+\gamma_l\mathcal{S}\mathcal{W}) - (\psi+\xi_{1,l}^*)\mathcal{I}_1\right]\\ &-\left(1-\frac{1}{\mathcal{I}_2}\right)\left[(1-\varrho)(\xi_l^*\mathcal{I}_1\mathcal{S}+\beta_l^*\mathcal{I}_2\mathcal{S}+\gamma_l\mathcal{S}\mathcal{W}) + \rho\xi_{1,l}^*\mathcal{I}_1 - (\delta_1^*+\xi_2^*+\psi)\mathcal{I}_2\right]\\ &-\left(1-\frac{1}{S}\right)\left[(1-\rho)\xi_{1,l}^*\mathcal{I}_1+\xi_2^*\mathcal{I}_2+\psi_l\mathcal{S}-\psi\mathcal{R}\right] - \left(1-\frac{1}{S}\right)\left[\omega_{1,l}\mathcal{I}_1+\omega_{2,l}\mathcal{I}_2-\delta_1^*\mathcal{W}\right]\\ &+\frac{\bar{\sigma}_{1,l}^2}{\mathcal{S}}+\frac{\bar{\sigma}_{2,l}^2}{\mathcal{I}_1}+\frac{\bar{\sigma}_{3,l}^2}{\mathcal{I}_2}\\ &+\frac{\bar{\sigma}_{4,l}^2}{\mathcal{R}}+\frac{\bar{\sigma}_{5,l}^2}{\mathcal{W}}+\int_{\tilde{z}}\frac{\mathscr{K}_{1,l}^2(\tilde{z})}{1+(\mathscr{K}_{1,l}(\tilde{z}))\,\mathcal{S}}\,v(d\tilde{z})+\int_{\tilde{z}}\frac{\mathscr{K}_{2,l}^2(\tilde{z})}{1+(\mathscr{K}_{2,l}(\tilde{z}))\,\mathcal{I}_1}\,v(d\tilde{z})\\ &+\int_{\tilde{z}}\frac{\mathscr{K}_{3,l}^2(\tilde{z})}{1+(\mathscr{K}_{3,l}(\tilde{z}))\,\mathcal{I}_2}\,v(d\tilde{z})\\ &+\int_{\tilde{z}}\frac{\mathscr{K}_{4,l}^2(\tilde{z})}{1+(\mathscr{K}_{4,l}(\tilde{z}))\,\mathcal{R}}\,v(d\tilde{z})+\int_{\tilde{z}}\frac{\mathscr{K}_{5,l}^2(\tilde{z})}{1+(\mathscr{K}_{5,l}(\tilde{z}))\,\mathcal{W}}\,v(d\tilde{z}).\end{aligned} \tag{2.25}$$

After some simplification, we obtain

$$\begin{aligned}\bar{\mathscr{L}}\bar{\mathscr{G}}_2 \le \Big[&4\phi_l+\varrho+\phi+\delta_{1,l}^*+\delta_1^*+\sum_{j=1}^{2}(\xi_{1,l}^*+\omega_{j,l}+\beta_l^*+\varrho)\frac{\check{\omega}}{\hat{\psi}}\\ &+\sum_{j=1}^{5}\left[\frac{\bar{\sigma}_{j,l}^2}{2}+\int_X\left(\mathscr{K}_{j,l}(\tilde{z})-\ln(1+\mathscr{K}_{j,l}(\tilde{z}))\right)v(d\tilde{z})\right],\end{aligned} \tag{2.26}$$

where $\mathscr{B}^* = \max\left\{\int_Z\left(\mathscr{K}_{j,l}(\tilde{z})-\ln(1+\mathscr{K}_{j,l}(\tilde{z}))\right)v(d\tilde{z})\right\}$, for $j=1,\ldots,5$) and utilize the value of $\mathscr{B}^*$ in (2.26), we get

$$\bar{\mathscr{L}}\bar{\mathscr{G}}_2 \le \Big[4\phi_l+\varrho+\phi+\delta_{1,l}^*+\delta_1^*+\sum_{j=1}^{2}(\xi_{1,l}^*+\omega_{j,l}+\beta_l^*+\varrho)\frac{\check{\omega}}{\hat{\psi}}+\sum_{j=1}^{5}\left(\frac{\bar{\sigma}_{j,l}^2}{2}+\mathscr{B}^*\right). \tag{2.27}$$

Substituting the Inequality (2.27) in (2.25) and integrating in the interval, $[0,\dot{\bar{\tau}}_a\wedge\mathbf{T}]$ and then taking expectation, we obtain

$$\bar{\mathscr{E}}\left[\int_0^{\hat{\bar{\tau}}_b\wedge T}\bar{\mathscr{L}}\bar{\mathscr{G}}_2(X(t))\,dt\right] \le \bar{\mathscr{E}}\left[\int_0^{\hat{\bar{\tau}}_b\wedge\mathbf{T}}\mathscr{B}^*\,dt\right]. \tag{2.28}$$

Since $\mathscr{B}^*>0$ is a constant, this simplifies to

$$0 \le \bar{\mathscr{E}}[\left(Y\bar{\mathscr{G}}_2\ \min\left(\hat{\bar{\tau}}_a,\mathbf{T}\right)\right), r\min(\left(\hat{\bar{\tau}}_a\mathbf{T}\right)] \le\ \bar{\mathscr{L}}\bar{\mathscr{G}}_2(Y(0),r(0))\min\left[\hat{\bar{\tau}}_a,\mathbf{T}\right]+\mathscr{B}^*\mathbf{T}. \tag{2.29}$$

This is very closely parallel to the argument which is used in Lemma (2.3) in [47]; therefore, the remaining details can be omitted. □

Lemma 2.6. *Let* $(Y(\mathtt{0}), \mathtt{r}(\mathtt{0})) \in \Omega$ *be any initial condition, and let* $i \in \{1, 2\}$. *If* $(Y(\mathtt{r}), \mathtt{r}(\mathtt{t}))$ *is the solution of the proposed model in* (2.4)*, we shall prove the following properties for* $\forall \mathtt{1} \geq 0$*:*

$$\lim_{\mathtt{t}\to\infty} \frac{\mathcal{S}(\mathtt{t})}{t} = \mathtt{0}, \quad \lim_{\mathtt{t}\to\infty} \frac{\mathcal{I}_1}{t}, \lim_{\mathtt{t}\to\infty} \frac{\mathcal{I}_2(t)}{t} = \mathtt{0}, \quad \lim_{\mathtt{t}\to\infty} \frac{\mathcal{R}(t)}{\mathtt{t}} = \mathtt{0}, \lim_{\mathtt{t}\to\infty} \frac{\mathcal{W}(\mathtt{t})}{\mathtt{t}} = \mathtt{0}.$$

and

$$\lim_{\mathtt{t}\to\infty} \frac{1}{\mathtt{t}} \int_0^{\mathtt{t}} \mathcal{S}(s)\, \mathtt{ds} = 0, \quad \lim_{\mathtt{t}\to\infty} \frac{1}{\mathtt{t}} \int_0^{\mathtt{t}} \mathcal{I}_i(\mathtt{s})\, \mathtt{ds} = 0, \quad \lim_{\mathtt{t}\to\infty} \frac{1}{\mathtt{t}} \int_0^{\mathtt{t}} \mathcal{R}(\mathtt{s})\, \mathtt{ds} = 0 \lim_{\mathtt{t}\to\infty} \frac{1}{\mathtt{t}} \int_0^{\mathtt{t}} \mathcal{W}(\mathtt{s})\, \mathtt{ds} = 0.$$

Lemma 2.7. *Assume Assumption 2.2 holds true. If* $(\mathcal{S}(0), \mathcal{I}_i(0), \mathcal{R}(0), \mathcal{W}(0)) \in \Omega$ *is the arbitrary initial condition and* $(\mathcal{S}(0), \mathcal{I}_i(0), \mathcal{R}(0), \mathcal{W}(0)) \in \Omega$*, is the trajectory of the stochastic model* (2.4)*, it suffices to prove that:*

$$\lim_{\mathtt{t}\to\infty} \frac{1}{\mathtt{t}} \int_0^{\mathtt{t}} \mathcal{S}(s)\, \mathtt{d}\mathscr{W}_{1(s)} = \lim_{\mathtt{t}\to\infty} \frac{1}{t} \int_0^{\mathtt{t}} \mathbf{I}_{i(s)}\, \mathtt{d}\mathscr{W}_{2(s)} = \lim_{\mathtt{t}\to\infty} \frac{1}{\mathtt{t}} \int_0^{\mathtt{t}} \mathcal{R}(s)\, \mathtt{d}\mathscr{W}_{4(s)} = \lim_{\mathtt{t}\to\infty} \frac{1}{\mathtt{t}} \int_0^{\mathtt{t}} \mathcal{W}(s)\, \mathtt{d}\mathscr{W}_{5(s)} = 0,$$

and

$$\lim_{\mathtt{t}\to\infty} \frac{1}{\mathtt{t}} \int_0^{\mathtt{t}} \int_Z \mathscr{K}_{1,l}(\mathcal{S}_{(s))}(\tilde{z})\, \tilde{Q}_{\mathtt{c}}(\mathtt{d}s, \mathtt{d}z) = \lim_{l\to\infty} \frac{1}{\mathtt{t}} \int_0^{\mathtt{t}} \int_Z \mathscr{K}_{i+1,l}(\mathcal{I}_i(s))(\tilde{z})\, \tilde{Q}_{\mathtt{c}}(\mathtt{d}s, \mathtt{d}z)$$

$$= \lim_{\mathtt{t}\to\infty} \frac{1}{\mathtt{t}} \int_0^{\mathtt{t}} \int_Z \mathscr{K}_{5,l}(\mathcal{W}_{(s)})(\tilde{z})\, \tilde{Q}_{\mathtt{c}}(\mathtt{d}s, \mathtt{d}z) = \lim_{\mathtt{t}\to\infty} \frac{1}{\mathtt{t}} \int_0^{\mathtt{t}} \int_Z \mathscr{K}_{5,l}(\mathcal{R}_{(s))}(\tilde{z})\, \tilde{Q}_{\mathtt{c}}(\mathtt{d}s, \mathtt{d}z) = \mathtt{0}.$$

For further details of Lemmas 2.6 and 2.7, see Lemma 3.1 in [48] and Lemma 2.2 in [49].

Definition 2.8. *Here, the mathematical definitions for the extinction and persistence of the diseases in the presumed model are defined.*

- $\forall\, i \in \{1, 2\}$ *in the stochastic coinfection model* (2.4)*, by extinction, we mean that* $\lim_{\mathtt{t}\to\infty} \mathcal{I}_i(\mathtt{t}) = \mathtt{0}$,
- *For* $\forall\, i \in \{1, 2\}$ *in the stochastic coinfection-model* (2.4)*, by persistence, we mean that* $\lim_{\mathtt{t}\to\infty} \mathcal{I}_1(t) + \mathcal{I}_{2(t)} \geq \mathtt{0}$.

Remark. Under this condition, $\mathscr{K}_{i+3,l} + 1(\tilde{z}) > 0, \quad \forall\, \tilde{z} \in \tilde{\Omega},$ and $l \in \mathscr{M}$, the basic reproductive number $\mathcal{R}_0^s$ for the stochastic model (2.4) is defined as follows:

$$\mathcal{R}_i = \frac{\sum_{l=1}^{m} \Theta_l \bar{\chi}\, \omega_l}{\sum_{l=1}^{m} \Theta_l\, \phi_{i,l}^*(\psi_l + \phi_l)},$$

where

$$\phi_{i,l}^* = \left\{\delta_1^* + \xi_{1,l}^* + \xi_{2,l}^* + 2\phi_l + \frac{\bar{\sigma}_{i+1,l}^2}{2}\right\} + \int_{\tilde{z}} \Big(\mathscr{K}_{i+1,l}(\tilde{z}) - \ln(1 + \mathscr{K}_{i+1,l}(\tilde{z}))\Big)\, \nu(d\tilde{z}).$$

Theorem 2.9. *Assume that Assumption 2.2 holds true. Let $Y(0), \hat{r}(0) \in \Omega$, be the initial values in the positive region. In this case, the system has a unique solution $Y(t), \hat{r}(t)$. Moreover, if $\mathcal{R}_i \leq 1,\ i \in \{1,2\}$, then both the diseases in Model* (2.4) *become extinct almost surely under the following condition:*

$$\limsup_{\mathtt{t}\to\infty} \frac{\ln \mathcal{I}_{i(t)}}{\mathtt{t}} \leq \Theta_l[\mathcal{R}_{\mathtt{i}}^s - 1] \sum_{l=1}^{m} \phi_{\mathtt{i},l}^* < 0, \text{ for } i \in \{1,2\},$$

and

$$0 \ \leq \ \lim_{\mathtt{t}\to\infty}\langle \mathcal{R}(t)\rangle \ \leq \ \frac{\hat{\phi}}{\check{\psi}}\frac{\hat{\omega}}{\check{\phi}+\check{\psi}}, \frac{\hat{\omega}}{\check{\phi}+\check{\psi}} \ \leq \ \lim_{\mathtt{t}\to\infty}\langle \mathcal{S}(\mathtt{t})\rangle \ \leq \ \frac{\hat{\omega}}{\check{\phi}+\check{\psi}}.$$

