AIMS Mathematics, 11(11): 38697–38720.
DOI: 10.3934/math.20261493
Received: 02 September 2026
Revised: 16 October 2026
Accepted: 27 October 2026
Published: 19 November 2026

https://www.aimspress.com/journal/Math

Research article

Topology and measures for soft sets based on parameterized families and application in MADM

Yuan Zou[1,2,*]

[1] School of Economics, Chongqing Technology and Business University, Chongqing 400067, China
[2] Chongqing Fintech Institute, Chongqing 401300, China

* **Correspondence:** Email: cqzouy@163.com.

Abstract: Soft set theory, characterized by approximate modeling (instead of precise modeling) for deriving approximate solutions with the aid of unrestricted parameterization tools, has been regarded as a versatile tool for handling diverse uncertainties in real-world scenarios. Despite the emergence of numerous versions of soft set operations, soft set theory founder Molodtsov noted that some deviated from the essence of soft sets as parameterized families, and thus formulated their correct rules. Since the parameterized family that determines the structure of a soft set can be regarded as an analogue of a topology, this paper investigates the topology generated by soft sets and defines some basic operations on the topology, such as intersection, union, sum, and Cartesian product, along with homeomorphism, with illustrative examples and proofs. Moreover, new distance and similarity measures based on the parameterized families of soft sets are proposed, and a corresponding algorithm for multi-attribute decision making (MADM) in environments with approximately descriptive information is developed. Finally, the rationality and applicability of the proposed algorithm are illustrated by a medical diagnosis case study and comparative analysis of the ranking results with other soft set similarity measures.

Keywords: soft set; parameterized family; topology; operations; measures; MADM
Mathematics Subject Classification: 54A05, 90B50

1. Introduction

Modeling various types of uncertainties is one of the most important and challenging issues in economics, engineering, environmental science, sociology, and other fields. Classical mathematical tools such as probability theory, fuzzy sets [1], intuitionistic fuzzy sets [2], rough sets [3], vague sets [4], interval mathematics [5], etc. are not always capable of handling such situations where uncertainties in these domains take diverse forms. As pointed out by Molodtsov [6], all these theories share one common inherent drawback, possibly the inadequacy of the parametrization tools.

The notion of soft sets was first proposed by Pawlak [7], the founder of rough set theory, and the Russian mathematician Molodtsov [6] then independently established soft sets as a completely new technique for handling uncertainties, which is free from the aforementioned inherent difficulty due to its use of adequate parametrization tools. Note that Pawlak's soft sets differ entirely from Molodtsov's because the former was introduced as a special case of rough sets, while the latter pursues approximate modeling rather than precise modeling of the concerned object: One only needs to search for an approximate solution rather than an exact one, and there are no restrictions on the use of parameterization tools. Molodtsov demonstrated that both Zadeh's fuzzy sets and neighborhood systems can be treated as soft sets, and he showed applications in several areas, including game theory, operations research, soft analysis, and soft probability. Specifically, fuzzy sets characterize the fuzziness of objects through membership functions with values in the interval $[0, 1]$, but the question of how to set the membership function in each specific case remains, and the basic operations using the universal cutting operation seem to contradict the individual nature of a fuzzy set; when point membership degrees are inadequate, fuzzy sets have been extended to intuitionistic fuzzy sets [2], interval-valued intuitionistic fuzzy sets [8], type-2 fuzzy sets [9], interval type-2 fuzzy sets [10], etc. Rough sets represent imperfect knowledge with the aid of equivalence relations that partition the initial universe to obtain knowledge granules, but equivalence relations sometimes are too strict to capture various types of imperfect knowledge, and thus extensions such as covering-based rough sets [11], rough fuzzy sets and fuzzy rough sets [12], and variable precision rough sets [13] have been developed. In conclusion, soft set theory adopts approximate modeling and uses an adequate parametrization to make it very flexible for dealing with various uncertainties under consideration. Later, Maji et al. [14] published a detailed study of basic operations and their properties for soft sets, and Chen et al. [15] studied the problem of parameter reduction for soft sets. Since the groundwork for soft set theory was established, a growing body of relevant research has emerged, which enriches soft set theory in terms of theoretical aspects such as operations on soft sets [16–18], parameter reduction [19, 20], soft topology [21], soft mappings [22], and combinations with other mathematical concepts such as fuzzy soft sets [23] and generalised fuzzy soft sets [24], soft rough sets [25], vague soft sets [26], etc., as well as application aspects such as decision making [27, 28], quality evaluation [29], rule mining [30], business failure prediction [31], assessment [32, 33], etc. Research on soft sets and their applications across diverse domains of science and technology is advancing at a rapid pace.

However, it is important to note that despite a large number of mathematical forms proposed for operations on soft sets, Molodtsov [34] commented that some existing studies defining operations based on parameter manipulations are incorrect, and the correct operations should be defined via families of sets related to soft sets, consistent with the essence of soft sets, i.e., a parameterized family over the initial universal set. Subsequently, Acharjee and Oza [35] followed this opinion and proposed operations on soft sets in matrix form. Inspired by these works, this paper further clarifies the mathematical essence of soft sets, namely, parameterized families over the initial universe. Due to the fact that the parameterized families can be regarded as an analogue of topology, it subsequently investigates the topology generated by a soft set, defines basic operations such as intersection, union, sum, and Cartesian product, as well as homeomorphism, accompanied by illustrative examples and proofs. Furthermore, it presents distance and similarity measures of soft sets based on parameterized families, together with the corresponding algorithm, and verifies the rationality and applicability of the algorithm through a medical diagnosis case and comparative analysis. This may be beneficial for

subsequent research on the structure and applications of soft sets.

The rest of the paper is organized as follows: Section 2 reviews preliminary knowledge about the basic notions and operations of soft sets, accompanied by illustrative examples, and provides a comparative analysis with fuzzy sets and rough sets. Section 3 introduces the concept of topology generated by a soft set, studies basic operations on such topologies (including intersection, union, sum, Cartesian product, and homeomorphism) with examples and proofs, and proposes new distance and similarity measures for soft sets based on parameterized families. Section 4 designs an algorithm based on the similarity measure of soft sets for multi-attribute decision making (MADM) with approximate information and solves a numerical example through the proposed algorithm. Section 5 discusses the structural characteristics of soft sets, their connection with topology, and measures for soft sets. Section 6 concludes the work with a summary and outlook for future research.

2. Preliminaries

This section begins with the definition of soft sets and then reviews some basic operations and propositions on soft sets, as well as providing some illustrative examples if necessary. Let X be a non-empty finite set that denotes the initial universe of discourse, and let A be a set of parameters that can take arbitrary type, i.e., words and sentences, real numbers, functions, mappings, etc. Let $\mathcal{P}(X)$ denote the power set of X, i.e., the collection of all subsets of X.

2.1. The concept of soft sets

Definition 2.1. [6] A pair (F, A) is called a soft set over X if and only if F is a mapping given by

$$F : A \to \mathcal{P}(X).$$

To be precise, a soft set is a parameterized family of subsets of X. It should be clarified that the term "soft" denotes approximation or inexactness. The fundamental distinction between soft sets and traditional uncertainty tools is that soft sets use approximate descriptions of initial objects to seek approximate solutions, while conventional tools rely on exact descriptions. Consequently, the structure of a soft set essentially depends on the family $\Im(F, A) = \{F(a) \mid a \in A\}$, which can be regarded as an approximate description for the initial universe. The set of parameters is chosen based on preferences and serves only as an auxiliary means to identify a specific element of the soft set.

Example 2.1. Consider a recruitment process at a company. Let X be the set of all six job applicants, i.e., $X = \{x_1, x_2, \ldots, x_6\}$, where each $x_i \in X$ represents an individual applicant. Let $A = \{a_1 : \textit{Education level}\ , a_2 : \textit{Work experience}\ , a_3 : \textit{Technical skills}\ , a_4 : \textit{Leadership}\}$ be the set of recruitment criteria (parameters) that is chosen according to the company's preferences for the ideal candidates. A pair (F, A) is a soft set over X. The mapping $F : A \to \mathcal{P}(X)$ is defined as follows:

$$(F, A) = \left\{ \begin{array}{l} F(a_1) = \{x_1, x_3, x_5\}, \\ F(a_2) = X, \\ F(a_3) = \{x_2, x_3, x_6\}, \\ F(a_4) = \{x_5, x_6\} \end{array} \right\}.$$

The soft set (F, A), represented by the parameterized family $\Im(F, A) = \{F(a) \mid a \in A\}$, serves as an approximate description of suitable candidates for the recruitment.

Definition 2.2. [34] Two soft sets (F, A) and (G, B) over the common universe X are called equivalent, denoted by $(F, A) \cong (G, B)$, if and only if $\Im(F, A) = \Im(G, B)$.

Let $\Im(F, A) = \{F_a : F(a), F_b : F(b), F_c : F(c), F_d : F(d)\}$ and $\Im(G, B) = \left\{G_\alpha : G(\alpha), G_\beta : G(\beta), G_\gamma : G(\gamma), G_\delta : G(\delta)\right\}$, where $F_a = G_\alpha$, $F_b = G_\beta$, $F_c = G_\gamma$, and $F_d = G_\delta$, i.e., $\Im(F, A) = \Im(G, B)$, which implies $(F, A) \cong (G, B)$, and thus (F, A) and (G, B) can be called equivalent. The representation of equivalent soft sets (F, A) and (G, B) over the common universe X is shown in Figure 1.

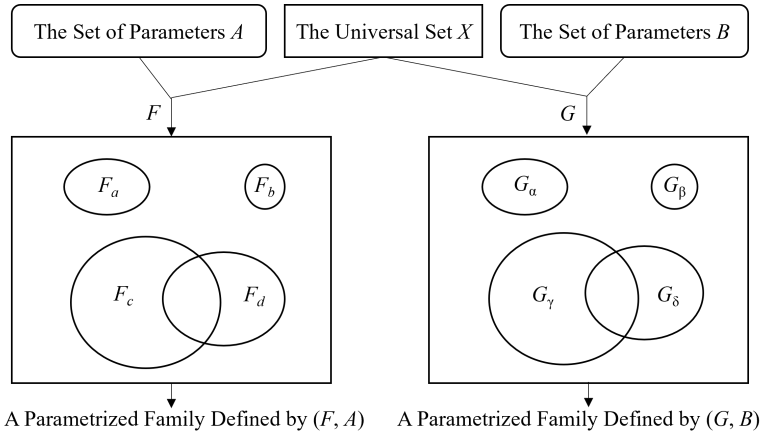


Figure 1. The representation of two equivalent soft sets.

Example 2.2. Let $X = \{x_1, x_2, x_3, x_4\}$ be the initial universal set, and let four sets of parameters be $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, and $C = \{\alpha, \beta, \gamma, \delta\}$, $D = \{high, medium, low\}$. Define four soft sets (F, A), (G, B), (H, C), and (K, D) as follows:

$$\begin{aligned} F(1) = \{x_1\}, F(2) = \{x_3\}, F(3) = \{x_2, x_4\}, \\ G(a) = \{x_1\}, G(b) = \{x_3\}, G(c) = \{x_2, x_4\}, \\ H(\alpha) = \{x_1\}, H(\beta) = \{x_3\}, H(\gamma) = \{x_2, x_4\}, H(\delta) = \{x_3\}, \\ K(high) = \{x_1\}, K(medium) = \{x_3\}, K(low) = \{x_2, x_4\}. \end{aligned}$$

Then the four soft sets (F, A), (G, B), (H, C), and (K, D) are equivalent because their parameterized families of subsets over X are identical, regardless of the forms of the parameters.

Definition 2.3. [34] Let (F, A) be a soft set over X. Then

(1) (F, A) is called a null soft set, denoted by Φ, if and only if $\Im(F, A) = \{\emptyset\}$.
(2) (F, A) is called an absolute soft set, denoted by $\mathcal{X}$, if and only if $\Im(F, A) = \{X\}$.

