AIMS Mathematics, 11(11): 38818–38838.
DOI:10.3934/math.20261497
Received: 16 September 2026
Revised: 02 November 2026
Accepted: 13 November 2026
Published: 20 November 2026

https://www.aimspress.com/journal/Math

Research article

On entire solutions of two kinds of Fermat-type functional equations

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Abstract: We study entire solutions of quadratic trinomial Fermat-type differential-difference and q-difference differential equations involving second-order derivatives. The differential-difference equation is considered under the condition $\rho_2(f) < 1$, while the q-difference equation is considered under a finite hyper-order condition. Gong and Yang [*Bull. Iran. Math. Soc.*, **17** (2025), 51] considered finite-order entire solutions of these equations and additionally imposed the condition $\rho(f) \leq 1$. We remove this additional condition and extend their results to the case where the exponential term involves a higher-degree polynomial. Several examples are provided to illustrate the conclusions.

Keywords: entire solutions; differential-difference equations; q-difference differential equations; existence; Fermat-type functional equations
2020 Mathematics Subject Classification: 30D35, 39B32

1. Introduction

Let f be a meromorphic function in the complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard notation and basic results of Nevanlinna theory, such as $m(r, f)$, $N(r, f)$, and $T(r, f)$; see [12] for further details. A meromorphic function g is called a small function of f if $T(r, g) = S(r, f)$, where $S(r, f) = o(T(r, f))$ as $r \to \infty$ outside a possible exceptional set of finite linear measure. We denote the order and hyper-order of f by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r, f)}{\log r}, \qquad \rho_2(f) = \limsup_{r\to\infty} \frac{\log^+ \log^+ T(r, f)}{\log r}.$$

To improve readability, we collect the principal notation used throughout the paper. Let

$$\mathcal{A} = \begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix}, \quad \mathcal{A}_1 = a_0b_1 - a_1b_0, \quad \mathcal{A}_2 = a_1b_2 - a_2b_1, \quad \mathcal{A}_3 = a_0b_2 - a_2b_0, \tag{1.1}$$
$$M_1(f) = a_0f(z) + a_1f'(z) + a_2f(z + c), \quad M_2(f) = b_0f(z) + b_1f'(z) + b_2f(z + c),$$

$$L_1(f) = a_0 f(z) + a_1 f''(z) + a_2 f(z+c), \quad L_2(f) = b_0 f(z) + b_1 f''(z) + b_2 f(z+c).$$

Here a_i and b_i $(i = 0, 1, 2)$ and $c \neq 0$ are constants. The rank of $\mathcal{A}$ is denoted by $r(\mathcal{A})$, and the symbols $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ always denote the three 2×2 minors of this matrix. The second-order q-difference differential operators are defined by

$$Q_1(f) = a_0 f(z) + a_1 f''(z) + a_2 f(qz), \qquad Q_2(f) = b_0 f(z) + b_1 f''(z) + b_2 f(qz). \tag{1.2}$$

For any function h, we write

$$h_c(z) = h(z+c), \qquad h_q(z) = h(qz).$$

For $\omega \in \mathbb{C} \setminus \{0, \pm 1\}$, set

$$\omega_1 = -\omega + \sqrt{\omega^2 - 1}, \qquad \omega_2 = -\omega - \sqrt{\omega^2 - 1}.$$

Equivalently, ω_1 and ω_2 are the two roots of $t^2 + 2\omega t + 1 = 0$.

In 1970, Yang [25] applied Nevanlinna theory to prove that if $\frac{1}{m} + \frac{1}{n} < \frac{2}{3}$, then the Fermat-type functional equation

$$a(z)h^n(z) + b(z)g^m(z) = 1 \tag{1.3}$$

does not admit non-constant meromorphic solutions $h(z)$ and $g(z)$, where $a(z)$ and $b(z)$ are small functions with respect to $h(z)$ and $g(z)$, and m and n are positive integers. Yang's results extend the findings of Montel [20]. Some important results related to Fermat-type functional equations can be found in [1, 2, 8, 9]. For the case $m = n = 2$ and $a = b = 1$, particularly when a relation between h and g is imposed in (1.3), extensive research has been conducted on the entire solutions of (1.3). For example, Yang et al. [26] considered the case $g = h'$; Liu et al. [15] studied the case $g = h(z+c)$; Hu et al. [14] explored the case $h = M_1(f)$ and $g = M_2(f)$, where $M_1(f)$ and $M_2(f)$ are defined in (1.1).

We next recall some results on the existence of solutions of the trinomial Fermat-type functional equation

$$f^m + h^p + g^n = 1, \tag{1.4}$$

where f, g, and h are meromorphic functions, and m, n, and p are positive integers. For equal exponents in (1.4), nonexistence results and examples were obtained by Hayman [13], Gundersen [4,5], Gundersen and Tohge [6], and Toda [22]. Yang and Zhang [28] considered unequal exponents. These results provide background for the quadratic case studied here. For the case $m = 2, n = 2, p = 1$, and $h = 2afg$ in (1.4), Saleeby [21] showed that the entire solutions of the quadratic trinomial Fermat-type equation

$$f^2 + 2afg + g^2 = 1, \ a^2 \neq 1, \ a \in \mathbb{C} \tag{1.5}$$

are $f(z) = \frac{1}{\sqrt{2}}\left(\frac{\cos(b)}{\sqrt{1+a}} + \frac{\sin(b)}{\sqrt{1-a}}\right)$ and $g(z) = \frac{1}{\sqrt{2}}\left(\frac{\cos(b)}{\sqrt{1+a}} - \frac{\sin(b)}{\sqrt{1-a}}\right)$, where b is any entire function. Later, Liu et al. [16] proved that if $g(z) = f'(z)$ in (1.5), then the equation

$$f^2 + 2aff' + f'^2 = 1, \ a^2 \neq 1, 0, \ a \in \mathbb{C} \tag{1.6}$$

has no transcendental meromorphic solutions. At the same time, they also proved that if $g(z) = f(z+c)$ in (1.5), then the finite-order transcendental entire solutions of the equation

$$f(z)^2 + 2af(z)f(z+c) + f^2(z+c) = 1,\ a^2 \neq 1, 0,\ a \in \mathbb{C} \tag{1.7}$$

must be of order one. Recently, many studies have examined the quadratic trinomial Fermat-type equations; see [7, 11, 19, 23, 24] and the references therein.

In 2020, Wang et al. [23] considered the more general forms of (1.6) and (1.7). They obtained the following conclusions.

Theorem A. *[23] Assume that M_1 and M_2 are defined in* (1.1)*, α, β, and $\omega(\neq 0, \pm 1)$ are complex numbers, and $r(\mathcal{A}) = 2$. If the equation*

$$(M_1(f))^2 + 2\omega M_1(f)M_2(f) + (M_2(f))^2 = e^{\alpha z+\beta} \tag{1.8}$$

admits an entire solution f of finite order, then f must take the form

$$f(z) = C_1 e^{(\alpha/2+\lambda)z} + C_2 e^{(\alpha/2-\lambda)z} + C_3 e^{C_0 z},$$

where $\lambda(\neq 0)$ and $C_i, i = 0, 1, 2, 3$, are constants satisfying $C_0 = C_3 = 0$ if $\mathcal{A}_2 = 0$, and $C_0 = -\dfrac{\mathcal{A}_3}{\mathcal{A}_2}$ if $\mathcal{A}_2 \neq 0$.

Remark 1.1. For simplicity, we do not include the expressions for the constants $C_i, i = 1, 2, 3$, in Theorem A. Readers can refer to the original paper for these expressions. However, Liu et al. [17, Example 1, 2] pointed out that there are some entire solutions which were missing from Theorem A. Liu et al. [17, Theorem 1] also gave all the entire solutions of (1.8) under the weakened growth condition when $\alpha z + \beta$ is replaced by a non-constant polynomial.

Later, Gong and Yang [7] extended the first-order derivatives f' in (1.8) to second-order derivatives f''. Upon carefully reviewing Gong and Yang [7, Theorem 1.5], we can summarize their result as follows.

Theorem B. *[7, Theorem 1.5] Assume that L_1 and L_2 are defined in* (1.1)*, α, β, and $\omega(\neq 0, \pm 1)$ are complex numbers, and $r(\mathcal{A}) = 2$. If the equation*

$$(L_1(f))^2 + 2\omega L_1(f)L_2(f) + (L_2(f))^2 = e^{\alpha z+\beta} \tag{1.9}$$

admits an entire solution f of finite order, then one of the following assertions holds:

(i) *If $\mathcal{A}_2 = 0$, then $f(z) = C_1 e^{(\lambda+\frac{\alpha}{2})z} + C_2 e^{(-\lambda+\frac{\alpha}{2})z}$, where λ and B are arbitrary constants, and $C_1 = \frac{(b_2\omega_2 - a_2)e^{B+\frac{\beta}{2}}}{(\omega_2-\omega_1)\mathcal{A}_3}$ and $C_2 = \frac{(a_2-b_2\omega_1)e^{-B+\frac{\beta}{2}}}{(\omega_2-\omega_1)\mathcal{A}_3}$ or $C_1 = \frac{(b_1\omega_2-a_1)e^{B+\frac{\beta}{2}}}{(\omega_2-\omega_1)\mathcal{A}_1}$ and $C_2 = \frac{(a_1-b_1\omega_1)e^{-B+\frac{\beta}{2}}}{(\omega_2-\omega_1)\mathcal{A}_1}$.*

(ii) *If $\mathcal{A}_2 \neq 0$ and $\rho(f) \leq 1$, then $f(z) = C_1 e^{(\lambda+\frac{\alpha}{2})z} + C_2 e^{(-\lambda+\frac{\alpha}{2})z} + C_3 e^{C_5 z} + C_4 e^{C_6 z}$, where λ, B, C_3, and C_4 are arbitrary constants, and C_1, C_2, C_5, and C_6 depend on $a_i, i = 1, 2, 3$, and α as follows:*

(1) *If $(\pm\lambda + \frac{\alpha}{2})^2 \neq \frac{-\mathcal{A}_3}{\mathcal{A}_2}$, then $C_5 = \sqrt{-\frac{\mathcal{A}_3}{\mathcal{A}_2}}, C_6 = -\sqrt{-\frac{\mathcal{A}_3}{\mathcal{A}_2}}$, $C_1 = \frac{4(b_2\omega_2-a_2)e^{\frac{\beta+2B}{2}}}{(\omega_2-\omega_1)[\mathcal{A}_2(2\lambda+\alpha)^2+4\mathcal{A}_3]}$, $C_2 = \frac{4(a_2-b_2\omega_1)e^{\frac{\beta-2B}{2}}}{(\omega_2-\omega_1)[\mathcal{A}_2(-2\lambda+\alpha)^2+4\mathcal{A}_3]}$;*