Proof. Let consider the recover class of the system of differential equations (2.4)

$$\begin{aligned} \mathtt{d}\mathcal{R}(t) = &\left\{\xi_{1,l}^*(1-\rho_l)\gamma_l \mathcal{I}_1(t) + \gamma_l \mathcal{I}_2(t) + \psi_l \mathcal{S}(\mathtt{t}) - \phi_l \mathcal{R}(t)\right\}\mathtt{dt} + \bar{\sigma}_{4,l}\mathcal{R}(t)\,\mathtt{d}\mathscr{W}_{4(\mathtt{t})} \\ &+ \int_Z \mathscr{K}_{4,l}\mathcal{R}(\mathtt{t}^-)(\tilde{z})\,\tilde{Q}_{\mathtt{c}}(\mathtt{dt}, \mathtt{t}\tilde{z}), \end{aligned} \tag{2.30}$$

assuming that $\bar{\rho}_{1,l} = \min\left\{(1-\rho)\xi_{1,l}^*. \xi_{2,l}^*\right\}$. Inserting the value of $\bar{\rho}_{1,l}$ in (2.30), we get

$$\mathtt{d}\mathcal{R}_{(t)} \geq \bar{\rho}_{1,l}(\mathcal{I}_1(t)\mathtt{dt} + \mathcal{I}_2(t)\mathtt{dt} + \mathcal{W}(\mathtt{t})\,\mathtt{dt} - \mathcal{R}(t)\,\mathtt{dt} + \bar{\sigma}_{6,l}\mathcal{R}(t)\,d\mathscr{W}_4\mathtt{dt} + \int_Z \mathscr{K}_{4,l}\mathcal{R}_{(\mathtt{t}^-)}(\tilde{z})\tilde{Q}_{\mathtt{c}}\mathtt{dt}, \mathtt{dx}), \tag{2.31}$$

which implies that

$$\mathtt{d}\mathcal{R}(t) \geq \mathcal{R}(t)\,\mathtt{dt} + \mathcal{W}(\mathtt{t})\,\mathtt{dt} + \bar{\sigma}_{6,l}\mathcal{R}(t)\,\mathscr{W}_4\mathtt{d}(\mathtt{t}) + \bar{\rho}_{1,l}\sum_{i=1}^{2} I_{i(t)}\,\mathtt{dt} + \int_Z \mathscr{K}_{4,l}\mathcal{R}(\mathtt{t}^-)(\tilde{z})\tilde{Q}_{\mathtt{c}}(\mathtt{dt}, \mathtt{dx}). \tag{2.32}$$

We integrate over the interval $[0, t]$ and multiply by $\frac{1}{t}$

$$\begin{aligned} \frac{\mathcal{R}(t) - R_{(0)}}{\mathtt{t}} \geq &\min\left\{(1-\rho)\xi_{1,l}^*,\ \xi_{2,l}^*\right\}\int_0^{\mathtt{t}}\left(\sum_{i=1}^{2}\mathcal{I}_{i(s)} + \mathcal{W}_{(s)}\right)\mathtt{d}s - \frac{k}{\mathtt{t}}\int_0^{\mathtt{t}} R_{(s)}\,\mathtt{d}s \\ &+ \frac{1}{\mathtt{t}}\int_0^{\mathtt{t}} \bar{\sigma}_{4,l}R(s)\,d\mathscr{W}_4(s) + \frac{1}{\mathtt{t}}\int_0^{\mathtt{t}}\int_Z \mathscr{K}_{4,l}\mathcal{R}(s^-)(x)\,\tilde{Q}_{\mathtt{c}}(\mathtt{d}s, \mathtt{d}x). \end{aligned} \tag{2.33}$$

Dividing both sides by ϕ_l and rearranging the resulting inequality yields the time average of $\mathcal{R}(t)$

$$\begin{aligned} \frac{1}{\mathtt{t}}\int_0^{\mathtt{t}} \mathcal{R}_{(s)}\,\mathtt{ds} \geq &\min\left\{(1-\rho)\xi_{1,l}^*,\ \xi_{2,l}^*\right\}.\int_0^{\mathtt{t}}\left(\sum_{i=1}^{3}\mathcal{I}_{i(s)} + \mathcal{W}_{(s)}\right)\mathtt{ds} \\ &+ \frac{1}{\phi_l\mathtt{t}}\left[\mathcal{R}_{(0)} - \mathcal{R}_{(t)} + \bar{\sigma}_{4,l}\int_0^t \mathcal{R}_{(s)}\,d\mathscr{W}_{4(s)} + \int_0^{\mathtt{t}}\int_X \mathscr{K}_{4,1}\mathcal{R}_{(s^-)}(x)\,\tilde{Q}_{\mathtt{c}}(\mathtt{ds}, \mathtt{dx})\right]. \end{aligned} \tag{2.34}$$

Suppose that

$$\mathscr{M}_1 := \frac{1}{\mathtt{t}}\left\{\mathcal{R}_{(\mathtt{0})} - \mathcal{R}(t) + \bar{\sigma}_{4,l}\int_0^{\mathtt{t}} \mathcal{R}_{(s)}\,\mathtt{d}\mathscr{W}_{4(s)} + \int_X \mathscr{K}_{4,l}\mathcal{R}_{(\mathtt{t}^-)}(\tilde{z})\tilde{Q}_{\mathtt{c}}(\mathtt{dt}, \mathtt{d}x)\right\}. \tag{2.35}$$

The above mentioned inequality in compact form is

$$\langle \mathcal{R}(t) \rangle \;\geq\; \frac{p_{1,l}}{\phi_l} \sum_{i=1}^{2} \langle \mathcal{I}_{i(\mathtt{t})} \rangle \, \mathtt{dt} + \mathscr{M}_1. \tag{2.36}$$

By using $\bar{\rho}_{2,l} = \max\left\{(1-\rho)\xi^*_{1,l},\, \xi^*_{2,l}\right\}$, then Inequality (2.30) can be expressed as follows

$$\begin{aligned} d\mathcal{R}_{(t)} \leq \rho_{2,l}\Big\{(1-\rho_l)\xi^*_{1,l}\gamma_l \mathcal{I}_1(t) + \gamma_l \mathcal{I}_2(t) + \psi_l \mathcal{S}(\mathtt{t}) - \phi_l \mathcal{R}_{(t)}\Big\}\, dt + \bar{\sigma}_{4,l}\mathcal{R}_{(t)}\, d\mathscr{W}_{4(t)} \\ + \int_Z \mathscr{K}_{4,l}\mathcal{R}(t^-)(\tilde{z})\, \tilde{Q}_c(dt d\tilde{z}). \end{aligned} \tag{2.37}$$

Integrating over the interval $[\mathtt{0}, \mathtt{t}]$ and dividing by $\mathtt{t}$ with some mathematical operations, we get

$$\langle \mathcal{R}(t) \rangle \leq \rho_{2,l}\left\{(1-\rho)\xi^*_{1,l},\, \xi^*_{2,l}\right\} \sum_{i=\mathtt{i}}^{2} \langle \mathcal{I}_{i(\mathtt{t})} \rangle + \frac{\psi_l}{\phi_l} \langle \mathcal{S}(\mathtt{t}) \rangle + \mathscr{M}_2. \tag{2.38}$$

Moreove

$$\mathscr{M}_2 = \frac{1}{\phi^*_l}\left(\int_0^{\mathtt{t}} (\mathcal{R}(t) - \mathcal{R}_{(0)})\, \mathtt{dt} + \int_0^{\mathtt{t}} \bar{\sigma}_{4,l}\, \mathtt{d}\mathscr{W}\mathtt{4(t)} + \int_0^{\mathtt{t}} \int_X \mathscr{K}_{4,l}\mathcal{R}(\mathtt{t}^-)(\tilde{z})\, \tilde{Q}_{\mathtt{c}}(\mathtt{dt}, \mathtt{d}\tilde{z}) \right). \tag{2.39}$$

By adding the first three equations of the stochastic model (2.4), we get

$$\begin{aligned} &\mathtt{d}\left(\mathcal{S}(\mathtt{t}) + \sum_{i=1}^{2} \mathcal{I}_{i(t)} \right) \\ &= \Big\{ \omega_l - \xi^*_l \mathcal{I}_{1(\mathtt{t})}\mathcal{S}(\mathtt{t}) - \beta^*_l \mathcal{I}_{2(\mathtt{t})}\mathcal{S}(\mathtt{t}) - \gamma_l \mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t}) - (\psi_l + \phi_l)\mathcal{S}(\mathtt{t}) \Big\} \mathtt{dt} + \bar{\sigma}_{1,l}\mathcal{S}(\mathtt{t})\, \mathtt{d}\mathscr{W}_{1(\mathtt{t})} \\ &\quad + \int_Z \mathscr{K}_{1,l}\mathcal{S}(\mathtt{t}^-)(\tilde{z})\, \tilde{Q}_{\mathtt{c}}(\mathtt{dt}, \mathtt{d}\tilde{z}) + \Big\{ \varrho(\xi^*_{r(\mathtt{t})}\mathcal{I}_1(\mathtt{t})\mathcal{S}(\mathtt{t}) + \beta^*_{r(\mathtt{t})}\mathcal{I}_2(\mathtt{t})\mathcal{S}(\mathtt{t}) + \gamma_{r(\mathtt{t})}\mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t})) \\ &\qquad - (\phi_{r(\mathtt{t})} + \xi^*_{1,\mathcal{R}(t)})\mathcal{I}_1(t) \Big\} \mathtt{dt} + \bar{\sigma}_{2,r(\mathtt{t})}\mathcal{I}_{\mathtt{i}}(\mathtt{t})\, \mathtt{d}\mathscr{W}_{2(\mathtt{t})} + \int_Z \mathscr{K}_{2,r(\mathtt{t})}\mathcal{I}_{1(t^-)}(\tilde{z})\, \tilde{Q}_{\mathtt{c}}(\mathtt{dt}, \mathtt{d}\tilde{z}) \\ &\quad + \Big\{ (1 - \varrho_{r(\mathtt{t})})(\xi^*_{r(\mathtt{t})}\mathcal{I}_1(t)\mathcal{S}(\mathtt{t}) + \beta^*_{r(\mathtt{t})}\mathcal{I}_2(t)\mathcal{S}(\mathtt{t}) + \gamma_{r(\mathtt{t})}\mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t})) + \rho\, \xi^*_{1,r(\mathtt{t})}\mathcal{I}_1(t) \\ &\qquad - (\delta_{1,r(\mathtt{t})} + \xi^*_{2,\mathcal{R}(t)} + \phi_{r(\mathtt{t})})\mathcal{I}_2(\mathtt{t}) \Big\} \mathtt{dt} + \bar{\sigma}_{3,r(\mathtt{t})}\mathcal{I}_2(t)\, \mathtt{d}\mathscr{W}_{3(\mathtt{t})} + \int_Z \mathscr{K}_{3,r(\mathtt{t})}\mathcal{I}_{2(\mathtt{t}^-)}(\tilde{z})\, \tilde{Q}_{\mathtt{c}}(\mathtt{dt}, \mathtt{d}\tilde{z}). \end{aligned} \tag{2.40}$$

After carrying out the simplification, Eq (2.40) takes the form of

$$\begin{aligned} &\mathtt{d}\left(\mathcal{S}(\mathtt{t}) + \sum_{i=1}^{2} \mathcal{I}_{i(\mathtt{t})} \right) \\ &= \omega_l\, \mathtt{dt} - \Big(\xi^*_l \mathcal{I}_1(t)\mathcal{S}(\mathtt{t}) + \beta^*_l \mathcal{I}_2(t)\mathcal{S}(\mathtt{t}) + \gamma_l \mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t}) \Big)\, \mathtt{dt} \\ &\quad - \Big((\psi_l + \phi_l)\mathcal{S}(\mathtt{t}) \Big)\, \mathtt{dt} + \varrho\Big(\xi_{r(\mathtt{t})}\mathcal{I}_1(t)\mathcal{S}(\mathtt{t}) + \beta^*_l \mathcal{I}_2(t)\mathcal{S}(\mathtt{t}) - \gamma_l \mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t}) \Big)\, \mathtt{dt} \\ &\quad - \Big((\phi_l + \xi^*_{1,l})\mathcal{I}_1(t) \Big)\, \mathtt{dt} \end{aligned} \tag{2.41}$$

$$
\begin{aligned}
&+ (1-\varrho_{r(\mathtt{t})})\left(\xi_l^* \mathcal{I}_1(t)\mathcal{S}(\mathtt{t}) + \beta_l^* \mathcal{I}_2(t)\mathcal{S}(\mathtt{t}) + \gamma_l \mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t})\right)\mathtt{dt} + \rho\,\xi_{1,l}^* \mathcal{I}_1(t)\,\mathtt{dt}\\
&- \left(\delta_{1,l}^* + \xi_{2,r(\mathtt{t})}^* + \phi_l\right)\mathcal{I}_2(t)\,\mathtt{dt} + \bar{\sigma}_{1,l}\mathcal{S}(\mathtt{t})\,\mathtt{d}\mathscr{W}_{1(\mathtt{t})} + \bar{\sigma}_{2,l}\mathcal{I}_1(t)\,\mathtt{d}\mathscr{W}_{2(\mathtt{t})} + \bar{\sigma}_{3,l}\mathcal{I}_2(t)\,\mathtt{d}\mathscr{W}_{3(\mathtt{t})}\\
&+ \int_Z \mathscr{K}_{1,l}\mathcal{S}_{(\mathtt{t}^-)}(\tilde{z})\,\tilde{Q}_{\mathtt{c}}(\mathtt{dt},\mathtt{d}\tilde{z}) + \int_Z \mathscr{K}_{2,\mathcal{R}(t)}\mathcal{I}_{1(\mathtt{t}^-)}(\tilde{z})\,\tilde{Q}_{\mathtt{c}}(\mathtt{dt},\mathtt{d}\tilde{z}) + \int_Z \mathscr{K}_{3,\mathcal{R}(t)}\mathcal{I}_{2(\mathtt{t}^-)}(\tilde{z})\,\tilde{Q}_{\mathtt{c}}(\mathtt{dt},\mathtt{d}\tilde{z}).
\end{aligned}
$$