Definition 2.4. [34] Let (F, A) be a soft set over X.

(1) The collection of minimal non-empty elements in $\Im(F, A)$ with respect to set inclusion takes the form

$$\mathrm{MIN}(F, A) = \left\{S \in \Im(F, A) \mid S \neq \emptyset, \nexists S' \in \Im(F, A) : \emptyset \neq S' \subsetneqq S\right\}.$$

(2) The collection of maximal non-empty elements in $\Im(F, A)$ with respect to set inclusion takes the form

$$\mathrm{MAX}(F, A) = \left\{S \in \Im(F, A) \mid S \neq X, \nexists S' \in \Im(F, A) : X \neq S' \supsetneqq S\right\}.$$

Example 2.3. Let the universal set be $X = \{x_1, x_2, x_3, x_4\}$ and the set of parameters be $A = \{a, b, c, d\}$. Define a soft set (F, A) over X such that

$$\begin{aligned} F(a) &= \{x_1, x_2\}, \\ F(b) &= \{x_2\}, \\ F(c) &= \{x_2, x_3, x_4\}, \\ F(d) &= \{x_3, x_4\}. \end{aligned}$$

The family of subsets $\Im(F, A)$ is then

$$\Im(F, A) = \{\{x_1, x_2\}, \{x_2\}, \{x_2, x_3, x_4\}, \{x_3, x_4\}\}.$$

By definition, $\mathrm{MIN}(F, A)$ consists of minimal non-empty elements in $\Im(F, A)$, thus $\mathrm{MIN}(F, A) = \{\{x_2\}, \{x_3, x_4\}\}$; $\mathrm{MAX}(F, A)$ consists of maximal elements in $\Im(F, A)$ excluding X, thus $\mathrm{MAX}(F, A) = \{\{x_1, x_2\}, \{x_2, x_3, x_4\}\}$.

2.2. Correct operations on soft sets

As Molodtsov [34] noted, many operations proposed in the literature are defined by manipulating parameters and thereby deviate from the essence of soft sets. The following notions therefore introduce a criterion of independence from the particular choice of parameters, so that the operations studied below are guaranteed to preserve the underlying family structure of soft sets.

Definition 2.5. [34] Two soft sets (F, A) and (F', A') over the common universe X are called equal, denoted by $(F, A) = (F', A')$, if and only if $F = F'$ and $A = A'$.

Definition 2.6. [34] Let (F, A), (F', A'), (G, B), and (G', B') be soft sets over the common universe X.

(1) A unary operation Ψ on soft sets is called correct if and only if $(F, A) \cong (F', A')$ implies $\Psi(F, A) \cong \Psi(F', A')$, where $\Psi(F, A) = (G, B)$ and $\Psi(F', A') = (G', B')$ denote soft sets over X, respectively.
(2) A binary operation Θ on soft sets is called correct if and only if $(F, A) \cong (F', A')$ and $(G, B) \cong (G', B')$ imply $\Theta((F, A), (G, B)) \cong \Theta\left((F', A'), (G', B')\right)$, where $\Theta((F, A), (G, B)) = (H, C)$ and $\Theta\left((F', A'), (G', B')\right) = (H', C')$ denote soft sets over X, respectively.

Definition 2.7. [34] Let (F, A) be a soft set over X. The correct unary operation "complement" of (F, A), denoted by $C(F, A) = (F_C, A)$, is defined as follows. The set of parameters remains unchanged, and the mapping is given by $F_C(a) = X \setminus F(a)$ for all $a \in A$.

Definition 2.8. [34] Let (F, A) and (G, B) be two soft sets over X.

(1) The correct binary operation "intersection" of (F, A) and (G, B), denoted by $\bigcap((F, A), (G, B)) = (H, A \times B)$, is defined as follows. The set of parameters is converted into a Cartesian product $A \times B$, and the mapping is given by $H(a, b) = F(a) \cap G(b)$ for all $(a, b) \in A \times B$.
(2) The correct binary operation "union" of (F, A) and (G, B), denoted by $\bigcup((F, A), (G, B)) = (H, A \times B)$, is defined as follows. The set of parameters is converted into a Cartesian product $A \times B$, and the mapping is given by $H(a, b) = F(a) \cup G(b)$ for all $(a, b) \in A \times B$.

Definition 2.9. [34] Let (F, A) and (G, B) be two soft sets over X. The binary operation "product" of (F, A) and (G, B), denoted by $(F, A) \times (G, B) = (H, A \times B)$, is defined as follows. The set of parameters is converted into a Cartesian product $A \times B$, and the mapping is given by $H(a, b) = F(a) \times G(b)$ for all $(a, b) \in A \times B$.

Note that the correct binary operations on soft sets may lead to an expansion of the parameter sets, this expansion gives a detailed description of the X. If necessary, we can use cutting operations which depend on the characteristics of the problem to simplify the Cartesian product of parameter sets.

Example 2.4. Let the universal set be $X = \{x_1, x_2, x_3, x_4\}$, with the set of parameters $A = \{a, b\}$ and $B = \{c, d\}$. Define two soft sets over X:

$$\begin{aligned} (F, A): &\quad F(a) = \{x_1, x_2\}, \quad F(b) = \{x_2, x_3\}, \\ (G, B): &\quad G(c) = \{x_2, x_3\}, \quad G(d) = \{x_3, x_4\}. \end{aligned}$$

(1) The complement of (F, A) is $C(F, A) = (F_C, A)$, where $F_C(a) = X \setminus F(a) = \{x_3, x_4\}$, $F_C(b) = X \setminus F(b) = \{x_1, x_4\}$. Thus, $C(F, A) = \{F_C(a) = \{x_3, x_4\}, F_C(b) = \{x_1, x_4\}\}$.

(2) The intersection of (F, A) and (G, B) is $\bigcap((F, A), (G, B)) = (H, A \times B)$, where $A \times B = \{(a, c), (a, d), (b, c), (b, d)\}$ and $H(a, c) = F(a) \cap G(c) = \{x_1, x_2\} \cap \{x_2, x_3\} = \{x_2\}$, $H(a, d) = F(a) \cap G(d) = \{x_1, x_2\} \cap \{x_3, x_4\} = \emptyset$, $H(b, c) = F(b) \cap G(c) = \{x_2, x_3\} \cap \{x_2, x_3\} = \{x_2, x_3\}$, $H(b, d) = F(b) \cap G(d) = \{x_2, x_3\} \cap \{x_3, x_4\} = \{x_3\}$. Thus,

$$\bigcap((F, A), (G, B)) = \begin{Bmatrix} H(a, c) = \{x_2\}, \ H(a, d) = \emptyset, \\ H(b, c) = \{x_2, x_3\}, \ H(b, d) = \{x_3\} \end{Bmatrix}.$$

(3) The union of (F, A) and (G, B) is $\bigcup((F, A), (G, B)) = (K, A \times B)$, where $K(a, c) = F(a) \cup G(c) = \{x_1, x_2\} \cup \{x_2, x_3\} = \{x_1, x_2, x_3\}$, $K(a, d) = F(a) \cup G(d) = \{x_1, x_2\} \cup \{x_3, x_4\} = X$, $K(b, c) = F(b) \cup G(c) = \{x_2, x_3\} \cup \{x_2, x_3\} = \{x_2, x_3\}$, $K(b, d) = F(b) \cup G(d) = \{x_2, x_3\} \cup \{x_3, x_4\} = \{x_2, x_3, x_4\}$. Thus,

$$\bigcup((F, A), (G, B)) = \begin{Bmatrix} K(a, c) = \{x_1, x_2, x_3\}, \ K(a, d) = X, \\ K(b, c) = \{x_2, x_3\}, \ K(b, d) = \{x_2, x_3, x_4\} \end{Bmatrix}.$$

(4) The product of (F, A) and (G, B) is $(F, A) \times (G, B) = (L, A \times B)$, where $L(a, c) = F(a) \times G(c) = \{x_1, x_2\} \times \{x_2, x_3\} = \{(x_1, x_2), (x_1, x_3), (x_2, x_2), (x_2, x_3)\}$, $L(a, d) = F(a) \times G(d) = \{x_1, x_2\} \times \{x_3, x_4\} = \{(x_1, x_3), (x_1, x_4), (x_2, x_3), (x_2, x_4)\}$, $L(b, c) = F(b) \times G(c) = \{x_2, x_3\} \times \{x_2, x_3\} = \{(x_2, x_2), (x_2, x_3), (x_3, x_2), (x_3, x_3)\}$, $L(b, d) = F(b) \times G(d) = \{x_2, x_3\} \times \{x_3, x_4\} = \{(x_2, x_3), (x_2, x_4), (x_3, x_3), (x_3, x_4)\}$. Thus,

$$(F, A) \times (G, B) = \begin{Bmatrix} L(a, c) = \{(x_1, x_2), (x_1, x_3), (x_2, x_2), (x_2, x_3)\}, \\ L(a, d) = \{(x_1, x_3), (x_1, x_4), (x_2, x_3), (x_2, x_4)\}, \\ L(b, c) = \{(x_2, x_2), (x_2, x_3), (x_3, x_2), (x_3, x_3)\}, \\ L(b, d) = \{(x_2, x_3), (x_2, x_4), (x_3, x_3), (x_3, x_4)\} \end{Bmatrix}.$$

Definition 2.10. [34] Let (F, A), (F', A'), (G, B), and (G', B') be soft sets over the common universe X.

(1) A relationship Ω between two soft sets (F, A) and (G, B) is defined as a mapping $\Omega : (F, A) \times (G, B) \to \{0, 1\}$. If $\Omega((F, A), (G, B)) = 1$, then it can be denoted by $(F, A)\Omega(G, B)$.

(2) A relationship Ω between soft sets is called correct if and only if $(F,A) \cong (F^{'},A^{'})$ and $(G,B) \cong (G^{'},B^{'})$ imply $\Omega((F,A),(G,B)) = \Omega\left((F^{'},A^{'}),(G^{'},B^{'})\right)$.

Definition 2.11. [34] Let (F,A) and (G,B) be two soft sets over X.

(1) The soft set (F,A) internally approximates the soft set (G,B), denoted by $(F,A) \subseteq (G,B)$, if for any $b \in B$ with $G(b) \neq \emptyset$, there exists $a \in A$ such that $\emptyset \neq F(a) \subseteq G(b)$.
(2) The soft set (F,A) externally approximates the soft set (G,B), denoted by $(F,A) \supseteq (G,B)$, if for any $b \in B$ with $G(b) \neq X$, there exists $a \in A$ such that $X \neq F(a) \supseteq G(b)$.

Example 2.5. Let the universal set be $X = \{x_1,x_2,x_3,x_4\}$, with the set of parameters $A = \{a,b\}$, $B = \{c,d,e\}$, and $C = \{\alpha,\beta\}$. Define three soft sets over X:

$$\begin{aligned}
(F,A):&\quad F(a) = \{x_2\},\quad F(b) = \{x_2,x_3\},\\
(G,B):&\quad G(c) = \{x_2,x_3,x_4\},\quad G(d) = \{x_2\},\quad G(e) = \{x_1,x_2\},\\
(H,C):&\quad H(\alpha) = \{x_1,x_2,x_3\},\quad H(\beta) = \{x_2,x_3,x_4\}.
\end{aligned}$$

(1) For $c \in B$ with $G(c) \neq \emptyset$, $G(c) = \{x_2,x_3,x_4\}$, exists $b \in A$ such that $F(b) = \{x_2,x_3\} \subseteq \{x_2,x_3,x_4\}$ and $F(b) \neq \emptyset$; for $d \in B$ with $G(d) \neq \emptyset$, $G(d) = \{x_2\}$, exists $a \in A$ such that $F(a) = \{x_2\} \subseteq \{x_2\}$ and $F(a) \neq \emptyset$; for $e \in B$ with $G(e) \neq \emptyset$, $G(e) = \{x_1,x_2\}$, exists $a \in A$ such that $F(a) = \{x_2\} \subseteq \{x_1,x_2\}$ and $F(a) \neq \emptyset$. Thus, $(F,A) \subseteq (G,B)$.
(2) For $c \in B$ with $G(c) \neq X$, $G(c) = \{x_2,x_3,x_4\}$, exists $\beta \in C$ such that $H(\beta) = \{x_2,x_3,x_4\} \supseteq \{x_2,x_3,x_4\}$ and $H(\beta) \neq X$; for $d \in B$ with $G(d) \neq X$, $G(d) = \{x_2\}$, exists $\beta \in C$ such that $H(\beta) = \{x_2,x_3,x_4\} \supseteq \{x_2\}$ and $H(\beta) \neq X$; for $e \in B$ with $G(e) \neq X$, $G(e) = \{x_1,x_2\}$, exists $\alpha \in C$ such that $H(\alpha) = \{x_1,x_2,x_3\} \supseteq \{x_1,x_2\}$ and $H(\alpha) \neq X$. Thus, $(H,C) \supseteq (G,B)$.

