AIMS Mathematics, 11(11): 38653–38675.
DOI:10.3934/math.20261491
Received: 11 September 2026
Revised: 9 November 2026
Accepted: 13 November 2026
Published: 19 November 2026

https://www.aimspress.com/journal/Math

Research article

Decision-making with entropy-driven particle swarm optimization for plithogenic Pythagorean fuzzy hypersoft sets

Mahrukh Irfan[1], **Usman Afzal**[1], **Shah Zeb**[2,*], **Khalil Ur Rehman**[3,*] **and Wasfi Shatanawi**[3]

[1] Department of Mathematics, University of Management and Technology, C-II Johar Town, Lahore, 54770 Punjab, Pakistan

[2] School of Distance Education, Universiti Sains Malaysia, Minden, USM, 11800, Penang, Malaysia

[3] Department of Mathematics and Sciences, College of Science and Humanities, Prince Sultan University, Riyadh 11586, Saudi Arabia

* **Correspondence:** Email: shahzeb@student.usm.my, kurrehman@psu.edu.sa.

Abstract: The computational complexity inherent in decision-making systems appears from the ontological uncertainty of the data. This complexity is further compounded by the intricacies of human brain activity, which include linguistic imprecision, internal conflicts, and the nuanced structure of perception, often resulting in the breakdown of conventional frameworks. The interdependent nature of multi-attribute environments poses challenges for modeling using traditional classical optimization and fuzzy sets. In this study, we proposed a more reliable framework, namely entropy-based particle swarm optimization, tailored for the plithogenic Pythagorean fuzzy hypersoft set environment. To effectively manage the PSO search process and balance exploration and exploitation in response to the evolving landscape of uncertainty, an entropy measure was developed to quantify the underlying informational ambiguity. Through the proposed integration, entropy was transformed from a passive measure of uncertainty into an active driver of optimization. The application of the proposed AI-driven optimization structure in a chatbot evaluation framework demonstrated improved transparency, robustness, and decision precision, highlighting the efficacy of the suggested framework in addressing complex real-world decision-making scenarios.

Keywords: particle swarm optimization; entropy; decision making; uncertainty; plithogenic Pythagorean fuzzy hypersoft set
Mathematics Subject Classification: 90B50, 03E72, 90C59, 94A17

1. Introduction

The empirical reality of conventional mathematical models and the human condition mostly show incompatibility. The bivalent concept of belonging is the basis for traditional set theory. Still, in reality, classes of things will have a graded level of membership anywhere from a group of words of a language to living organisms. For example, how do you define a person's height? Tallness is not a well-defined number, nor is there a well-defined number for height. Therefore, one could say that tallness and height are not sets according to the classical definition of a set, but these categories do enable people to categorize things in a loose sense. Here, we are trying to develop a solid base for dealing with the inherent uncertainty found within natural phenomena, by looking at the basic characteristics of a conceptual scheme to deal with a non-discrete set of categories.

In 1965, Zadeh [1] developed an innovative idea of fuzzy sets as a class of sets that can handle different types of ambiguity by introducing the degree of membership. Building on this definition, Shahzad et al. [2] developed fuzzy mappings and examines when a fuzzy set remains stable under them, along with some applications of the results. Atanassov [3] subsequently extended fuzzy sets to form intuitionistic fuzzy sets (IFS). This set consists of an ordered pair of values (λ, γ), which represent the two types of membership of an element and satisfy the inequalities $\lambda + \gamma \leq 1$. The use of IFS provides a systematic means to determine a resolution for any value-based conflicts that may arise in multi-attribute decision analysis. In terms of making decisions with multiple attributes in the real world, in most instances, the sum of the truth membership of an alternative and the false membership of that same alternative can exceed 1. In these cases, the data cannot be dealt with by traditional concepts such as intuitionistic fuzzy sets. To address this problem, Yager [4] proposed the Pythagorean Fuzzy Set. Moreover, Ejegwa developed model operators and their properties in a Pythagorean fuzzy set [5].

