AIMS Mathematics, 11(11): 38839–38849.
DOI: 10.3934/math.20261498
Received: 25 June 2026
Revised: 23 October 2026
Accepted: 09 November 2026
Published: 20 November 2026

https://www.aimspress.com/journal/Math

Research article

Structure and characterization of derivations on lattice-ordered effect algebras

Lijun Liu*

School of Mathematics and Computer Science, Shaanxi University of Technology, Hanzhong, 723000, Shaanxi, China

* **Correspondence:** Email: lljgsjys@163.com.

Abstract: This paper investigated derivations on lattice-ordered effect algebras, an important partial-algebraic framework for unsharp quantum logic. We distinguished lattice derivations from effect algebra derivations and established their interconnections. Using the dual partial operation $\diamond$, we derived several structural properties for effect algebra derivations and systematically studied the corresponding fixed-point sets. Under suitable assumptions, we showed that the fixed-point set of a derivation can be equipped with an effect algebra structure by reversing the zero and unit elements. Furthermore, we clarified the relationship between lattice-ordered effect algebras and lattice-ordered orthoalgebras from the viewpoint of derivations.

Keywords: effect algebras; lattice-ordered effect algebras; derivations; fixed-point set
Mathematics Subject Classification: 03G27, 06A11, 16W25

1. Introduction

Lattice-ordered effect algebras originate from quantum mechanics, nonclassical logic, and abstract algebra. Early quantum mechanics showed that classical mathematical structures cannot describe uncertain and superposed quantum-measurement phenomena. This observation motivated the development of effect algebras [1,2]. Meanwhile, many-valued and fuzzy logics required suitable algebraic semantics to evaluate truth values. Lattice theory, whose main framework was completed in the 1930s, together with effect algebra theory, laid the corresponding algebraic foundation. Lattice-ordered effect algebras are constructed by endowing effect algebras with a lattice order. They provide precise descriptions of the relationships between quantum states and measurement outputs and have been widely applied in various domains [3–7].

Derivations first emerged from differential calculus. The additive and product rules of differentiation discovered by Newton and Leibniz formed the primitive template for abstract derivations. With the

advance of algebra, derivations were abstracted as mappings subject to axiomatic constraints and generalized to broader algebraic contexts. Posner [8] introduced ring derivations in 1957 and proved that prime rings admitting nonzero centralizing derivations are commutative. Derivations satisfying the Leibniz rule have since played a vital role in the structural theory of rings. This success motivated researchers to extend the notion of derivations to other algebraic systems. Lattices combine order-theoretic and algebraic operations, so they serve as a natural object for such generalization. Szăsz [9] pioneered research on lattice derivations in 1975; Xin et al. [10] later established equivalent conditions for characterizing isotone derivations on lattices with a greatest element.

In recent years, derivations have been investigated for numerous algebraic varieties: derivations of basic algebras were treated in [11], and the theory was further extended to residuated lattices [12], L-algebras [13], and MV-algebras (many-valued algebras) [14,15]. Despite their apparent differences, MV-algebras, basic algebras, and L-algebras can be unified within the framework of lattice-ordered effect algebras. Every lattice-ordered effect algebra decomposes into MV-algebras, and both basic algebras and L-algebras exhibit strong operational analogies to lattice-ordered effect algebras [16,17]. Consequently, investigating derivations on lattice-ordered effect algebras is of considerable interest.

Motivated by the above observations, the present paper develops the theory of derivations on lattice-ordered effect algebras based on the dual partial operation $\diamond$. We distinguish lattice derivations from effect algebra derivations and compare their respective properties. We further analyze isotone, contractive, and ideal derivations, together with the algebraic structure of their fixed-point sets.

The paper is organized as follows. Section 2 recalls basic definitions and known facts about effect algebras, orthoalgebras, and the dual partial operation $\diamond$. Section 3 introduces lattice derivations and effect algebra derivations. It formulates a standing domain convention for identities built on partial operations and presents our main theorems, propositions, examples, and remarks. Section 4 concludes the paper and outlines potential directions for further investigation.

2. Definitions and properties

We recall standard definitions and preliminary results on effect algebras and related structures, following [1].

Definition 2.1. (see [1]) Suppose $\mathscr{A}$ is a nonempty set furnished with two special constants 0 and 1, and $\oplus$ denotes a partial binary operation on $\mathscr{A}$. The quadruple $(\mathscr{A}, \oplus, 0, 1)$ forms an effect algebra provided the following axioms hold for arbitrary $\xi, \varrho, \kappa \in \mathscr{A}$:

(1) Commutativity: Whenever $\xi \oplus \varrho$ is admissible, $\varrho \oplus \xi$ is likewise admissible and $\xi \oplus \varrho = \varrho \oplus \xi$;

(2) Associativity: If $\xi \oplus \varrho$ and $(\xi \oplus \varrho) \oplus \kappa$ are both admissible, then $\varrho \oplus \kappa$ and $\xi \oplus (\varrho \oplus \kappa)$ are admissible as well, with $(\xi \oplus \varrho) \oplus \kappa = \xi \oplus (\varrho \oplus \kappa)$;

(3) Orthosupplementation: For each $\xi \in \mathscr{A}$, there exists a unique element $\varrho \in \mathscr{A}$ such that $\xi \oplus \varrho = 1$. Then ϱ is the orthosupplement of ξ, that is, $\varrho = \overline{\xi}$;

(4) Zero-one property: If $\xi \oplus 1$ is defined, then necessarily $\xi = 0$.

If $\xi \oplus \varrho$ exists, the elements ξ and ϱ are said to be orthogonal, written $\xi \perp \varrho$.

Furthermore, a partial order on $\mathscr{A}$ can be introduced by

$$\xi \leq \varrho \iff \exists\, \kappa \in \mathscr{A} \text{ such that } \xi \oplus \kappa = \varrho.$$

If the ordered set $(\mathscr{A}, \leq)$ forms a lattice, that is, every pair $\xi, \varrho \in \mathscr{A}$ admits a join $\xi \vee \varrho$ and a meet $\xi \wedge \varrho$, then $(\mathscr{A}, \oplus, 0, 1)$ can be regarded as a lattice-ordered effect algebra.

Example 2.1. Let $\xi, \varrho \in [0,1] \subset \mathbb{R}$. Define the partial binary operation $\oplus$ on $[0,1]$ by

$$\xi \oplus \varrho = \xi + \varrho \text{ whenever } \xi + \varrho \leq 1.$$

Under this operation, $([0,1], \oplus, 0, 1)$ yields a lattice-ordered effect algebra. Its ortho-supplement takes the form $\overline{\xi} = 1 - \xi$, and the relation $\xi \perp \varrho$ holds exactly when $\xi + \varrho \leq 1$.

Definition 2.2. (see [14]) An orthoalgebra is an algebraic structure $(\mathscr{A}, \oplus, 0, 1)$, where $\mathscr{A}$ denotes a nonempty set furnished with two special constants 0 and 1, while $\oplus$ stands for a partial binary operation on $\mathscr{A}$. The structure obeys the axioms listed below for all elements $\xi, \varrho, \kappa \in \mathscr{A}$:

(1) Commutativity: If $\xi \oplus \varrho$ is admissible, then $\varrho \oplus \xi$ is admissible too, and $\xi \oplus \varrho = \varrho \oplus \xi$;

(2) Associativity: Provided $(\xi \oplus \varrho) \oplus \kappa$ exists, then $\xi \oplus (\varrho \oplus \kappa)$ is also defined and $(\xi \oplus \varrho) \oplus \kappa = \xi \oplus (\varrho \oplus \kappa)$;

(3) Ortho-supplementation: Given $\xi \in \mathscr{A}$, there exists exactly one $\varrho \in \mathscr{A}$ such that $\xi \oplus \varrho = 1$;

(4) Consistency condition: If the expression $\xi \oplus \xi$ is defined, then necessarily $\xi = 0$.