This becomes

$$
\begin{aligned}
\mathtt{d}\left(\mathcal{S}(\mathtt{t}) + \sum_{i=1}^{2}\mathcal{I}_{i(\mathtt{t})}\right) = \Big[&\omega_l\,\mathtt{dt} - (\psi_l+\phi_l)\mathcal{S}(\mathtt{t})\,\mathtt{dt} - (\phi_l + \xi_{1,l}^*)\mathcal{I}_1(t)\,\mathtt{dt}\\
&- (\delta_1^* + \xi_{2,l}^* + \phi_l)\mathcal{I}_2(t)\,\mathtt{dt} + \rho\,\gamma_{1,\mathcal{R}(t)})\mathcal{I}_1(t)\\
&+ \bar{\sigma}_{1,l}\mathcal{S}(\mathtt{t})\,\mathtt{d}\mathscr{W}_{1(t)} + \bar{\sigma}_{2,\mathcal{R}(t)}\mathcal{I}_1(\mathtt{t})\,\mathtt{d}\mathscr{W}_{2(\mathtt{t})} + \bar{\sigma}_{3,\mathcal{R}(t)}\mathcal{I}_2(\mathtt{t})\,\mathtt{d}\mathscr{W}_{3(\mathtt{t})}\\
&+ \int_Z \mathscr{K}_{1,l}\mathcal{S}_{(\mathtt{t}^-)}(\tilde{z})\,\tilde{Q}_{\mathtt{c}}(\mathtt{dcd}\tilde{z}) + \int_Z \mathscr{K}_{2,\mathcal{R}(t)}\mathcal{I}_{1(\mathtt{t}^-)}(\tilde{z})\,\tilde{Q}_{\mathtt{c}}(\mathtt{dt},\mathtt{d}\tilde{z})\\
&+ \int_Z \mathscr{K}_{3,\mathcal{R}(t)}\mathcal{I}_{2(\mathtt{t}^-)}(\tilde{z})\,\tilde{Q}_{\mathtt{c}}(\mathtt{dc},\mathtt{d}\tilde{z}).
\end{aligned}
\tag{2.42}
$$

By removing the negative $-(\phi_l + \xi_{1,l}^*)\mathcal{I}_1(t)\,\mathtt{dt} - (\delta_1^* + \xi_{2,l}^* + \phi_l)\mathcal{I}_2(t)$ terms, the inequality (2.42) can be expressed as

$$
\begin{aligned}
\mathtt{d}\left(\mathcal{S}(\mathtt{t}) + \sum_{i=1}^{2}\mathcal{I}_{i(\mathtt{t})}\right) \geq \Big[&\omega_l\,\mathtt{dt} - (\psi_l+\phi_l)\mathcal{S}(\mathtt{t})\,\mathtt{dt} - \sum_{i=1}^{2}(\phi_{2,l}^* + ((i-1)\delta_{(i-1),l}^* + \xi_l^*))\mathcal{I}_i(\mathtt{t})\,\mathtt{dt}\\
&+ \bar{\sigma}_{1,l}\mathcal{S}(\mathtt{t})\,d\mathscr{W}_{1(\mathtt{t})} + \bar{\sigma}_{2,r(\mathtt{t})}\mathcal{I}_1(t)\,\mathtt{d}\mathscr{W}_{2(\mathtt{t})} + \bar{\sigma}_{3,r(\mathtt{t})}\mathcal{I}_2(t)\,d\mathscr{W}_{3(\mathtt{t})}\\
&+ \int_X \mathscr{K}_{1,l}\mathcal{S}_{(\mathtt{t}^-)}(\tilde{z})\,\tilde{Q}_{\mathtt{c}}(\mathtt{dt},\mathtt{d}\tilde{z}) + \int_X \mathscr{K}_{2,\mathcal{R}(t)}\mathcal{I}_{1(\mathtt{t}^-)}(\tilde{z})\,\tilde{Q}_{\mathtt{c}}(\mathtt{dtd}\tilde{z})\\
&+ \int_X \mathscr{K}_{3,r(\mathtt{t})}\mathcal{I}_{2(\mathtt{t}^-)}(\tilde{z})\,\tilde{Q}_{\mathtt{c}}(\mathtt{dtd}\tilde{z}).
\end{aligned}
\tag{2.43}
$$

To simplify our results, we introduce some auxiliary notations for convenience.

$$
\begin{aligned}
\bar{\mathscr{E}}_1 = &\bar{\sigma}_{1,l}\mathcal{S}(\mathtt{t})\,\mathtt{d}\mathscr{W}_{1(\mathtt{t})} + \sum_{i=2}^{2}\bar{\sigma}_{i,l}\mathcal{I}_{i-1(\mathtt{t})}\,dW_{i(\mathtt{t})} + \int_X \mathscr{K}_{3,r(\mathtt{t})}\mathcal{S}_{(\mathtt{t}^-)}(\tilde{z})\,\tilde{Q}_c(\mathtt{dt},\mathtt{d}\tilde{z})\\
&+ \int_X \mathscr{K}_{2,r(\mathtt{t})}\mathcal{I}_{1(t^-)}(\tilde{z})\,\tilde{Q}_{\mathtt{c}}\mathscr{W}_2(\mathtt{dt},\mathtt{d}\tilde{z}) + \int_X \mathscr{K}_{3,r(\mathtt{t})}\mathcal{I}_{2(t^-)}(\tilde{z})\,\tilde{Q}_{\mathtt{c}}\mathscr{W}_2(\mathtt{dt},\mathtt{d}\tilde{z}).
\end{aligned}
\tag{2.44}
$$

Then (2.43) can be expressed as

$$
\mathtt{d}\left(\mathcal{S}(\mathtt{t}) + \sum_{i=1}^{2}\mathcal{I}_{i(\mathtt{t})}\right) \geq \Big[\omega_l\,\mathtt{dt} - (\psi_l+\phi_l)\mathcal{S}(\mathtt{t})\,\mathtt{dt} - \sum_{i=1}^{2}(\phi_{2,l} + ((i-1)\delta_{(i-1),l}^* + \xi_l^*))\mathcal{I}_{i(\mathtt{t})}\,\mathtt{dt} + \bar{\mathscr{E}}_1. \tag{2.45}
$$

If the Inequality (2.43) is divided by $\mathtt{t}$ and integrating $\int_0^t$, we have

$$\frac{1}{\mathtt{t}}\Bigg(\mathcal{S}(t)-\mathcal{S}(\mathbb{0})+\sum_{i=1}^{2}(\mathcal{I}_{i(\mathtt{t})}-\mathcal{I}_{i(\mathbb{0})})\Bigg)\geq \omega_l-(\psi_l+\phi_l)\frac{1}{\mathtt{t}}\int_0^{\mathtt{t}}\mathcal{S}_{(s)}\,\mathtt{ds} \\ -\sum_{i=1}^{2}(\phi_{2,l}+(i-1)\delta^*_{(i-1),l}+\xi^*_l)\frac{1}{\mathtt{t}}\int_{\mathbb{0}}^{\mathtt{t}}\mathcal{I}_{i(s)}\,\mathtt{ds}+\frac{1}{\mathtt{t}}\int_0^{\mathtt{t}}\bar{\mathscr{E}}_1\,\mathtt{ds}. \tag{2.46}$$

$$\langle\mathcal{S}(\mathtt{t})\rangle\geq\frac{\omega}{\phi_l+\phi_l}-\sum_{i=1}^{2}\frac{(\langle\phi_{i,l}+(i-1)\delta^*_{(i-1),l}+\xi^*_l\rangle)}{\phi_l+\phi_l}\langle\mathcal{I}_{i(\mathtt{t})}\rangle+\mathscr{M}_3, \tag{2.47}$$

$$\mathscr{M}_3=\frac{1}{(\phi_l+\phi_l)t}\Bigg\{\mathcal{S}_{(\mathbb{0})}+\sum_{i=1}^{2}\mathcal{I}_{i(\mathbb{0})}-\mathcal{S}(\mathtt{t})-\sum_{i=1}^{2}\mathcal{I}_{i(\mathtt{t})}+\int_{\mathbb{0}}^{\mathtt{t}}\bar{\sigma}_{1,l}\mathcal{S}_{(s)}\,\mathrm{d}\mathscr{W}_{1(s)}+\sum_{i=2}^{2}\int_{\mathbb{0}}^{\mathtt{t}}\bar{\sigma}_{i,l}\mathcal{I}_{i-1}(s)\,\mathrm{d}W_{i(s)} \\ +\int_{\mathbb{0}}^{\mathtt{t}}\int_Z\mathscr{K}_{1,l}\mathcal{S}_{(\mathtt{t}^-)}(\tilde{z})\,\tilde{Q}_{\mathtt{c}}(\mathrm{d}s,\mathrm{d}x)+\sum_{i=2}^{3}\int_0^{\mathtt{t}}\int_Z\mathscr{K}_{1,l}\mathcal{I}_{i-1(\mathtt{t}^-)}(\tilde{z})\,\tilde{Q}_{\mathtt{c}}(\mathrm{d}s,\mathrm{d}z)\Bigg\}. \tag{2.48}$$

By applying some basic mathematical operations, the Inequality (2.40) yields

$$\langle\mathcal{S}(\mathtt{t})\rangle\leq\frac{\omega_l}{(\psi_l+\phi_l)t}+\frac{\rho_l\xi^*_{1,l}\langle\mathcal{I}_i\rangle}{(\psi_l+\phi_l)}-\sum_{i=1}^{2}\frac{(\phi_l+\xi^*_l)\langle\mathcal{I}_i\rangle}{(\psi_l+\phi_l)}+\mathscr{M}_4, \tag{2.49}$$

where

$$\mathscr{M}_4=\frac{1}{(\phi_l+\phi_l)\mathtt{t}}\Bigg\{\mathcal{S}(\mathbb{0})+\sum_{i=1}^{2}\mathcal{I}_i(\mathbb{0})-\mathcal{S}(\mathtt{t})-\sum_{i=1}^{2}Y_i(t)+\int_{\mathbb{0}}^{\mathtt{t}}\bar{\sigma}_{1,l}\mathcal{S}(s)\,\mathrm{d}\mathscr{W}_1(s)+\sum_{i=2}^{2}\int_{\mathbb{0}}^{\mathtt{t}}\bar{\sigma}_{i,l}\mathcal{I}_{i-1}(s)\,dW_i(s) \\ +\int_0^t\int_Z\mathscr{K}_{1,l}\mathcal{S}_{(\mathtt{t}^-)}(\tilde{z})\,\tilde{Q}_c(ds,dx)+\sum_{i=2}^{3}\int_0^t\int_Z\mathscr{K}_{1,l}\mathcal{I}_{i-1}(t^-)(\tilde{z})\,\tilde{Q}_c(ds,dx)\Bigg\}. \tag{2.50}$$

Consider the function $\bar{\mathscr{G}}_3:\mathscr{R}^n\times\mathbf{S}\to\mathscr{R}$ defined by $\bar{\mathscr{G}}(\mathcal{I}_i,\kappa)=\ln(\mathcal{I}_i)+\Psi(\kappa)$, and Ito's formula given by

$$\mathrm{d}\bar{\mathscr{G}}_3=\bar{\mathscr{L}}\bar{\mathscr{G}}_3\,\mathtt{dt}+\bar{\sigma}_{i+1,l}\,\mathrm{d}\mathscr{W}_{i+1}+\int_{\tilde{z}}\ln\left(1+\mathscr{K}_{i+1,l}(\tilde{z})\right)\tilde{Q}_{\mathtt{c}}(\mathtt{dt},\mathrm{d}\tilde{z}). \tag{2.51}$$

$$\bar{\mathscr{L}}\bar{\mathscr{G}}_3(Y,\mathtt{t},l)=\mathscr{G}_{\mathtt{t}}(Y,\mathtt{t},l)+\sum_{j=1}^{m}\theta_{lj}\mathscr{G}(Y,\mathtt{t},j)+\mathscr{G}_Y(Y,\mathtt{t},l)\,\mathbf{P}_1(Y,\mathtt{t},l)+\frac{1}{2}\Big[\mathbf{P}_2^{\top}(Y,\mathtt{t},l)\,\mathscr{G}_{YY}(Y,\mathtt{t},l)\,\mathbf{P}_2(Y,\mathtt{t},l)\Big] \\ +\int_{\tilde{Z}}\Big[Y(Y+Y_3(Y,\mathtt{t},\tilde{z},l),\mathtt{t},l)-\mathscr{G}(Y,\mathtt{t},l)-\mathscr{G}_Y(Y,\mathtt{t},l)\,\mathbf{P}_3(Y,\mathtt{t},\tilde{z},l)\Big]\,\nu(\mathrm{d}x). \tag{2.52}$$

First, we evaluate the following. By using Itô's formula, we have

$$\begin{aligned}
\mathscr{G}_{3t}(Y,\mathtt{t},l) &= \frac{\partial}{\partial \mathtt{t}}(\ln(\mathcal{I}_i)+\Psi(\kappa))=0,\\
\mathscr{G}_{3Y}(Y,\mathtt{t},l) = \mathscr{G}_{3Y_i}(Y_i) &= \frac{\partial}{\partial \mathcal{I}_i}(\ln(\mathcal{I}_i)+\Psi(\kappa)) = -\frac{1}{\mathcal{I}_i},\\
\mathscr{G}_{3Y}(Y,\mathtt{t},l)\,\mathbf{P}_1(Y,\mathtt{t},l) &= \frac{\varrho}{\mathcal{I}_i}\Big(\xi_l^*\mathcal{I}_1(t)\mathcal{S}(\mathtt{t})+\beta_l^*\mathcal{I}_2(t)\mathcal{S}(\mathtt{t})+\gamma_l\mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t})\Big)\\
&\quad+\frac{1-\varrho}{\mathcal{I}_2}\Big(\xi_l^*\mathcal{I}_1(t)\mathcal{S}(\mathtt{t})+\beta_l^*\mathcal{I}_2(t)\mathcal{S}(\mathtt{t})+\gamma_l\mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t})\Big)\\
\frac{1}{2}\mathbf{P}_2^{\top}(Y,\mathtt{t},l)\,\mathscr{G}_{3YY}(Y,\mathtt{t},l)\,\mathbf{P}_2\mathtt{t}(Y,\mathtt{t},l) &= -\frac{1}{2}\bar{\sigma}_{i+1,l}^2\,\mathbf{P}_3^{\top}(Y,\mathtt{t},\tilde{z},l)\\
&= \mathscr{K}_{i+1,l}(\tilde{z})\,\mathcal{I}_i.
\end{aligned} \tag{2.53}$$