(2) If $(\lambda+\frac{\alpha}{2})^2=\frac{-\mathcal{A}_3}{\mathcal{A}_2}$ and $\lambda\neq 0$, then $C_5=\lambda+\frac{\alpha}{2}, C_6=-\lambda-\frac{\alpha}{2}, C_1=\frac{(a_2-b_2\omega_2)e^{\frac{\beta+2B}{2}z}}{(2\lambda+\alpha)(\omega_2-\omega_1)\mathcal{A}_2}$, $C_2=\frac{(a_2-b_2\omega_1)e^{\frac{\beta-2B}{2}}}{2\alpha\lambda(\omega_2-\omega_1)\mathcal{A}_2}$;

(3) If $(-\lambda+\frac{\alpha}{2})^2=\frac{-\mathcal{A}_3}{\mathcal{A}_2}$ and $\lambda\neq 0$, then $C_5=-\lambda+\frac{\alpha}{2}, C_6=\lambda-\frac{\alpha}{2}, C_1=\frac{(b_2\omega_2-a_2)e^{\frac{\beta+2B}{2}}}{-2\alpha\lambda(\omega_2-\omega_1)\mathcal{A}_2}$, $C_2=\frac{(a_2-b_2\omega_1)e^{\frac{\beta-2B}{2}z}}{(2\lambda-\alpha)(\omega_2-\omega_1)\mathcal{A}_2}$;

(4) If $\frac{\alpha^2}{4}\mathcal{A}_2=-\mathcal{A}_3$, then $C_5=C_6=\frac{\alpha}{2}, C_1=\frac{(b_2\omega_2-a_2)e^{\frac{\beta+2B}{2}z}}{\alpha(\omega_2-\omega_1)\mathcal{A}_2}, C_2=\frac{(a_2-b_2\omega_1)e^{\frac{\beta-2B}{2}z}}{\alpha(\omega_2-\omega_1)\mathcal{A}_2}$,

where $\omega_1=-\omega+\sqrt{\omega^2-1}, \omega_2=-\omega-\sqrt{\omega^2-1}$.

By replacing L_1 with $Q_1(f)$ and $L_2(f)$ with $Q_2(f)$ in (1.9), where Q_1 and Q_2 are defined in (1.2), Gong and Yang [7] obtained the q-difference analogue of Theorem B.

Theorem C. *[7, Theorem 1.7] Assume that $\alpha,\beta,\omega(\neq 0,\pm1)$, and q are complex numbers such that $q^n\neq 0,\pm1$, $n\in\mathbb{N}^+$, and $r(\mathcal{A})=2$. If the equation*

$$(Q_1(f))^2+2\omega Q_1(f)Q_2(f)+(Q_2(f))^2=e^{\alpha z+\beta} \tag{1.10}$$

admits an entire solution f with $\rho(f)<\infty$, then one of the following assertions holds:

(i) If $\mathcal{A}_2=0$, then the conclusion of Theorem B-(i) holds.

(ii) If $\mathcal{A}_2\neq 0$ and $\rho(f)\leq 1$, then $f(z)=C_1e^{(\lambda+\frac{\alpha}{2})z}+C_2e^{(-\lambda+\frac{\alpha}{2})z}+C_3e^{C_5z}+C_4e^{C_6z}$, where λ, B, C_3, and C_4 are arbitrary constants, and C_1, C_2, C_5, and C_6 are given as in (1)–(3) of Theorem B-(ii).

Inspired by Theorem B and Theorem C, we ask the following question.

Question 1. Can the condition $\rho(f)\leq 1$ be removed from assertion (ii) of Theorem B and Theorem C?

The following example illustrates that when $\alpha z+\beta$ in (1.9) is replaced by a polynomial $g(z)=z^2$, there exists an entire solution of Eq (1.9).

Example 1.1. Let $c\neq 0$, $\omega=2$, and

$$\mathcal{A}=\begin{pmatrix}-(2+\sqrt{3}) & 0 & 2-\sqrt{3}\\ 1 & 0 & -1\end{pmatrix},\qquad g(z)=z^2,\qquad f(z)=-\frac{\sqrt{3}}{6}e^{z^2-cz}.$$

Compare [17, Example 3]. We have $\mathcal{A}_1=\mathcal{A}_2=0$, $\mathcal{A}_3=2\sqrt{3}\neq 0$, and

$$f_c(z)=-\frac{\sqrt{3}}{6}e^{z^2+cz},$$

$$L_1(f)^2+4L_1(f)L_2(f)+L_2(f)^2=12f(z)f(z+c)=e^{2z^2}=e^{2g(z)}.$$

Inspired by Example 1.1, we ask the following question.

Question 2. Can we find all possible forms of entire solutions when $\alpha z+\beta$ in (1.9) and (1.10) is replaced by a polynomial $g(z)$ with $\deg g>1$?

This paper mainly addresses Questions 1 and 2 in Theorems 2.1 and 3.1. Specifically, our Theorems 2.1 and 3.1 remove the condition $\rho(f) \leq 1$ from assertion (ii) of Theorems B and C. Furthermore, we determine all possible forms of entire solutions when the term $\alpha z + \beta$ in (1.9) and (1.10) is replaced by a polynomial $g(z)$ with $\deg g > 1$. Moreover, our main theorems improve the results of Theorems A and B by extending them from finite-order entire solutions to those with hyper-order less than 1 or with finite hyper-order. Since finite-order functions have hyper-order 0, our growth condition is weaker.

The remainder of this paper is organized as follows. Our main results, Theorems 2.1 and 3.1, are given in Sections 2 and 3, respectively. In Section 2, we study all possible forms of entire solutions f of the quadratic trinomial Fermat-type differential-difference equation $(L_1(f))^2 + 2\omega L_1(f)L_2(f) + (L_2(f))^2 = e^{g(z)}$; in Section 3, we study all possible forms of entire solutions f of the q-difference differential equation $(Q_1(f))^2 + 2\omega Q_1(f)Q_2(f) + (Q_2(f))^2 = e^{g(z)}$. Some examples are provided in Section 4 to illustrate the results in Sections 2 and 3.

2. Quadratic trinomial Fermat-type differential-difference equations

In this section, we study the existence of solutions to the Fermat-type differential-difference equation $(L_1(f))^2 + 2\omega L_1(f)L_2(f) + (L_2(f))^2 = e^{2g(z)}$. The parts related to Theorem B in Questions 1 and 2 are answered here.

Theorem 2.1. *Assume that L_1 and L_2 are defined in* (1.1)*, $\omega(\neq \pm 1)$ is a nonzero complex number, $\omega_1 = -\omega + \sqrt{\omega^2 - 1}, \omega_2 = -\omega - \sqrt{\omega^2 - 1}$, g is a non-constant polynomial, and $r(\mathcal{A}) = 2$. If the equation*

$$(L_1(f))^2 + 2\omega L_1(f)L_2(f) + (L_2(f))^2 = e^{2g(z)} \tag{2.1}$$

admits an entire solution f with $\rho_2(f) < 1$, then one of the following cases holds:

(1) $g = \alpha z + \beta$, $\mathcal{A}_2 = 0$, $f(z) = C_1 e^{(\lambda+\alpha)z} + C_2 e^{(-\lambda+\alpha)z}$, where $\alpha(\neq 0)$, β, λ, and B are constants, and $C_1 = \frac{(b_2\omega_2 - a_2)e^{B+\beta}}{(\omega_2-\omega_1)\mathcal{A}_3}$ and $C_2 = \frac{(a_2 - b_2\omega_1)e^{-B+\beta}}{(\omega_2-\omega_1)\mathcal{A}_3}$ or $C_1 = \frac{(b_1\omega_2 - a_1)e^{B+\beta}}{(\omega_2-\omega_1)\mathcal{A}_1}$ and $C_2 = \frac{(a_1 - b_1\omega_1)e^{-B+\beta}}{(\omega_2-\omega_1)\mathcal{A}_1}$.

(2) $g = \alpha z + \beta$, $\mathcal{A}_2 \neq 0$ and $\mathcal{A}_3 = 0$,

$$f(z) = \frac{1}{\mathcal{A}_2(\omega_2 - \omega_1)} \cdot \begin{cases} \left(U_1 e^{(\alpha+\lambda)z} + U_2 e^{(\alpha-\lambda)z}\right) + C_3 z + C_4, & \alpha \neq \pm\lambda, \\ \left(D_1 e^{2\lambda z} + D_2 z^2\right) + C_3 z + C_4, & \alpha = \lambda, \\ \left(C_1 z^2 + C_2 e^{-2\lambda z}\right) + C_3 z + C_4, & \alpha = -\lambda, \end{cases}$$

where $\alpha(\neq 0)$, β, λ, B, C_3, and C_4 are constants and

$U_1 = \frac{(b_2\omega_2 - a_2)e^{\beta+B}}{(\alpha+\lambda)^2}, U_2 = \frac{(a_2 - b_2\omega_1)e^{\beta-B}}{(\alpha-\lambda)^2}, D_1 = \frac{(b_2\omega_2 - a_2)e^{\beta+B}}{4\lambda^2},$

$D_2 = \frac{(a_2 - b_2\omega_1)e^{\beta-B}}{2}, C_1 = \frac{(b_2\omega_2 - a_2)e^{\beta+B}}{2}, C_2 = \frac{(a_2 - b_2\omega_1)e^{\beta-B}}{4\lambda^2}.$

(3) $g = \alpha z + \beta$, $\mathcal{A}_2 \neq 0$ and $\mathcal{A}_3 \neq 0$,

$$f(z) = C_1 e^{\mu z} + C_2 e^{-\mu z} + \begin{cases} P_1 e^{k_1 z} + P_2 e^{k_2 z}, & k_1 \neq \pm\mu, k_2 \neq \pm\mu, \\ R_1 z e^{k_1 z} + P_2 e^{k_2 z}, & k_1 = \pm\mu, k_2 \neq \pm\mu, \\ P_1 e^{k_1 z} + R_2 z e^{k_2 z}, & k_1 \neq \pm\mu, k_2 = \pm\mu, \end{cases}$$

where $\alpha(\neq 0), \beta, C_1, C_2, \lambda$, and B are constants, $\mu = \sqrt{-\frac{\mathcal{A}_3}{\mathcal{A}_2}}, k_1 = \alpha + \lambda, k_2 = \alpha - \lambda$,

$$P_1 = \frac{(b_2\omega_2 - a_2)e^{\beta+B}}{(\omega_2 - \omega_1)[\mathcal{A}_2(\alpha+\lambda)^2 + \mathcal{A}_3]}, P_2 = \frac{(a_2 - b_2\omega_1)e^{\beta-B}}{(\omega_2 - \omega_1)[\mathcal{A}_2(\alpha-\lambda)^2 + \mathcal{A}_3]},$$
$$R_1 = \frac{(b_2\omega_2 - a_2)e^{\beta+B}}{2\mathcal{A}_2(\omega_2 - \omega_1)(\alpha+\lambda)}, R_2 = \frac{(a_2 - b_2\omega_1)e^{\beta-B}}{2\mathcal{A}_2(\omega_2 - \omega_1)(\alpha-\lambda)}.$$

(4) $\deg g > 1$, $\mathcal{A}_2 = \mathcal{A}_1 = 0$, $f(z) = \frac{(a_2 - b_2\omega_1)e^{g-p}}{\mathcal{A}_3(\omega_2 - \omega_1)}$ or $f(z) = \frac{(b_2\omega_2 - a_2)e^{g(z)+p(z)}}{\mathcal{A}_3(\omega_2 - \omega_1)}$, where p is a polynomial with $\deg p = \deg g - 1$.