An orthoalgebra is termed a lattice-ordered orthoalgebra when it carries a lattice order induced by the above operation.

Proposition 2.1. (see [14]) The statements listed below are mutually equivalent for all $\xi, \varrho \in \mathscr{A}$:

(1) The structure $\mathscr{A}$ is an orthoalgebra;

(2) Whenever $\xi, \varrho \in \mathscr{A}$ are orthogonal (i.e., $\xi \perp \varrho$), their sum $\xi \oplus \varrho$ serves as the least upper bound (join) of ξ and ϱ;

(3) For every element $\xi \in \mathscr{A}$, one has $\xi \wedge \overline{\xi} = 0$;

(4) The mapping $\xi \longmapsto \overline{\xi}$ defines an ortho-complementation on $\mathscr{A}$.

Property 2.1. (see [1]) For arbitrary elements $\xi, \varrho, \kappa \in \mathscr{A}$, the following assertions hold:

(1) The partial sum $\xi \oplus \varrho$ is defined if, and only if, $\xi \leq \overline{\varrho}$, which is equivalent to the condition $\varrho \leq \overline{\xi}$;

(2) Whenever $\xi \oplus \varrho$ exists, then $s \oplus t$ is also defined for all $s \leq \xi$ and $t \leq \varrho$;

(3) $\xi \leq \varrho$ if, and only if, $\overline{\varrho} \leq \overline{\xi}$;

(4) $\overline{\overline{\varrho}} = \varrho, \varrho \oplus 0 = \varrho, \varrho \oplus \overline{\varrho} = 1$;

(5) If $\xi \leq \varrho$ and $\varrho \oplus \kappa$ exists, then $\xi \oplus \kappa$ exists and $\xi \oplus \kappa \leq \varrho \oplus \kappa$;

(6) $\kappa \oplus \xi = \varrho$ iff $\overline{\kappa} = \xi \oplus \overline{\varrho}$ iff $\kappa = \overline{(\xi \oplus \overline{\varrho})}$;

(7) $\varrho \wedge \overline{\varrho} = 0, \varrho \vee \overline{\varrho} = 1$;

(8) If both $\xi \oplus \varrho$ and $\xi \oplus \kappa$ exist and satisfy $\xi \oplus \varrho = \xi \oplus \kappa$, then it follows that $\varrho = \kappa$.

Definition 2.3. (see [4]) A derived (partial) operation $\diamond$ on $\mathscr{A}$ is defined as follows:

$\xi \diamond \varrho$ is defined exactly when $\overline{\xi} \perp \overline{\varrho}$ (i.e., $\overline{\xi} \oplus \overline{\varrho}$ is defined), and then $\xi \diamond \varrho := \overline{(\overline{\xi} \oplus \overline{\varrho})}$, for all $\xi, \varrho \in \mathscr{A}$.

By combining Property 2.1 with the above definition, it directly follows that:

· $\xi \diamond \varrho$ is defined iff $\overline{\xi} \oplus \overline{\varrho}$ is defined iff $\overline{\xi} \leq \varrho$ (equivalently, $\overline{\varrho} \leq \xi$).

· In the standard effect algebra [0,1], $\xi \diamond \varrho = \xi + \varrho - 1$ whenever $\xi + \varrho \geq 1$, and is undefined otherwise.

Proposition 2.2. Let $\xi, \varrho, \kappa \in \mathscr{A}$. All statements below hold whenever the involved partial operations are defined:

(1) The existence of $\xi \diamond \varrho$ is equivalent to that of $\varrho \diamond \xi$, and whenever they exist, they are equal;

(2) The expression $\xi \diamond (\varrho \diamond \kappa)$ is defined exactly when $(\xi \diamond \varrho) \diamond \kappa$ is defined, and both yield the same result;

(3) For every $\xi \in \mathscr{A}$, the elements $\xi \diamond 1$ and $1 \diamond \xi$ are defined, and $1 \diamond \xi = \xi = \xi \diamond 1$;

(4) If $\xi \oplus \varrho = 0$, then $\xi = \varrho = 0$. Likewise, if $\xi \diamond \varrho = 1$, then $\xi = \varrho = 1$;

(5) The identity $\kappa \diamond \overline{\kappa} = \overline{\kappa} \diamond \kappa = 0$ holds whenever the involved expressions are defined;

(6) The inequality $\xi \diamond \varrho \leq \kappa$ is equivalent to $\varrho \leq \overline{\xi} \oplus \kappa$;

(7) $\varrho \diamond (\overline{\varrho} \oplus \xi) \leq \xi$ and $(\varrho \diamond \overline{\xi}) \oplus \xi \leq \varrho$;

(8) If $\xi \leq \varrho$,then $\xi \diamond \kappa \leq \varrho \diamond \kappa$ and $\xi \diamond \varrho \leq \xi$;

Remark 2.1. All identities and inequalities above are asserted only for those arguments for which the corresponding partial operations are well-defined.

3. Derivations on Lattice-ordered effect algebras

Given that lattice-ordered effect algebras incorporate both lattice and effect algebra structures, we define lattice derivations and effect algebra derivations and study their connection in this section.

Definition 3.1. The function $\Gamma : \mathscr{A} \to \mathscr{A}$ is termed a lattice derivation if, for arbitrary elements $\xi, \varrho \in \mathscr{A}$, one has

$$\Gamma(\xi \wedge \varrho) = (\Gamma(\xi) \wedge \varrho) \vee (\xi \wedge \Gamma(\varrho)).$$

Proposition 3.1. Let Γ be a lattice derivation on $\mathscr{A}$. Then,

(1) $\Gamma(0) = 0$;

(2) If $\xi \leq \varrho$, then $\Gamma(\xi) \leq \Gamma(\varrho) \vee \varrho$;

(3) For any $u \in \mathscr{A}$, the mapping $\Gamma_u(\xi) = u \wedge \xi$ is a lattice derivation.

Proof. (1) Let $\xi = \varrho = 0$, then $\Gamma(0) = (\Gamma(0) \wedge 0) \vee (0 \wedge \Gamma(0)) = 0 \vee 0 = 0$.

(2) If $\xi \leq \varrho$, then $\xi \wedge \varrho = \xi$, so $\Gamma(\xi) = \Gamma(\xi \wedge \varrho) = (\Gamma(\xi) \wedge \varrho) \vee (\xi \wedge \Gamma(\varrho))$, since $\Gamma(\xi) \wedge \varrho \leq \varrho \leq \varrho \vee \Gamma(\varrho)$, and $\xi \wedge \Gamma(\varrho) \leq \Gamma(\varrho) \leq \varrho \vee \Gamma(\varrho)$. Therefore, $(\Gamma(\xi) \wedge \varrho) \vee (\xi \wedge \Gamma(\varrho)) \leq \varrho \vee \Gamma(\varrho)$, that is, $\Gamma(\xi) \leq \Gamma(\varrho) \vee \varrho$.

(3) For all $\xi, \varrho \in \mathscr{A}$, $\Gamma_u(\xi \wedge \varrho) = u \wedge (\xi \wedge \varrho) = (u \wedge \xi) \wedge \varrho = ((u \wedge \xi) \wedge \varrho) \vee (\xi \wedge (u \wedge \varrho)) = (\Gamma_u(\xi) \wedge \varrho) \vee (\xi \wedge \Gamma_u(\varrho))$. Therefore, Γ_u is a lattice derivation.