Using $\mathbf{P}_3^{\top}(Y,\mathtt{t},\tilde{z},l)$ in the free, we define the function

$$\bar{\mathscr{G}_3}(Y\ +\mathbf{P}_3(Y,t,\tilde{z},l) = ln(1+\mathscr{K}_{i+1,l}\tilde{z}+ln(\mathcal{I})+\Psi(\kappa).$$

In addition

$$\bar{\mathscr{G}_3}Y(Y\ +\mathbf{P}_3(Y,\mathtt{t},\tilde{z},l) = \frac{1}{\mathcal{I}_i}\mathscr{K}_{i+1,l}\mathcal{I}_{\rangle}\tilde{z} = \mathscr{K}_{i+1,l}(\tilde{z}).$$

Moreover

$$\int_{\tilde{z}}\Big(\bar{\mathscr{G}_3}(Y+\mathbf{P}_3(Y,t,\tilde{z},l),t,l)-\bar{\mathscr{G}_3}(Y,t,l)-\bar{\mathscr{G}_{3Y}}(Y,t,l)\,\mathbf{P}_3(Y,t,\tilde{z},l)\Big)v(d\tilde{z}) = \int_{\tilde{z}}(ln(1+\mathscr{K}_{i+1,l}\tilde{z})-\mathscr{K}_{i+1,l}\tilde{z})\,v(d\tilde{z}). \tag{2.54}$$

We have

$$\sum_{j=1}^{m}\theta_{lj}\bar{\mathscr{G}_3}(Y,t,j) = \sum_{j=1}^{m}\theta_{lj}\ln(\mathcal{I}_i)+\sum_{j=1}^{m}\theta_{lj}\,\Psi(j). \tag{2.55}$$

Note that the sum of all rows in the generating matrix is $\theta=(\theta_{lj})_{\mathtt{m}\times\mathtt{m}}\ \sum_{j=1}^{\mathtt{m}}\theta_{lj}=\mathbb{0}$ and also

$$\sum_{j=1}^{\mathtt{m}}\theta_{lj}\bar{\mathscr{G}_3}(Y,\mathtt{t},j) = \sum_{j=1}^{\mathtt{m}}\theta_{lj}\,\Psi(j). \tag{2.56}$$

By utilizing (2.53), (2.55), and (2.56) in (2.52) , we get the following result:

$$\begin{aligned}
\bar{\mathscr{L}}\bar{\mathscr{G}}(Y,\mathtt{t},l) &= \frac{1}{\mathcal{I}_1}\Big(\varrho\big(\xi_l^*\mathcal{I}_{1(\mathtt{t})}\mathcal{S}(\mathtt{t})+\beta_l^*\mathcal{I}_2(t)\mathcal{S}(\mathtt{t})+\gamma_l\mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t})+\frac{\rho_l\xi_{1,l}^*}{\mathcal{I}_2}\mathcal{I}_{1(\mathtt{t})}\big)\Big)\\
&\quad+\frac{1}{\mathcal{I}_2}\Big(1-\rho)(\xi_l^*\mathcal{I}_1(\mathtt{t})\mathcal{S}(\mathtt{t})+\beta_l^*\mathcal{I}_2(\mathtt{t})\mathcal{S}(\mathtt{t})+\gamma_l\mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t})+\sum_{j=1}^{\mathtt{m}}\theta_{lj}\,\Psi(j))\Big)\\
&\quad-\Big\{(\delta_1^*+\xi_{2,l}^*+2\phi_l)+\frac{1}{2}\bar{\sigma}_{i+1,l}^2+\int\ln(1+\mathscr{K}_{i+1,l}\tilde{z})+\mathscr{K}_{i+1,l}\tilde{z}\Big\}\,v(d\tilde{z})
\end{aligned} \tag{2.57}$$

By using this result $\bar{\mathscr{G}}(\mathcal{I}_i, \kappa) = \ln(\mathcal{I}_i) + \Psi(\kappa)$ and applying mathematical calculation to the expression above, it can be written as

$$\begin{aligned}\bar{\mathscr{L}}\bar{\mathscr{G}}_3(Y, \mathtt{t}, l) \leq & \Big(\varrho(\xi_l^* \mathcal{I}_1(\mathtt{t})\mathcal{S}(\mathtt{t}) + \beta_l^* \mathcal{I}_2(t)\mathcal{S}(\mathtt{t}) + \gamma_l \mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t}) + \rho\xi_{1,l}^* \mathcal{I}_1(\mathtt{t}))\Big) \\ & - \Big(\rho(\xi_l^* \mathcal{I}_1(\mathtt{t})\mathcal{S}(\mathtt{t}) + \beta_l^* \mathcal{I}_2(\mathtt{t})\mathcal{S}(\mathtt{t}) + \gamma_l \mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t}))\Big) \\ & + \Big(\rho(\xi_l^* \mathcal{I}_1(t)\mathcal{S}(\mathtt{t}) + \beta_l^* \mathcal{I}_2(t)\mathcal{S}(\mathtt{t}) + \gamma_l \mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t}) + \sum_{j=1}^{m} \theta_{lj}\, \Psi(j))\Big) \\ & - \Big\{(\delta_1^* + \xi_{2,l}^* + 2\phi_l) + \frac{1}{2}\bar{\sigma}_{i+1,l}^2 + \int \ln(1 + \mathscr{K}_{i+1,l}\tilde{z}) + \mathscr{K}_{i+1,l}\tilde{z}\Big\}\, \nu(\mathrm{d}\tilde{z})\,.\end{aligned} \tag{2.58}$$

This implies that

$$\begin{aligned}\bar{\mathscr{L}}\bar{\mathscr{G}}_3(Y, \mathtt{t} \leq & \Big(\varrho(\xi_l^* \mathcal{I}_1(t)\mathcal{S}(\mathtt{t}) + \beta_l^* \mathcal{I}_2(t)\mathcal{S}(\mathtt{t}) + \gamma_l \mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t}) + \rho_{2,l}\left\{(1-\rho)\xi_{1,l}^*,\, \xi_{2,l}^*\right\} \mathcal{I}_1(t))\Big) + \sum_{j=1}^{m} \theta_{lj}\, \Psi(j))\Big) \\ & - \Big\{(\delta_1^* + \xi_{2,l}^* + 2\phi_l) + \frac{1}{2}\bar{\sigma}_{i+1,l}^2 + \int \ln(1 + \mathscr{K}_{i+1,l}\tilde{z}) + \mathscr{K}_{i+1,l}\tilde{z}\Big\}\, \nu(d\tilde{z})\,.\end{aligned} \tag{2.59}$$

Using (2.6) in Inequality (2.59), we get

$$\begin{aligned}\bar{\mathscr{L}}\bar{\mathscr{G}}_3(Y, \mathtt{t}, l) \leq & \left(\xi_l \frac{\breve{\omega}}{\hat{\psi}} + \beta^* \frac{\breve{\omega}}{\hat{\psi}} + \gamma \frac{\breve{\omega}}{\hat{\psi}}\right)\mathcal{S} + \rho_{1,l}\left\{(1-\rho)\xi_{1,l}^*,\, \xi_{2,l}^*\right\} \mathcal{I}_i + \sum_{j=1}^{m} \theta_{\kappa j}\, \Psi(\kappa_j) \\ & - (\delta_1^* + \xi_{1,l}^* + \xi_{2,l}^* + 2\phi_l) + \frac{\bar{\sigma}_{i+1,l}^2}{2} + \int_{\tilde{z}} \ln\Big(1 + \mathscr{K}_{i+1,l}\tilde{z}\Big) + \mathscr{K}_{i+1,l}\tilde{z}\, \nu(d\tilde{z}).\end{aligned} \tag{2.60}$$

To derive an auxiliary equation, let us assume that $\bar{\chi}_{0,l} = \left(\xi_l \frac{\breve{\omega}}{\hat{\psi}} + \beta^* \frac{\breve{\omega}}{\hat{\psi}} + \gamma \frac{\breve{\omega}}{\hat{\psi}}\right)$. The Inequality (2.60) can be written as

$$\begin{aligned}\bar{\mathscr{L}}\bar{\mathscr{G}}_3(Y, \mathtt{t}, l) \leq & \bar{\chi}_{0,l}\mathcal{S} + \rho\xi_{1,l}^* \mathcal{I}_1 + \sum_{j=1}^{\mathtt{m}} \theta_{\kappa j}\, \Psi(\kappa_j) \\ & - (\delta_1^* + \xi_{1,l}^* + \xi_{2,l}^* + 2\phi_l) + \frac{\bar{\sigma}_{i+1,l}^2}{2} + \int_{\tilde{z}} \ln\Big(1 + \mathscr{K}_{i+1,l}\tilde{z}\Big) + \mathscr{K}_{i+1,l}\tilde{z}\, \nu(d\tilde{z}).\end{aligned} \tag{2.61}$$

Putting $(\delta_1^* + \xi_{1,l}^* + \xi_{2,l}^* + 2\phi_l) + \frac{\bar{\sigma}_{i+1,l}^2}{2} + \int_{\tilde{z}} \ln\Big(1 + \mathscr{K}_{i+1,l}\tilde{z}\Big) + \mathscr{K}_{i+1,l}\tilde{z}\, \nu(d\tilde{z}) = \phi_{i,l}^*$, we get

$$\bar{\mathscr{L}}\bar{\mathscr{G}}_3(Y, \mathtt{t}, l) \leq \bar{\chi}_{\mathtt{0},l} S - \phi_{i,l}^* + \sum_{j=1}^{\mathtt{m}} \theta_{\kappa j}\, \Psi(j) + \rho_l \xi_l^* \mathcal{I}_i \tag{2.62}$$

and

$$\bar{\mu}_{0,i}(l) = \bar{\chi}_{0,l} S_l - \phi_{i,l}^*,$$

Evaluating (2.62) at $s = s_l$, we get:

$$\bar{\mathscr{L}}\bar{\mathscr{G}}_3(Y, t, l) \leq \bar{\mu}_{0,i}(l) + \sum_{j=1}^{m} \theta_{\kappa j}\, \Psi(j) + \rho_l \xi_l \mathcal{I}_i. \tag{2.63}$$

Suppose that $\Delta \in \mathscr{R}^{\mathtt{m}\times\mathtt{m}}$ is an irreducible matrix, and let $\bar{\mu}_{\mathtt{0},i} = (\bar{\mu}_{\mathtt{0},i}(1), \ldots, \bar{\mu}_{\mathtt{0},i}(\mathtt{m}))^T$ denote a given vector. We seek a vector$\mathbf{\Psi} = (\Psi(1), \ldots, \Psi(m))^T$ satisfying the Poisson system.

$$\bar{\mu}_{\mathtt{0},i} = \begin{pmatrix} \bar{\mu}_{\mathtt{0},i}(1) \\ \bar{\mu}_{\mathtt{0},i}(2) \\ \bar{\mu}_{\mathtt{0},i}(3) \\ \bar{\mu}_{\mathtt{0},i}(4) \\ \vdots \\ \bar{\mu}_{\mathtt{0},i}(m) \end{pmatrix}, \quad \Psi = \begin{pmatrix} \Psi(1) \\ \Psi(2) \\ \Psi(3) \\ \Psi(4) \\ \vdots \\ \Psi(m) \end{pmatrix} \Longrightarrow \Delta\Psi = \sum_{j=1}^{m} \phi_j \bar{\mu}_{\mathtt{0},i}(j) \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} - \bar{\mu}_{1,i}.$$

$\bar{\mathscr{L}}\bar{\mathscr{G}}_3(Y, \mathtt{t}, l) \leq \sum_{l=1}^{m} \Theta_l \bar{\mu}_{\mathtt{0},i}(l) + \rho_l \xi_l \mathcal{I}_i$. By using $\bar{\mu}_{0,i}(l) = \bar{\chi}_{0,l} S_l - \phi^*_{i,l}$, we get the following:

$$\bar{\mathscr{L}}\bar{\mathscr{G}}_3(Y, t, l) \leq \sum_{l=1}^{m} \Theta_l \bar{\chi}_{0,l} S_l - \sum_{l=1}^{m} \phi^*_l \bar{\mu}_{0,i}(l) + \rho_l \xi_l \mathcal{I}_i. \tag{2.64}$$

Take the common $\sum_{l=1}^{\mathtt{m}} \phi_l \Theta_{i,p}$ and simplify as follows:

$$\bar{\mathscr{L}}\bar{\mathscr{G}}_3(Y, \mathtt{t}, l) \leq \left(\frac{\sum_{l=1}^{\mathtt{m}} \Theta_l \bar{\chi}_{\mathtt{0},l} S_l}{\sum_{l=1}^{\mathtt{m}} \Theta_l \phi^*_{i,l}} - 1 \right) \sum_{l=1}^{\mathtt{m}} \Theta_l \phi^*_{i,l} + \rho_l \xi_l \mathcal{I}_i. \tag{2.65}$$

By putting Inequality (2.65) in (2.51), we obtain

$$\begin{aligned} \mathtt{d}\bar{\mathscr{G}}_3 \leq & \left[\left\{ \frac{\sum_{l=1}^{\mathtt{m}} \Theta_1 \bar{\chi}_{\mathtt{0},l} S_l}{\sum_{1=\mathtt{m}}^{\mathtt{m}} \Theta_{i,l} \phi^*_l} - 1 \right\} \sum_{l=1}^{m} \phi^*_l \Theta_{i,l} + \rho_l \xi_l \mathcal{I}_i \right] \mathtt{dt} + \bar{\sigma}_{i+1,l}\, \mathtt{d}W_{i+1} \\ & + \int \ln(1 + \mathscr{K}_{i+1,l}\tilde{z}) \Big\} \widetilde{Q}_c(\mathtt{dt}, \mathtt{d}\tilde{z}) \,. \end{aligned} \tag{2.66}$$