Remark 2.1. Theorem 2.1 removes the condition $\rho(f) \leq 1$ from assertion (ii) of Theorem B and determines all possible forms of an entire solution f with $\rho_2(f) < 1$ when $\alpha z + \beta$ in (1.9) is replaced by a polynomial $g(z)$ with $\deg g > 1$. Therefore, Theorem 2.1 answers the parts related to Theorem B in Question 1 and Question 2.

Remark 2.2. Obviously, Theorem 2.1 improves the condition $\rho(f) < \infty$ in Theorem B to $\rho_2(f) < 1$.

Proof of Theorem 2.1. The function f is transcendental; otherwise, the left-hand side of (2.1) would be a polynomial, whereas e^{2g} is transcendental because g is a non-constant polynomial. To make the use of [17, Lemma 1] explicit, factor (2.1) as

$$(L_1(f) - \omega_1 L_2(f))(L_1(f) - \omega_2 L_2(f)) = e^{2g}.$$

Both factors are zero-free entire functions. Their product allows us to write the factors as e^{g+p} and e^{g-p}, respectively, for an entire function p. Solving these two linear equations gives

$$L_1(f) = a_0 f + a_1 f'' + a_2 f_c = \frac{\omega_2 e^{g+p} - \omega_1 e^{g-p}}{\omega_2 - \omega_1}, \tag{2.2}$$
$$L_2(f) = b_0 f + b_1 f'' + b_2 f_c = \frac{e^{g+p} - e^{g-p}}{\omega_2 - \omega_1}, \tag{2.3}$$

where f_c denotes $f(z + c)$, $\omega_1 = -\omega + \sqrt{\omega^2 - 1}, \omega_2 = -\omega - \sqrt{\omega^2 - 1}$, and p is an entire function. Therefore, the two factors above have hyper-order at most $\rho_2(f)$ and $\rho(p) \leq \rho_2(f) < 1$.

Taking, respectively, $b_0 L_1 - a_0 L_2$, $b_1 L_1 - a_1 L_2$, and $b_2 L_1 - a_2 L_2$ in (2.2) gives

$$-\mathcal{A}_1 f'' - \mathcal{A}_3 f_c = \frac{(b_0\omega_2 - a_0)e^{g+p} + (a_0 - b_0\omega_1)e^{g-p}}{\omega_2 - \omega_1}, \tag{2.4}$$

$$\mathcal{A}_1 f - \mathcal{A}_2 f_c = \frac{(b_1\omega_2 - a_1)e^{g+p} + (a_1 - b_1\omega_1)e^{g-p}}{\omega_2 - \omega_1}, \tag{2.5}$$

$$\mathcal{A}_3 f + \mathcal{A}_2 f'' = \frac{(b_2\omega_2 - a_2)e^{g+p} + (a_2 - b_2\omega_1)e^{g-p}}{\omega_2 - \omega_1}. \tag{2.6}$$

Differentiating equation (2.5) twice yields

$$\mathcal{A}_1 f'' - \mathcal{A}_2 f_c'' = \frac{(b_1\omega_2 - a_1)[(g' + p')^2 + g'' + p'']e^{g+p} + (a_1 - b_1\omega_1)[(g' - p')^2 + g'' - p'']e^{g-p}}{\omega_2 - \omega_1}. \tag{2.7}$$

Adding (2.7) and the shift of (2.6) gives

$$\begin{aligned}\mathcal{A}_3 f_c + \mathcal{A}_1 f'' = &\frac{(b_1\omega_2 - a_1)[(g' + p')^2 + g'' + p'']e^{g+p} + (a_1 - b_1\omega_1)[(g' - p')^2 + g'' - p'']e^{g-p}}{\omega_2 - \omega_1} \\ &+ \frac{(b_2\omega_2 - a_2)e^{g_c+p_c} + (a_2 - b_2\omega_1)e^{g_c-p_c}}{\omega_2 - \omega_1}.\end{aligned} \tag{2.8}$$

Adding (2.4) and (2.8) gives

$$B_1 e^p + B_2 e^{-p} + B_3 e^{p_c+g_c-g} + B_4 e^{-p_c+g_c-g} = 0, \tag{2.9}$$

where

$$\begin{aligned}B_1 &= b_0\omega_2 - a_0 + (b_1\omega_2 - a_1)[(g' + p')^2 + g'' + p''], \\ B_2 &= a_0 - b_0\omega_1 + (a_1 - b_1\omega_1)[(g' - p')^2 + g'' - p''], \\ B_3 &= b_2\omega_2 - a_2, \\ B_4 &= a_2 - b_2\omega_1.\end{aligned}$$

We now distinguish two cases according to whether $a_2 = b_2 = 0$.

Case 1. $|a_2| + |b_2| = 0$. It is easy to see $\mathcal{A}_1 \neq 0$ from $r(\mathcal{A}) = 2$ and $\mathcal{A}_2 = \mathcal{A}_3 = 0$. Then combining (2.7) and (2.4) implies

$$\begin{aligned}\{(b_1\omega_2 - a_1)[(g' + p')^2 + g'' + p''] + (b_0\omega_2 - a_0)\}e^{2p} + (a_1 - b_1\omega_1)[(g' - p')^2 + g'' - p''] \\ +(a_0 - b_0\omega_1) = 0.\end{aligned} \tag{2.10}$$

If p is constant and $\deg g = 1$, write $g = \alpha z + \beta$ and $p = B$. Equation (2.5), with $\mathcal{A}_1 \neq 0$, gives conclusion (1), with $\lambda = 0$. We may therefore assume, in the constant-p case, that $\deg g > 1$. The polynomial $(g')^2 + g''$ is then non-constant, so comparing its coefficient and the constant term in (2.10) gives

$$(b_1\omega_2 - a_1)e^{2p} + (a_1 - b_1\omega_1) = (b_0\omega_2 - a_0)e^{2p} + (a_0 - b_0\omega_1) = 0. \tag{2.11}$$

If $b_1\omega_2 - a_1 = 0$, then (2.11) gives $a_1 - b_1\omega_1 = 0$. Substituting these into Eq (2.5) leads to the conclusion that f is a constant, which is impossible. Thus, $b_1\omega_2 - a_1 \neq 0$. Similarly, $a_0 - b_0\omega_2 \neq 0$. Thus, (2.11) gives $\frac{a_1 - b_1\omega_1}{a_1 - b_1\omega_2} = \frac{a_0 - b_0\omega_1}{a_0 - b_0\omega_2} = e^{2p}$, which implies $(a_0b_1 - a_1b_0)(\omega_2 - \omega_1) = 0$, contradicting $\mathcal{A}_1 \neq 0$. Thus, the constant-p case has been settled. When p is non-constant, exponential independence in (2.10) gives

$$\begin{aligned}(b_1\omega_2 - a_1)[(g' + p')^2 + g'' + p''] + (b_0\omega_2 - a_0) = 0, \\ (a_1 - b_1\omega_1)[(g' - p')^2 + g'' - p''] + (a_0 - b_0\omega_1) = 0.\end{aligned} \tag{2.12}$$

If $b_1\omega_2 - a_1 = 0$, then $a_0 - b_0\omega_2 = 0$, which gives $a_0b_1 - a_1b_0 = 0$, contradicting $\mathcal{A}_1 \neq 0$. Thus, $b_1\omega_2 - a_1 \neq 0$. Similarly, $a_1 - b_1\omega_1 \neq 0$. From (2.12), we get

$$\begin{aligned} 2(g')^2 + 2(p')^2 + 2g'' &= \frac{a_0 - b_0\omega_2}{b_1\omega_2 - a_1} + \frac{a_0 - b_0\omega_1}{b_1\omega_1 - a_1}, \\ 4g'p' + 2p'' &= \frac{a_0 - b_0\omega_2}{b_1\omega_2 - a_1} - \frac{a_0 - b_0\omega_1}{b_1\omega_1 - a_1}. \end{aligned} \tag{2.13}$$

We claim $\deg g = 1$. Otherwise, if $\deg g > 1$, then the first equation of (2.13) implies p is a polynomial and $\deg g = \deg p > 1$. Combining this with the second equation of (2.13), we get $\deg g'p' = \deg p''$, which is impossible. Thus, $\deg g = 1$ and the first equation of (2.13) implies p is a polynomial of $\deg p = 1$.

Write $g = \alpha z+\beta$ and $p = \lambda z+B$. Since $\mathcal{A}_1 \neq 0$ and $\mathcal{A}_2 = 0$, division of (2.5) by $\mathcal{A}_1$ gives conclusion (1) directly, with the displayed expressions involving a_1 and b_1.

Case 2. $|a_2| + |b_2| \neq 0$. Since $\omega_1 \neq \omega_2$, it is impossible for $B_3 = 0$ and $B_4 = 0$ to hold simultaneously.