Definition 3.2. A mapping $\Gamma : \mathscr{A} \to \mathscr{A}$ is said to be an effect algebra derivation (briefly, a derivation) provided that for any $\xi, \varrho \in \mathscr{A}$ for which $\xi \diamond \varrho$, $\Gamma(\xi) \diamond \varrho$, and $\xi \diamond \Gamma(\varrho)$ are all defined, the equality

$$\Gamma(\xi \diamond \varrho) = (\Gamma(\xi) \diamond \varrho) \vee (\xi \diamond \Gamma(\varrho))$$

holds.

Moreover, the assignment sending every element $\xi \in \mathscr{A}$ to 0 yields the zero derivation. When $\Gamma(\xi) = \xi$ holds for all $\xi \in \mathscr{A}$, Γ is termed the identity derivation.

Convention 3.1. (Standing domain convention) Unlike derivations on lattices and rings defined for total binary operations over all element pairs of the underlying algebra, the derivation considered in this paper is defined for partial binary operations; the derivation identity from Definition 3.2 holds exclusively on the common domain

$$\mathcal{D}_\Gamma := \{(\xi, \varrho) \,\big|\, \xi \diamond \varrho,\ \Gamma(\xi) \diamond \varrho,\ \text{and } \xi \diamond \Gamma(\varrho) \text{ are simultaneously defined}\}.$$

This common-domain convention is deliberately chosen because the dual partial operation $\diamond$ is not total in a lattice-ordered effect algebra, and requiring the derivation identity to hold on all pairs would force the algebra to be total (e.g., an MV-algebra), which would exclude the broader class of lattice-ordered effect algebras. The weak common-domain condition ensures that the derivation identity is meaningful wherever all three terms are defined, without imposing unnecessary totality assumptions.

We note that this is the natural partial-algebra analogue of the $(\odot, \vee)$-derivation studied by Zhao, Gan, and Yang on MV-algebras [14], where $\odot$ is a total operation and the identity $d(x \odot y) = (d(x) \odot y) \vee (x \odot d(y))$ holds for all pairs. The present work extends this notion from total MV-operations to the

partial dual operation $\diamond$ on lattice-ordered effect algebras. The key difference is that in the MV-algebra setting, the identity is imposed universally, whereas here it is restricted to the common domain $\mathcal{D}_\Gamma$; this restriction is necessary because the partial operation $\diamond$ is not defined for all pairs in a general lattice-ordered effect algebra. Stronger domain-preserving notions of derivations on partial algebras (e.g., requiring that Γ map the domain of $\diamond$ into itself) are also possible and will be explored in future work.

Unless otherwise specified, every identity and inequality involving the partial operation $\diamond$ throughout Section 3 is stated solely for pairs $(\xi, \varrho) \in \mathcal{D}_\Gamma$, and the derivation equality $\Gamma(\xi \diamond \varrho) = (\Gamma(\xi) \diamond \varrho) \vee (\xi \diamond \Gamma(\varrho))$ is not assumed and may fail for pairs outside $\mathcal{D}_\Gamma$.

Example 3.1. Lattice derivations and effect algebra derivations are not equivalent in general.

Take the standard effect algebra $\mathscr{A} = [0, 1]$ from Example 2.1, where $\xi \diamond \varrho = \xi + \varrho - 1$ whenever $\xi + \varrho \geq 1$. Fix $u \in (0, 1)$ and define $\Gamma_u(\xi) = u \wedge \xi = min\{u, \xi\}$. By Proposition 3.1(3), Γ_u is a lattice derivation.

Take $u = \frac{1}{2}$ and $\xi = \varrho = \frac{3}{4}$. One checks that $(\xi, \varrho) \in \mathcal{D}_\Gamma$, so this pair lies within the common domain of Convention 3.1. Compute $\xi \diamond \varrho = \frac{3}{4} + \frac{3}{4} - 1 = \frac{1}{2}$, $\Gamma_u(\xi) = min\{\frac{1}{2}, \frac{3}{4}\} = \frac{1}{2}$, $\Gamma_u(\xi) \diamond \varrho = \frac{1}{2} \diamond \frac{3}{4} = \frac{1}{2} + \frac{3}{4} - 1 = \frac{1}{4}$, $\xi \diamond \Gamma_u(\varrho) = \frac{3}{4} \diamond \frac{1}{2} = \frac{1}{4}$. The left-hand side is $\Gamma_u(\xi \diamond \varrho) = \Gamma_u(\frac{1}{2}) = min\{\frac{1}{2}, \frac{1}{2}\} = \frac{1}{2}$, while the right-hand side is $\frac{1}{4} \vee \frac{1}{4} = \frac{1}{4}$. Since $\frac{1}{2} \neq \frac{1}{4}$, the effect algebra derivation identity fails. Hence, Γ_u is a lattice derivation but not an effect algebra derivation.

This example illustrates that lattice derivations generally fail to be effect-algebra derivations; the two notions coincide only under additional constraints, such as within lattice-ordered orthoalgebras.

Note that $[0, 1]$ is not a lattice-ordered orthoalgebra: for $\xi \in (0, 1)$, $\xi \wedge \overline{\xi} = min\{\xi, 1 - \xi\} \neq 0$. This fact is consistent with Theorem 3.1 below.

Theorem 3.1. For all $\xi, \varrho \in \mathscr{A}$. Consider the following conditions on $\mathscr{A}$:

(1) Every lattice derivation Γ on $\mathscr{A}$ is an effect algebra derivation (i.e., the derivation identity holds for all $(\xi, \varrho) \in \mathcal{D}_\Gamma$);

(2) $\xi \wedge \varrho = \xi \diamond \varrho$ whenever $\xi \diamond \varrho$ is defined;

(3) $\mathscr{A}$ is a lattice-ordered orthoalgebra.

Then (2) $\Leftrightarrow$ (3) and (3) $\Rightarrow$ (1).

Proof. (2) $\Rightarrow$ (3) Since $\xi \diamond \overline{\xi}$ is always defined (since $\overline{\xi} \perp \xi$ always holds in an effect algebra), and $\xi \diamond \overline{\xi} = 0$. By (2), $\xi \wedge \overline{\xi} = \xi \diamond \overline{\xi} = 0$. Hence, by Proposition 2.1(3), $\mathscr{A}$ forms a lattice-ordered orthoalgebra.

(3) $\Rightarrow$ (2) If $\mathscr{A}$ is a lattice-ordered orthoalgebra, by Proposition 2.1(2), whenever $\xi \perp \varrho$, $\xi \oplus \varrho = \xi \vee \varrho$. Now suppose $\xi \diamond \varrho$ is defined, i.e., $\overline{\xi} \perp \overline{\varrho}$. Then $\overline{\xi} \oplus \overline{\varrho} = \overline{\xi} \vee \overline{\varrho}$. By De Morgan's law in an ortholattice (which holds in lattice-ordered orthoalgebras by Proposition 2.1(4)), $\overline{\overline{\xi} \vee \overline{\varrho}} = \xi \wedge \varrho$. Hence, $\xi \diamond \varrho = \overline{\overline{\xi} \oplus \overline{\varrho}} = \overline{\overline{\xi} \vee \overline{\varrho}} = \xi \wedge \varrho$.

(3) $\Rightarrow$ (1) Let Γ be a lattice derivation on $\mathscr{A}$, for any $(\xi, \varrho) \in \mathcal{D}_\Gamma$. Using (2), $\Gamma(\xi \diamond \varrho) = \Gamma(\xi \wedge \varrho) = (\Gamma(\xi) \wedge \varrho) \vee (\xi \wedge \Gamma(\varrho)) = (\Gamma(\xi) \diamond \varrho) \vee (\xi \diamond \Gamma(\varrho))$. Thus, Γ satisfies the effect algebra derivation identity on D_Γ, and is therefore an effect algebra derivation under Convention 3.1.