Take $\int_0^t$ of Eq (2.66) and divide by $\mathtt{t}$ as follows

$$\begin{aligned} \bar{\mathscr{G}}_3(\mathtt{t}) - \bar{\mathscr{G}}_3(\mathtt{0}) \leq & \frac{1}{t} \int_0^t \left[\left\{ \frac{\sum_{l=1}^{\mathtt{m}} \Theta_l \bar{\chi}_{\mathtt{0},l} S_l}{\sum_{l=1}^{\mathtt{m}} \phi^*_l \Theta_{i,l}} - 1 \right\} \sum_{l=1}^{m} \phi^*_l \Theta_{i,l} + \rho_l \xi^*_l \mathcal{I}_i \right] \mathtt{d}s + \frac{1}{t} \int_0^t \bar{\sigma}_{1,l} \mathtt{s}\, \mathtt{d}\mathscr{W}_1(s) \\ & + \frac{1}{\mathtt{t}} \int_{\mathtt{0}}^{\mathtt{t}} \int_{\tilde{Z}} \Big[\ln(1 + \mathscr{K}_{i+1,l}\tilde{z}) \Big] \tilde{Q}_{\mathtt{c}}(d\tilde{z}) \mathtt{ds}. \end{aligned} \tag{2.67}$$

Using Inequality (2.67) in $G_3(X_i, \kappa) = \ln(X_i) + \Psi(\kappa)$, we have:

$$\begin{aligned} \frac{1}{\mathtt{t}} \Big\{ \ln(\mathcal{I}_{i(\mathtt{t})}) + \Psi(\kappa) - (\ln(\mathcal{I}_{i(\mathtt{0})}) + \Psi(\kappa)) \Big\} \leq & \frac{1}{\mathtt{t}} \int_{\mathtt{0}}^{\mathtt{t}} \left[\left\{ \frac{\sum_{l=1}^{\mathtt{m}} \Theta_l \bar{\chi}_{\mathtt{0},l} S_l}{\sum_{l=1}^{\mathtt{m}} \phi^*_l \Theta_{i,l}} - 1 \right\} \sum_{l=1}^{\mathtt{m}} \phi^*_l \Theta_{i,l} \right] \mathtt{ds} \\ & + \frac{1}{\mathtt{t}} \int_{\mathtt{0}}^{\mathtt{t}} \bar{\sigma}_{i+1,l}\, dW_{i+1}(\mathtt{s}) \\ & + \frac{1}{\mathtt{t}} \int_{\mathtt{o}}^{\mathtt{t}} \int_{\tilde{Z}} \Big[\ln(\ln(1 + \mathscr{K}_{i+1,l}\tilde{z}) + \mathscr{K}_{i+1,l}\tilde{z} \Big] \tilde{Q}_{\mathtt{c}}(dt, d\tilde{z}) \mathtt{ds}. \end{aligned} \tag{2.68}$$

Using some auxiliary variables in (2.68), we get

$$\frac{\ln(\mathcal{I}_{i(\mathtt{t})})}{\mathtt{t}} \leq \left\{ \frac{\sum_{l=1}^{\mathtt{m}} \Theta_l \bar{\chi}_{\mathtt{0},l} \langle S_l \rangle}{\sum_{l=1}^{\mathtt{m}} \phi_l^* \Theta_{i,l}} - 1 \right\} \sum_{l=1}^{\mathtt{m}} \phi_l^* \Theta_{i,l} + \mathscr{M}_5, \tag{2.69}$$

where

$$\mathscr{M}_5 = \frac{1}{\mathtt{t}} \left\{ \int_{\mathtt{0}}^{\mathtt{t}} \bar{\sigma}_{i+1,l}\, dW_{i+1} \int_{\bar{Z}} \left[\ln(_1 + \mathscr{K}_{i+1,l}\tilde{z}) + \mathscr{K}_{i+1,l}\tilde{z} \right] \tilde{Q}_5\, \nu(d\tilde{z})\, \mathrm{d}s \right\}.$$

It follows from Lemmas 2.6 and 2.7, as $\lim_{t\to\infty} \mathscr{M}_5 = \mathtt{0}$. Moreover as $t \to \infty$, we have $\mathscr{M}_i = \mathtt{0} \forall i \in [1, 2, 3, 4]$. Since $\mathcal{R}_i < 1$ for $i \in \{1, 2\}$, taking the superior limit on both sides of (2.69) and applying the result $\langle \mathcal{S}(\mathtt{t}) \rangle \leq \frac{\omega}{(\psi_l + \phi_l)t}$. which is given in the Inequality in Eq (2.49), we obtain

$$\sup \lim_{t\to\infty} \frac{\mathcal{I}_{i(t)}}{\mathtt{t}} \leq \left\{ \frac{\sum_{l=1}^{\mathtt{m}} \Theta_l \bar{\chi}_{\mathtt{0},l} \omega_l}{\sum_{l=1}^{\mathtt{m}} \phi_l^* \Theta_{i,l} (\psi_l + \phi_l)} - 1 \right\} \sum_{l=1}^{\mathtt{m}} \phi_l^* \Theta_{i,l},$$

or

$$\sup \lim_{t\to\infty} \frac{\bar{\mathscr{G}}_3(\mathcal{I}_i)(\mathtt{t})}{\mathtt{t}} \leq \{\mathcal{R}_{\mathtt{0}} - 1\} \sum_{l=1}^{\mathtt{m}} \phi_l^* \Theta_{i,l}.$$

The following conclusions follow from the previous expression. If $t \to \infty$, both $\mathcal{I}_{i(t)}$ and $\mathcal{W}(\mathtt{t})$ converge to zero almost surely for $i \in \{1, 2\}$. In addition, letting $t \to \infty$ in (2.36), (2.37), (2.47), and (2.49) shows that these expressions also converge to zero.

$$0 \leq \lim_{t\to\infty} \langle \mathcal{R}(t) \rangle \leq \frac{\hat{\phi}}{\check{\psi}} \frac{\hat{\omega}}{\check{\phi} + \check{\psi}} \leq \lim_{t\to\infty} \langle \mathcal{S}(\mathtt{t}) \rangle \leq \frac{\hat{\omega}}{\check{\phi} + \check{\psi}}.$$

□

Theorem 2.10. *Suppose that Assumption 2.2 holds. Let* $(Y(\mathtt{0}), \hat{r}(\mathtt{0})) \in \Omega$ *be the initial values, and let* $(Y(\mathtt{t}), \hat{r}(\mathtt{t}))$ *be the corresponding solution of the stochastic model* (2.4). *Then the infection persists in the model* (2.4) *almost surely if either of the following conditions holds.*

- *If* $\mathcal{R}_0 > 1$, *then* $\bar{\chi}_{0,l} \left(\frac{\check{\psi}}{\check{\omega}}\right)^3 \sum_{i=1}^{2} \frac{\left(\phi_i^* + \xi_{i,l}^* + (i-1)\delta_{i-1,l}^*\right)}{\psi_l + \phi_l} = \bar{\mu}_{i,l} > 0\ \forall i \in \{1, 2\}$ *and both brucellosis and tuberculosis will persist and will satisfy the condition*

$$\inf \lim_{t\to\infty} (\mathcal{I}_i(t)) \geq \frac{(\mathcal{R}_{\mathtt{i}} - 1) \sum_{l=1}^{\mathtt{m}} \phi_l \Theta_l}{\bar{\bar{\mu}}_i} = \underline{\mathcal{I}_{\mathtt{i}}}.$$

- *If* $\mathcal{R}_2 < 1 < \mathcal{R}_1$ *and* $\bar{\mu}_l > \mathtt{0}$, *the infection* $\mathcal{I}_2$ *for tuberculosis be die out and* $\mathcal{I}_1$ *shows that brucellosis will be present in the population, satisfy the result is as follows*

$$\inf \lim_{\mathtt{t}\to\infty} (\mathcal{I}_{1(\mathtt{t})}) \geq \frac{(\mathcal{R}_1 - 1) \sum_{l=1}^{\mathtt{m}} \phi_l \Theta_l}{\bar{\bar{\mu}}_1} = \underline{\mathcal{I}_1}.$$

- *If* $\mathcal{R}_1 < 1 < \mathcal{R}_2$ *and* $\bar{\mu}_l > 0$, *the infection* $\mathcal{I}_1$ *die out and the* $\mathcal{I}_2$ *persist and satisfies the following:*

$$\inf \lim_{\mathtt{t}\to\infty} (\mathcal{I}_{2(\mathtt{t})}) \geq \frac{(\mathcal{R}_2 - 1) \sum_{l=1}^{\mathtt{m}} \phi_l \Theta_l}{\bar{\bar{\hat{\mu}}}_2} = \underline{\mathcal{I}_2}.$$

Furthermore, under the preceding assumptions, the following results hold:

$$\underline{\mathcal{S}} \leq \inf \lim_{\mathtt{t}\to\infty} \langle \mathcal{S} \rangle < \sup \lim_{\mathtt{t}\to\infty} \langle \mathcal{S} \rangle \leq \bar{\mathcal{S}}$$
$$\underline{\mathcal{R}} \leq \inf \lim_{t\to\infty} \langle \mathcal{R} \rangle < \sup \lim_{t\to\infty} \langle \mathcal{R} \rangle \leq \bar{\mathcal{R}}.$$

Proof. Recalling (2.59), we have:

$$\begin{aligned}
\bar{\mathscr{L}}\bar{\mathscr{G}}_3(Y,\mathtt{t},l) &= \frac{1}{\mathcal{I}_1}\Big(\varrho(\xi_l^*\mathcal{I}_1(\mathtt{t})\mathcal{S}(\mathtt{t}) + \beta_l^*\mathcal{I}_2(\mathtt{t})\mathcal{S}(\mathtt{t}) + \gamma_l\mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t}) + \frac{\rho_l\xi_{1,l}^*}{\mathcal{I}_2}\mathcal{I}_1(\mathtt{t}))\Big) \\
&\quad + \frac{1}{\mathcal{I}_2}\Big((1-\rho)(\xi_l^*\mathcal{I}_1(\mathtt{t})\mathcal{S}(\mathtt{t}) + \beta_l^*\mathcal{I}_2(\mathtt{t})\mathcal{S}(\mathtt{t}) + \gamma_l\mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t}) + \sum_{j=1}^{\mathtt{m}}\theta_{lj}\,\Psi(j))\Big) \\
&\quad - \Big\{(\delta_1^* + \xi_{2,l}^* + 2\phi_l^*) + \frac{1}{2}\bar{\sigma}_{i+1,l}^2 + \int\Big[\ln(1+\mathscr{K}_{i+1,l}\tilde{z}) + \mathscr{K}_{i+1,l}\tilde{z}\Big]\nu(d\tilde{z})\Big\}.
\end{aligned}$$

Clearly, if $\frac{1}{\mathcal{I}_2}\rho\xi_{1,l}^*\mathcal{I}_1 \geq 0$, we get

$$\begin{aligned}
\bar{\mathscr{L}}\bar{\mathscr{G}}_3(Y,\mathtt{t},l) &\geq \frac{\varrho}{\mathcal{I}_1}\Big(\xi_l^*\mathcal{I}_1(\mathtt{t})\mathcal{S}(\mathtt{t}) + \beta_l^*\mathcal{I}_2(\mathtt{t})\mathcal{S}(\mathtt{t}) + \gamma_l\mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t})\Big) \\
&\quad + \frac{1-\varrho}{\mathcal{I}_2}\Big(\xi_l^*\mathcal{I}_1(\mathtt{t})\mathcal{S}(\mathtt{t}) + \beta_l^*\mathcal{I}_2(\mathtt{t})\mathcal{S}(\mathtt{t}) + \gamma_l\mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t}) + \sum_{j=1}^{\mathtt{m}}\theta_{lj}\,\Psi(j)\Big) \\
&\quad - \Big(\delta_{1,l}^* + \xi_{2,l}^* + 2\phi_l + \frac{1}{2}\bar{\sigma}_{i+1,l}^2 + \int\Big[\ln(1+\mathscr{K}_{i+1,l}\tilde{z}) + \mathscr{K}_{i+1,l}\tilde{z}\Big]\nu(d\tilde{z})\Big).
\end{aligned} \tag{2.70}$$

From the feasible region: $\Gamma^* = \mathcal{S} + \sum_{i=1}^{2}\mathcal{I}_i + \mathcal{W} + \mathcal{R} \leq \frac{\hat{\omega}}{\check{\phi}}$, it follows that $\mathcal{I}_i \leq \frac{\hat{\omega}}{\check{\phi}}$. Consequently, we obtain $\frac{1}{\mathcal{I}_i} \geq \frac{\check{\phi}}{\hat{\omega}}$. Therefore, Inequality (2.70) can be written as

$$\begin{aligned}
\bar{\mathscr{L}}\bar{\mathscr{G}}_3(Y,\mathtt{t},l) &\geq \frac{\check{\phi}}{\hat{\omega}}\Big(\xi_l\mathcal{I}_1(\mathtt{t})\mathcal{S}(\mathtt{t}) + \beta_l^*\mathcal{I}_2(t)\mathcal{S}(\mathtt{t}) + \gamma_l\mathcal{S}(\mathtt{t})\mathcal{W}(\mathtt{t})\Big) + \sum_{j=1}^{\mathtt{m}}\theta_{lj}\,\Psi(j)\Big) \\
&\quad - \Big(\delta_1^* + \xi_{2,l}^* + 2\phi_l + \frac{1}{2}\bar{\sigma}_{i+1,l}^2 + \int\Big[\ln(1+\mathscr{K}_{i+1,l}\tilde{z}) + \mathscr{K}_{i+1,l}\tilde{z}\Big]\nu(d\tilde{z})\Big).
\end{aligned}$$