First, suppose that p is constant. We claim $\deg g = 1$. Otherwise, if $\deg g > 1$, then $\deg(g_c - g) = \deg g - 1 \geq 1$, since $c \neq 0$. Therefore, from (2.9), we get

$$B_1e^p + B_2e^{-p} = 0, B_3e^p + B_4e^{-p} = 0.$$

That implies $B_1B_4 - B_2B_3 = \{b_0\omega_2 - a_0 + (b_1\omega_2 - a_1)[(g')^2 + g'']\}(a_2 - b_2\omega_1) - \{a_0 - b_0\omega_1 + (a_1 - b_1\omega_1)[(g')^2 + g'']\}(b_2\omega_2 - a_2) = 0$. By comparing the coefficients of the polynomials, we can obtain

$$\begin{aligned} (b_0\omega_2 - a_0)(a_2 - b_2\omega_1) &= (a_0 - b_0\omega_1)(b_2\omega_2 - a_2), \\ (b_1\omega_2 - a_1)(a_2 - b_2\omega_1) &= (a_1 - b_1\omega_1)(b_2\omega_2 - a_2). \end{aligned}$$

Hence, $\mathcal{A}_2 = \mathcal{A}_3 = 0$. Since the last column $(a_2, b_2)^T$ is nonzero, both other columns would then be proportional to it, contradicting $r(\mathcal{A}) = 2$. Thus, $\deg g = 1$.

Now, suppose that $\deg g = 1$. If $\mathcal{A}_2 = 0$, the nonzero last column and rank-two assumption imply $\mathcal{A}_3 \neq 0$. Dividing (2.6) by $\mathcal{A}_3$ gives conclusion (1), with $\lambda = 0$ and $B = p$.

We next consider $\mathcal{A}_2 \neq 0$. Let $g = \alpha z + \beta$, where $\alpha(\neq 0)$ and β are constants. If $\mathcal{A}_3 = 0$, then (2.6) becomes

$$\mathcal{A}_2 f'' = \frac{(b_2\omega_2 - a_2)e^{\alpha z+\beta+p} + (a_2 - b_2\omega_1)e^{\alpha z+\beta-p}}{\omega_2 - \omega_1}.$$

That is,

$$f(z) = \frac{e^{\alpha z+\beta}}{\alpha^2\mathcal{A}_2(\omega_2 - \omega_1)}\left[(b_2\omega_2 - a_2)e^p + (a_2 - b_2\omega_1)e^{-p}\right] + C_3z + C_4,$$

where C_3 and C_4 are constants. This is conclusion (2) of Theorem 2.1, in which $\lambda = 0$ and $B = p$.

If $\mathcal{A}_3 \neq 0$, then (2.6) becomes

$$\mathcal{A}_3 f + \mathcal{A}_2 f'' = \frac{(b_2\omega_2 - a_2)e^{\alpha z+\beta+p} + (a_2 - b_2\omega_1)e^{\alpha z+\beta-p}}{\omega_2 - \omega_1}. \tag{2.14}$$

Solving this equation gives

$$f(z) = C_1e^{kz} + C_2e^{-kz} + \frac{De^{\beta}}{\mathcal{A}_2} \cdot \begin{cases} \dfrac{e^{\alpha z}}{\alpha^2 - k^2}, & \alpha \neq \pm k, \\ \dfrac{ze^{\alpha z}}{2\alpha}, & \alpha = \pm k, \end{cases}$$

where $k = \sqrt{-\frac{\mathcal{A}_3}{\mathcal{A}_2}}$, $D = \frac{1}{\omega_2 - \omega_1}[(b_2\omega_2 - a_2)e^{p} + (a_2 - b_2\omega_1)e^{-p}]$, and C_1 and C_2 are constants. This is conclusion (3) of Theorem 2.1, in which $\lambda = 0$ and $B = p$.

Following the previous discussion, we now proceed to analyze the case in which p is not a constant. It is easy to see $\rho(B_i) < 1, i = 1, 2, 3, 4$. Applying [27, Theorem 1.51] to (2.9) gives that one of $p - p_c - g_c + g, p + p_c - g_c + g, -p - p_c - g_c + g,$ and $-p + p_c - g_c + g$ must be a constant.

Subcase 2.1. $p - p_c - g_c + g = b$ is a constant. Here $\rho(p + g) < 1$ and $(p + g) - (p + g)_c = b$, with $c \neq 0$. Thus, [18, Theorem 3.1] applies and shows that $p + g$ is a polynomial. Comparing degrees in its constant difference shows that its degree is at most one. Thus, $\deg g = \deg p$. (2.9) becomes $B_1e^{b} + B_3 + (B_2e^{b} + B_4e^{2p-2p_c})e^{-2p} = 0$, which implies

$$B_1e^{b} + B_3 = 0, B_2e^{b} + B_4e^{2p-2p_c} = 0. \tag{2.15}$$

If $\deg p = \deg g > 1$, then the fact that $g' + p'$ is a constant implies that $g' - p'$ is not a constant; otherwise, g' is a constant, which contradicts $\deg g > 1$. Therefore, (2.15) gives $B_2 = a_0 - b_0\omega_1 + (a_1 - b_1\omega_1)[(g'-p')^2+g''-p''] = 0$ and $B_4 = a_2-b_2\omega_1 = 0$. This implies $a_0-b_0\omega_1 = a_1-b_1\omega_1 = a_2-b_2\omega_1 = 0$, which contradicts $r(\mathcal{A}) = 2$. Thus, $\deg p = \deg g = 1$.

Let $g = \alpha z+\beta$ and $p = \lambda z+B$, where $\alpha(\neq 0), \lambda(\neq 0), B,$ and β are constants. If $\mathcal{A}_2 = 0$, the nonzero last column and rank-two assumption imply $\mathcal{A}_3 \neq 0$. Division of (2.6) by $\mathcal{A}_3$ gives conclusion (1) directly.

We next consider $\mathcal{A}_2 \neq 0$. If $\mathcal{A}_3 = 0$, then (2.6) becomes

$$\mathcal{A}_2f'' = \frac{(b_2\omega_2 - a_2)e^{\alpha z+\beta+\lambda z+B} + (a_2 - b_2\omega_1)e^{\alpha z+\beta-(\lambda z+B)}}{\omega_2 - \omega_1}. \tag{2.16}$$

That is,

$$f(z) = \frac{1}{\mathcal{A}_2(\omega_2 - \omega_1)} \cdot \begin{cases} \left(U_1e^{(\alpha+\lambda)z} + U_2e^{(\alpha-\lambda)z}\right) + C_3z + C_4, & \alpha \neq \pm\lambda, \\ \left(D_1e^{2\lambda z} + D_2z^2\right) + C_3z + C_4, & \alpha = \lambda, \\ \left(C_1z^2 + C_2e^{-2\lambda z}\right) + C_3z + C_4, & \alpha = -\lambda, \end{cases}$$

where C_3 and C_4 are constants and

$$\begin{aligned} U_1 &= \frac{(b_2\omega_2 - a_2)e^{\beta+B}}{(\alpha + \lambda)^2}, U_2 = \frac{(a_2 - b_2\omega_1)e^{\beta-B}}{(\alpha - \lambda)^2}, \\ D_1 &= \frac{(b_2\omega_2 - a_2)e^{\beta+B}}{4\lambda^2}, D_2 = \frac{(a_2 - b_2\omega_1)e^{\beta-B}}{2}, \\ C_1 &= \frac{(b_2\omega_2 - a_2)e^{\beta+B}}{2}, C_2 = \frac{(a_2 - b_2\omega_1)e^{\beta-B}}{4\lambda^2}. \end{aligned}$$

This is conclusion (2) of Theorem 2.1.

If $\mathcal{A}_3 \neq 0$, then (2.6) becomes

$$\mathcal{A}_3 f + \mathcal{A}_2 f'' = \frac{(b_2\omega_2 - a_2)e^{\alpha z+\beta+\lambda z+B} + (a_2 - b_2\omega_1)e^{\alpha z+\beta-(\lambda z+B)}}{\omega_2 - \omega_1}. \tag{2.17}$$

Solving this equation gives

$$f(z) = C_1 e^{\mu z} + C_2 e^{-\mu z} + \begin{cases} P_1 e^{k_1 z} + P_2 e^{k_2 z}, & k_1 \neq \pm\mu, k_2 \neq \pm\mu, \\ R_1 z e^{k_1 z} + P_2 e^{k_2 z}, & k_1 = \pm\mu, k_2 \neq \pm\mu, \\ P_1 e^{k_1 z} + R_2 z e^{k_2 z}, & k_1 \neq \pm\mu, k_2 = \pm\mu, \end{cases}$$

where

$$\begin{aligned} &\mu = \sqrt{-\frac{\mathcal{A}_3}{\mathcal{A}_2}}, k_1 = \alpha + \lambda, k_2 = \alpha - \lambda, \\ &P_1 = \frac{(b_2\omega_2 - a_2)e^{\beta+B}}{(\omega_2 - \omega_1)[\mathcal{A}_2(\alpha+\lambda)^2 + \mathcal{A}_3]}, P_2 = \frac{(a_2 - b_2\omega_1)e^{\beta-B}}{(\omega_2 - \omega_1)[\mathcal{A}_2(\alpha-\lambda)^2 + \mathcal{A}_3]}, \\ &R_1 = \frac{(b_2\omega_2 - a_2)e^{\beta+B}}{2\mathcal{A}_2(\omega_2 - \omega_1)(\alpha+\lambda)}, R_2 = \frac{(a_2 - b_2\omega_1)e^{\beta-B}}{2\mathcal{A}_2(\omega_2 - \omega_1)(\alpha-\lambda)}. \end{aligned}$$

This is conclusion (3) of Theorem 2.1.

Subcase 2.2. $p + p_c - g_c + g = b$ is a constant. The relation $p + p_c = g_c - g + b$ has a polynomial right-hand side, and $\rho(p) < 1$ and $c \neq 0$. Hence the conclusion of [18, Theorem 3.1] applies and p is a polynomial. Thus, $\deg p = \deg g - 1$ and (2.9) becomes $B_1 e^b + B_4 + B_2 e^{-2p+b} + B_3 e^{2p_c} = 0$. Then applying [27, Theorem 1.51] to this equation implies $B_2 = B_3 = 0$ and $B_1 e^b + B_4 = 0$. Therefore, $a_0 - b_0\omega_1 = a_1 - b_1\omega_1 = b_1\omega_2 - a_1 = b_2\omega_2 - a_2 = 0$. This gives $\mathcal{A}_2 = \mathcal{A}_1 = 0$, so $\mathcal{A}_3 \neq 0$ by $r(\mathcal{A}) = 2$. Substituting this into (2.6) gives

$$f(z) = \frac{(a_2 - b_2\omega_1)e^{g-p}}{\mathcal{A}_3(\omega_2 - \omega_1)}.$$

This is conclusion (4) of Theorem 2.1.