Definition 2.12. [34] Let (F,A) and (G,B) be two soft sets over X.

(1) The soft set (F,A) is internally equivalent to the soft set (G,B), denoted by $(F,A) \overset{\subseteq}{\approx} (G,B)$, if $(F,A) \subseteq (G,B)$ and $(G,B) \subseteq (F,A)$.
(2) The soft set (F,A) is externally equivalent to the soft set (G,B), denoted by $(F,A) \overset{\supseteq}{\approx} (G,B)$, if $(F,A) \supseteq (G,B)$ and $(G,B) \supseteq (F,A)$.
(3) The soft set (F,A) is weakly equivalent to the soft set (G,B), denoted by $(F,A) \approx (G,B)$, if $(F,A) \overset{\subseteq}{\approx} (G,B)$ and $(F,A) \overset{\supseteq}{\approx} (G,B)$.

Example 2.6. Let the universal set be $X = \{x_1,x_2,x_3,x_4\}$, with the set of parameters $A = \{a,b,c,d\}$, $B = \{1,2,3\}$. Define two soft sets over X:

$$\begin{aligned}
(F,A):&\quad F(a) = \{x_1,x_2,x_3\},\quad F(b) = \{x_1,x_2\},\quad F(c) = \{x_1\},\quad F(d) = \{x_2\},\\
(G,B):&\quad G(1) = \{x_1,x_2,x_3\},\quad G(2) = \{x_1\},\quad G(3) = \{x_2\}.
\end{aligned}$$

(1) It is straightforward to check that $(F,A) \subseteq (G,B)$ and $(G,B) \subseteq (F,A)$ by the definition of internal approximation. Thus, $(F,A) \overset{\subseteq}{\approx} (G,B)$ holds.
(2) It is straightforward to check that $(F,A) \supseteq (G,B)$ and $(G,B) \supseteq (F,A)$ by the definition of external approximation. Thus, $(F,A) \overset{\supseteq}{\approx} (G,B)$ holds.
(3) Because $(F,A) \overset{\subseteq}{\approx} (G,B)$ and $(F,A) \overset{\supseteq}{\approx} (G,B)$, thus, $(F,A) \approx (G,B)$ holds.

Definition 2.13. A binary relation $\sim$ on a set X is called an equivalence relation if it is reflexive, symmetric, and transitive, i.e., for all $a,b,c \in X$,

(i) $a \sim a$ (reflexive);
(ii) $a \sim b \implies b \sim a$ (symmetric);
(iii) $a \sim b$ and $b \sim c \implies a \sim c$ (transitive).

If $a, b \in X$, then a and b are called to be in the same equivalence class if $a \sim b$.

Proposition 2.1. [34] Let (F, A) and (G, B) be two soft sets over X.

(1) The relationships $(F, A) \subseteq (G, B)$ and $(F, A) \supseteq (G, B)$ are reflexive and transitive.
(2) The relationships $(F, A) \cong (G, B)$, $(F, A) \overset{\subseteq}{\approx} (G, B)$, $(F, A) \overset{\supseteq}{\approx} (G, B)$, and $(F, A) \approx (G, B)$ are equivalence relations, satisfying reflexivity, symmetry, and transitivity.

Proof. The proof is straightforward from Definitions 2.2, 2.11–2.13. □

Proposition 2.2. [34] Let (F, A) and (G, B) be two soft sets over X, and let the families $\Im(F, A)$ and $\Im(G, B)$ be finite and nontrivial.

(1) The relationship $(F, A) \overset{\subseteq}{\approx} (G, B)$ holds if and only if $\mathrm{MIN}(F, A) = \mathrm{MIN}(G, B)$.
(2) The relationship $(F, A) \overset{\supseteq}{\approx} (G, B)$ holds if and only if $\mathrm{MAX}(F, A) = \mathrm{MAX}(G, B)$.
(3) The relationship $(F, A) \approx (G, B)$ holds if and only if $\mathrm{MIN}(F, A) = \mathrm{MIN}(G, B)$ and $\mathrm{MAX}(F, A) = \mathrm{MAX}(G, B)$.

Proof. The proof is straightforward from Definitions 2.4 and 2.12. □

Proposition 2.3. [34] Let (F, A) and (G, B) be two soft sets over X.

(1) If $X \in \Im(F, A)$, then $\bigcap((F, A), (G, B)) \subseteq (G, B)$.
(2) $\bigcup((F, A), (G, B)) \supseteq (F, A)$, $\bigcup((F, A), (G, B)) \supseteq (G, B)$.

Proof. The proof is straightforward from Definitions 2.8 and 2.11. □

2.3. A comparison of soft sets with fuzzy sets and rough sets

In this section, we compare soft sets with fuzzy sets and rough sets, which are common tools for handling uncertainties.

2.3.1. Relations and differences between soft sets and fuzzy sets

Fuzzy sets handle boundary vagueness via membership functions mapping to $[0, 1]$, with Cantor sets as special cases (membership taking the values 0 and 1). A fuzzy set is equivalent to nested α-level sets: $A_\mu(\alpha) = \{x \mid \mu_A(x) \geq \alpha\}$, with $\mu_A(x) = \sup\left\{\alpha \mid x \in A_\mu(\alpha), \alpha \in [0, 1]\right\}$. A soft set (F, E) maps parameters E to $\mathcal{P}(X)$, forming a parameterized subset family. When $E = [0, 1]$, a fuzzy set can be represented as a soft set $\left(A_\mu, [0, 1]\right)$. In [36], three key differences are summarized as follows:

(1) Fuzzy sets restrict $[0, 1]$ as the parameter set, while soft sets allow arbitrary parameter types.
(2) Binary operations for fuzzy sets use universal cutting on parameters, which may contradict with their individual nature. In contrast, soft sets apply problem-specific partial cutting operations.
(3) Fuzzy sets use scalar-valued mappings (membership functions) to describe fuzziness of objects, whereas soft sets employ set-valued mappings from parameters to the universe to approximately describe parameterized subset families.

In addition, Molodtsov provided an intuitive example to illustrate the fundamental difference between the equivalence of soft sets and fuzzy sets [34]. Consider two fuzzy sets A and B over the universal set $X = [0, 1]$ with membership functions $\mu_A(x) = x^n$ and $\mu_B(x) = x^{1/n}$ for $n \geq 2$ and $n \in \mathbb{N}^+$. It is straightforward to demonstrate that the two fuzzy sets are equivalent from the standpoint of soft set theory but are not equivalent in terms of fuzzy set theory due to their distinct membership functions. Note that this example highlights that the concept of equivalence in soft sets diverges fundamentally from that in fuzzy sets, as the latter's equivalence relies on numerical membership degrees, while the former's equivalence is defined via parameterized subset families regardless of parameter form.

2.3.2. Relations and differences between soft sets and rough sets

Rough set theory characterizes imprecision of knowledge through a set's boundary region instead of membership degrees. A set is crisp if its boundary is empty; otherwise, it is rough (inexact). A non-empty boundary indicates insufficient knowledge to define the set precisely. Formally speaking, the rough set handles boundary roughness via indiscernibility relations, expressed as a pair of lower and upper approximations $\left(\underline{Apr}_A(X), \overline{Apr}_A(X)\right)$, where $\underline{Apr}_A(X)$ denotes the set of all elements surely belonging to X and $Bnd_A(X) = \overline{Apr}_A(X) - \underline{Apr}_A(X)$ denotes the set of all elements possibly belonging to X. From the viewpoint of soft set theory, a rough set is a special soft set $(F(X), \{\approx_A, \simeq_A\})$, where parameters are indiscernibility relations and F is an identity mapping to a topological space. Two key differences are summarized in [36] as follows:

(1) Soft sets use general mappings, while rough sets employ identity mappings.
(2) Soft sets allow arbitrary parameters, but rough sets restrict to equivalence relations.

Defining equivalence relations that partition the entire universe into disjoint classes is often challenging. Moreover, equivalence relations are sometimes inadequate for representing the imprecision of knowledge. These difficulties reduce the flexibility of rough sets and limit their applicability compared to soft sets, which allow more general parameterizations.

3. Parameterized family-based topology and measures for soft sets

From the perspective in which a parameterized family defines the structure of a soft set, this family can be regarded as an analogue to topology or used to generate a topology. The same parameterized family that underlies the structure of a soft set thus connects the topology and its operations, as well as the distance and similarity measures and the decision making application.

3.1. Topology generated by a soft set

Definition 3.1. [37] Let X be a non-empty universal set. A set τ of subsets of X is called a topology on X if it satisfies:

(i) X and $\emptyset$ belong to τ.
(ii) The union of any (finite or infinite) number of sets in τ belongs to τ.
(iii) The intersection of any two sets in τ belongs to τ.

The pair (X, τ) is called a topological space.

Definition 3.2. [37] Let X be a non-empty universal set.

(1) If τ is the collection of all subsets of X, then τ is called the discrete topology on X, and the topological space (X, τ) is called a discrete space.
(2) If $\tau = \{\emptyset, X\}$, then τ is called the indiscrete topology on X, and (X, τ) is called an indiscrete space.

Definition 3.3. [37] Let (X, τ) be a topological space.

(1) Then all the members of τ are called open sets.
(2) A collection $\mathcal{B}$ of open subsets of X is called a basis for the topology τ if every open set is a union of members of $\mathcal{B}$.

Note that a soft set (F, A) is essentially a parameterized family of subsets $\Im(F, A) = \{F(a) \mid a \in A\} \subseteq \mathcal{P}(X)$. To generate a topology, one needs to expand this family of subsets into a minimal family of subsets τ that satisfies the three axioms of topology through the "closure operation" of the topology axioms. First, define the union of its elements as the new universal set $X' \subseteq X$, i.e., $\bigcup_{a\in A} F(a) = X'$. The formal definition is provided as follows.

Definition 3.4. Let (F, A) be a soft set over X and let $X' = \bigcup_{a\in A} F(a)$. The collection of subsets of X' denoted by $\tau_{(F,A)}$ is called a topology generated by (F, A), defined as the collection of all arbitrary unions of sets in the basis, as follows:

$$\tau_{(F,A)} = \left\{ \bigcup_{i\in I} B_i \,\middle|\, I \text{ is an index set, } B_i \in \mathcal{B} \right\} \cup \{\emptyset\},$$

where $\mathcal{B} = \left\{ \bigcap_{i=1}^{k} F(a_i) \,\middle|\, k \in \mathbb{N}^+, a_i \in A \right\}$ forms a basis for the topology $\tau_{(F,A)}$. The pair $(X', \tau_{(F,A)})$ is called a topological space generated by (F, A).