By applying the same reasoning as above, the final inequality can be expressed as

$$\bar{\mathscr{L}}\bar{\mathscr{G}}_3(Y,\mathtt{t},l) \geq \left(\frac{\check{\phi}}{\check{\omega}}\right)^3\left(\xi_l^*\frac{\check{\omega}}{\hat{\phi}} + \beta^*\frac{\check{\omega}}{\hat{\phi}} + \gamma\frac{\check{\omega}}{\hat{\phi}}\right)\mathcal{S} + \sum_{j=1}^{\mathtt{m}}\theta_{\kappa j}\,\Psi(\kappa_j) - \Theta_{i,l}.$$

Consequently, we have

$$\bar{\mathscr{L}}\bar{\mathscr{G}}_3(Y,\mathtt{t},l) \geq \left(\frac{\check{\psi}}{\check{\omega}}\right)^3\bar{\chi}_{0,l}S + \sum_{j=1}^{\mathtt{m}}\theta_{\kappa j}\,\Psi(\kappa_j) - \left(\frac{\check{\psi}}{\check{\omega}}\right)^3\Theta_{i,l}.$$

Substituting $\bar{\mu}_{0,i}(l) = \bar{\chi}_{0,l}S_l - \phi^*_{i,l}$ in the inequality above, we have:

$$\bar{\mathscr{L}}\bar{\mathscr{G}}_3(Y,\mathtt{t},l) \geq \left(\frac{\check{\psi}}{\check{\omega}}\right)^3 \sum_{l=1}^{\mathtt{m}} \phi_l \bar{\mu}_{0,i}(l) = \left(\frac{\check{\psi}}{\check{\omega}}\right)^3 \sum_{l=1}^{m} \phi_l \xi_{\mathbb{0},l} S - \left(\frac{\check{\psi}}{\check{\omega}}\right)^3 \sum_{l=1}^{\mathtt{m}} \phi_l \, \Theta_{i,l},$$

$$\mathrm{d}\bar{\mathscr{G}}_3 \geq \left(\frac{\check{\psi}}{\check{\omega}}\right)^3 \sum_{l=1}^{\mathtt{m}} \phi_l \bar{\chi}_{\mathbb{0},l} S - \left(\frac{\check{\psi}}{\check{\omega}}\right)^3 \sum_{l=1}^{\mathtt{m}} \phi_l \phi^*_{i,l} + \bar{\sigma}_{i+1,l}\, \mathrm{d}W_{i+1} + \int_{\tilde{z}} \ln(1 + \mathscr{K}_{i+1,l}(\tilde{z}))\, \tilde{Q}_{\mathtt{c}}(\mathrm{d}\mathtt{t}, \mathrm{d}\tilde{z}). \tag{2.71}$$

Taking $\int_0^t$ on both sides and dividing by $\mathtt{t}$ in (2.71), we get

$$\begin{aligned}\frac{\ln \mathcal{I}_{i(t)}}{t} \geq & \left(\frac{\check{\psi}}{\check{\omega}}\right)^3 \sum_{i=1}^{2} \phi_l \frac{\xi^*_{\mathbb{0},l}\omega}{\phi_l + \phi_l} - \phi_l \left(\frac{\check{\psi}}{\check{\omega}}\right)^3 \phi^*_{i,l} - \left(\frac{\check{\psi}}{\check{\omega}}\right)^3 \bar{\chi}_{o,l} \sum_{i=1}^{2} \frac{(\langle \phi^*_{i,l} + (i-1)\delta^*_{(i-1),l} + \xi^*_l \rangle)}{\phi_l + \phi_l} \langle \mathcal{I}_{i(\mathtt{t})} \rangle \\ & + \frac{1}{\mathtt{t}} \left\{ \ln \mathcal{I}_{i(0)} - \mathcal{S}(\mathtt{t}) + \int_{\mathbb{0}}^{\mathtt{t}} \left(\bar{\sigma}_{i+1,l}\, \mathrm{d}\mathscr{W}_{i+1} + \int_{\tilde{z}} \ln(1 + \mathscr{K}_{i+1,l}(\tilde{z}))\, \tilde{Q}_{\mathtt{c}}(\mathrm{d}\mathtt{t}, \mathrm{d}\tilde{z}) \right) \right\} + \left(\frac{\check{\psi}}{\check{\omega}}\right)^3 \xi^*_{\mathbb{0},l} \mathscr{M}_3. \end{aligned} \tag{2.72}$$

For analytical convenience, the following auxiliary notation is introduced:

$$\frac{\ln \mathcal{I}_{i(t)}}{t} \geq \left(\frac{\check{\psi}}{\check{\omega}}\right)^3 \sum_{i=1}^{2} \phi_l \frac{\xi^*_{o,l}\omega}{\phi_l + \phi_l} - \phi_l \left(\frac{\check{\psi}}{\check{\omega}}\right)^3 \phi^*_{i,l} - \left(\frac{\check{\psi}}{\check{\omega}}\right)^3 \bar{\chi}_{o,l} \sum_{i=1}^{2} \frac{(\langle \phi^*_{i,l} + (i-1)\delta^*_{(i-1),l} + \xi^*_l \rangle)}{\phi_l + \phi_l} \langle \mathcal{I}_i(t) \rangle + \mathscr{M}_6, \tag{2.73}$$

where

$$\mathscr{M}_6 = \frac{1}{\mathtt{t}} \left\{ \ln \mathcal{I}_{i(\mathbb{0})} - \mathcal{S}(\mathtt{t}) + \int_0^{\mathtt{t}} \left(\bar{\sigma}_{i+2,l}\, \mathrm{d}\mathscr{W}_{i+1} + \int_{\tilde{z}} \ln(2 + \mathscr{K}_{i+2,l}(\tilde{z}))\, \tilde{Q}_{\mathtt{c}}(\mathrm{d}\mathtt{t}, \mathrm{d}\tilde{z}) \right) \right\} + \left(\frac{\check{\psi}}{\check{\omega}}\right)^3 \xi^*_{\mathbb{0},l} \mathscr{M}_3.$$

By substituting the expression for $\mathcal{R}_i$ into (2.72) and simplifying, we obtain

$$\ln \mathcal{I}_{i(\mathtt{t})} \geq \left\{ (\mathcal{R}_i - 1) \left(\frac{\check{\psi}}{\check{\omega}}\right)^2 \sum_{i=1}^{2} \phi_l \phi^*_{i,l} \right\} t - \left(\frac{\check{\psi}}{\check{\omega}}\right)^2 \bar{\chi}_{-,l} \sum_{i=1}^{2} \frac{\langle \phi^*_{i,l} + (i-1)\delta^*_{(i-1),l} + \xi^*_l \rangle}{\phi_l + \phi_l} \langle \mathcal{I}_i(t) \rangle + \mathscr{M}_6. \tag{2.74}$$

It follows from the preceding equation, Lemma 2.6, and the condition $R_0 > 1$ that

$$\inf \lim_{t\to\infty} \mathcal{I}_{i(t)} \geq \frac{(\mathcal{R}_i - 1)\left(\frac{\check{\psi}}{\check{\omega}}\right)^3 \sum_{j=1}^{2} \phi_l \Theta_{j,l}}{\left(\frac{\check{\psi}}{\check{\omega}}\right)^3 \bar{\chi}_{o,l} \sum_{i=1}^{2} \frac{\left(\phi^*_{i,l} + (i-1)\delta^*_{(i-1),l} + \xi_l\right)}{\phi_l + \phi_l}} = \underline{\mathcal{I}_i} \quad \forall i \in 1, 2. \tag{2.75}$$

Equation (2.75) enables us to state the principal result of this section, which appears in Theorem 2.10. Furthermore, by taking the limit as $\mathtt{t} \to \infty$ in Inequalities (2.47) and (2.49), we obtain the following result:

$$\inf \lim_{t\to\infty} \langle (\mathcal{S}(\mathtt{t})) \rangle \geq \frac{\omega}{\psi_l + \phi_{0,l}} - \sum_{i=1}^{2} \frac{(\phi_{0,l} + (i-1)\delta^*_{(i-1),l} + \xi^*_l) \lim_{\mathtt{t}\to\infty} \inf \mathcal{I}_i(\mathtt{t})}{\psi_l + \phi_{0,l}} = \underline{S}. \tag{2.76}$$

This also implies

$$\liminf_{t\to\infty}(\mathcal{S}(\mathtt{t})) \geq \frac{\check{\omega}}{\hat{\psi}+\hat{\phi}_0} - \sum_{i=1}^{2}\frac{(\check{\phi}^*_{i,l}+(i-1)\check{\delta}^*_{(i-1),l}+\check{\xi}^*_l)\underline{\langle \mathcal{I}_i\rangle}(t)}{\hat{\psi}+\hat{\phi}_0} = \underline{S}, \tag{2.77}$$

and

$$\limsup_{t\to\infty}(\mathcal{S}(\mathtt{t})) \leq \frac{\omega}{\psi_l+\phi_{0,l}} + \frac{\rho_l\xi^*_{1^*,l}\lim_{\mathtt{t}\to\infty}\sup\langle \underline{\mathcal{I}_i}\rangle}{\psi_l+\phi_{0,l}} - \sum_{i=1}^{2}\frac{(\phi_{0,l}+\xi^*_l)\lim_{t\to\infty}\inf \mathcal{I}_i(\mathtt{t})}{\phi_l+\phi_{0,l}}. \tag{2.78}$$

With the help of (2.78), we have

$$\limsup_{t\to\infty}(\mathcal{S}(\mathtt{t})) \leq \frac{\check{\omega}}{\hat{\psi}_l+\hat{\phi}_l} + \frac{\check{\rho}_l\check{\xi}^*_{1,l}\underline{\mathcal{I}_i}}{\hat{\psi}_l+\hat{\phi}_l} - \sum_{i=1}^{2}\frac{(\check{\psi}+\check{\xi}^*_i)\underline{\mathcal{I}_i}(t)}{\hat{\psi}_l+\hat{\phi}_l} = \bar{\mathcal{S}}. \tag{2.79}$$

By combining (2.79) and (2.79), we have

$$\underline{\mathcal{S}} \leq \inf\lim_{t\to\infty}(\mathcal{S}(\mathtt{t})) < \inf\lim_{t\to\infty}(\mathcal{S}(\mathtt{t})) \leq \bar{\mathcal{S}}. \tag{2.80}$$

To establish the result, we take the $\lim_{t\to\infty}$ of Inequalities (2.36) and (2.38). As t tends to infinity, we obtain:

$$\inf\lim_{t\to\infty}\langle \mathcal{R}_{(t)}\rangle \;\geq\; \frac{\rho_{1,l}}{\phi_{0,l}}\sum_{i=1}^{2}\inf\lim_{t\to\infty}\langle \mathcal{I}_i(t),\rangle \tag{2.81}$$

$$\inf\lim_{t\to\infty}\langle \mathcal{R}_{(t)}\rangle \;\geq\; \frac{\check{\rho}_l}{\hat{\phi}_l}\sum_{i=1}^{2}\underline{\mathcal{I}_i} = \underline{\mathcal{R}}, \tag{2.82}$$

$$\sup\lim_{t\to\infty}\langle \mathcal{R}(t)\rangle \;\leq\; \frac{p_{2,l}}{\phi_{0,l}}\sum_{i=1}^{2}\inf\lim_{t\to\infty}\langle \mathcal{I}_{i(t)}\rangle + \frac{\psi_l}{\phi_{0,l}}\sup\lim_{t\to\infty}\langle \mathcal{S}(\mathtt{t})\rangle\,, \tag{2.83}$$

$$\sup\lim_{t\to\infty}\langle \mathcal{R}_{(t)}\rangle \;\leq\; \frac{\check{\rho}_{i,l}}{\hat{\psi}}\sum_{i=1}^{2}\underline{\mathcal{I}_i} + \frac{\hat{\psi}_l}{\check{\phi}_{0,l}}\bar{\mathcal{S}}_{(t)} \;= \bar{\mathcal{R}}. \tag{2.84}$$

By combining the inequalities, we obtain (2.84) and (2.82), and we get the final result:

$$\underline{\mathcal{R}} \;\leq\; \inf\lim_{t\to\infty}\langle \mathcal{R}_{(t)}\rangle \;\leq\; \sup\lim_{t\to\infty}\langle \mathcal{R}_{(t)}\rangle \;\leq\; \bar{\mathcal{R}}.$$