Proposition 3.2. Let Γ be a derivation on $\mathscr{A}$. Then:

(1) For all $\xi \in \mathscr{A}$ with $(\xi, 1) \in \mathcal{D}_\Gamma$, we have $\xi \diamond \Gamma(1) \leq \Gamma(\xi)$;

(2) If $\Gamma(0) = 0$, and $(\xi, \overline{\xi}) \in \mathcal{D}_\Gamma$, then $\Gamma(\overline{\xi}) = \overline{\Gamma(\xi)}$;

(3) For integer $n \geq 1$, suppose all pairs involved in the powers satisfy the common domain condition $\mathcal{D}_\Gamma$. Then $\Gamma(\xi^n) = \xi^{n-1} \diamond \Gamma(\xi)$, where $\xi^1 = \xi$, $\xi^n = \xi^{n-1} \diamond \xi$, and $\xi^0 := 1$.

Proof. (1) Since $(\xi, 1) \in \mathcal{D}_\Gamma$. By Convention 3.1, we apply the derivation definition: $\Gamma(\xi) = \Gamma(\xi \diamond 1) = (\Gamma(\xi) \diamond 1) \vee (\xi \diamond \Gamma(1)) = \Gamma(\xi) \vee (\xi \diamond \Gamma(1))$, hence $\xi \diamond \Gamma(1) \leq \Gamma(\xi)$.

(2) Since $\xi \diamond \overline{\xi} = 0$ is always defined and $(\xi, \overline{\xi}) \in \mathcal{D}_\Gamma$, by Convention 3.1, $\Gamma(0) = \Gamma(\xi \diamond \overline{\xi}) = (\Gamma(\xi) \diamond \overline{\xi}) \vee (\xi \diamond \Gamma(\overline{\xi}))$. Since $\Gamma(0) = 0$, we have $\Gamma(\xi) \diamond \overline{\xi} = 0$ and $\xi \diamond \Gamma(\overline{\xi}) = 0$. By definition of $\diamond$, $\Gamma(\xi) \diamond \overline{\xi} = 0$ means $\overline{\Gamma(\xi)} \oplus \overline{\overline{\xi}} = \overline{\Gamma(\xi)} \oplus \xi = 1$, so $\overline{\Gamma(\xi)}$ is the ortho-complement of ξ, i.e., $\overline{\Gamma(\xi)} = \overline{\xi}$. Taking ortho-complementation on both sides gives $\Gamma(\xi) = \xi$. Similarly, $\xi \diamond \Gamma(\overline{\xi}) = 0$ yields $\overline{\xi} \oplus \overline{\Gamma(\overline{\xi})} = 1$, so $\Gamma(\overline{\xi}) = \overline{\xi}$. Combined with $\Gamma(\xi) = \xi$, we get $\Gamma(\overline{\xi}) = \overline{\xi} = \overline{\Gamma(\xi)}$.

(3) By mathematical induction. For $n = 1$: $\Gamma(\xi^1) = \Gamma(\xi) = \xi^0 \diamond \Gamma(\xi)$ where we define $\xi^0 = 1$ (since $1 \diamond \Gamma(\xi) = \Gamma(\xi)$). For $n = 2$: $\Gamma(\xi^2) = \Gamma(\xi \diamond \xi) = \xi \diamond \Gamma(\xi)$ (since both terms are equal by commutativity of $\diamond$ when defined). Assume $\Gamma(\xi^k) = \xi^{k-1} \diamond \Gamma(\xi)$ and all required pairs lie in $\mathcal{D}_\Gamma$ for $n = k$. Then for $n = k+1$: $\Gamma(\xi^{k+1}) = \Gamma(\xi^k \diamond \xi) = (\Gamma(\xi^k) \diamond \xi) \vee (\xi^k \diamond \Gamma(\xi)) = ((\xi^{k-1} \diamond \Gamma(\xi)) \diamond \xi) \vee (\xi^k \diamond \Gamma(\xi))$. By associativity of $\diamond$ (Proposition 2.2(2)), $(\xi^{k-1} \diamond \Gamma(\xi)) \diamond \xi = \xi^{k-1} \diamond (\Gamma(\xi) \diamond \xi) = \xi^{k-1} \diamond (\xi \diamond \Gamma(\xi)) = (\xi^{k-1} \diamond \xi) \diamond \Gamma(\xi) = \xi^k \diamond \Gamma(\xi)$. Hence, both terms equal $\xi^k \diamond \Gamma(\xi)$, and the join is $\xi^k \diamond \Gamma(\xi)$. Then $\Gamma(\xi^n) = \xi^{n-1} \diamond \Gamma(\xi)$ for $n \geq 1$.

Definition 3.3. (see [10]) Let Γ stand for a derivation on $\mathscr{A}$.

(1) Γ is termed an isotone derivation whenever $\xi \leq \varrho$ implies $\Gamma(\xi) \leq \Gamma(\varrho)$.

(2) Γ is called a contractive derivation provided $\Gamma(\xi) \leq \xi$ holds for all elements.

(3) If Γ simultaneously satisfies the isotone and contractive properties, it is referred to as an ideal derivation.

Remark 3.1. These notions originate from lattice derivation literature [10]; here they are adapted to effect algebra derivations on partial operation structures.

Definition 3.4. Let Γ denote a derivation on $\mathscr{A}$. The set $F_\Gamma(\mathscr{A}) = \{a \in \mathscr{A} \mid \Gamma(a) = a\}$ is called the fixed-point set of Γ.

Proposition 3.3. Suppose that $\Gamma : \mathscr{A} \to \mathscr{A}$ is both a lattice derivation and an effect algebra derivation obeying $\Gamma(0) = 0$. Then $F_\Gamma(\mathscr{A})$ is closed under the meet operation and contains 0.

Proof. For any $\xi, \varrho \in F_\Gamma(\mathscr{A})$, then $\Gamma(\xi) = \xi$ and $\Gamma(\varrho) = \varrho$. Since Γ is a lattice derivation, we compute $\Gamma(\xi \wedge \varrho) = (\Gamma(\xi) \wedge \varrho) \vee (\xi \wedge \Gamma(\varrho)) = (\xi \wedge \varrho) \vee (\xi \wedge \varrho) = \xi \wedge \varrho$. Thus, $\xi \wedge \varrho \in F_\Gamma(\mathscr{A})$, so $F_\Gamma(\mathscr{A})$ is closed under the meet operation. Moreover, $\Gamma(0) = 0$ implies $0 \in F_\Gamma(\mathscr{A})$.

Remark 3.2. In general $F_\Gamma(\mathscr{A})$ need not be closed under the joins. For example, let $\mathscr{A} = \{0, \xi, \varrho, 1\}$ be the four-element Boolean lattice, where ξ and ϱ are complementary atoms: $\xi \wedge \varrho = 0$ and $\xi \vee \varrho = 1$. Define the mapping Γ by $\Gamma(0) = 0, \Gamma(\xi) = \xi, \Gamma(\varrho) = \varrho, \Gamma(1) = 0$. A direct check verifies the lattice derivation identity. For the derivation condition of effect algebras, the only nontrivial pair is (ξ, ϱ). Since $\xi \oplus \varrho$ is defined in $\mathscr{A}$, this pair belongs to $\mathcal{D}_\Gamma$. Hence, all nontrivial pairs within $\mathcal{D}_\Gamma$ satisfy the derivation identity under Convention 3.1. Here, $F_\Gamma(\mathscr{A}) = \{0, \xi, \varrho\}$. Since $\xi \vee \varrho = 1 \notin F_\Gamma(\mathscr{A})$, $F_\Gamma(\mathscr{A})$ fails join-closure.