By definition, it is straightforward to verify that $\tau_{(F,A)} = \bigcap \{\tau \mid \tau \text{ is a topology on } X' \text{ and } \Im(F, A) \subseteq \tau\}$. Namely, the generated topology $\tau_{(F,A)}$ on X' is the smallest topology on X' that contains $\Im(F, A)$, which in turn determines the structure of the soft set (F, A).

Remark 3.1. *The topology $\tau(F, A)$ is a classical topology on the universe X' (a subset of X). It should be distinguished from the "soft topology" of [21], which is defined on the collection of all soft sets over a universe rather than on the universe itself. The two notions are complementary: [21] topologizes the family of soft sets, whereas here the parameterized family $S(F, A)$ of a single soft set is used to topologize the underlying universe. Throughout this paper, "the topology generated by a soft set" refers to the latter.*

Example 3.1. Let $X = \{x_1, x_2, x_3, x_4\}$, $A = \{a, b\}$, and define a soft set (F, A) over X by:

$$F(a) = \{x_1, x_2\}, \quad F(b) = \{x_2, x_3\}.$$

The union of elements in $\Im(F, A) = \{F(a), F(b)\}$ is

$$X' = \bigcup_{c\in A} F(c) = F(a) \cup F(b) = \{x_1, x_2, x_3\}.$$

The basis $\mathcal{B}$ for $\tau_{(F,A)}$ consists of finite intersections

$$\mathcal{B} = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_1, x_2\} \cap \{x_2, x_3\} = \{x_2\}\}.$$

Thus the topology on $X' \subseteq X$ generated by (F, A) is

$$\tau_{(F,A)} = \{\emptyset, \{x_2\}, \{x_1, x_2\}, \{x_2, x_3\}, X'\}.$$

It is readily verified that this satisfies all three axioms of topology, with $X' \in \tau_{(F,A)}$ as the universal set.

Definition 3.5. Let (F, A) be a soft set over X with $X' = \bigcup_{a\in A} F(a)$.

(1) The topology $\tau_{(F,A)}$ generated by (F, A) is called the discrete topology on X' if it contains all subsets of X', i.e.,

$$\tau_{(F,A)} = \mathcal{P}(X'),$$

which occurs when the basis $\mathcal{B} = \left\{\bigcap_{i=1}^{k} F(a_i) \,\middle|\, k \in \mathbb{N}^+,\ a_i \in A\right\}$ allows every singleton $\{x\} \subseteq X'$ to be expressed as a union of elements in $\mathcal{B}$.

(2) The topology $\tau_{(F,A)}$ generated by (F, A) is called the indiscrete topology on X' if it consists only of $\emptyset$ and X' itself, i.e.,

$$\tau_{(F,A)} = \{\emptyset, X'\},$$

which holds when the basis $\mathcal{B}$ cannot generate non-trivial open sets, meaning the only finite intersections of $F(a_i)$ are $\emptyset$ and X'.

Example 3.2. Let $X = \{x_1, x_2, x_3\}$, $A = B = \{a, b\}$, and define two soft sets (F, A) and (G, B) over X:

$$\begin{aligned}(F, A): &\quad F(a) = \{x_1\}, \quad F(b) = \{x_2\},\\ (G, B): &\quad G(a) = \{x_2, x_3\}, \quad G(b) = \{x_2, x_3\}.\end{aligned}$$

The union $X' = \bigcup_{e\in A} F(e) = \{x_1, x_2\}$. The basis is $\mathcal{B} = \{\{x_1\}, \{x_2\}\}$ generating the discrete topology

$$\tau_{(F,A)} = \mathcal{P}(X') = \{\emptyset, \{x_1\}, \{x_2\}, \{x_1, x_2\}\}.$$

The union $X'' = \bigcup_{e\in B} G(e) = \{x_2, x_3\}$. The basis $\mathcal{B} = \{\{x_2, x_3\}\}$ generates the indiscrete topology

$$\tau_{(G,B)} = \{\emptyset, X''\} = \{\emptyset, \{x_2, x_3\}\}.$$

Proposition 3.1. Let (F, A) be a soft set over X, and let $\mathcal{B} \supseteq \Im(F, A)$ be a collection of subsets of X', where $X' = \bigcup_{a\in A} F(a) \subseteq X$. Then $\mathcal{B}$ is a basis for the topology $\tau_{(F,A)}$ on X' generated by (F, A) if and only if $\mathcal{B}$ satisfies:

(i) $X' = \bigcup_{B\in\mathcal{B}} B$.
(ii) For any $B_i, B_j \in \mathcal{B}$, the set $B_i \cap B_j$ is a union of members of $\mathcal{B}$.

Proof. If $\mathcal{B}$ is a basis for $\tau_{(F,A)}$, then by the definition of a topology basis, the topology $\tau_{(F,A)}$ must cover X'. Thus $X' = \bigcup_{B\in\mathcal{B}} B$. For any $B_i, B_j \in \mathcal{B}$, their intersection $B_i \cap B_j$ is an open set in $\tau_{(F,A)}$. As $\mathcal{B}$ is a basis, $B_i \cap B_j$ must be expressed as a union of $\mathcal{B}$-elements.

Conversely, assume that $\mathcal{B}$ satisfies properties (i) and (ii). Then, by (i), $X' = \bigcup_{B\in\mathcal{B}} B \in \tau_{(F,A)}$. And $\emptyset$ is the empty union of $\mathcal{B}$-elements, so $\emptyset \in \tau_{(F,A)}$. Thus axiom (i) of topology is established. Let $\{T_j\}_{j\in J}$

be a family of open sets in $\tau_{(F,A)}$. By definition of $\tau_{(F,A)}$, each T_j is a union of $\Im(F,A)$-elements, since $\mathcal{B} \supseteq \Im(F,A)$. The union of all T_j is thus a union of $\mathcal{B}$-elements, so it belongs to $\tau_{(F,A)}$. Axiom (ii) is satisfied. Let C and D be in $\tau_{(F,A)}$. One needs to verify that $C \cap D \in \tau_{(F,A)}$. $C = \bigcup_{i\in I} B_i$ for some index set I and sets $B_i \in \mathcal{B}$. Also $D = \bigcup_{j\in J} B_j$ for some index set J and $B_j \in \mathcal{B}$. Therefore

$$C \cap D = \left(\bigcup_{i\in I} B_i\right) \bigcap \left(\bigcup_{j\in J} B_j\right) = \bigcup_{i\in I,\, j\in J} \left(B_i \cap B_j\right).$$

By (ii), each $B_i \cap B_j$ is a union of basis elements in $\mathcal{B}$, so $C \cap D$ is a union of members of $\mathcal{B}$. Thus $C \cap D \in \tau_{(F,A)}$. So $\tau_{(F,A)}$ satisfies axiom (iii) of Definition 3.1. Since $\tau_{(F,A)}$ satisfies all three axioms of topology, it is a topology on X'. By construction, $\tau_{(F,A)}$ consists of all unions of $\Im(F,A)$-elements. Hence $\tau_{(F,A)}$ is the topology generated by (F,A), and $\mathcal{B} \supseteq \Im(F,A)$ is its basis. □

Proposition 3.2. Let $(X', \tau_{(F,A)})$ be the topological space generated by a soft set (F,A), with $X' = \bigcup_{a\in A} F(a) \subseteq X$. Let $\mathcal{B} \supseteq \Im(F,A)$ be a collection of subsets of X'. Then $\mathcal{B}$ is a basis for $\tau_{(F,A)}$ if and only if for every open set $Y \in \tau_{(F,A)}$ and every $x \in Y$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq Y$.

Proof. Let $\mathcal{B}$ be a basis for $\tau_{(F,A)}$. For any $Y \in \tau_{(F,A)}$ and $x \in Y$, by definition of a basis, Y is a union of elements in $\mathcal{B}$, i.e., $Y = \bigcup_{j\in J} B_j$ for some index set J and $B_j \in \mathcal{B}$. Since $x \in Y$, there exists $j_0 \in J$ such that $x \in B_{j_0}$. By construction, $B_{j_0} \subseteq Y$, so $x \in B_{j_0} \subseteq Y$.

Conversely, suppose for every $Y \in \tau_{(F,A)}$ and $x \in Y$, there exists $B_x \in \mathcal{B}$ with $x \in B_x \subseteq Y$. For any $Y \in \tau_{(F,A)}$, define $\mathcal{U} = \{B_x \in \mathcal{B} \mid x \in Y, B_x \subseteq Y\}$. By definition, each $B_x \subseteq Y$, so their union is also contained in Y, i.e., $\bigcup \mathcal{U} \subseteq Y$. For any $x \in Y$, $B_x \in \mathcal{U}$ implies $x \in B_x \in \mathcal{U} \subseteq \bigcup \mathcal{U}$, i.e., $Y \subseteq \bigcup \mathcal{U}$. Hence $Y = \bigcup \mathcal{U}$, so Y is a union of elements in $\mathcal{B}$. Since Y is arbitrary, $\mathcal{B}$ generates $\tau_{(F,A)}$ and is therefore a basis. □

3.2. Operations on the topologies generated by soft sets

In analogy with the classical operations on topological spaces, the following introduces the intersection, union, sum, and product of the topologies generated by soft sets.

Definition 3.6. Let (F,A) and (G,B) be two non-null soft sets over X. Assume that $X_1 = \bigcup_{a\in A} F(a)$ and $X_2 = \bigcup_{b\in B} G(b)$, $X_1 \cap X_2 \neq \emptyset$. Let $\tau_{(F,A)}$ and $\tau_{(G,B)}$ be two topologies generated by soft sets (F,A) and (G,B) on X_1 and X_2, respectively.

(1) The intersection of $\tau_{(F,A)}$ and $\tau_{(G,B)}$ is defined on the common universe $Y = X_1 \cap X_2$ as the set of all subsets of Y that are open in both the restrictions of $\tau_{(F,A)}$ and $\tau_{(G,B)}$ to Y, which takes the form:

$$\tau_{(F,A)} \cap \tau_{(G,B)} = \{W \subseteq Y \mid W \in \tau_{(F,A)}|_Y \text{ and } W \in \tau_{(G,B)}|_Y\},$$

where $\tau|_Y = \{U \cap Y \mid U \in \tau\}$ denotes the restriction of a topology τ to Y.

(2) The union of $\tau_{(F,A)}$ and $\tau_{(G,B)}$, denoted $\tau_{(F,A)} \cup \tau_{(G,B)}$, is the collection of all open sets from either topology (F,A) or (G,B) and takes the form:

$$\tau_{(F,A)} \cup \tau_{(G,B)} = \{U \mid U \in \tau_{(F,A)}\} \cup \{V \mid V \in \tau_{(G,B)}\}.$$

Definition 3.7. Let (F,A) and (G,B) be non-null soft sets over a common universe X. Assume that $X_1 = \bigcup_{a\in A} F(a)$ and $X_2 = \bigcup_{b\in B} G(b)$, $X_1 \cap X_2 \neq \emptyset$. Let $\tau_{(F,A)}$ and $\tau_{(G,B)}$ be the topologies on X_1 and X_2 generated by (F,A) and (G,B), respectively.

(1) The sum of $\tau_{(F,A)}$ and $\tau_{(G,B)}$, denoted $\tau_{(F,A)} \oplus \tau_{(G,B)}$, is defined on $Y = X_1 \cup X_2$ and takes the form:

$$\tau_{(F,A)} \oplus \tau_{(G,B)} = \{W \cup V \mid W \in \tau_{(F,A)},\ V \in \tau_{(G,B)}\}.$$

(2) The product of $\tau_{(F,A)}$ and $\tau_{(G,B)}$, denoted $\tau_{(F,A)} \times \tau_{(G,B)}$, is defined on the Cartesian product $Y = X_1 \times X_2$ as the topology generated by the basis:

$$\mathcal{B} = \{U \times V \mid U \in \tau_{(F,A)},\ V \in \tau_{(G,B)}\},$$

i.e.,

$$\tau_{(F,A)} \times \tau_{(G,B)} = \left\{ \bigcup_{i \in I} (U_i \times V_i) \mid U_i \in \tau_{(F,A)},\ V_i \in \tau_{(G,B)},\ I \text{ is an index set} \right\},$$

where the collection $\mathcal{B} = \{U \times V \mid U \in \tau_{(F,A)},\ V \in \tau_{(G,B)}\}$ is a basis for the product topology $\tau_{(F,A)} \times \tau_{(G,B)}$.