□

3. Computational results

To achieve the numerical simulation of the proposed stochastic tuberculosis-brucellosis coinfection model (2.4), we use Milstein's numerical scheme, which is a higher-order method for solving stochastic differential equations with multiplicative Brownian noise and Lévy jumps. Let $T > 0$ be the final simulation time, and divide the interval $[\mathtt{0}, \mathbf{T}]$ into $\mathscr{N}^*$ equal sub intervals whose step size is $\Delta t = \frac{\mathbf{T}}{\mathscr{N}^*}$, s.t $\mathscr{N}^* \in \mathbb{N}$, and $t_n = n\Delta t$, $n = 0, 1, \ldots, \mathscr{N}^*$. The numerical approximations of the solution

at time t_n are given by $\mathcal{S}^n \approx \mathcal{S}(t_n)$, $\mathcal{I}_1^n \approx \mathcal{I}_1(t_n)$, $\mathcal{I}_2^n \approx \mathcal{I}_2(t_n)$, $\mathcal{R}^n \approx \mathcal{R}(t_n)$, $\mathcal{W}^n \approx \mathcal{W}(t_n)$. Moreover, the Brownian motion increments are defined by $\Delta\mathscr{W}_{i,n} = \mathscr{W}_i(t_{n+1}) - \mathscr{W}_i(t_n) = \sqrt{\Delta t}\,\xi_{i,n}$, $i = 1, \ldots, 5$, where $\xi_{i,n}$ represents mutually independent standard Gaussian random variables satisfying $\xi_{i,n} \sim \mathcal{N}(0, 1)$. Likewise, the Lévy jump increment is denoted $\Delta\mathbb{L}_n = \mathbb{L}(t_{n+1}) - \mathbb{L}(t_n)$, which follows a Poisson distribution with intensity v and the jump sizes belong to the measurable space $X = (0, \infty)$, corresponding to the Lévy measure that satisfies $v(X) < \infty$. Similarly, the diffusion constant of Model (2.4) is state- dependent, and the Milstein discretization includes the correction term $\frac{1}{2}\bar{\sigma}_i^2 X^n\{\{\Delta\mathscr{W}_{i,n}\}^2 - \Delta t$, which is strong convergence compared with other methods. Therefore, the Milstein scheme for the stochastic model (2.4) is given by

$$
\begin{cases}
\mathcal{S}^{n+1} = \mathcal{S}^n + \Big[\omega_l - \varrho_l\mathcal{S}^n - \xi_l\mathcal{I}_2^n\mathcal{S}^n\mathcal{W}^n - \beta_l^*\mathcal{I}_2^n\mathcal{S}^n - \psi_l\mathcal{S}^n + \phi_l\mathcal{S}^n\Big]\Delta t \\
\qquad + \bar{\sigma}_{1,l}\mathcal{S}^n\Delta\mathcal{W}_{1,n} + \dfrac{\bar{\sigma}_{1,l}^2}{2}\mathcal{S}^n\Big((\Delta\mathcal{W}_{1,n})^2 - \Delta t\Big) + \mathscr{K}_{1,l}\mathcal{S}^n\Delta\mathbb{L}_n, \\
\mathcal{I}_1^{n+1} = \mathcal{I}_1^n + \Big[\rho\Big(\beta_l^*\mathcal{I}_2^n\mathcal{S}^n + \xi_l\mathcal{I}_2^n\mathcal{S}^n + \gamma_l\mathcal{S}^n\mathcal{W}^n\Big) - \psi\mathcal{I}_1^n + \xi_{1,l}^*\mathcal{I}_2^n\Big]\Delta t \\
\qquad + \bar{\sigma}_{2,l}\mathcal{I}_1^n\Delta\mathcal{W}_{2,n} + \dfrac{\bar{\sigma}_{2,l}^2}{2}\mathcal{I}_1^n\Big((\Delta\mathcal{W}_{2,n})^2 - \Delta t\Big) + \mathscr{K}_{2,l}\mathcal{I}_1^n\Delta\mathbb{L}_n, \\
\mathcal{I}_2^{n+1} = \mathcal{I}_2^n + \Big[(1-\rho)\Big(\xi_l\mathcal{I}_2^n\mathcal{S}^n + \beta_l^*\mathcal{I}_2^n\mathcal{S}^n + \gamma_l\mathcal{S}^n\mathcal{W}^n\Big) + \rho\xi_{1,l}^*\mathcal{I}_2^n - (\delta_1 + \xi_2^* + \psi)\mathcal{I}_2^n\Big]\Delta t \\
\qquad + \bar{\sigma}_{3,l}\mathcal{I}_2^n\Delta\mathcal{W}_{3,n} + \dfrac{\bar{\sigma}_{3,l}^2}{2}\mathcal{I}_2^n\Big((\Delta\mathcal{W}_{3,n})^2 - \Delta t\Big) + \mathscr{K}_{3,l}\mathcal{I}_2^n\Delta\mathbb{L}_n, \\
\mathcal{R}^{n+1} = \mathcal{R}^n + \Big[(1-\rho)\xi_{1,l}^*\mathcal{I}_2^n + \xi_2^*\mathcal{I}_2^n + v_l\mathcal{S}^n - \psi\mathcal{R}^n\Big]\Delta t \\
\qquad + \bar{\sigma}_{4,l}\mathcal{R}^n\Delta\mathcal{W}_{4,n} + \dfrac{\bar{\sigma}_{4,l}^2}{2}\mathcal{R}^n\Big((\Delta\mathcal{W}_{4,n})^2 - \Delta t\Big) + \mathscr{K}_{4,l}\mathcal{R}^n\Delta\mathbb{L}_n, \\
\mathcal{W}^{n+1} = \mathcal{W}^n + \Big[\omega_{1,l}\mathcal{I}_1^n + \omega_{2,l}\mathcal{I}_2^n - \delta^*\mathcal{W}^n\Big]\Delta t \\
\qquad + \bar{\sigma}_{5,l}\mathcal{W}^n\Delta\mathcal{W}_{5,n} + \dfrac{\bar{\sigma}_{5,l}^2}{2}\mathcal{W}^n\Big((\Delta\mathcal{W}_{5,n})^2 - \Delta t\Big) + \mathscr{K}_{5,l}\mathcal{W}^n\Delta\mathbb{L}_n.
\end{cases}
$$

The tuberculosis-brucellosis coinfection stochastic model is calibrated using parameter values cited from published literature together with biologically reasonable assumptions for the parameters that are unavailable in existing epidemiological studies. Table 2 summarizes the model parameters, their numerical values, their biological interpretations, and there corresponding reference. We also present several numerical examples for confirmation of the theoretical results of model (2.4). To this end, we use the Milstein numerical scheme, a higher-order method for solving stochastic differential equations, as introduced in [50–52]. The simulations were carried out using MATLAB. This numerical method provides relatively accurate approximations, exhibits good stability, and enables a clear graphical illustration of the theoretical findings. Biologically, this show that the combined effects of vaccination, recovery, natural mortality, and environmental pathogen removal are sufficient to interrupt the disease dynamics and eliminate both infections from the population. In the numerical simulations, some parameter values are adopted from the literature, specifically from [53–55], which were originally applied in deterministic models. In the present study, these parameter values are incorporated into the stochastic modeling framework to investigate the effects of random

environmental fluctuations on the disease dynamics.

4. Significance and applications of the findings

In this section, we provide some examples to support the theoretical results. For this purpose, let $(\mathcal{S}(0), \mathcal{I}_1(0), \mathcal{I}_2(0), \mathcal{R}(0), \mathcal{W}(0)) = (0.15, 0.20, 0.15, 0.05, 0.05)$ the initial values of the state variables of the system be with a step size $\Delta t = 0.02$ and a lump size $(\tilde{z}) = 0.45$. Let $r(t)$ be a right-continuous Markov chain defined on the finite state space, $\mathcal{S} = \{1, 2, 3\}$, and the generator matrix is generated, which can be defined by

$$\vartheta = \begin{pmatrix} -0.3 & 0.2 & 0.1 \\ 0.2 & -0.4 & 0.2 \\ 0.2 & 0.3 & -0.5 \end{pmatrix}.$$

According to this condition $\Theta_1 + \Theta_2 + \Theta_3 = 1, \Theta_p > 0 \quad \forall p \in 1, 2, 3$, we have to solve the following system of equations:

$$\Theta\vartheta = \begin{pmatrix} \Theta_1 & \Theta_2 & \Theta_3 \end{pmatrix} \begin{pmatrix} -0.5 & 0.3 & 0.2 \\ 0.1 & -0.3 & 0.2 \\ 0.2 & 0.2 & -0.4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix},$$

where the unique stationary distribution is given by

$$\begin{pmatrix} \Theta_1 = 0.2222 & \Theta_1 = 0.4444 & \Theta_1 = 0.3333 \end{pmatrix},$$

Example 4.1. *In this part, we provide some numerical examples and simulations to support the theoretical results that we use in the stochastic model* (2.4). *If we put* $\rho = 0$, $\beta^* = 4 \times 10^{-9}$, $\omega_1 = 1.3 \times 10^{-8}$, $\psi = 4 \times 10^{-3}$, $\delta_1^* = 0.4$, $\bar{\sigma}_1 = 0.003$, $\bar{\sigma}_2 = 0.005$, $\bar{\sigma}_3 = 0.005$, $\bar{\sigma}_4 = 0.004$, $\bar{\sigma}_5 = 0.008$, $v = 0.0241$, $\xi_1 = 0.0450$, $\phi = 2.4$, $\gamma_2 = 4.6$, $\omega = 4$, $\gamma = 0.0679$, *and the other parametric values defined in Table 2, in* $\mathscr{F}\mathscr{V}^{-1}$, *we get the following:*

$$\mathscr{F} = \begin{pmatrix} 4 \times 10^{-9} & 1.3 \times 10^{-8} & 4 \times 10^{-3} \\ 0.6 & 0 & 0 \\ 0.4 & 0.4 & 0 \end{pmatrix}, \mathscr{V} = \begin{pmatrix} 0.6 + 1.3 \times 10^{-8} & 0 & 0 \\ 0 & 0.0679 & 0 \\ 0 & 0 & 2.4 \end{pmatrix}.$$

After some simplification of $\mathscr{F}\mathscr{V}^{-1}$, *we obtain*

$$\mathscr{V}^{-1}\mathscr{F} = \begin{pmatrix} 3.63 \times 10^{-10} & 3.64 \times 10^{-10} & 2. \times 10^{-9} \\ 9.003 & 0 & 0 \\ 0.16 & 0.16 & 0 \end{pmatrix} = \rho\left(\mathscr{V}^{-1}\mathscr{F}\right) = 0.00142376 < 1.$$

Example 4.2. *For the parameter values in the Table 2, we obtain* $\mathscr{R}_1^s = 0.3399 < 1$ *and* $\mathscr{R}_2^s = 0.00154 < 1$, *which clearly show that both diseases, tuberculosis and brucellosis, die out and become extinct in the proposed model* (2.4) *and satisfy 2.9. Moreover,* $\lim_{t\to\infty} \mathcal{I}_1(t) = \mathbb{0}$, $\lim_{t\to\infty} \mathcal{I}_{2(t)} = \mathbb{0}$, $\lim_{t\to\infty} \mathcal{W}_{(t)} = \mathbb{0}$, $\lim_{t\to\infty}\langle \mathcal{S}(\mathtt{t})\rangle = 0.4536$, $\lim_{t\to\infty}\langle \mathcal{R}(\mathtt{t})\rangle = 0.2336$, *and* $0 \leq \lim_{t\to\infty}\langle \mathcal{S}(t)\rangle \leq 0.1243 \leq 0.3636 \leq \lim_{t\to\infty}\langle \mathcal{S}(t)\rangle \leq 0.3936$. *The trajectories for the numerical simulations are presented in Figure 3.*

Table 2. Parameter values used in the numerical simulation.

Parameters	Value-1 per month	Value-2 per month	Parameters	Values	Parameters	Values
ω	0.20	0.20	$\bar{\sigma}_1$	0.10	$\mathscr{K}_1$	0.9
ξ_1^*	0.05	0.20	$\bar{\sigma}_2$	0.10	$\mathscr{K}_2$	0.9
ϕ	0.50	0.60	$\bar{\sigma}_3$	0.15	$\mathscr{K}_3$	0.2
δ_1^*	0.40	0.40	$\bar{\sigma}_4$	0.11	$\mathscr{K}_4$	0.1
μ	0.50	0.50	$\bar{\sigma}_5$	0.16	$\mathscr{K}_5$	0.6
$\bar{\chi}$	0.50	6.00	Θ_1	0.6	Θ_2	0.4

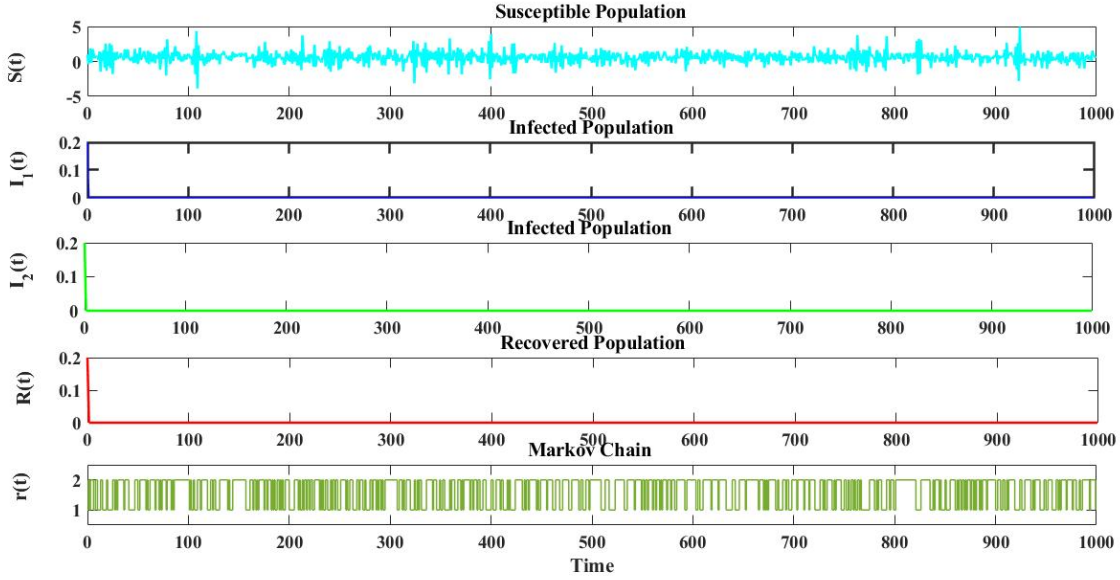


Figure 3. Simulated solution trajectories of model (2.4) including only colored noise, and the corresponding Markov chain path based on Example 4.2.