Proposition 3.4. Let Γ be a derivation on $\mathscr{A}$ with $\Gamma(0) = 0$, If Γ preserves ortho-complementation ($\Gamma(\overline{\xi}) = \overline{\Gamma(\xi)}$ for all $\xi \in \mathscr{A}$), then $F_\Gamma(\mathscr{A})$ forms a sub-effect-algebra of $\mathscr{A}$.

Proof. Since $\Gamma(0) = 0$, $0 \in F_\Gamma(\mathscr{A})$. Because Γ preserves ortho-complementation, $\Gamma(1) = \Gamma(\overline{0}) = \overline{\Gamma(0)} = \overline{0} = 1$, which implies $1 \in F_\Gamma(\mathscr{A})$. Let $\xi \in F_\Gamma(\mathscr{A})$, and we have $\Gamma(\overline{\xi}) = \overline{\Gamma(\xi)} = \overline{\xi}$, giving $\overline{\xi} \in F_\Gamma(\mathscr{A})$.

Now take $\xi, \varrho \in F_\Gamma(\mathscr{A})$ and assume $\xi \oplus \varrho$ is defined, i.e., $\xi \perp \varrho$. Recall the identity $\overline{\xi \oplus \varrho} = \overline{\xi} \diamond \overline{\varrho}$. Since $\xi, \varrho \in F_\Gamma(\mathscr{A})$, we have $\overline{\xi}, \overline{\varrho} \in F_\Gamma(\mathscr{A})$, so $\Gamma(\overline{\xi}) = \overline{\xi}$ and $\Gamma(\overline{\varrho}) = \overline{\varrho}$. Whenever $\overline{\xi} \diamond \overline{\varrho}$ is defined, $(\overline{\xi}, \overline{\varrho}) \in \mathcal{D}_\Gamma$. By Convention 3.1: $\overline{\Gamma(\xi \oplus \varrho)} = \Gamma(\overline{\xi \oplus \varrho}) = \Gamma(\overline{\xi} \diamond \overline{\varrho}) = (\Gamma(\overline{\xi}) \diamond \overline{\varrho}) \vee (\overline{\xi} \diamond \Gamma(\overline{\varrho})) = (\overline{\xi} \diamond \overline{\varrho}) \vee (\overline{\xi} \diamond \overline{\varrho}) = \overline{\xi} \diamond \overline{\varrho} = \overline{\xi \oplus \varrho}$, which implies $\Gamma(\xi \oplus \varrho) = \xi \oplus \varrho$. Therefore, $\xi \oplus \varrho \in F_\Gamma(\mathscr{A})$.

We have verified that $0, 1 \in F_\Gamma(\mathscr{A})$, that $F_\Gamma(\mathscr{A})$ is closed under ortho-complementation, and that $F_\Gamma(\mathscr{A})$ is closed under the partial sum $\oplus$ whenever $\oplus$ is defined. Consequently, $F_\Gamma(\mathscr{A})$ satisfies all the axioms for a sub-effect-algebra of $\mathscr{A}$.

Remark 3.3. Proposition 3.4 provides a sufficient but not necessary condition for $F_\Gamma(\mathscr{A})$ to form a sub-effect-algebra, This proposition requires that Γ preserves ortho-complementation globally on the whole algebra $\mathscr{A}$. If $F_\Gamma(\mathscr{A})$ is a sub-effect-algebra, it only ensures that the ortho-complement of every element in $F_\Gamma(\mathscr{A})$ also belongs to $F_\Gamma(\mathscr{A})$, i.e., the ortho-complementation holds for all fixed-point elements. This yields ortho-complementation preservation restricted to $F_\Gamma(\mathscr{A})$, which is strictly weaker than the global preservation assumed in Proposition 3.4. We raise an open question: does there exist a nontrivial derivation such that $F_\Gamma(\mathscr{A})$ is a sub-effect-algebra, while Γ fails to preserve ortho-complementation globally on $\mathscr{A}$. Additional assumptions on Γ are necessary to guarantee global ortho-complementation preservation over the entire algebra $\mathscr{A}$.

Theorem 3.2. Let Γ be a contractive derivation on $\mathscr{A}$. For $\xi, \varrho \in \mathscr{A}$, with $(\xi, \varrho) \in \mathcal{D}_\Gamma$, the following hold:

(1) $\Gamma(\overline{\varrho}) \leq \overline{\varrho} \leq \overline{\Gamma(\varrho)}$;

(2) $\Gamma(\xi)^n \leq \Gamma(\xi^n)$, assuming all pairs involved in the powers belong to $\mathcal{D}_\Gamma$;

(3) $\Gamma(\xi \diamond \varrho) \leq \Gamma(\xi) \vee \Gamma(\varrho)$;

Proof. (1) Since Γ is a contractive derivation, we have $\Gamma(\varrho) \leq \varrho$, which implies $\overline{\varrho} \leq \overline{\Gamma(\varrho)}$. Furthermore, by contractivity, it follows that $\Gamma(\overline{\varrho}) \leq \overline{\varrho}$, and consequently, $\Gamma(\overline{\varrho}) \leq \overline{\varrho} \leq \overline{\Gamma(\varrho)}$.

(2) Since $\Gamma(\xi) \leq \xi$, it follows that $\Gamma(\xi)^{n-1} \leq \xi^{n-1}$ whenever these powers are defined and the relevant pairs lie in $\mathcal{D}_\Gamma$. Using Proposition 3.2(3): $\Gamma(\xi)^n = \Gamma(\xi)^{n-1} \diamond \Gamma(\xi) \leq \xi^{n-1} \diamond \Gamma(\xi) = \Gamma(\xi^n)$.

(3) Under $(\xi, \varrho) \in \mathcal{D}_\Gamma$, by Proposition 2.2(8), $\Gamma(\xi) \diamond \varrho \leq \Gamma(\xi) \diamond 1 = \Gamma(\xi)$ and $\xi \diamond \Gamma(\varrho) \leq 1 \diamond \Gamma(\varrho) = \Gamma(\varrho)$. Also, by the derivation identity: $\Gamma(\xi \diamond \varrho) = (\Gamma(\xi) \diamond \varrho) \vee (\xi \diamond \Gamma(\varrho))$. Thus, $\Gamma(\xi \diamond \varrho) \leq \Gamma(\xi) \vee \Gamma(\varrho)$.

Theorem 3.3. Let Γ be an isotone derivation on $\mathscr{A}$. For all $\xi, \varrho, \omega \in \mathscr{A}$, the following hold:

(1) Suppose $(\xi, \omega) \in \mathcal{D}_\Gamma$, $\xi \diamond \omega \leq \varrho$, and both partial sums $\overline{\Gamma(\xi)} \oplus \Gamma(\rho)$ and $\overline{\Gamma(\omega)} \oplus \Gamma(\rho)$ are defined. Then, $\omega \leq \overline{\Gamma(\xi)} \oplus \Gamma(\varrho)$ and $\xi \leq \overline{\Gamma(\omega)} \oplus \Gamma(\varrho)$;

(2) Suppose $\overline{\varrho} \oplus \xi$ is defined, $(\varrho, \overline{\varrho} \oplus \xi) \in \mathcal{D}_\Gamma$, and both partial sums $\Gamma(\xi) \oplus \Gamma(\bar{\rho})$ and $\bar{\rho} \oplus \Gamma(\xi)$ are defined. Then, $\xi \oplus \overline{\varrho} \leq \Gamma(\xi) \oplus \overline{\Gamma(\varrho)}$ and $\Gamma(\xi \oplus \overline{\varrho}) \leq \overline{\varrho} \oplus \Gamma(\xi)$;

(3) Suppose $(\xi, \varrho) \in \mathcal{D}_\Gamma$, $\xi \diamond \varrho$ is defined, and both partial sums $\overline{\Gamma(\rho)} \oplus \Gamma(\xi)$ and $\overline{\Gamma(\xi)} \oplus \Gamma(\rho)$ are defined. Then, $\xi \leq \overline{\Gamma(\varrho)} \oplus \Gamma(\xi)$ and $\varrho \leq \overline{\Gamma(\xi)} \oplus \Gamma(\varrho)$.