Example 3.3. Let $X = \{x_1, x_2, x_3, x_4\}$, $A = \{a_1, a_2\}$, and $B = \{b_1, b_2\}$. Define two soft sets (F, A) and (G, B) over X:

$$(F, A): \quad F(a_1) = \emptyset, \quad F(a_2) = \{x_1, x_2\},$$
$$(G, B): \quad G(b_1) = \emptyset, \quad G(b_2) = \{x_2, x_3\}.$$

(1) $Y = X_1 \cap X_2 = \{x_2\}$. $\tau_{(F,A)} = \{\emptyset, \{x_1, x_2\}\}$, $\tau_{(G,B)} = \{\emptyset, \{x_2, x_3\}\}$. The restrictions are $\tau_{(F,A)}|_Y = \{\emptyset, \{x_2\}\}$ and $\tau_{(G,B)}|_Y = \{\emptyset, \{x_2\}\}$. Then the intersection of $\tau_{(F,A)}$ and $\tau_{(G,B)}$ is $\{\emptyset, \{x_2\}\}$, which is a topology on Y by definition.

(2) The union of $\tau_{(F,A)}$ and $\tau_{(G,B)}$ is $\tau_{(F,A)} \cup \tau_{(G,B)} = \{\emptyset, \{x_1, x_2\}, \{x_2, x_3\}\}$. Since $\{x_1, x_2\} \cap \{x_2, x_3\} = \{x_2\} \notin \tau_{(F,A)} \cup \tau_{(G,B)}$ and $\{x_1, x_2\} \cup \{x_2, x_3\} = \{x_1, x_2, x_3\} \notin \tau_{(F,A)} \cup \tau_{(G,B)}$, $\tau_{(F,A)} \cup \tau_{(G,B)}$ is not a topology by definition.

(3) $Y = X_1 \cup X_2 = \{x_1, x_2, x_3\}$. The sum of $\tau_{(F,A)}$ and $\tau_{(G,B)}$ is $\tau_{(F,A)} \oplus \tau_{(G,B)} = \{\emptyset, \{x_1, x_2\}, \{x_2, x_3\}, \{x_1, x_2, x_3\}\}$. Since $\{x_1, x_2\} \cap \{x_2, x_3\} = \{x_2\} \notin \tau_{(F,A)} \oplus \tau_{(G,B)}$, $\tau_{(F,A)} \oplus \tau_{(G,B)}$ is not a topology by definition.

(4) $X_1 \times X_2 = \{(x_1, x_2), (x_1, x_3), (x_2, x_2), (x_2, x_3)\}$. The basis of the product topology is $\mathcal{B} = \{U \times V \mid U \in \tau_{(F,A)}, V \in \tau_{(G,B)}\} = \{\emptyset, X_1 \times X_2\}$, and the product topology generated by this basis is $\tau_{(F,A)} \times \tau_{(G,B)} = \{\emptyset, X_1 \times X_2\}$, which is a topology on $X_1 \times X_2$ by definition.

Remark 3.2. *As shown in Example 3.3, $\tau(F, A) \cup \tau(G, B)$ and $\tau(F, A) \oplus \tau(G, B)$ are not necessarily topologies, as they may fail to be closed under finite intersections. However, they form subbases; the smallest topology containing both is generated by the subbasis $\tau(F, A) \cup \tau(G, B)$. The union is a topology if and only if it is closed under finite intersections, which is particularly true when one topology is coarser than the other. Similarly, the sum is a topology if and only if $W_1 \cap W_2$ belongs to the sum for all W_1, W_2 in it. We retain these definitions because they arise naturally when combining two induced topologies, and the failure of closure itself provides insight into how the two parameterized families interact.*

Proposition 3.3. Let (F, A) and (G, B) be non-null soft sets over a common universe X. Assume that $X_1 = \bigcup_{a \in A} F(a)$ and $X_2 = \bigcup_{b \in B} G(b)$, $X_1 \cap X_2 \neq \emptyset$. Let $\tau_{(F,A)}$ and $\tau_{(G,B)}$ be the topologies on X_1 and X_2 generated by (F, A) and (G, B), respectively.

(1) The intersection of $\tau_{(F,A)}$ and $\tau_{(G,B)}$ is a topology on $X_1 \cap X_2$.

(2) The collection $\mathcal{B} = \{U \times V \mid U \in \tau_{(F,A)},\ V \in \tau_{(G,B)}\}$ is a basis for the product topology $\tau_{(F,A)} \times \tau_{(G,B)}$ defined in Definition 3.7; consequently, $\tau_{(F,A)} \times \tau_{(G,B)}$ is a topology on $X_1 \times X_2$.

Proof. (1) Let $Y = X_1 \cap X_2$. The intersection of $\tau_{(F,A)}$ and $\tau_{(G,B)}$ is defined as

$$\tau_{(F,A)} \cap \tau_{(G,B)} = \tau_{(F,A)}|_Y \cap \tau_{(G,B)}|_Y,$$

where $\tau|_Y = \{U \cap Y \mid U \in \tau\}$ denotes the restriction of a topology τ to Y. We verify that this intersection is a topology on Y by checking the three axioms of topology: $\emptyset \in \tau_{(F,A)}$ and $\emptyset \in \tau_{(G,B)}$, so $\emptyset = \emptyset \cap Y \in \tau_{(F,A)}|_Y \cap \tau_{(G,B)}|_Y$. Since $X_1 \in \tau_{(F,A)}$ and $X_2 \in \tau_{(G,B)}$, we have $Y = X_1 \cap Y \in \tau_{(F,A)}|_Y$ and $Y = X_2 \cap Y \in \tau_{(G,B)}|_Y$, so $Y \in \tau_{(F,A)} \cap \tau_{(G,B)}$. Let $\{W_i \mid i \in I\} \subseteq \tau_{(F,A)} \cap \tau_{(G,B)}$. By definition, $W_i \in \tau_{(F,A)}|_Y$ and $W_i \in \tau_{(G,B)}|_Y$ for all $i \in I$. Since $\tau_{(F,A)}|_Y$ and $\tau_{(G,B)}|_Y$ are topologies on Y, their unions are closed under arbitrary unions

$$\bigcup_{i\in I} W_i \in \tau_{(F,A)}|_Y \quad \text{and} \quad \bigcup_{i\in I} W_i \in \tau_{(G,B)}|_Y.$$

Thus, $\bigcup_{i\in I} W_i \in \tau_{(F,A)} \cap \tau_{(G,B)}$. Let $W_1, W_2 \in \tau_{(F,A)} \cap \tau_{(G,B)}$, then $W_1, W_2 \in \tau_{(F,A)}|_Y$ and $W_1, W_2 \in \tau_{(G,B)}|_Y$. Since topologies are closed under finite intersections

$$W_1 \cap W_2 \in \tau_{(F,A)}|_Y \quad \text{and} \quad W_1 \cap W_2 \in \tau_{(G,B)}|_Y,$$

thus $W_1 \cap W_2 \in \tau_{(F,A)} \cap \tau_{(G,B)}$. All axioms are satisfied, so $\tau_{(F,A)} \cap \tau_{(G,B)}$ is a topology on $Y = X_1 \cap X_2$.

(2) Let $\mathcal{B} = \{U \times V \mid U \in \tau_{(F,A)},\ V \in \tau_{(G,B)}\}$. We verify that $\mathcal{B}$ satisfies the two conditions of a basis. (i) Since $X_1 \in \tau_{(F,A)}$ and $X_2 \in \tau_{(G,B)}$, we have $X_1 \times X_2 \in \mathcal{B}$, and hence $\bigcup_{\mathcal{B}_i\in\mathcal{B}} \mathcal{B}_i = X_1 \times X_2$; moreover, $\emptyset = \emptyset \times \emptyset \in \mathcal{B}$. (ii) For any $\mathcal{B}_1 = U_1 \times V_1$ and $\mathcal{B}_2 = U_2 \times V_2$ in $\mathcal{B}$, where $U_1, U_2 \in \tau_{(F,A)}$ and $V_1, V_2 \in \tau_{(G,B)}$, since $\tau_{(F,A)}$ and $\tau_{(G,B)}$ are topologies, we have $U_1 \cap U_2 \in \tau_{(F,A)}$ and $V_1 \cap V_2 \in \tau_{(G,B)}$, and thus

$$\mathcal{B}_1 \cap \mathcal{B}_2 = (U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B},$$

which is trivially a union of members of $\mathcal{B}$. Let $\tilde{\tau} = \{\bigcup_{i\in I} \mathcal{B}_i \mid \mathcal{B}_i \in \mathcal{B},\ I \text{ is an index set}\}$ denote the collection of all unions of members of $\mathcal{B}$. Then $\emptyset$ and $X_1 \times X_2$ belong to $\tilde{\tau}$; any union of members of $\tilde{\tau}$ is a union of members of $\mathcal{B}$, and hence belongs to $\tilde{\tau}$; and for any $W_1 = \bigcup_{i\in I} \mathcal{B}_i$ and $W_2 = \bigcup_{j\in J} \mathcal{B}_j$ in $\tilde{\tau}$,

$$W_1 \cap W_2 = \bigcup_{(i,j)\in I\times J} (\mathcal{B}_i \cap \mathcal{B}_j),$$

which is a union of members of $\mathcal{B}$ by (ii), and hence belongs to $\tilde{\tau}$. Thus $\tilde{\tau}$ satisfies the three axioms of topology, $\mathcal{B}$ is a basis for $\tilde{\tau}$, and $\tilde{\tau}$ is precisely the product topology $\tau_{(F,A)} \times \tau_{(G,B)}$ defined in Definition 3.7. Consequently, $\tau_{(F,A)} \times \tau_{(G,B)}$ is a topology on $X_1 \times X_2$. □

Definition 3.8. Let $(X, \tau_{(F,A)})$ and $(Y, \tau_{(G,B)})$ be topological spaces generated by soft sets (F, A) and (G, B), respectively, where $X = \bigcup_{a\in A} F(a)$ and $Y = \bigcup_{b\in B} G(b)$. They are said to be homeomorphic, denoted by $(X, \tau_{(F,A)}) \cong (Y, \tau_{(G,B)})$, if there exists a bijection $f : X \to Y$ such that

(i) For every open set $V \in \tau_{(G,B)}$, the preimage $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ is open in X, i.e., $f^{-1}(V) \in \tau_{(F,A)}$.
(ii) For every open set $U \in \tau_{(F,A)}$, the image $f(U) = \{f(x) \mid x \in U\}$ is open in Y, i.e., $f(U) \in \tau_{(G,B)}$.

The bijection f is then called a homeomorphism.