Subcase 2.3. $-p - p_c - g_c + g = b$ is a constant. As in Subcase 2.2, the relation $p + p_c = g - g_c - b$, together with $\rho(p) < 1$ and $c \neq 0$, implies that p is a polynomial. Comparing leading terms gives $\deg p = \deg g - 1$. Then (2.9) becomes $B_1 e^{2p+b} + B_2 e^b + B_3 + B_4 e^{-2p_c} = 0$. Applying [27, Theorem 1.51] to this equation implies $B_1 = B_4 = 0$ and $B_2 e^b + B_3 = 0$. Therefore, $a_2 - b_2\omega_1 = a_1 - b_1\omega_1 = b_0\omega_2 - a_0 = b_1\omega_2 - a_1 = 0$. This gives $\mathcal{A}_2 = \mathcal{A}_1 = 0$, so $\mathcal{A}_3 \neq 0$ by $r(\mathcal{A}) = 2$. Substituting this into (2.6) gives

$$f(z) = \frac{(b_2\omega_2 - a_2)e^{g(z)+p(z)}}{\mathcal{A}_3(\omega_2 - \omega_1)}.$$

This is conclusion (4) of Theorem 2.1.

Subcase 2.4. $-p + p_c - g_c + g = b$ is a constant. Here $\rho(g-p) < 1$ and $c \neq 0$, so [18, Theorem 3.1] applied to $(g-p) - (g-p)_c = b$ gives a polynomial $g-p$ of degree at most one. Hence p is a polynomial. If $\deg g > 1$, the second coefficient group in the resulting exponential identity gives $B_1 = B_3 = 0$. Since $g' + p'$ is then non-constant, these identities force $b_j\omega_2 - a_j = 0$ for $j = 0, 1, 2$, contradicting rank two. Consequently, g and p are affine. Substituting them into (2.6) gives conclusions (1), (2), or (3), according as $\mathcal{A}_2 = 0$, $\mathcal{A}_2 \neq 0 = \mathcal{A}_3$, or $\mathcal{A}_2\mathcal{A}_3 \neq 0$, exactly as in Subcase 2.1.

This completes the stated argument for Theorem 2.1. □

3. Quadratic trinomial Fermat-type q-difference differential equations

In this section, the existence of solutions to the Fermat-type q-difference differential equation $(Q_1(f))^2 + 2\omega Q_1(f)Q_2(f) + (Q_2(f))^2 = e^{2g(z)}$ is studied, thereby answering the parts related to Theorem C in Question 1 and Question 2.

Theorem 3.1. *Assume that Q_1 and Q_2 are defined in* (1.2)*, q and ω are nonzero constants such that $q^n \neq \pm 1, n \in \mathbb{N}^+$, $\omega \neq \pm 1$ and $\omega_1 = -\omega + \sqrt{\omega^2 - 1}, \omega_2 = -\omega - \sqrt{\omega^2 - 1}$, g is a non-constant polynomial, and $r(\mathcal{A}) = 2$. If the equation*

$$(Q_1(f))^2 + 2\omega Q_1(f)Q_2(f) + (Q_2(f))^2 = e^{2g(z)} \tag{3.1}$$

admits an entire solution f with $\rho_2(f) < \infty$, then $g = \alpha z + \beta$, where $\alpha(\neq 0)$ and β are constants, and one of the following cases holds:

(1) $\mathcal{A}_2 = 0$, $f(z) = C_1 e^{(\lambda+\alpha)z} + C_2 e^{(-\lambda+\alpha)z}$, $C_1 = \frac{(b_2\omega_2 - a_2)e^{B+\beta}}{(\omega_2-\omega_1)\mathcal{A}_3}$ and $C_2 = \frac{(a_2 - b_2\omega_1)e^{-B+\beta}}{(\omega_2-\omega_1)\mathcal{A}_3}$ or $C_1 = \frac{(b_1\omega_2 - a_1)e^{B+\beta}}{(\omega_2-\omega_1)\mathcal{A}_1}$ and $C_2 = \frac{(a_1 - b_1\omega_1)e^{-B+\beta}}{(\omega_2-\omega_1)\mathcal{A}_1}$, where λ and B are arbitrary constants.

(2) $\mathcal{A}_2 \neq 0$ and $\mathcal{A}_3 = 0$, $f(z) = \frac{(b_2\omega_2 - a_2)e^{(\alpha+\lambda)z+\beta+B}}{\mathcal{A}_2(\omega_2-\omega_1)(\alpha+\lambda)^2} + C_1 z + C_2$ or $f(z) = \frac{(a_2 - b_2\omega_1)e^{(\alpha-\lambda)z+\beta-B}}{\mathcal{A}_2(\omega_2-\omega_1)(\alpha-\lambda)^2} + C_1 z + C_2$, where λ, B, C_1, and C_2 are constants and $\alpha - \lambda = q(\alpha + \lambda)$ or $q(\alpha - \lambda) = \alpha + \lambda$, respectively.

(3) $\mathcal{A}_2 \neq 0$ and $\mathcal{A}_3 \neq 0$,

$$f(z) = C_1 e^{\mu z} + C_2 e^{-\mu z} + \begin{cases} P_1 e^{k_1 z} + P_2 e^{k_2 z}, & k_1 \neq \pm\mu, k_2 \neq \pm\mu, \\ R_1 z e^{k_1 z} + P_2 e^{k_2 z}, & k_1 = \pm\mu, k_2 \neq \pm\mu, \\ P_1 e^{k_1 z} + R_2 z e^{k_2 z}, & k_1 \neq \pm\mu, k_2 = \pm\mu, \end{cases}$$

where C_1, C_2, λ, and B are constants, $\mu = \sqrt{-\frac{\mathcal{A}_3}{\mathcal{A}_2}}, k_1 = \alpha + \lambda, k_2 = \alpha - \lambda$,

$$P_1 = \frac{(b_2\omega_2 - a_2)e^{\beta+B}}{(\omega_2-\omega_1)[\mathcal{A}_2(\alpha+\lambda)^2 + \mathcal{A}_3]}, P_2 = \frac{(a_2 - b_2\omega_1)e^{\beta-B}}{(\omega_2-\omega_1)[\mathcal{A}_2(\alpha-\lambda)^2 + \mathcal{A}_3]},$$
$$R_1 = \frac{(b_2\omega_2 - a_2)e^{\beta+B}}{2\mathcal{A}_2(\omega_2-\omega_1)(\alpha+\lambda)}, R_2 = \frac{(a_2 - b_2\omega_1)e^{\beta-B}}{2\mathcal{A}_2(\omega_2-\omega_1)(\alpha-\lambda)}.$$

Remark 3.1. Theorem 3.1 states a nonexistence conclusion for (3.1) when $\deg g > 1$ under the assumption $\rho_2(f) < \infty$. It is intended to address the parts of Questions 1 and 2 corresponding to Theorem C, including the additional restriction $\rho(f) \leq 1$ in assertion (ii) of that theorem.

Remark 3.2. Theorem 3.1 improves the condition $\rho(f) < \infty$ in Theorem C to $\rho_2(f) < \infty$.

The following example illustrates the role of the restriction on q.

Example 3.1. [3, Example 1.1] Let $q = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, so that $q^3 = 1$, and put

$$f(z) = \frac{1}{\sqrt{6}}e^{z^3/2}, \qquad g(z) = \frac{1}{2}z^3, \qquad Q_1(f) = f(qz), \quad Q_2(f) = f(z).$$

Then $f(qz) = f(z)$, and hence,

$$Q_1(f)^2 + 4Q_1(f)Q_2(f) + Q_2(f)^2 = 6f(z)^2 = e^{z^3} = e^{2g(z)}.$$

This example does not satisfy the restrictions on q in Theorem 3.1.

Proof of Theorem 3.1. Factoring (3.1) as $(Q_1 - \omega_1 Q_2)(Q_1 - \omega_2 Q_2) = e^{2g}$ gives two zero-free entire factors e^{g+p} and e^{g-p}, where p is an entire function. Thus, $\rho(p) < \infty$. Taking $b_jQ_1 - a_jQ_2$, for $j = 0, 1, 2$, gives

$$-\mathcal{A}_1 f'' - \mathcal{A}_3 f_q = \frac{(b_0\omega_2 - a_0)e^{g+p} + (a_0 - b_0\omega_1)e^{g-p}}{\omega_2 - \omega_1}, \tag{3.2}$$

$$\mathcal{A}_1 f - \mathcal{A}_2 f_q = \frac{(b_1\omega_2 - a_1)e^{g+p} + (a_1 - b_1\omega_1)e^{g-p}}{\omega_2 - \omega_1}, \tag{3.3}$$

$$\mathcal{A}_3 f + \mathcal{A}_2 f'' = \frac{(b_2\omega_2 - a_2)e^{g+p} + (a_2 - b_2\omega_1)e^{g-p}}{\omega_2 - \omega_1}, \tag{3.4}$$

where p is entire with $\rho(p) < \infty$, and $f_q = f(qz)$. Throughout this proof, $h_q^{(j)}$ means $h^{(j)}(qz)$, so $(h_q)^{(j)} = q^j h_q^{(j)}$. Differentiating equation (3.3) twice yields

$$\mathcal{A}_1 f'' - q^2\mathcal{A}_2 f_q'' = \frac{(b_1\omega_2 - a_1)[(g' + p')^2 + g'' + p'']e^{g+p} + (a_1 - b_1\omega_1)[(g' - p')^2 + g'' - p'']e^{g-p}}{\omega_2 - \omega_1}. \tag{3.5}$$

Adding (3.2) and (3.5) gives

$$-\mathcal{A}_3 f_q - q^2\mathcal{A}_2 f_q'' = \frac{(b_1\omega_2 - a_1)[(g' + p')^2 + g'' + p'']e^{g+p} + (a_1 - b_1\omega_1)[(g' - p')^2 + g'' - p'']e^{g-p}}{\omega_2 - \omega_1} + \frac{(b_0\omega_2 - a_0)e^{g+p} + (a_0 - b_0\omega_1)e^{g-p}}{\omega_2 - \omega_1}. \tag{3.6}$$

Taking the q-shift of (3.4) and combining it with (3.6) gives

$$(1 - q^2)\mathcal{A}_2 f_q'' = \frac{(b_2\omega_2 - a_2)e^{g_q+p_q} + (a_2 - b_2\omega_1)e^{g_q-p_q}}{\omega_2 - \omega_1} + \frac{(b_0\omega_2 - a_0)e^{g+p} + (a_0 - b_0\omega_1)e^{g-p}}{\omega_2 - \omega_1} \tag{3.7}$$

$$+\frac{(b_1\omega_2-a_1)[(g'+p')^2+g''+p'']e^{g+p}+(a_1-b_1\omega_1)[(g'-p')^2+g''-p'']e^{g-p}}{\omega_2-\omega_1},$$

$$(q^2-1)\mathcal{A}_3 f_q=\frac{(b_1\omega_2-a_1)[(g'+p')^2+g''+p'']e^{g+p}+(a_1-b_1\omega_1)[(g'-p')^2+g''-p'']e^{g-p}}{\omega_2-\omega_1} \tag{3.8}$$
$$+\frac{(b_0\omega_2-a_0)e^{g+p}+(a_0-b_0\omega_1)e^{g-p}}{\omega_2-\omega_1}+q^2\frac{(b_2\omega_2-a_2)e^{g_q+p_q}+(a_2-b_2\omega_1)e^{g_q-p_q}}{\omega_2-\omega_1}.$$

If $|a_2|+|b_2|=0$, then $\mathcal{A}_1\neq 0$ by $r(\mathcal{A})=2$ and $\mathcal{A}_2=\mathcal{A}_3=0$. Thus, the remaining proof is similar to that in Case 1 in the proof of Theorem 2.1, and we get Theorem 3.1-(1).