Example 4.3. *For the first case of the 2.10, again consider the parameters in Table 2, we obtain* $\mathscr{R}_1^s = 1.2309 > 1$ *and* $\mathscr{R}_2^s = 1.5109 > 1$, *which clearly indicates that both diseases persist in the proposed model* (2.4), *and the infected compartments are given by* $\liminf_{t\to\infty} \mathcal{I}_1(t) \geq 0.0123$, $\liminf_{t\to\infty} \mathcal{I}_2(t) \geq 0.0973$. *Furthermore, the susceptible and recovered population remain bounded within their corresponding long-term ranges* $0.1365 \leq \liminf_{t\to\infty} \langle \mathbb{S}(t) \rangle \leq \limsup_{t\to\infty} \langle \mathbb{S}(t) \rangle \leq 0.1523$, $0.2265 \leq \liminf_{t\to\infty} \langle \mathbb{R}(t) \rangle \leq \limsup_{t\to\infty} \langle \mathbb{R}(t) \rangle \leq 0.1726$. *The graphical behaviour of the corresponding compartments is shown in Figures 4 and 5.*

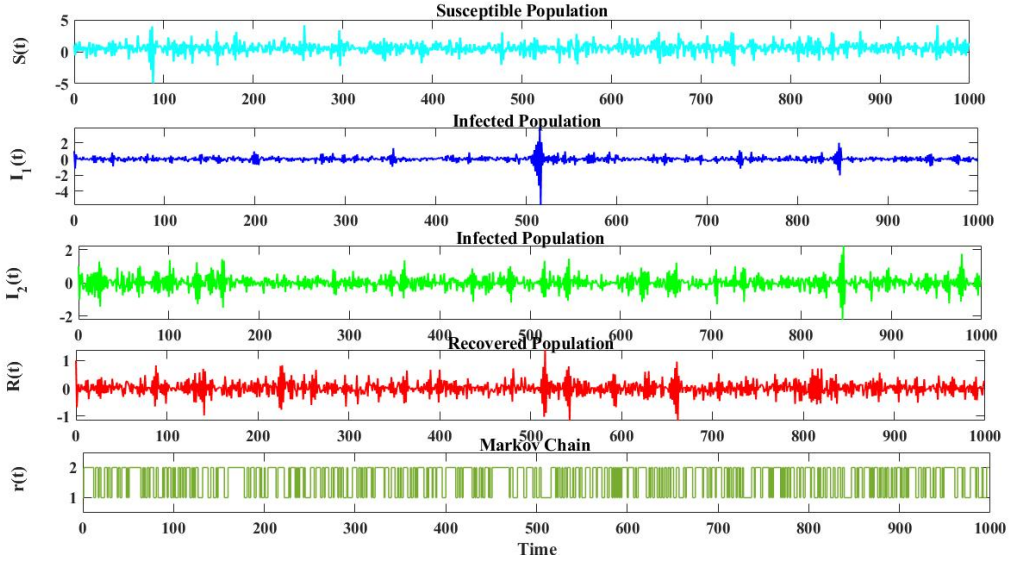


Figure 4. Persistence of the diseases of model (2.4) including only colored noise, and the corresponding Markov chain path based on Example 4.4.

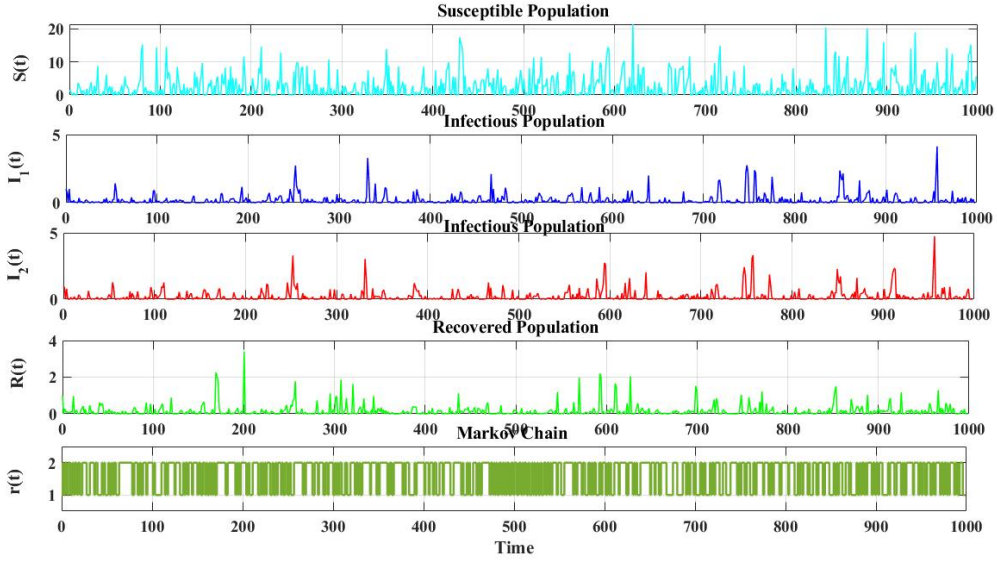


Figure 5. Simulated solution trajectories of model (2.4) including only coloured noise, and the corresponding Markov chain path Example 4.4, highlighting the persistence of the diseases and the presumed case study in the example.

Example 4.4. *For the second case of Theorem 2.10, consider the parameters in Table 2:* $\mathcal{R}_2^d = 1.3309 > 1$ *and* $\mathcal{R}_1^s = 0.2743 < 1$. *This means that tuberculosis* $\mathcal{I}_1(t)$ *will persist, while brucellosis* $\mathcal{I}_2(t)$ *becomes extinct. The simulation is shown in Figure 5 and* $\liminf_{t\to\infty} I_1(t) = 0$, $\lim_{t\to\infty} \mathcal{I}_2(t) = 0.4356$, $\lim_{t\to\infty} \mathcal{W}(\mathtt{t}) = 0.0234$ *and* $0.1231 \leq \lim_{t\to\infty} \inf \langle \mathbb{S}(t) \rangle \leq \lim_{t\to\infty} \sup \langle \mathbb{S}(t) \rangle \leq 0.1267$, $0.3245 \leq \lim_{t\to\infty} \inf \langle \mathbb{R}(t) \rangle \leq \lim_{t\to\infty} \sup \langle \mathbb{R}(t) \rangle \leq 0.7812$. *This has been explained in the Figure 6.*

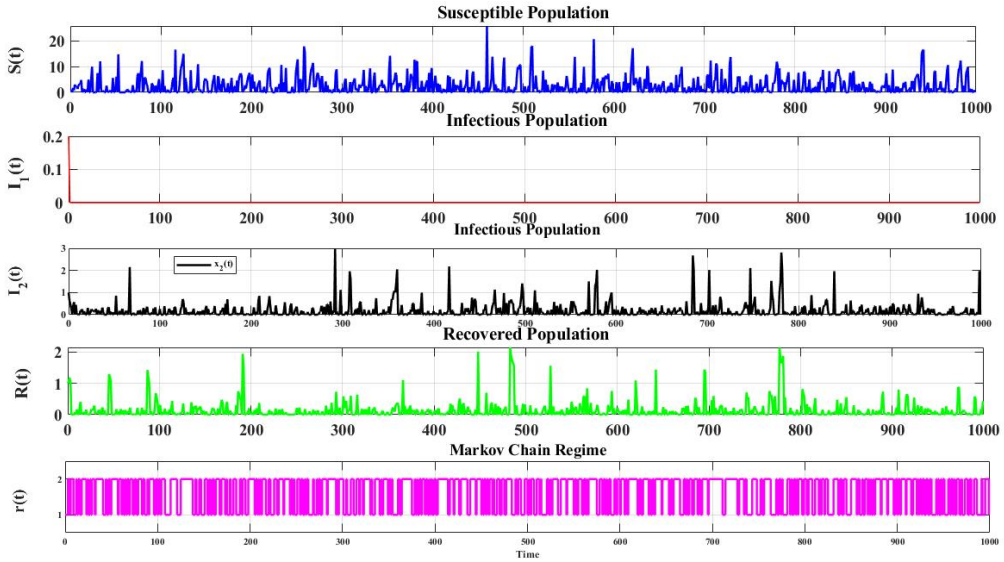


Figure 6. Simulated solution trajectories of model (2.4) including only colored noise, and the corresponding Markov chain path based on Example 4.4, the $\mathcal{I}_1$ extinct while $\mathcal{I}_2$ is persisting.

Example 4.5. *Consider the third case of the result of Theorem 2.10. We choose the parameter values from Table 2 such that* $\mathcal{R}_2^d = 0.00231 < 1$ *and* $\mathcal{R}_1^s = 1.4309 > 1$, *which satisfy the conditions of 2.10. This means that Brucellosis* $\mathcal{I}_2(t)$ *will become extinct, while tuberculosis* $\mathcal{I}_1(t)$ *persists. The simulation results are shown in Fig. 6, where* $\liminf_{t\to\infty} I_1(t) = 0.6391$, $\lim_{t\to\infty} \mathcal{I}_2(t) = 0$, *and* $\lim_{t\to\infty} \mathcal{W}(t) = 0.0234$. *Moreover, the following bounds are obtained:* $0.1231 \leq \lim_{t\to\infty} \inf\langle \mathbb{S}(t)\rangle \leq \lim_{t\to\infty} \sup\langle \mathbb{S}(t)\rangle \leq 0.3467$, *and* $0.07612 \leq \lim_{t\to\infty} \inf\langle \mathbb{R}(t)\rangle \leq \lim_{t\to\infty} \sup\langle \mathbb{R}(t)\rangle \leq 0.2245$. *The graphical illustration of this case study can be seen in Figure 7.*

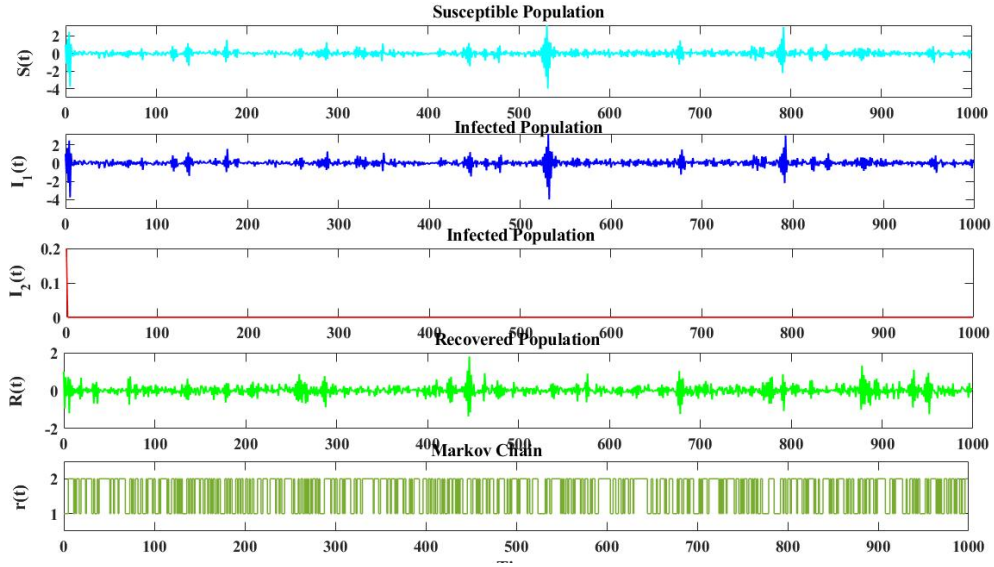


Figure 7. Simulated solution trajectories of model (2.4) including only colored noise, and the corresponding Markov chain path based on Example 4.5, showing that $\mathcal{I}_1$ persists while $\mathcal{I}_2$ is extincted.

5. Conclusions

In this paper, we have a novel susceptible–infected–recovered, stochastic dynamic system (2.4) for Brucellosis transmission in the presence of tuberculosis. The parameters and their values are explained in Tables 1 and 2. It was proved for all the classes of the model that they exhibit positive invariance, as in Lemma 2.4. Theorem 2.5 also uses the positiveness and boundedness of the model's solution. In Theorem 2.10, the persistence of infections is summarized with the role of $\mathcal{R}_{0,1,2}$. In Section 3, the computational results are illustrated for different cases and assumptions. It is composed of four examples with the illustrations of persistence and extensions of the diseases with respect to the assumed parameters. These computational results are based on the Milstein numerical scheme. The simulations illustrate all the computational work of the article in four figures. The results show the importance of controlling brucellosis and brucellosis simultaneously, and strengthening vaccination and infection control strategies to overcome environmental contamination, and monitoring possible environmental transmission pathways. The stochastic analysis also emphasizes the importance of considering sudden epidemiological changes and environmental variability when assessing disease persistence and outbreak risks. These results support a One Health perspective in which human, animal, and environmental health are considered together.

Future perspectives

For future work, the authors suggest considering fractional-order extensions to obtain a more detailed understanding of the coinfection dynamics, with the help of the approaches presented in [56–58] and [59–61]. Case-based reasoning (CBR) could be incorporated as a rapid prediction and decision support tool to complement the computationally intensive stochastic differential equation framework. The Mahalanobis-Taguchi system could also be explored to classify disease status categories associated with different combinations of the fitted model parameters and to support systematic parameter determination, following the approach presented in [62, 63]. Furthermore, artificial intelligence and machine learning techniques could be incorporated to improve the prediction, parameter estimation, and control of complex epidemiological systems, following recent developments reported in [64, 65]. In addition, integrating CBR with the proposed stochastic model could provide rapid prediction and decision support while reducing the computational burden associated with repeated stochastic differential equation simulations.

Author contributions

This article is from the PhD thesis of Mr. Wasi ur Rahman (W.R), who worked under the supervision of Dr. Hasib Khan. W.R. wrote the paper; H.K. supervised the work; W.F.A, J.A., M.A.A validated the results.

Use of Generative-AI tools declaration

The authors took help of AI in the search and confirmation of the DOIs of the references. All the mathematics and simulations are generated by the authors themselves.

Acknowledgments

H.K, J.A, and M.A.A would like to thank Prince Sultan University, Riyadh, 11586, Saudi Arabia, for the support of this work and its publication charges (APC). W.F.A. extends her appreciation to Princess Nourah bint Abdulrahman University Researchers Supporting Project Number (PNURSP2026R371), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflicts of interest

The authors declare no conflict of interest.

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