Example 3.4. Let the universe $X = \mathbb{R}$, i.e., the set of real numbers. And let $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Define two soft sets (F, A) and (G, B) over X:

$$\begin{aligned}(F, A): &\quad F(a_1) = (0, 2), \quad F(a_2) = (1, 3),\\ (G, B): &\quad G(b_1) = (1, 3), \quad G(b_2) = (2, 4).\end{aligned}$$

It is straightforward to obtain that the topological spaces $(X_1, \tau_{(F,A)})$ and $(X_2, \tau_{(G,B)})$ generated by soft sets (F, A) and (G, B) are as follows:

$$(X_1, \tau_{(F,A)}) = ((0, 3), \{\emptyset, (0, 3), (0, 2), (1, 3), (1, 2)\}) \quad (X_2, \tau_{(G,B)}) = ((1, 4), \{\emptyset, (1, 4), (1, 3), (2, 4), (2, 3)\}).$$

Define a linear function $f : X_1 = (0, 3) \to X_2 = (1, 4)$ by $f(x) = x + 1$. Obviously, f is injective and surjective, as its inverse is $f^{-1}(y) = y - 1$, which maps $(1, 4)$ back to $(0, 3)$. So f is a bijection from X_1 to X_2. For every open set $V \in \tau_{(G,B)}$, the preimage $f^{-1}(V)$ is as follows:

$$\begin{aligned}&f^{-1}(\emptyset) = \emptyset \in \tau_{(F,A)}, f^{-1}((1, 4)) = (0, 3) = X_1 \in \tau_{(F,A)}, f^{-1}((1, 3)) = (0, 2) = F(a_1) \in \tau_{(F,A)},\\ &f^{-1}((2, 4)) = (1, 3) = F(a_2) \in \tau_{(F,A)}, f^{-1}((2, 3)) = (1, 2) \in \tau_{(F,A)}.\end{aligned}$$

For every open set $U \in \tau_{(F,A)}$, the image $f(U)$ is as follows:

$$\begin{aligned}&f(\emptyset) = \emptyset \in \tau_{(G,B)}, \quad f((0, 3)) = (1, 4) = X_2 \in \tau_{(G,B)}, \quad f((0, 2)) = (1, 3) = G(b_1) \in \tau_{(G,B)},\\ &f((1, 3)) = (2, 4) = G(b_2) \in \tau_{(G,B)}, \quad f((1, 2)) = (2, 3) \in \tau_{(G,B)}.\end{aligned}$$

Then the function $f(x) = x + 1$ is a homeomorphism, and $(X_1, \tau_{(F,A)}) \cong (X_2, \tau_{(G,B)})$.

Proposition 3.4. Let $(X_1, \tau_{(F,A)})$ and $(X_2, \tau_{(G,B)})$ be topological spaces generated by soft sets (F, A) and (G, B), respectively. Then the relationship $(X_1, \tau_{(F,A)}) \cong (X_2, \tau_{(G,B)})$ is an equivalence relation, satisfying reflexivity, symmetry, and transitivity.

Proof. Reflexivity holds via the identity map. Symmetry holds because a homeomorphism has a continuous inverse: If f satisfies conditions (i) and (ii) of Definition 2.13, then f^{-1} satisfies them as well. Transitivity holds because the composition of two homeomorphisms is again a homeomorphism. Thus the relation satisfies reflexivity, symmetry, and transitivity and is therefore an equivalence relation by Definition 3.8. □

Hence, homeomorphism is an equivalence relation on the class of topological spaces generated by soft sets. Two such spaces are equivalent precisely when there exists a homeomorphism between them. This equivalence relation depends only on the induced topological structures, while the parameters of the soft sets serve merely as auxiliary labels for the associated families of subsets.

3.3. Parameterized family-based distance and similarity of soft sets

The topological framework characterizes the structure of soft sets qualitatively. However, many applications require a quantitative comparison of soft sets, for instance, to rank alternatives in decision making. In this section, we propose axioms for distance and similarity measures of soft sets and construct concrete measures directly from the parameterized families that determine their structure.

Definition 3.9. Let (F, A) and (G, B) be two soft sets over the common universe X. Let the families $\mathfrak{S}(F, A) = \{F(a) \mid a \in A\}$ and $\mathfrak{S}(G, B) = \{G(b) \mid b \in B\}$. Then the distance measure between (F, A) and (G, B) is defined as $d((F, A), (G, B))$, which satisfies the following properties:

(i) $0 \leq d((F, A), (G, B)) \leq 1$.
(ii) $d((F, A), (G, B)) = 0$ if and only if $\mathfrak{S}(F, A) = \mathfrak{S}(G, B)$.
(iii) $d((F, A), (G, B)) = d((G, B), (F, A))$.

Definition 3.10. Let (F, A) and (G, B) be two soft sets over the common universe X. Let the families $\mathfrak{S}(F, A) = \{F(a) \mid a \in A\}$ and $\mathfrak{S}(G, B) = \{G(b) \mid b \in B\}$. Then the similarity measure between (F, A) and (G, B) is defined as $s((F, A), (G, B))$, which satisfies the following properties:

(i) $0 \leq s((F, A), (G, B)) \leq 1$.
(ii) $s((F, A), (G, B)) = 1$ if and only if $\mathfrak{S}(F, A) = \mathfrak{S}(G, B)$.
(iii) $s((F, A), (G, B)) = s((G, B), (F, A))$.

Proposition 3.5. Let a function $d : \mathfrak{S} \times \mathfrak{S} \to [0, 1]$ be a distance measure. Then d is a correct relationship, i.e., for any four soft sets (F, A), (F', A'), (G, B), and (G', B') over X with $(F, A) \cong (F', A')$ and $(G, B) \cong (G', B')$, the equality $d((F, A), (G, B)) = d((F', A'), (G', B'))$ holds.

Proof. By definition, $d((F, A), (G, B)) = d(\mathfrak{S}(F, A), \mathfrak{S}(G, B))$ and $d((F', A'), (G', B')) = d(\mathfrak{S}(F', A'), \mathfrak{S}(G', B'))$. For any soft sets $(F, A) \cong (F', A')$ and $(G, B) \cong (G', B')$, equivalence of soft sets implies $\mathfrak{S}(F, A) = \mathfrak{S}(F', A')$ and $\mathfrak{S}(G, B) = \mathfrak{S}(G', B')$. Thus, $d(\mathfrak{S}(F, A), \mathfrak{S}(G, B)) = d(\mathfrak{S}(F', A'), \mathfrak{S}(G', B'))$, which implies $d((F, A), (G, B)) = d((F', A'), (G', B'))$. Hence, d is a correct relationship. □

For the same reason, if s is a similarity measure for soft sets as defined above, then s is also a correct relationship.

Proposition 3.6. Let (F, A) and (G, B) be two soft sets over the common universe X. Let the families $\mathfrak{S}(F, A) = \{F(a) \mid a \in A\}$ and $\mathfrak{S}(G, B) = \{G(b) \mid b \in B\}$. A function $d : \mathfrak{S} \times \mathfrak{S} \to [0, 1]$ is a distance measure if for $\mathfrak{S}(F, A)$, and $\mathfrak{S}(G, B)$ over X, it takes the form:

$$d((F, A), (G, B)) = \frac{|\mathfrak{S}(F, A) \triangle \mathfrak{S}(G, B)|}{|\mathfrak{S}(F, A) \cup \mathfrak{S}(G, B)|},$$

where $\triangle$ denotes the symmetric difference of sets, i.e., $S \triangle T = (S \setminus T) \cup (T \setminus S)$, and $|\cdot|$ denotes the cardinality of a set.

Proof. (1) The symmetric difference $\mathfrak{S}(F, A) \triangle \mathfrak{S}(G, B)$ is a set, and its cardinality $|\cdot|$ is non-negative. Similarly, the union $\mathfrak{S}(F, A) \cup \mathfrak{S}(G, B)$ is also a set with non-negative cardinality. Furthermore, $\mathfrak{S}(F, A) \triangle \mathfrak{S}(G, B) \subseteq \mathfrak{S}(F, A) \cup \mathfrak{S}(G, B)$. Hence, $0 \leq d((F, A), (G, B)) \leq 1$. This completes the proof of (i).

(2) If $\mathfrak{S}(F, A) = \mathfrak{S}(G, B)$, then the symmetric difference of two identical families is empty, so $|\mathfrak{S}(F, A) \triangle \mathfrak{S}(G, B)| = 0$. Therefore, $d((F, A), (G, B)) = \frac{0}{|\mathfrak{S}(F,A) \cup \mathfrak{S}(G,B)|} = 0$. If $d((F, A), (G, B)) = 0$, then $|\mathfrak{S}(F, A) \triangle \mathfrak{S}(G, B)| = 0$, implying $\mathfrak{S}(F, A) \triangle \mathfrak{S}(G, B) = \emptyset$. This means $\mathfrak{S}(F, A) = \mathfrak{S}(G, B)$. This completes the proof of (ii).

(3) The symmetric difference is commutative: $\mathfrak{S}(F, A) \triangle \mathfrak{S}(G, B) = \mathfrak{S}(G, B) \triangle \mathfrak{S}(F, A)$. Thus $|\mathfrak{S}(F, A) \triangle \mathfrak{S}(G, B)| = |\mathfrak{S}(G, B) \triangle \mathfrak{S}(F, A)|$. Additionally, the union is commutative: $|\mathfrak{S}(F, A) \cup \mathfrak{S}(G, B)| = |\mathfrak{S}(G, B) \cup \mathfrak{S}(F, A)|$. Therefore, $d((F, A), (G, B)) = d((G, B), (F, A))$. So the proof is concluded. □

Proposition 3.7. Let (F, A) and (G, B) be two soft sets over the common universe X. Let the families $\mathfrak{S}(F, A) = \{F(a) \mid a \in A\}$ and $\mathfrak{S}(G, B) = \{G(b) \mid b \in B\}$. A function $s : \mathfrak{S} \times \mathfrak{S} \to [0, 1]$ is a similarity measure if for $\mathfrak{S}(F, A)$ and $\mathfrak{S}(G, B)$ over X, it takes the form:

$$s((F, A), (G, B)) = 1 - \frac{|\mathfrak{S}(F, A) \triangle \mathfrak{S}(G, B)|}{|\mathfrak{S}(F, A) \cup \mathfrak{S}(G, B)|} = \frac{|\mathfrak{S}(F, A) \cap \mathfrak{S}(G, B)|}{|\mathfrak{S}(F, A) \cup \mathfrak{S}(G, B)|},$$

where $\triangle$ denotes the symmetric difference of sets, i.e, $S \triangle T = (S \setminus T) \cup (T \setminus S)$, and $|\cdot|$ denotes the cardinality of a set.

Proof. The proof is similar to that of Proposition 3.6. □

Remark 3.3. *When $\mathfrak{S}(F, A)$ and $\mathfrak{S}(G, B)$ are finite, the quantity in Proposition 3.7 is the classical Jaccard similarity of the two families regarded as sets of sets, and the distance in Proposition 3.6 is the complementary Jaccard distance. The added value of the present formulation is twofold: The objects being compared are the parameterized families (subsets of the universe) that determine the structure of soft sets, rather than individual elements or membership degrees; and, because soft-set equivalence is defined by equality of these families, both measures are automatically correct relationships in Molodtsov's sense.*

Example 3.5. Let the universe $X = \{x_1, x_2, x_3\}$. Let $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Define two soft sets (F, A) and (G, B) over X:

$$\begin{aligned} (F, A): &\quad F(a_1) = \{x_1\}, \quad F(a_2) = \{x_1, x_2\}, \\ (G, B): &\quad G(b_1) = \{x_1, x_2\}, \quad G(b_2) = \{x_2, x_3\}. \end{aligned}$$

Thus the families for (F, A) and (G, B) are as follows:

$$\mathfrak{S}(F, A) = \{\{x_1\}, \{x_1, x_2\}\}, \quad \mathfrak{S}(G, B) = \{\{x_1, x_2\}, \{x_2, x_3\}\}.$$

Subsequently, one can obtain

$$\begin{aligned} \mathfrak{S}(F, A) \triangle \mathfrak{S}(G, B) &= \{\{x_1\}, \{x_2, x_3\}\}, & |\mathfrak{S}(F, A) \triangle \mathfrak{S}(G, B)| &= 2, \\ \mathfrak{S}(F, A) \cap \mathfrak{S}(G, B) &= \{\{x_1, x_2\}\}, & |\mathfrak{S}(F, A) \cap \mathfrak{S}(G, B)| &= 1, \\ \mathfrak{S}(F, A) \cup \mathfrak{S}(G, B) &= \{\{x_1\}, \{x_1, x_2\}, \{x_2, x_3\}\}, & |\mathfrak{S}(F, A) \cup \mathfrak{S}(G, B)| &= 3. \end{aligned}$$

Hence,

$$d((F, A), (G, B)) = 2/3, \quad s((F, A), (G, B)) = 1/3.$$

Note that these distance and similarity measures are straightforward to compute and rely on the parameterized families themselves, thus offering a practical tool for comparing soft sets. We next apply them to a decision making problem with approximately described information.