Proof. (1) Suppose $\xi \diamond \omega \leq \varrho$, isotonicity yields $\Gamma(\xi \diamond \omega) \leq \Gamma(\varrho)$. Since $(\xi, \omega) \in \mathcal{D}_\Gamma$, Convention 3.1 gives: $\Gamma(\xi \diamond \omega) = (\Gamma(\xi) \diamond \omega) \vee (\xi \diamond \Gamma(\omega)) \leq \Gamma(\varrho)$, and consequently, $\Gamma(\xi) \diamond \omega \leq \Gamma(\varrho)$ and $\xi \diamond \Gamma(\omega) \leq \Gamma(\varrho)$. Thus, $\omega \leq \overline{\Gamma(\xi)} \oplus \Gamma(\varrho)$ and $\xi \leq \overline{\Gamma(\omega)} \oplus \Gamma(\varrho)$, by Proposition 2.2(6).

(2) Assume $\overline{\varrho} \oplus \xi$ is defined and $(\varrho, \overline{\varrho} \oplus \xi) \in \mathcal{D}_\Gamma$. By Proposition 2.2(7), we have $\varrho \diamond (\xi \oplus \overline{\varrho}) \leq \xi$, and hence $\Gamma(\varrho \diamond (\xi \oplus \overline{\varrho})) \leq \Gamma(\xi)$, Since $(\varrho, \bar{\varrho} \oplus \xi) \in D_\Gamma$, Convention 3.1 yields $\Gamma(\varrho \diamond (\xi \oplus \overline{\varrho})) = (\Gamma(\varrho) \diamond (\xi \oplus \overline{\varrho})) \vee (\varrho \diamond \Gamma(\xi \oplus \overline{\varrho})) \leq \Gamma(\xi)$. Hence, $\Gamma(\varrho) \diamond (\xi \oplus \overline{\varrho}) \leq \Gamma(\xi)$ and $\varrho \diamond \Gamma(\xi \oplus \overline{\varrho}) \leq \Gamma(\xi)$. Since both partial sums $\Gamma(\xi) \oplus \Gamma(\bar{\rho})$ and $\bar{\rho} \oplus \Gamma(\xi)$ are defined, applying Proposition 2.2(6) gives $\xi \oplus \overline{\varrho} \leq \Gamma(\xi) \oplus \overline{\Gamma(\varrho)}$ and $\Gamma(\xi \oplus \overline{\varrho}) \leq \overline{\varrho} \oplus \Gamma(\xi)$.

(3) Suppose $\xi \diamond \varrho$ is defined and $(\xi, \varrho) \in \mathcal{D}_\Gamma$. Since $\xi \diamond \varrho \leq \xi \diamond 1 = \xi$, isotonicity gives $\Gamma(\xi \diamond \varrho) \leq \Gamma(\xi)$. Moreover, since $(\xi, \varrho) \in D_\Gamma$, Convention 3.1 gives $\Gamma(\xi \diamond \varrho) = (\Gamma(\xi) \diamond \varrho) \vee (\xi \diamond \Gamma(\varrho)) \leq \Gamma(\xi)$, which implies $\xi \diamond \Gamma(\varrho) \leq \Gamma(\xi)$ and $\Gamma(\xi) \diamond \varrho \leq \Gamma(\xi)$. Since both partial sums $\overline{\Gamma(\varrho)} \oplus \Gamma(\xi)$ and $\overline{\Gamma(\xi)} \oplus \Gamma(\varrho)$ are defined by hypothesis, applying Proposition 2.2(6) yields $\xi \leq \overline{\Gamma(\varrho)} \oplus \Gamma(\xi) \quad \text{and} \quad \varrho \leq \overline{\Gamma(\xi)} \oplus \Gamma(\varrho)$.

Theorem 3.4. Suppose Γ is an ideal derivation on $\mathscr{A}$.

(1) Let $\xi, \varrho \in \mathscr{A}$. Suppose $(\overline{\varrho}, \xi) \in \mathcal{D}_\Gamma$, $\overline{\varrho} \oplus \xi$ is defined, and $(\varrho, \overline{\varrho} \oplus \xi) \in \mathcal{D}_\Gamma$. Then $\Gamma(\overline{\varrho} \oplus \xi) \leq \overline{\Gamma(\varrho)} \oplus \Gamma(\xi) \leq \overline{\Gamma(\varrho)} \oplus \xi$;

(2) Let $s, t \in F_\Gamma(\mathscr{A})$, if $s \diamond t$ is defined and $(s, t) \in \mathcal{D}_\Gamma$, then $s \diamond t \in F_\Gamma(\mathscr{A})$.

Proof. (1) Assume $\overline{\varrho} \oplus \xi$ is defined. By Proposition 2.2(7), we have $\varrho \diamond (\overline{\varrho} \oplus \xi) \leq \xi$. Since Γ is an ideal derivation on $\mathscr{A}$, we obtain $\Gamma(\varrho \diamond (\overline{\varrho} \oplus \xi)) \leq \Gamma(\xi)$, $\Gamma(\overline{\varrho} \oplus \xi) \leq \overline{\varrho} \oplus \xi$, and $\Gamma(\varrho) \leq \varrho$, for all $\varrho, \xi \in \mathscr{A}$. It follows that $\Gamma(\xi \oplus \overline{\varrho}) \diamond \Gamma(\varrho) \leq (\xi \oplus \overline{\varrho}) \diamond \Gamma(\varrho)$ and $\Gamma(\xi \oplus \overline{\varrho}) \diamond \Gamma(\varrho) \leq \Gamma(\xi \oplus \overline{\varrho}) \diamond \varrho$. Hence, $\Gamma(\overline{\varrho} \oplus \xi) \diamond \Gamma(\varrho) \leq (\Gamma(\overline{\varrho} \oplus \xi) \diamond \varrho) \vee ((\overline{\varrho} \oplus \xi) \diamond \Gamma(\varrho)) = \Gamma((\overline{\varrho} \oplus \xi) \diamond \varrho) \leq \Gamma(\xi)$, that is, $\Gamma(\overline{\varrho} \oplus \xi) \diamond \Gamma(\varrho) \leq \Gamma(\xi)$, by Proposition 2.2(6), $\Gamma(\overline{\varrho} \oplus \xi) \leq \overline{\Gamma(\varrho)} \oplus \Gamma(\xi)$. Moreover, $\overline{\Gamma(\varrho)} \oplus \Gamma(\xi) \leq \overline{\Gamma(\varrho)} \oplus \xi$. Therefore, $\Gamma(\overline{\varrho} \oplus \xi) \leq \overline{\Gamma(\varrho)} \oplus \Gamma(\xi) \leq \overline{\Gamma(\varrho)} \oplus \xi$.