Next, we consider $|a_2|+|b_2|\neq 0$. If $\mathcal{A}_3=\mathcal{A}_2=0$, then $|a_2|+|b_2|\neq 0$ implies $\mathcal{A}_1=0$, which is a contradiction. Thus, we discuss three cases: $\mathcal{A}_3=0,\mathcal{A}_2\neq 0$; $\mathcal{A}_3\neq 0,\mathcal{A}_2=0$; and $\mathcal{A}_3\neq 0,\mathcal{A}_2\neq 0$.

Case 1. $\mathcal{A}_3=0,\mathcal{A}_2\neq 0$. (3.8) reduces to

$$B_1e^{p}+B_2e^{-p}+B_3e^{p_q+g_q-g}+B_4e^{-p_q+g_q-g}=0, \tag{3.9}$$

where

$$\begin{aligned}B_1&=b_0\omega_2-a_0+(b_1\omega_2-a_1)[(g'+p')^2+g''+p''],\\ B_2&=a_0-b_0\omega_1+(a_1-b_1\omega_1)[(g'-p')^2+g''-p''],\\ B_3&=q^2(b_2\omega_2-a_2),\\ B_4&=q^2(a_2-b_2\omega_1).\end{aligned}$$

We claim that it is impossible for $B_3=0$ and $B_4=0$ to hold simultaneously. Otherwise, $\omega_1=\omega_2$ or $a_2=b_2=0$, which is a contradiction.

If p is a constant, then (3.9) becomes

$$B_1e^{p}+B_2e^{-p}+\left[B_3e^{p}+B_4e^{-p}\right]e^{g_q-g}=0.$$

Since g_q-g is non-constant, by $q^n\neq 1$, $B_1e^{p}+B_2e^{-p}=0$ and $B_3e^{p}+B_4e^{-p}=0$. Eliminating e^{p} and e^{-p} gives

$$0=B_1B_4-B_2B_3=-q^2(\omega_2-\omega_1)\{\mathcal{A}_3+\mathcal{A}_2[(g')^2+g'']\}.$$

Here $\mathcal{A}_3=0$ and $\mathcal{A}_2\neq 0$, so $(g')^2+g''=0$. This is impossible for a non-constant polynomial g: The left-hand side has a nonzero constant value if $\deg g=1$, and a nonzero leading term of degree $2\deg g-2$ otherwise. Therefore, p is non-constant.

Since $\rho(B_i)<\infty, i=1,2,3,4$, applying [27, Theorem 1.51] to (3.9) gives that one of $p-p_q-g_q+g, p+p_q-g_q+g, -p-p_q-g_q+g,$ and $-p+p_q-g_q+g$ must be a polynomial. Therefore, from [17, Lemma 5], we find that p is a polynomial and $\deg g=\deg p$. Now, $B_i, i=1,2,3,4$, are polynomials; applying [27, Theorem 1.51] to (3.9) again gives that one of $p-p_q-g_q+g, p+p_q-g_q+g, -p-p_q-g_q+g,$ and $-p+p_q-g_q+g$ must be a constant.

Subcase 1.1. $p-p_q-g_q+g=b$ is a constant. Then (3.9) becomes

$$B_1e^{b}+B_3+B_2e^{-2p+b}+B_4e^{-2p_q}=0.$$

By the second main theorem, we have $B_4=q^2(a_2-b_2\omega_1)=0$. Substituting this and $\mathcal{A}_3=0$ into (3.4) implies $\mathcal{A}_2f''=\frac{(b_2\omega_2-a_2)e^{g+p}}{\omega_2-\omega_1}$. Since $p-p_q-g_q+g=p+g-(p+g)_q=b$, then $p+g$ is a constant. Therefore, $\mathcal{A}_2f''=\frac{(b_2\omega_2-a_2)e^{g+p}}{\omega_2-\omega_1}$ is a constant. This contradicts the fact that f is transcendental.

Subcase 1.2. $p + p_q - g_q + g = b$ is a constant. Then (3.9) becomes

$$B_1 e^b + B_2 e^{b-2p} + B_3 e^{2p_q} + B_4 = 0.$$

Because p is a non-constant polynomial and $q^n \neq -1$, the three exponents 0, $-2p$, and $2p_q$ have pairwise non-constant differences. By Borel's uniqueness theorem [27, Theorem 1.51], applied to the exponential equation, we obtain

$$B_2 = B_3 = 0, \qquad B_1 e^b + B_4 = 0.$$

Let $h = g' - p'$. From $p + p_q - g_q + g = b$, we have $g' + p' = qh_q$. Suppose, to the contrary, that $\deg h \geq 1$. Then $B_2 = 0$ forces

$$a_1 - b_1\omega_1 = a_0 - b_0\omega_1 = 0.$$

Also, $B_3 = 0$ gives $a_2 = b_2\omega_2$. Since $\mathcal{A}_2 \neq 0$, this implies $b_2 \neq 0$ and $b_1\omega_2 - a_1 \neq 0$. On the other hand, $B_1 e^b + B_4 = 0$ implies that B_1 is constant. However, by the definition of B_1,

$$B_1 = b_0\omega_2 - a_0 + (b_1\omega_2 - a_1)[(g' + p')^2 + g'' + p''],$$

and

$$(g' + p')^2 + g'' + p'' = q^2(h^2 + h')_q$$

is non-constant. Since $b_1\omega_2 - a_1 \neq 0$, this shows that B_1 is non-constant. This contradicts the fact that B_1 is constant. Hence h is constant, and so is $g' - p'$. Consequently, g and p are affine; since both are non-constant, $\deg g = \deg p = 1$. Let $g = \alpha z + \beta$ and $p = \lambda z + B$, where $\alpha(\neq 0), \lambda(\neq 0), B$, and β are constants. Substituting $\mathcal{A}_3 = 0, g, p$, and $B_3 = q^2(b_2\omega_2 - a_2) = 0$ into (3.4) gives $\mathcal{A}_2 f'' = \frac{(a_2 - b_2\omega_1)e^{\alpha z+\beta-(\lambda z+B)}}{\omega_2-\omega_1}$. It is easy to get

$$f(z) = \frac{(a_2 - b_2\omega_1)e^{(\alpha-\lambda)z+\beta-B}}{\mathcal{A}_2(\omega_2 - \omega_1)(\alpha - \lambda)^2} + C_1 z + C_2,$$

where C_1 and C_2 are constants. This is conclusion (2) of Theorem 3.1.

Subcase 1.3. $-p - p_q - g_q + g = b$ is a constant. Then (3.9) becomes

$$B_1 e^{b+2p} + B_2 e^b + B_3 + B_4 e^{-2p_q} = 0.$$

Applying [27, Theorem 1.51] to this equation implies $B_1 = B_4 = 0$ and $B_2 e^b + B_3 = 0$. $-p - p_q - g_q + g = b$ implies $g' - p' = q(g' + p')_q$. Therefore, as in the case where $p + p_q - g_q + g$ is a constant, we can get $g = \alpha z + \beta$ and $p = \lambda z + B$, where $\alpha(\neq 0), \lambda(\neq 0), B$, and β are constants. Substituting $\mathcal{A}_3 = 0, g, p$, and $B_4 = 0$ into (3.4) gives $\mathcal{A}_2 f'' = \frac{(b_2\omega_2 - a_2)e^{\alpha z+\beta+(\lambda z+B)}}{\omega_2-\omega_1}$. Thus,

$$f(z) = \frac{(b_2\omega_2 - a_2)e^{(\alpha+\lambda)z+\beta+B}}{\mathcal{A}_2(\omega_2 - \omega_1)(\alpha + \lambda)^2} + C_1 z + C_2,$$

where C_1 and C_2 are constants. This is conclusion (2) of Theorem 3.1.

Subcase 1.4. $-p+p_q-g_q+g=b$ is a constant. Then we find that $g-p$ is a constant and (3.9) becomes

$$B_1e^{b+2p}+B_2e^{b}+B_3e^{2p_q}+B_4=0.$$

This implies $B_1=B_3=0$. Then using the same methods as in the case where $p-p_q-g_q+g=b$ is a constant, we also get a contradiction.

Case 2. $\mathcal{A}_3\neq 0,\mathcal{A}_2=0$. (3.7) reduces to

$$D_1e^{p}+D_2e^{-p}+D_3e^{p_q+g_q-g}+D_4e^{-p_q+g_q-g}=0, \tag{3.10}$$

where

$$\begin{aligned}
D_1&=b_0\omega_2-a_0+(b_1\omega_2-a_1)[(g'+p')^2+g''+p''],\\
D_2&=a_0-b_0\omega_1+(a_1-b_1\omega_1)[(g'-p')^2+g''-p''],\\
D_3&=b_2\omega_2-a_2,\\
D_4&=a_2-b_2\omega_1.
\end{aligned}$$

We claim that it is impossible for $D_3=0$ and $D_4=0$ to hold simultaneously; otherwise, $\omega_1=\omega_2$ or $a_2=b_2=0$, which is a contradiction. As in Case 1, we conclude that p is non-constant.

Applying [27, Theorem 1.51] to (3.10) gives that one of $p-p_q-g_q+g, p+p_q-g_q+g, -p-p_q-g_q+g$, and $-p+p_q-g_q+g$ must be a polynomial. Therefore, from [17, Lemma 5], we find that p is a polynomial and $\deg g=\deg p$. Applying [27, Theorem 1.51] to (3.10) again shows that $p-p_q-g_q+g, p+p_q-g_q+g, -p-p_q-g_q+g, -p+p_q-g_q+g$ must be a constant. Using the same proof method as in Case 1, we can show that $\deg g=1$. By $\mathcal{A}_2=0$, (3.4) shows that f is of finite order. Since $\deg g=1$ and $\mathcal{A}_2=0$, by Theorem C-(i), we get Theorem 3.1-(1).