4. An application in MADM based on the similarity of soft sets

MADM is a critical branch of decision analysis that involves selecting the optimal alternative from a finite set of candidates based on multiple attributes. In real-world scenarios, such as supplier selection,

project evaluation, medical diagnosis, or product recommendation, decision-makers frequently face the challenges that the structure of information is usually characterized by approximate descriptions, and thus rigorous mathematical models are inappropriate for handling this kind of problem. In this section, we propose a novel decision making approach based on the similarity measure of soft sets to deal with MADM in an environment with approximate descriptions. This framework of the MADM problem can be expressed as follows:

Suppose $A = \{a_1, a_2, \cdots, a_n\}$ is a collection of attributes used as evaluation criteria, and $X = \{x_1, x_2, \cdots, x_m\}$ is the set of decision alternatives evaluated under these attributes. Due to subjective judgments and imprecise information, evaluations of attributes and their results are often approximate. To handle this, for each alternative $x_k \in X$, a baseline soft set (F_k^*, A) over X is constructed, where $F_k^* : A \to \mathcal{P}(X)$ assigns each attribute $a \in A$ a subset $F_k^*(a) \subseteq X$ representing the typical results of x_k under a. Correspondingly, an actual soft set (F, A) over X is also constructed, with the mapping $F : A \to \mathcal{P}(X)$ describing the actual results of the decision object under each attribute $a \in A$. The decision maker ranks the alternatives by comparing the similarity values between each baseline soft set (F_k^*, A) and the actual soft set (F, A). The alternative with the highest similarity value between its baseline soft set and actual soft set is chosen as the optimal decision. This framework incorporates the approximate nature of information into the soft set structure, using similarity to objectively measure the degree to which each alternative fits the baseline standards, thus establishing a rational decision procedure.

Algorithm 4.1. The proposed algorithm aims to handle this kind of MADM problem by quantifying the similarity value between each baseline soft set (BSS) and the actual soft set (ASS) using the above similarity measure. The steps are illustrated as follows:

Step 1: Input the set of parameters A and the discrete universe of alternatives X.

Step 2: Construct the BSSs (F_k^*, A) $(k = 1, 2, \ldots, m)$ over X that represent the baseline standard, and construct the ASS (F, A) that represents the actual performance of the decision object.

Step 3: Extract the families of BSSs and the ASS, thus obtaining $\Im(F_k^*, A) = \{F_k^*(a) \mid a \in A\}$ $(k = 1, 2, \ldots, m)$ and $\Im(F, A) = \{F(a) \mid a \in A\}$.

Step 4: Compute the similarity measure between each (F_k^*, A) $(k = 1, 2, \ldots, m)$ and (F, A).

Step 5: Rank all alternatives based on $s\left((F_k^*, A), (F, A)\right)$ and select the alternative with the highest similarity value as the optimal choice.

The flowchart of the algorithm above is shown in Figure 2.

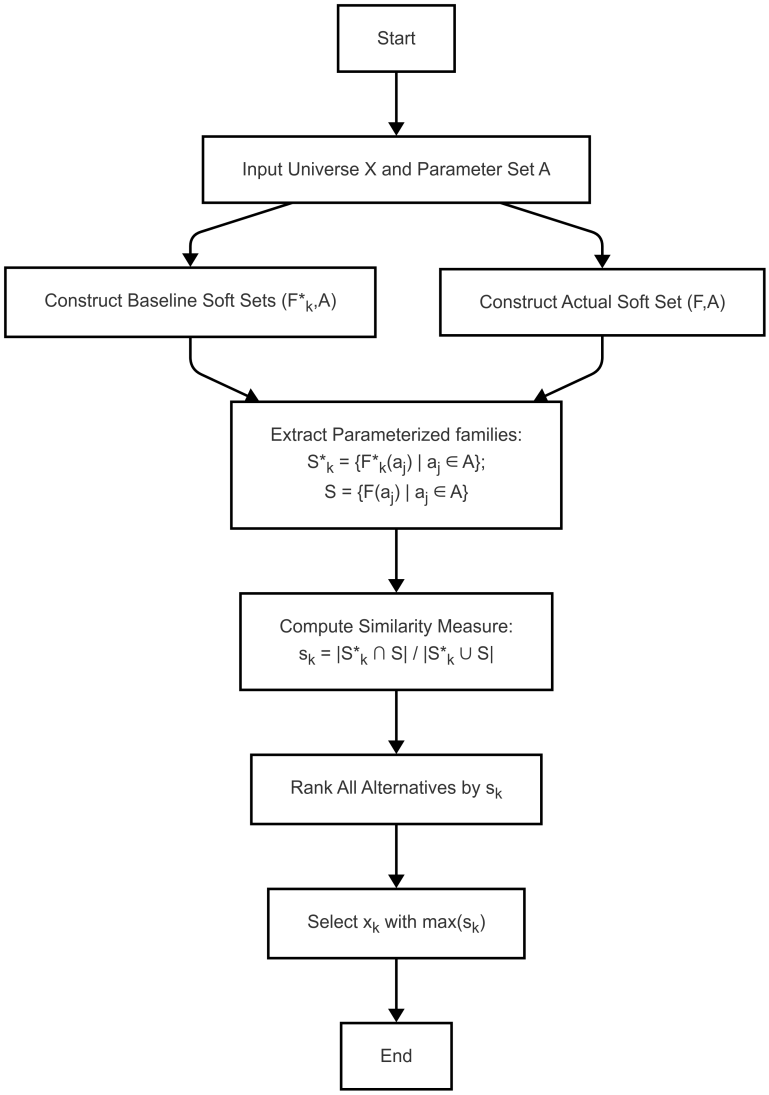


Figure 2. The flowchart of the MADM problem algorithm based on similarity of soft sets.

In order to test the validity and effectiveness of this method, an illustrative example is provided as follows.

Example 4.1. Consider a medical diagnosis scenario with the universe of diseases $X = \{d_1, d_2, d_3, d_4\}$, where d_1: *Common cold*; d_2: *Influenza*; d_3: *Bacterial pneumonia*; d_4: *Bronchitis*. Let $A = \{a_1, a_2, a_3\}$ be a set of symptom parameters, where a_1: *fever*; a_2: *cough*; a_3: *fatigue*. Assume that there exists a patient who probably suffers from one of diseases in X and shows some symptoms from A. Based on the patient's description of exhibited symptoms, the ASS (F, A) is constructed, in which the sets of diseases causing each symptom are

$F(a_1) = \{d_1, d_2, d_3\}$(*fever* may be caused by *Common cold*, *Influenza*, *Bacterial pneumonia*),
$F(a_2) = \{d_1, d_2, d_3, d_4\}$(*cough* may be caused by all four diseases),
$F(a_3) = \{d_2, d_3\}$(*fatigue* may be caused by *Influenza*, *Bacterial pneumonia*).

Typical symptoms with regard to each disease $d_k \in X$ are described in the form of the BSSs as follows:

The BSS (F_1^*, A) with regard to d_1 : $F_1^*(a_1) = \{d_1, d_2\}$, $F_1^*(a_2) = \{d_1, d_4\}$, $F_1^*(a_3) = \{d_1\}$,
The BSS (F_2^*, A) with regard to d_2 : $F_2^*(a_1) = \{d_2, d_3\}$, $F_2^*(a_2) = \{d_2, d_3\}$, $F_2^*(a_3) = \{d_2, d_3\}$,
The BSS (F_3^*, A) with regard to d_3 : $F_3^*(a_1) = \{d_2, d_3\}$, $F_3^*(a_2) = \{d_1, d_2, d_3, d_4\}$, $F_3^*(a_3) = \{d_2, d_3, d_4\}$,
The BSS (F_4^*, A) with regard to d_4 : $F_4^*(a_1) = \{d_3, d_4\}$, $F_4^*(a_2) = \{d_1, d_3, d_4\}$, $F_4^*(a_3) = \emptyset$.

The aim is to diagnose the most likely disease for this patient by comparing the similarity between the patient's ASS and the BSSs of potential diseases, using the proposed similarity measure.

Implementing the proposed algorithm above, one can obtain the family with regard to the ASS (F, A), expressed as

$$\mathfrak{S}(F, A) = \{\{d_1, d_2, d_3\}, \{d_1, d_2, d_3, d_4\}, \{d_2, d_3\}\},$$

and the families with regard to the BSSs (F_k^*, A) $(k = 1, 2, \ldots, m)$, which are expressed as

$$\begin{aligned}
&\mathfrak{S}(F_1^*, A) = \{\{d_1, d_2\}, \{d_1, d_4\}, \{d_1\}\},\\
&\mathfrak{S}(F_2^*, A) = \{\{d_2, d_3\}\},\\
&\mathfrak{S}(F_3^*, A) = \{\{d_2, d_3\}, \{d_1, d_2, d_3, d_4\}, \{d_2, d_3, d_4\}\},\\
&\mathfrak{S}(F_4^*, A) = \{\{d_3, d_4\}, \{d_1, d_3, d_4\}, \emptyset\}.
\end{aligned}$$

By calculating similarities using the formula $s_k = \frac{|\mathfrak{S}(F_k^*, A) \cap \mathfrak{S}(F, A)|}{|\mathfrak{S}(F_k^*, A) \cup \mathfrak{S}(F, A)|}$, one can obtain

$$s_1 = \frac{0}{6} = 0, \quad s_2 = \frac{1}{3} \approx 0.33, \quad s_3 = \frac{2}{4} = 0.50, \quad s_4 = \frac{0}{6} = 0.$$

Thus the similarities are $s_3(0.50) > s_2(0.33) > s_1(0) = s_4(0)$. Therefore, the patient is most likely suffering from d_3 (Bacterial pneumonia).

To verify the rationality of the proposed algorithm, a comparative analysis is conducted between the decision making results based on the same data as in Example 4.1 using the similarity defined in this paper, and those from the models proposed by Acharjee and Oza [34], Aygün and Kamacı [38], Majumdar and Samanta [39], and Kharal [40]. The rankings of alternatives generated by these models are presented in Table 1.

Table 1. Comparison of ranking results from different approaches.

Different approaches	Ranking results
Acharjee and Oza [34]	$s_3(0.67) > s_2(0.50) > s_1(0.33) = s_4(0.33)$
Aygün and Kamacı [38]	$s_3(0.83) > s_2(0.75) > s_1(0.50) = s_4(0.50)$
Majumdar and Samanta [39]	$s_3(0.80) > s_2(0.60) > s_1(0.40) > s_4(0.36)$
Kharal [40]	$s_3(0.42) > s_2(0.31) > s_1(0.21) = s_4(0.21)$
The proposed approach	$s_3(0.50) > s_2(0.33) > s_1(0) = s_4(0)$

As shown in Table 1, the ranking results and the optimal alternative (i.e., d_3) are entirely consistent across all the compared methods. The only minor difference lies in the ranking of d_1 and d_4 from Majumdar and Samanta's method, where d_1 is shown to slightly dominate d_4, whereas the other four

methods indicate their equivalence. The consistency of the ranking results indicates the credibility of the proposed approach. It should be noted, however, that a single illustrative example cannot establish the general superiority or robustness of the algorithm. What the proposed measure ensures is theoretical adequacy and computational simplicity: On the one hand, it is defined directly on the parameterized families that determine the structures of soft sets; on the other hand, it requires only the cardinalities of the intersection and union of the two families, i.e., elementary set operations, without auxiliary weights or membership functions, and hence possesses the advantages of low computational complexity and simple implementability.