(2) Let $s, t \in F_\Gamma(\mathscr{A})$, and suppose $s \diamond t$ is defined and $(s, t) \in \mathcal{D}_\Gamma$. By definition of derivation, $\Gamma(s \diamond t) = (\Gamma(s) \diamond t) \vee (s \diamond \Gamma(t))$, then $\Gamma(s) \diamond t \leq \Gamma(s \diamond t)$. Besides, $\Gamma(s) = s$ and $\Gamma(t) = t$, so $\Gamma(s) \diamond t = s \diamond t$ and $s \diamond \Gamma(t) = s \diamond t$, both of which are defined. Hence, the pair (s, t) lies in the common domain of the derivation identity. It follows that $s \diamond t = \Gamma(s) \diamond t \leq \Gamma(s \diamond t) \leq s \diamond t$, where the inequality $\Gamma(s \diamond t) \leq s \diamond t$ holds because Γ is contractive (being an ideal derivation). Consequently, $\Gamma(s \diamond t) = s \diamond t$, which yields, $s \diamond t \in F_\Gamma(\mathscr{A})$.

Theorem 3.5. Let $\mathscr{A}$ be a lattice-ordered effect algebra and let Γ be a derivation on $\mathscr{A}$ with $\Gamma(1) = 1$. For arbitrary $\varrho \in \mathscr{A}$, assume $(\varrho, 1) \in \mathcal{D}_\Gamma$. Then, Γ is a contractive derivation if, and only if, Γ is the identity derivation ($F_\Gamma(\mathscr{A}) = \mathscr{A}$).

Proof. ($\Rightarrow$) For arbitrary $\varrho \in \mathscr{A}$, assume $(\varrho, 1) \in \mathcal{D}_\Gamma$. Applying Definition 3.2 explicitly using the hypothesis $\Gamma(1) = 1$, we have $\Gamma(\varrho) = \Gamma(\varrho \diamond 1) = (\Gamma(\varrho) \diamond 1) \vee (\varrho \diamond \Gamma(1)) = \Gamma(\varrho) \vee (\varrho \diamond 1) = \Gamma(\varrho) \vee \varrho$. Hence, $\varrho \leq \Gamma(\varrho)$. Since Γ is contractive, $\Gamma(\varrho) \leq \varrho$. Therefore, $\Gamma(\varrho) = \varrho$.

($\Leftarrow$) Suppose $\Gamma(\varrho) = \varrho$ for every $\varrho \in \mathscr{A}$. Clearly $\Gamma(\varrho) \leq \varrho$, so Γ is contractive.

Remark 3.4. The hypothesis $\Gamma(1) = 1$ cannot be omitted. The zero derivation furnishes a trivial counterexample with $\Gamma(1) = 0 \neq 1$. We give a nonzero, nonidentity counterexample.

Take the four-element lattice-ordered effect algebra $\mathscr{A} = \{0, \xi, \varrho, 1\}$ with orthogonal atoms ξ and ϱ satisfying $\xi \oplus \varrho = 1$, $\overline{\xi} = \varrho$, $\overline{\varrho} = \xi$. Define $\Gamma(0) = 0$, $\Gamma(\xi) = 0$, $\Gamma(\varrho) = \varrho$, $\Gamma(1) = \varrho$. One verifies directly that Γ satisfies the effect algebra derivation under Convention 3.1. Here $\Gamma(1) = \varrho \neq 1$, Γ is nonzero and not the identity mapping. This demonstrates that nontrivial derivations with $\Gamma(1) \neq 1$ genuinely exist.

Theorem 3.6. Let Γ be a derivation on $\mathscr{A}$ with fixed-point set $F_\Gamma(\mathscr{A})$. Assume that Γ preserves orthocomplementation on $F_\Gamma(\mathscr{A})$ (i.e., $\Gamma(\overline{\xi}) = \overline{\Gamma(\xi)}$ for all $\xi \in F_\Gamma(\mathscr{A})$) and that $0, 1 \in F_\Gamma(\mathscr{A})$. Endowed with the partial operation $\diamond$ (restricted to pairs where it is defined and $(\xi, \varrho) \in \mathcal{D}_\Gamma$), zero element $0_\Gamma = 1$, and unit element $1_\Gamma = 0$. Then $(F_\Gamma(\mathscr{A}), \diamond, 0_\Gamma, 1_\Gamma)$ is an effect algebra.

Proof. First, we show that $F_\Gamma(\mathscr{A})$ is closed under $\diamond$. Let $\phi, \vartheta \in F_\Gamma(\mathscr{A})$ and suppose $\phi \diamond \vartheta$ is defined with $(\phi, \vartheta) \in \mathcal{D}_\Gamma$. Since $\Gamma(\phi) = \phi$ and $\Gamma(\vartheta) = \vartheta$, by Convention 3.1: $\Gamma(\phi \diamond \vartheta) = (\Gamma(\phi) \diamond \vartheta) \vee (\phi \diamond \Gamma(\vartheta)) = (\phi \diamond \vartheta) \vee (\phi \diamond \vartheta) = \phi \diamond \vartheta$, which shows, $\phi \diamond \vartheta \in F_\Gamma(\mathscr{A})$.

We now verify the four effect-algebra axioms.

(E1) Commutativity: Let $\phi, \vartheta \in F_\Gamma(\mathscr{A})$ and $\phi \diamond \vartheta$ is defined. By Definition 2.3, $\overline{\phi} \perp \overline{\vartheta}$ holds, and orthogonality is symmetric, hence $\overline{\vartheta} \perp \overline{\phi}$, then $\vartheta \diamond \phi$ is defined. Using the definition of $\diamond$ and commutativity of $\oplus$: $\phi \diamond \vartheta = \overline{\overline{\phi} \oplus \overline{\vartheta}} = \overline{\overline{\vartheta} \oplus \overline{\phi}} = \vartheta \diamond \phi$, commutativity holds on $F_\Gamma(\mathscr{A})$.

(E2) Associativity: Assume that $(\phi \diamond \vartheta) \diamond \varsigma$ is defined for $\phi, \vartheta, \varsigma \in F_\Gamma(\mathscr{A})$. By Definition 2.3, we have $\overline{\phi \diamond \vartheta} \perp \overline{\varsigma}$, i.e., $\overline{\phi \diamond \vartheta} = \overline{\phi} \oplus \overline{\vartheta}$ is orthogonal to $\overline{\varsigma}$, so $(\overline{\phi} \oplus \overline{\vartheta}) \oplus \overline{\varsigma}$ exists in $\mathscr{A}$. By associativity of the partial sum $\oplus$ in $\mathscr{A}$, $\overline{\phi} \oplus (\overline{\vartheta} \oplus \overline{\varsigma})$ is defined and $(\overline{\phi} \oplus \overline{\vartheta}) \oplus \overline{\varsigma} = \overline{\phi} \oplus (\overline{\vartheta} \oplus \overline{\varsigma})$. Hence $\overline{\vartheta} \perp \overline{\varsigma}$, so $\vartheta \diamond \varsigma$ is defined. Moreover, $\overline{\vartheta} \oplus \overline{\varsigma} = \overline{\vartheta \diamond \varsigma}$, and therefore $\phi \diamond (\vartheta \diamond \varsigma)$ is defined and, by the closure established above,

belongs to $F_\Gamma(\mathscr{A})$. Finally, $(\phi \diamond \vartheta) \diamond \varsigma = \overline{\overline{\phi \diamond \vartheta} \oplus \overline{\varsigma}} = \overline{(\overline{\phi} \oplus \overline{\vartheta}) \oplus \overline{\varsigma}} = \overline{\overline{\phi} \oplus (\overline{\vartheta} \oplus \overline{\varsigma})} = \overline{\overline{\phi} \oplus \overline{\vartheta \diamond \varsigma}} = \phi \diamond (\vartheta \diamond \varsigma)$.