Case 3. $\mathcal{A}_3\neq 0,\mathcal{A}_2\neq 0$. Differentiate (3.8) twice, multiply by $\mathcal{A}_2$, and add $q^2\mathcal{A}_3$ times (3.7). The terms involving f_q'' cancel. After division by $q^2e^g/(\omega_2-\omega_1)$, this gives

$$E_1e^{p}+E_2e^{-p}+E_3e^{(p+g)_q-g}+E_4e^{(g-p)_q-g}=0, \tag{3.11}$$

where

$$\begin{aligned}
E_1&=q^{-2}\mathcal{A}_2(b_1\omega_2-a_1)[u''''+4u'u'''+3(u'')^2+6(u')^2u''+(u')^4]\\
&\quad+[q^{-2}\mathcal{A}_2(b_0\omega_2-a_0)+\mathcal{A}_3(b_1\omega_2-a_1)][u''+(u')^2]+\mathcal{A}_3(b_0\omega_2-a_0),\\
E_2&=q^{-2}\mathcal{A}_2(a_1-b_1\omega_1)[v''''+4v'v'''+3(v'')^2+6(v')^2v''+(v')^4]\\
&\quad+[q^{-2}\mathcal{A}_2(a_0-b_0\omega_1)+\mathcal{A}_3(a_1-b_1\omega_1)][v''+(v')^2]+\mathcal{A}_3(a_0-b_0\omega_1),\\
E_3&=\mathcal{A}_2q^2(b_2\omega_2-a_2)[u_q''+(u_q')^2]+\mathcal{A}_3(b_2\omega_2-a_2),\\
E_4&=\mathcal{A}_2q^2(a_2-b_2\omega_1)[v_q''+(v_q')^2]+\mathcal{A}_3(a_2-b_2\omega_1), u=g+p, v=g-p.
\end{aligned}$$

Subcase 3.1. $E_i\equiv 0, i=1,2,3,4$. Since $\omega_1\neq\omega_2$, it is impossible for $a_2-b_2\omega_1=0$ and $b_2\omega_2-a_2=0$ to hold simultaneously.

If $b_2\omega_2-a_2\neq 0$ and $a_2-b_2\omega_1\neq 0$, then $E_3=0$ implies $u_q''+(u_q')^2=c_1$, where c_1 is a nonzero constant. Thus, $2T(r,u_q')\leq m(r,\frac{u_q''}{u_q'})+m(r,u_q')+O(1)$. Therefore, u_q' is a polynomial. $u_q''+(u_q')^2=c_1$ means u_q' is a constant. Similarly, $E_4=0$ implies v_q' is a constant. Thus, $u_q'+v_q'=2g_q'$ and $u_q'-v_q'=2p_q'$

are constants. $E_3 = E_4 = 0$ gives $2p_q'' + 4p_q'g_q' = 0$. Combining this and $\deg g > 0$, we get $p_q' = 0$. Thus, p is a constant. Let $g = \alpha z + \beta$, where $\alpha(\neq 0)$ and β are constants. Substituting p and g into (3.4), we also get (2.14). Solving Equation (2.14) gives conclusion (3) of Theorem 3.1, in which $\lambda = 0$ and $p = B$.

If $b_2\omega_2 - a_2 = 0$, then $a_2 - b_2\omega_1 \neq 0$ by $\omega_1 \neq \omega_2$. If $a_2 - b_2\omega_1 = 0$, then $b_2\omega_2 - a_2 \neq 0$ by $\omega_1 \neq \omega_2$. Without loss of generality, consider the case $b_2\omega_2 - a_2 = 0$ and $a_2 - b_2\omega_1 \neq 0$. From $\mathcal{A}_2 \neq 0$ and $b_2\omega_2 - a_2 = 0$, we get $b_1\omega_2 - a_1 \neq 0$. The identity $E_4 = 0$ gives

$$v'' + (v')^2 = -\frac{\mathcal{A}_3}{q^2\mathcal{A}_2}.$$

Thus, e^v solves a nontrivial second-order homogeneous linear differential equation with constant coefficients. It has order at most one; since it is zero-free, Hadamard factorization implies that v is affine. Similarly, writing $c_0 = b_0\omega_2 - a_0$ and $c_1 = b_1\omega_2 - a_1 \neq 0$, we see that the identity $E_1 = 0$ is equivalent to

$$(q^{-2}\mathcal{A}_2D^2 + \mathcal{A}_3)(c_1D^2 + c_0)e^u = 0, \qquad D = \frac{d}{dz}.$$

This is a nontrivial constant-coefficient differential equation. Again e^u has order at most one and is zero-free, so u is affine. Therefore, $u' + v' = 2g'$ and $u' - v' = 2p'$ are constants. Let $g = \alpha z + \beta$ and $p = \lambda z + B$, where $\alpha(\neq 0), \lambda(\neq 0), B$, and β are constants. Substituting p and g into (3.4), we also get (2.17), in which $b_2\omega_2 - a_2 = 0$. Solving Eq (2.17) gives conclusion (3) of Theorem 3.1, in which $b_2\omega_2 - a_2 = 0$.

Subcase 3.2. At least one of the E_i is not zero. Then (3.11) shows that at least two of the E_i are not zero, $i = 1, 2, 3, 4$.

First, consider the case where p is a constant. (3.11) gives

$$E_1e^p + E_2e^{-p} = 0, E_3e^p + E_4e^{-p} = 0. \tag{3.12}$$

If g' is not constant, then (3.12) yields

$$(b_1\omega_2 - a_1)e^p + (a_1 - b_1\omega_1)e^{-p} = 0, \quad (b_2\omega_2 - a_2)e^p + (a_2 - b_2\omega_1)e^{-p} = 0.$$

This implies $\mathcal{A}_2 = 0$. It contradicts the assumption $\mathcal{A}_2 \neq 0$. Thus, $\deg g = 1$. Let $g = \alpha z + \beta$, where $\alpha(\neq 0)$ and β are constants. Substituting these affine functions into (3.4) gives (2.14); solving it as in Subcase 3.1 gives conclusion (3) of Theorem 3.1, with $\lambda = 0$ and $B = p$.

Next, we consider the case where p is non-constant. Applying [27, Theorem 1.51] and [17, Lemma 5] to (3.11) shows that one of $p - p_q - g_q + g, p + p_q - g_q + g, -p - p_q - g_q + g,$ and $-p + p_q - g_q + g$ is a constant, p is a polynomial, and $\deg g = \deg p$.

If $p - p_q - g_q + g = b$ is a constant, then (3.11) becomes

$$E_1e^b + E_3 + E_2e^{-2p+b} + E_4e^{-2p_q} = 0.$$

This gives $E_1e^b + E_3 = E_2 = E_4 = 0$. If $a_2 - b_2\omega_1 \neq 0$, then $E_4 = 0$ implies that $g' - p'$ is a constant. If $a_2 - b_2\omega_1 = 0$, by $a_2b_1 - a_1b_2 = -\mathcal{A}_2 \neq 0$, then $a_1 - b_1\omega_1 \neq 0$. Thus, $E_2 \equiv 0$ implies that $g' - p'$ is a constant. Since $p - p_q - g_q + g = p + g - (p + g)_q = b$, $p' + g'$ is also a constant.

This means $\deg p = \deg g = 1$. Let $g = \alpha z + \beta$ and $p = \lambda z + B$, where $\alpha(\neq 0), \lambda(\neq 0), B,$ and β are constants. Substituting p and g into (3.4), we also get (2.17). Solving this equation gives conclusion (3) of Theorem 3.1.

If $p + p_q - g_q + g = b$ is a constant, then (3.11) becomes

$$E_1 e^b + E_2 e^{b-2p} + E_3 e^{2p_q} + E_4 = 0.$$

Applying [27, Theorem 1.51] to this equation implies $E_2 = E_3 = 0$ and $E_1 e^b + E_4 = 0$. $p + p_q - g_q + g = b$ implies $g' + p' = q(g' - p')_q$. If $a_2 - b_2\omega_2 \neq 0$, then $E_3 = 0$ implies that $g' + p'$ is a constant. Thus, $g' + p' = q(g' - p')_q$ implies $g' - p'$ is also a constant. If $a_2 - b_2\omega_2 = 0$, then $a_1 - b_1\omega_2 \neq 0$ by $a_2 b_1 - a_1 b_2 = -\mathcal{A}_2 \neq 0$. Thus, $E_1 e^b + E_4 = 0$ implies that $g' - p'$ is a constant. $p + p_q - g_q + g = b$ implies $g' + p' = q(g' - p')_q$ is also a constant. In summary, $\deg p = \deg g = 1$. Let $g = \alpha z + \beta$ and $p = \lambda z + B$, where $\alpha(\neq 0), \lambda(\neq 0), B,$ and β are constants. Substituting p and g into (3.4), we also get (2.17). Solving this equation gives conclusion (3) of Theorem 3.1.

For the remaining two relations, replace p by $-p$ and interchange ω_1 and ω_2. This interchanges the two unshifted exponentials and the two shifted exponentials in (3.11); the corresponding coefficients are interchanged up to a common sign. The minors $\mathcal{A}_2$ and $\mathcal{A}_3$, the rank assumption, and the restrictions on q are unchanged. Thus, the preceding two calculations apply with the two signs exchanged and lead to the same displayed solution forms in conclusion (3).

This completes the stated argument for Theorem 3.1.

□

4. Examples

Example 4.1. Let $c = 2\pi i$, $\omega = 2$, and

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad g(z) = z + \frac{1}{2}\log 6, \qquad f(z) = e^z.$$

The matrix has rank two, $\mathcal{A}_2 = 0$ and $\mathcal{A}_3 = 1$. Since $f_c = e^z$, we have $L_1(f) = L_2(f) = e^z$, whence

$$L_1(f)^2 + 4L_1(f)L_2(f) + L_2(f)^2 = 6e^{2z} = e^{2g(z)}.$$

Thus, f solves (2.1) and illustrates the affine $\mathcal{A}_2 = 0$ alternative in Theorem 2.1.