5. Discussion

Fundamentally, a soft set is a parameterized family of subsets over a universal set. It provides an approximate description of objects through a parameter set, which serves as an auxiliary tool without inherent restrictions on its type or structure, merely indicating subsets of the universe. A considerable body of research has focused on defining operations for soft sets based on parameter manipulations. However, such operations may often diverge from the core idea of soft sets, and thus fail to fully reflect their inherent advantages. Inspired by the foundational work [34] and [35], this study further clarifies the characteristics of soft sets and systematizes their fundamental operations which match the essential nature of soft sets, accompanied by corresponding examples.

It is worth noting that the family of a soft set plays a central role in determining its structure. While this family is not necessarily a topology itself, it can be regarded as an analogue of topology. Alternatively, it can play the role of a basis to generate a topology, namely the topology generated by the soft set. Investigating this topology and its fundamental operations facilitates a deeper understanding of the structural properties of a soft set.

This study also defines distance and similarity measures directly based on the parameterized families, which determine the structures of soft sets. A comparative analysis of ranking results obtained by the other four similarity measures illustrates the reasonableness and applicability of these definitions. Furthermore, these measures naturally extend to defining distances and similarities between topologies generated by different soft sets. This extension enables quantitative assessment of differences or consistencies in topological structures that represent uncertain information generated by soft sets. Such techniques may hold significant potential for applications in fields where measuring structural uncertainty is crucial, such as pattern recognition and data analysis.

6. Conclusions

To clarify the essential structure of soft sets, this paper reviews the concept and basic operations of soft sets from the perspective of parameterized families, providing illustrative examples for explanation. A brief comparative analysis with fuzzy sets and rough sets is also included. Since the family defining a soft set determines its structure and serves as a topological analogue, a detailed study of the topology generated by soft sets is presented, including some basic concepts and operations with examples and proofs. Furthermore, distance and similarity measures based on families of soft sets are also proposed, and an algorithm for MADM in environments with approximate descriptive information is developed accordingly. Finally, an illustrative numerical example in medical diagnosis and a

comparative analysis of decision results based on other similarity measures of soft sets demonstrate the rationality and applicability of the proposed algorithm. It should be noted that these results are obtained from a single illustrative case rather than broader-scale experiments; the aim of the example is to verify the correctness and feasibility of the algorithmic procedure, and large-scale validation on real-world datasets will be carried out in future work.

This study provides a starting point for exploring soft sets from a topological perspective, which is consistent with their nature as parameterized families. Since the framework represents uncertain information by set-valued approximations under unrestricted parameterization, it can be applied to domains where information is approximate, qualitative, or incomplete, such as medical diagnosis, risk assessment, and decision making with incomplete information. In medical diagnosis, symptoms serve directly as parameters, and comparing a patient's soft set with the baseline soft sets of candidate diseases parallels the matching of presentations to typical disease profiles. In risk assessment, heterogeneous qualitative factors are handled without membership functions, and the similarity measure supports comparison with reference risk profiles. In decision making under incomplete information, approximate descriptions can be used to handle imprecise or missing evaluations, while ranking uses only set-theoretic operations and has low computational cost. Future work may investigate the generated topology for quantifying information uncertainty, compare and integrate the topological features of soft sets with those of fuzzy and rough sets, and develop distance and similarity measures aligned with the structure of parameterized families, as well as conduct broader-scale experiments on real-world datasets to further validate the proposed algorithm, together with their applications in these domains.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The work is sponsored by Scientific Research Foundation for Advanced Talents of Chongqing Technology and Business University, China (2155011). Many thanks to the editor and the anonymous reviewers for their constructive comments and sincere assistance.

Conflict of interest

The author declares no competing interests to disclose.

References

1. L. A. Zadeh, Fuzzy sets, *Inf. Control*, **8** (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X
2. K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets Syst.*, **20** (1986), 87–96. https://doi.org/10.1016/S0165-0114(86)80034-3

3. Z. Pawlak, Rough sets, *Int. J. Comput. Inf. Sci.*, **11** (1982), 341–356. https://doi.org/10.1007/BF01001956
4. W. L. Gau, D. J. Buehrer, Vague sets, *IEEE Trans. Syst. Man Cybern.*, **23** (1993), 610–614. https://doi.org/10.1109/21.229476
5. M. B. Gorzałczany, A method of inference in approximate reasoning based on interval-valued fuzzy sets, *Fuzzy Sets Syst.*, **21** (1987), 1–17. https://doi.org/10.1016/0165-0114(87)90148-5
6. D. Molodtsov, Soft set theory-first results, *Comput. Math. Appl.*, **37** (1999), 19–31. https://doi.org/10.1016/S0898-1221(99)00056-5
7. Z. Pawlak, Hard and soft sets, In: *Rough sets, fuzzy sets and knowledge discovery*, London: Springer, 1994, 130–135. https://doi.org/10.1007/978-1-4471-3238-7_15
8. K. Atanassov, G. Gargov, Interval valued intuitionistic fuzzy sets, *Fuzzy Sets Syst.*, **31** (1989), 343–349. https://doi.org/10.1016/0165-0114(89)90205-4
9. J. M. Mendel, R. I. B. John, Type-2 fuzzy sets made simple, *IEEE Trans. Fuzzy Syst.*, **10** (2002), 117–127. https://doi.org/10.1109/91.995115
10. J. M. Mendel, R. I. John, F. Liu, Interval type-2 fuzzy logic systems made simple, *IEEE Trans. Fuzzy Syst.*, **14** (2006), 808–821. https://doi.org/10.1109/TFUZZ.2006.879986
11. Y. Yao, B. Yao, Covering based rough set approximations, *Inf. Sci.*, **200** (2012), 91–107. https://doi.org/10.1016/j.ins.2012.02.065
12. D. Dubois, H. Prade, Rough fuzzy sets and fuzzy rough sets, *Int. J. Gen. Syst.*, **17** (1990), 191–209. https://doi.org/10.1080/03081079008935107
13. W. Ziarko, Variable precision rough set model, *J. Comput. Syst. Sci.*, **46** (1993), 39–59. https://doi.org/10.1016/0022-0000(93)90048-2
14. P. K. Maji, R. Biswas, A. R. Roy, Soft set theory, *Comput. Math. Appl.*, **45** (2003), 555–562. https://doi.org/10.1016/S0898-1221(03)00016-6
15. D. G. Chen, E. C. C. Tsang, D. S. Yeung, X. Wang, The parameterization reduction of soft sets and its applications, *Comput. Math. Appl.*, **49** (2005), 757–763. https://doi.org/10.1016/j.camwa.2004.10.036
16. M. I. Ali, F. Feng, X. Liu, W. K. Min, M. Shabir, On some new operations in soft set theory, *Comput. Math. Appl.*, **57** (2009), 1547–1553. https://doi.org/10.1016/j.camwa.2008.11.009
17. A. Sezgin, A. O. Atagün, On operations of soft sets, *Comput. Math. Appl.*, **61** (2011), 1457–1467. https://doi.org/10.1016/j.camwa.2011.01.018
18. F. Feng, Y. Li, Soft subsets and soft product operations, *Inf. Sci.*, **232** (2013), 44–57. https://doi.org/10.1016/j.ins.2013.01.001
19. Z. Kong, L. Gao, L. Wang, S. Li, The normal parameter reduction of soft sets and its algorithm, *Comput. Math. Appl.*, **56** (2008), 3029–3037. https://doi.org/10.1016/j.camwa.2008.07.013
20. J. M. Zhan, J. C. R. Alcantud, A survey of parameter reduction of soft sets and corresponding algorithms, *Artif. Intell. Rev.*, **52** (2019), 1839–1872. https://doi.org/10.1007/s10462-017-9592-0
21. N. Çağman, S. Karataş, S. Enginoğlu, Soft topology, *Comput. Math. Appl.*, **62** (2011), 351–358. https://doi.org/10.1016/j.camwa.2011.05.016
22. P. Majumdar, S. K. Samanta, On soft mappings, *Comput. Math. Appl.*, **60** (2010), 2666–2672. https://doi.org/10.1016/j.camwa.2010.09.004

23. A. R. Roy, P. K. Maji, A fuzzy soft set theoretic approach to decision making problems, *J. Comput. Appl. Math.*, **203** (2007), 412–418. https://doi.org/10.1016/j.cam.2006.04.008
24. P. Majumdar, S. K. Samanta, Generalised fuzzy soft sets, *Comput. Math. Appl.*, **59** (2010), 1425–1432. https://doi.org/10.1016/j.camwa.2009.12.006
25. F. Feng, X. Liu, V. Leoreanu-Fotea, Y. B. Jun, Soft sets and soft rough sets, *Inf. Sci.*, **181** (2011), 1125–1137. https://doi.org/10.1016/j.ins.2010.11.004
26. W. Xu, J. Ma, S. Wang, G. Hao, Vague soft sets and their properties, *Comput. Math. Appl.*, **59** (2010), 787–794. https://doi.org/10.1016/j.camwa.2009.10.015
27. N. Çağman, S. Enginoğlu, Soft set theory and uni-int decision making, *Eur. J. Oper. Res.*, **207** (2010), 848–855. https://doi.org/10.1016/j.ejor.2010.05.004
28. Y. Zou, Bayesian decision making with soft probabilities under statistical regularity, *AIMS Mathematics*, **11** (2026), 10883–10907. https://doi.org/10.3934/math.2026447
29. Y. Zou, Z. Xiao, Data analysis approaches of soft sets under incomplete information, *Knowl.-Based Syst.*, **21** (2008), 941–945. https://doi.org/10.1016/j.knosys.2008.04.004
30. F. Feng, J. Cho, W. Pedrycz, H. Fujita, T. Herawan, Soft set based association rule mining, *Knowl.-Based Syst.*, **111** (2016), 268–282. https://doi.org/10.1016/j.knosys.2016.08.020
31. W. Xu, Z. Xiao, X. Dang, D. Yang, X. Yang, Financial ratio selection for business failure prediction using soft set theory, *Knowl.-Based Syst.*, **63** (2014), 59–67. https://doi.org/10.1016/j.knosys.2014.03.007
32. K. H. Chang, A more general risk assessment methodology using a soft set-based ranking technique, *Soft Comput.*, **18** (2014), 169–183. https://doi.org/10.1007/s00500-013-1045-3
33. D. Orhan, N. Demirta, Algorithms for Covid-19 outbreak using soft set theory: Estimation and application, *Soft Comput.*, **27** (2023), 3203–3211. https://doi.org/10.1007/s00500-022-07519-5
34. D. Molodtsov, Equivalence and correct operations for soft sets, *Int. Rob. Auto. J.*, **4** (2018), 18–21. https://doi.org/10.15406/iratj.2018.04.00086
35. S. Acharjee, A. Oza, Correct structures and similarity measures of soft sets along with historic comments of Prof. D. A. Molodtsov, *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki*, **33** (2023), 32–53. https://doi.org/10.48550/arXiv.2206.03725
36. Z. Xiao, Y. Zou, A comparative study of soft sets with fuzzy sets and rough sets, *J. Intell. Fuzzy Syst.*, **27** (2014), 425–434. https://doi.org/10.3233/IFS-131010
37. S. A. Morris, *Topology without tears*, 2018.
38. E. Aygün, H. Kamacı, Some new algebraic structures of soft sets, *Soft Comput.*, **25** (2021), 8609–8626. https://doi.org/10.1007/s00500-021-05744-y
39. P. Majumdar, S. K. Samanta, Similarity measure of soft sets, *New Math. Nat. Comput.*, **4** (2008), 1–12. https://doi.org/10.1142/S1793005708000908
40. A. Kharal, Distance and similarity measures for soft sets, *New Math. Nat. Comput.*, **6** (2010), 321–334. https://doi.org/10.1142/S1793005710001724



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)