(E3) Ortho-supplementation: Take arbitrary $\varsigma \in F_\Gamma(\mathscr{A})$. Since Γ preserves ortho-complementation on the fixed-point set, $\Gamma(\overline{\varsigma}) = \overline{\Gamma(\varsigma)} = \overline{\varsigma}$, so $\overline{\varsigma} \in F_\Gamma(\mathscr{A})$. By the ortho-supplementation axiom of $\mathscr{A}$, $\varsigma \oplus \overline{\varsigma} = 1$; by commutativity, $\overline{\varsigma} \oplus \varsigma = 1$ is defined. Hence $\overline{\varsigma} \perp \varsigma$, so $\varsigma \diamond \overline{\varsigma}$ is defined and $\varsigma \diamond \overline{\varsigma} = \overline{(\overline{\varsigma} \oplus \varsigma)} = \overline{1} = 0 = 1_\Gamma$.

Moreover, this ortho-supplement is unique: if $\varsigma \diamond \kappa = 1_\Gamma = 0$ for some $\kappa \in F_\Gamma(\mathscr{A})$, then $\varsigma \diamond \kappa = 0$ implies $\overline{\varsigma} \oplus \overline{\kappa} = 1$, so $\overline{\kappa}$ is the ortho-supplement of $\overline{\varsigma}$, whence $\kappa = \overline{\varsigma}$.

(E4) Zero-one property: Assume $\varsigma \diamond 1_\Gamma$ is defined. Since $1_\Gamma = 0$, we have $\varsigma \diamond 1_\Gamma = \varsigma \diamond 0$. By definition of $\diamond$, $\varsigma \diamond 0$ is defined iff $\overline{\varsigma} \perp \overline{0}$, i.e., $\overline{\varsigma} \oplus 1$ is admissible. By the zero-one property of the original effect algebra $\mathscr{A}$, $\overline{\varsigma} \oplus 1$ can be defined only if $\overline{\varsigma} = 0$, that is $\varsigma = 1 = 0_\Gamma$. Therefore, if $\varsigma \diamond 1_\Gamma$ is defined, then $\varsigma = 0_\Gamma$.

Since $0, 1 \in F_\Gamma(\mathscr{A})$, both distinguished elements $0_\Gamma = 1$ and $1_\Gamma = 0$ belong to $F_\Gamma(\mathscr{A})$.

To sum up, $(F_\Gamma(\mathscr{A}), \diamond, 0_\Gamma, 1_\Gamma)$ is an effect algebra.

Remark 3.5. Observe that the zero element 0_Γ and the unit element 1_Γ of the constructed effect algebra are reversed relative to the original lattice-ordered effect algebra $\mathscr{A}$. This reversal is only a notational feature of the dual structure and does not interfere with the validity of the effect-algebra axioms.

4. Conclusions

Motivated by intrinsic structural features of lattice-ordered effect algebras, this paper has investigated derivations on these partial algebraic systems and obtained several structural characterizations. As algebraic models for quantum logic, effect algebras provide a natural framework for studying derivations, a concept originating in analysis and ring theory. Research on effect algebra derivations fosters cross-fertilization between quantum logic and abstract algebra, supplying new tools for analyzing substructures of effect algebras.

The partial operation $\diamond$ employed here is the standard dual partial operation induced by ortho-supplementation and the partial sum $\oplus$ (Definition 2.3). While derivations have been studied for lattices, residuated lattices, basic algebras, L-algebras, and MV-algebras, the present work systematically defines and studies derivations on lattice-ordered effect algebras using $\diamond$, and compares them with lattice derivations. Unlike earlier derivation concepts defined for total operations, Definition 3.2 is formulated for a partial binary operation; accordingly, identities hold only within the common domain specified by Convention 3.1, and every main theorem explicitly lists the required domain-membership hypotheses. This common-domain approach extends the $(\odot, \vee)$-derivation theory of Zhao, Gan, and Yang on MV-algebras [14] from total operations to the partial setting, and we have identified the key differences arising from the partial nature of the dual operation in a general lattice-ordered effect algebra. Stronger domain-preserving notions of derivations on partial algebras remain to be explored, and we hope the present work will stimulate further research in this direction.

Use of Generative-AI tools declaration

The author declares they have not used Artificial Intelligence (AI) tools in the creation of this article.

Funding

This work was supported by the Key Project of the Chinese Society of Higher Education (Grant No.25JK0203) and the Research Project of Shaanxi University of Technology (Grant No.XJG-2553).

Conflict of interest

The author declares that there is no conflict of interest.

References

1. D. J. Foulis, M. K. Bennett, Effect algebras and unsharp quantum logics, *Found. Phys.*, **24** (1994), 1331–1352. https://doi.org/10.1007/BF02283036
2. Z. Riečanová, Generalization of blocks for D-lattices and lattice-ordered effect algebras, *Int. J. Theor. Phys.*, **39** (2000), 231–237. https://doi.org/10.1023/A:1003619806024
3. Y. J. Xie, Y. M. Li, A. L. Yang, Pasting of lattice-ordered effect algebras, *Fuzzy Sets Syst.*, **260** (2015), 77–96. https://doi.org/10.1016/j.fss.2014.03.015
4. B. L. Meng, X. L. Xin, A note of filters in effect algebras, *Chin. J. Math.*, **2013** (2013), 570496. https://doi.org/10.1155/2013/570496
5. J. Wei, Fuzzy implications in lattice effect algebras, *Fuzzy Sets Syst.*, **405** (2021), 40–46. https://doi.org/10.1016/j.fss.2020.04.021
6. W. Ji, H. Xie, Hyper-commutators in effect algebras, *Soft Comput.*, **23** (2019), 5967–5973. https://doi.org/10.1007/s00500-018-3516-z
7. G. Jenča, S. Pulmannová, Ideals and quotients in lattice ordered effect algebras, *Soft Comput.*, **5** (2001), 376–380. https://doi.org/10.1007/s005000100139
8. E. C. Posner, Derivations in prime rings, *Proc. Amer. Math. Soc.*, **8** (1957), 1093–1100.
9. G. Szász, Derivations of lattices, *Acta Sci. Math.*, **37** (1975), 149–154.
10. X. L. Xin, T. Y. Li, J. H. Lu, On derivations of lattices, *Inf. Sci.*, **178** (2008), 307–316. https://doi.org/10.1016/j.ins.2007.08.018
11. J. Krnávek, J. Kůhr, A note on derivations on basic algebras, *Soft Comput.*, **19** (2015), 1765–1771. https://doi.org/10.1007/s00500-014-1586-0
12. P. F. He, X. L. Xin, J. M. Zhan, On derivations and their fixed point sets in residuated lattices, *Fuzzy Sets Syst.*, **303** (2016), 97–113. https://doi.org/10.1016/j.fss.2016.01.006
13. X. J. Hua, State L-algebras and derivations of L-algebras, *Soft Comput.*, **25** (2021), 4201–4212. https://doi.org/10.1007/s00500-021-05651-2
14. X. T. Zhao, A. P. Gan, Y. C. Yang, $(\odot, \vee)$-derivations on MV-algebras, *Soft Comput.*, **28** (2024), 1833–1849. https://doi.org/10.1007/s00500-023-09384-2

15. J. T. Wang, P. F. He, Y. H. She, Some results on derivations of MV-algebras, *Appl. Math. J. Chin. Univ. Ser. B*, **38** (2023), 126–143. https://doi.org/10.1007/s11766-023-4054-8

16. Y. L. Wu, X. Li, J. Wang, Notes on sharp and principal elements in effect algebras, *Order*, **40** (2023), 525–536. https://doi.org/10.1007/s11083-022-09619-1

17. Y. L. Wu, J. Wang, Y. C. Yang, Lattice-ordered effect algebras and L-algebras, *Fuzzy Sets Syst.*, **369** (2019), 103–113. https://doi.org/10.1016/j.fss.2018.08.013



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)