Example 4.2. Let $c = 2\pi i$, $\omega = 2$, and

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \qquad g(z) = z + \log 2, \qquad f(z) = e^z.$$

Here $r(\mathcal{A}) = 2$, $\mathcal{A}_2 = -2$, and $\mathcal{A}_3 = 0$. The identities $f'' = f_c = e^z$ give

$$L_1(f) = f'' + f_c = 2e^z, \qquad L_2(f) = f'' - f_c = 0.$$

Consequently,

$$L_1(f)^2 + 4L_1(f)L_2(f) + L_2(f)^2 = 4e^{2z} = e^{2g(z)}.$$

Example 4.3. Let $c = 2\pi i$, $\omega = \pi/(2i)$, and

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}, \qquad g(z) = z + \log 2, \qquad f(z) = ze^z.$$

Then $\mathcal{A}_2 = -1$, $\mathcal{A}_3 = 1$, and $r(\mathcal{A}) = 2$. Direct differentiation and translation yield

$$f''(z) = (z+2)e^z, \qquad f_c(z) = (z + 2\pi i)e^z,$$

and therefore,

$$L_1(f) = f - f_c = -2\pi i e^z, \qquad L_2(f) = f - f'' = -2e^z.$$

Since $2\omega = \pi/i$,

$$\begin{aligned} L_1(f)^2 + 2\omega L_1(f)L_2(f) + L_2(f)^2 &= \left(-4\pi^2 + 4\pi^2 + 4\right)e^{2z} \\ &= 4e^{2z} = e^{2g(z)}. \end{aligned}$$

Example 4.4. Let $q = 2$, $\omega = 2$, and

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad g(z) = z, \qquad f(z) = \frac{1}{\sqrt{6}}e^z.$$

Then $r(\mathcal{A}) = 2$, $\mathcal{A}_1 = 1$, and $\mathcal{A}_2 = 0$.

$$Q_1(f) = f = \frac{e^z}{\sqrt{6}}, \qquad Q_2(f) = f'' = \frac{e^z}{\sqrt{6}},$$

so that

$$Q_1(f)^2 + 4Q_1(f)Q_2(f) + Q_2(f)^2 = e^{2z} = e^{2g(z)}.$$

Here the polynomial g is z.

Example 4.5. Let $q \neq 0$ satisfy $q^n \neq \pm 1$ for every positive integer n, let $\omega = 2$, and

$$\mathcal{A} = \begin{pmatrix} 1 & -\frac{1}{4} & 2 \\ 1 & -\frac{1}{4} & 1 \end{pmatrix}, \qquad g(z) = 2qz + \frac{1}{2}\log 13, \qquad f(z) = e^{2z}.$$

The minors are $\mathcal{A}_2 = 1/4$ and $\mathcal{A}_3 = -1$, so the matrix has rank two. Since $f'' = 4e^{2z}$ and $f_q = e^{2qz}$,

$$Q_1(f) = f - \frac{1}{4}f'' + 2f_q = 2e^{2qz}, \qquad Q_2(f) = f - \frac{1}{4}f'' + f_q = e^{2qz}.$$

Thus,

$$Q_1(f)^2 + 4Q_1(f)Q_2(f) + Q_2(f)^2 = (4 + 8 + 1)e^{4qz} = e^{2g(z)}.$$

5. Conclusions

We have studied quadratic trinomial Fermat-type equations containing second derivatives together with either a translation or a dilation. The differential-difference equation is treated under the growth condition $\rho_2(f) < 1$, whereas the q-difference equation is considered under a finite hyper-order condition and restrictions on q.

For the differential-difference equation, we removed the additional condition $\rho(f) \leq 1$ imposed by Gong and Yang [7] and determined all possible forms of entire solutions with $\rho_2(f) < 1$. In particular, when the exponential term is replaced by a polynomial g with $\deg g > 1$, we obtained explicit solution forms, thereby extending the results of Gong and Yang to the higher-degree polynomial case. For the q-difference equation, we also removed the condition $\rho(f) \leq 1$ and proved a nonexistence result when $\deg g > 1$ under the assumption $\rho_2(f) < \infty$, showing that the polynomial g must be affine in this setting.

The growth assumptions and the restrictions on q delimit the scope of the present work. Solutions outside these growth classes, the cases $q^n = 1$ or $q^n = -1$ for some positive integer n, and analogous equations involving higher derivatives are left for future investigation.

Author contributions

Jiayi Dan: Formal analysis, methodology, writing–original draft; Xuxu Xiang: Conceptualization, supervision, editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Funding

This research was supported by the Graduate Research Fund Project of Guizhou Province (2025YJSKYJJ107) and the National Natural Science Foundation of China (Grant Nos. 12261023 and 11861023).

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. I. N. Baker, On a class of meromorphic functions, *Proc. Amer. Math. Soc.*, **17** (1966), 819–822. https://doi.org/10.1090/S0002-9939-1966-0197732-X
2. W. Chen, Q. Han, J. B. Liu, On Fermat Diophantine functional equations, little Picard theorem and beyond, *Aequat. Math.*, **93** (2019), 425–432. https://doi.org/10.1007/s00010-018-0614-z

3. Z. G. Gao, L. Y. Gao, M. L. Liu, Entire solutions of two certain types of quadratic trinomial q-difference differential equations, *AIMS Math.*, **8** (2023), 27659–27669. https://doi.org/10.3934/math.20231415
4. G. G. Gundersen, Meromorphic solutions of $f^6 + g^6 + h^6 = 1$, *Analysis*, **18** (1998), 285–290. https://doi.org/10.1524/anly.1998.18.3.285
5. G. G. Gundersen, Meromorphic solutions of $f^5 + g^5 + h^5 = 1$, *Complex Variables, Theory and Application*, **43** (2001), 293–298. https://doi.org/10.1080/17476930108815320
6. G. G. Gundersen, K. Tohge, Entire and meromorphic solutions of $f^5+g^5+h^5 = 1$, In: *Symposium on complex differential and functional equations*, University of Joensuu, Department of Mathematics, 2004, 57–67.
7. Y. H. Gong, Q. Yang, On entire solutions of two Fermat-type differential-difference equations, *Bull. Iran. Math. Soc.*, **51** (2025), 17. https://doi.org/10.1007/s41980-024-00942-4
8. F. Gross, On the equation $f^n + g^n = 1$, *Bull. Amer. Math. Soc.*, **72** (1966), 86–88. https://doi.org/10.1090/S0002-9904-1966-11429-5
9. F. Gross, On the functional equation $f^n + g^n = h^n$, *American Mathematical Monthly*, **73** (1966), 1093–1096. https://doi.org/10.2307/2314644
10. R. G. Halburd, R. J. Korhonen, K. Tohge, Holomorphic curves with shift-invariant hyperplane preimages, *Trans. Amer. Math. Soc.*, **366** (2014), 4267–4298. https://doi.org/10.1090/S0002-9947-2014-05949-7
11. G. Haldar, M. B. Ahamed, Entire solutions of several quadratic binomial and trinomial partial differential-difference equations in $\mathbb{C}^2$, *Anal. Math. Phys.*, **12** (2022), 113. https://doi.org/10.1007/s13324-022-00722-5
12. W. K. Hayman, *Meromorphic functions*, Oxford: Clarendon Press, 1964.
13. W. K. Hayman, Waring's problem für analytische funktionen, *Bayer. Akad. Wiss. Math.-Natur. Kl. Sitzungsber.*, **1984** (1985), 1–13.
14. P. C. Hu, Q. Wang, On meromorphic solutions of functional equations of Fermat type, *Bull. Malays. Math. Sci. Soc.*, **42** (2019), 2497–2515. https://doi.org/10.1007/s40840-018-0613-1
15. K. Liu, T. B. Cao, H. Z. Cao, Entire solutions of Fermat type differential-difference equations, *Arch. Math.*, **99** (2012), 147–155. https://doi.org/10.1007/s00013-012-0408-9
16. K. Liu, L. Z. Yang, A note on meromorphic solutions of Fermat types equations, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. (N.S.)*, **62** (2016), 317–325.
17. X. Y. Liu, F. Lü, W. R. Lü, J. Wang, On entire solutions of two types of differential-difference equations, *Comput. Methods Funct. Theory*, **26** (2026), 459–480. https://doi.org/10.1007/s40315-025-00588-1
18. F. Lü, W. R. Lü, C. P. Li, J. F. Xu, Growth and uniqueness related to complex differential and difference equations, *Results Math.*, **74** (2019), 30. https://doi.org/10.1007/s00025-018-0945-z
19. Z. Y. Tai, J. R. Long, X. X. Xiang, On entire solutions for several systems of quadratic trinomial Fermat type functional equations in $\mathbb{C}^2$, *Rev. Real Acad. Cienc. Exactas, Fís. Nat. Ser. A-Mat.*, **118** (2024), 156. https://doi.org/10.1007/s13398-024-01655-3

20. P. Montel, *Leçons sur les familles normales de fonctions analytiques et leurs applications*, Paris: Gauthier-Villars, 1927.

21. E. G. Saleeby, On complex analytic solutions of certain trinomial functional and partial differential equations, *Aequat. Math.*, **85** (2013), 553–562. https://doi.org/10.1007/s00010-012-0154-x

22. N. Toda, On the functional equation $\sum_{i=0}^{p} a_i f_i^{n_i} = 1$, *Tôhoku Math. J.*, **23** (1971), 289–299. https://doi.org/10.2748/tmj/1178242646

23. Q. Wang, W. Chen, P. C. Hu, On entire solutions of two certain Fermat-type differential-difference equations, *Bull. Malays. Math. Sci. Soc.*, **43** (2020), 2951–2965. https://doi.org/10.1007/s40840-019-00848-z

24. H. Y. Xu, Y. Y. Jiang, Results on entire and meromorphic solutions for several systems of quadratic trinomial functional equations with two complex variables, *RACSAM Rev. R. Acad. A*, **116** (2022), 8. https://doi.org/10.1007/s13398-021-01154-9

25. C. C. Yang, A generalization of a theorem of P. Montel on entire functions, *Proc. Amer. Math. Soc.*, **26** (1970), 332–334. https://doi.org/10.1090/S0002-9939-1970-0264080-X

26. C. C. Yang, P. Li, On the transcendental solutions of a certain type of nonlinear differential equations, *Arch. Math.*, **82** (2004), 442–448. https://doi.org/10.1007/s00013-003-4796-8

27. C. C. Yang, H. X. Yi, *Uniqueness theory of meromorphic functions*, Dordrecht: Kluwer Academic Publishers Group, 2003.

28. L. Z. Yang, J. L. Zhang, Non-existence of meromorphic solutions of a Fermat type functional equation, *Aequat. Math.*, **76** (2008), 140–150. https://doi.org/10.1007/s00010-007-2913-7



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