To address uncertainty, Molodtsov [6] established the notion of soft sets. Numerous researchers developed new concepts after applying this theory in various ways [7]. This definition has been used to investigate several innovative uses for soft set theory. Fuzzy soft set applications were given by Roy and Maji [8]. Smarandache [9] introduced hypersoft sets, expanding soft sets by applying different parameters and relationships to incorporate complex uncertainties. This facilitates multilevel membership, different degrees of relevance uncertainty modeling. The study of hypersoft sets can be applied in multiple fields, encompassing decision theory, and data analysis artificial intelligence, where complicated and uncertain fields require effective modeling. Smarandache also provided the definition of a fuzzy hypersoft set, which is primarily used in decision-making, data analysis, and the sophisticated modeling of uncertainty and ambiguity. Fuzzy sets generalize classical sets by enabling degrees of membership, specifying that an element can partially belong to a set rather than exclusively being a member. The concept of hypersoft sets utilizes soft sets by applying a collection of attributes or criteria to elaborate uncertainty in decision-making processes. Fuzzy hypersoft sets use the principles of fuzzy sets and hypersoft sets, enabling an enhanced representation of uncertainty. These can model scenarios in which both criteria and degrees of membership are fuzzy. As a result, they can be applied in scenarios where decisions must be based on incomplete or imprecise information.

A hybrid structure of Pythagorean fuzzy hypersoft set [10] that incorporates the advantages of two strong concepts, one being the hypersoft set and the second is the Pythagorean fuzzy set, was developed by Zulqarnain et al. The Pythagorean fuzzy hypersoft set, an appropriate extension of the Pythagorean fuzzy soft set [11]. Inappropriate ambiguity and reluctance throughout the decision-making process

can be accurately assessed using the Pythagorean fuzzy hypersoft set [12]. When making assessments, it is the most crucial concept for drawing connections between ambiguous data.

The concept of plithogenic sets was created by Smarandache [13]. It was meant to be a refined version of the neutrosophic set, intuitionistic fuzzy set, and fuzzy set. One or more characteristics, each of which may have multiple sub-attributes, define a plithogenic set. To boost accuracy and generate more precise exclusion (partial order), the degree of disagreement (dissimilarity) of each attribute value with the value of dominant attribute is considered. In this work, we introduce a specialized and unique mathematical structure called the plithogenic Pythagorean fuzzy hypersoft set, a theoretical and logical combination of Pythagorean fuzzy hypersoft set and the plithogenic set. The foundation of Pythagorean fuzzy sets is adopted in this method, which utilize membership and non-membership degrees using an adjusted approach. It also handles the case of uncertainty when each attribute is subdivided into attribute values and all alternatives are evaluated against each criterion. It can model many types of uncertainty: membership, non-membership, hesitation, multiple attributes, sub-attributes, and contradictions. This is especially helpful when you have complex information with many interrelated parameters.

Entropy measures are a basic computational tool used in fuzzy mathematics and information theory to assess the level of uncertainty, fuzziness, or lack of information in a particular dataset or system. First, entropy was defined by Claude Shannon [14], and nonprobabilistic entropy on the fuzzy set was interpreted by De Luca et al. in 1972 [15]. Entropy measures for the extensions of fuzzy sets have been widely used to determine criteria weights in multi-criteria decision-making problems. A novel approach to entropy measures based on plithogenic set was first introduced by Quek [16]. Here, we propose an innovative entropy measure for PPFHSS that may assess composite uncertainty resulting from attribute contradiction and fuzzy hesitation. Biased or suboptimal results may arise from manual or subjective measurement of certain entropy factors. To address this limitation, an entropy-guided regularized optimization framework is formulated, with PSO introduced by Kennedy and Eberhart in 1995, inspired by the way birds flock and fish school together [17]. Each particle updates its position using its own best experience and the best solution found by the whole swarm, which keeps PSO easy to implement and has made it popular for tasks such as tuning PV controllers [18] and selecting features for microRNA-based cancer classification [19]. By providing a more comprehensive evaluation of contradictory and hesitant information, the suggested entropy overcomes the shortcomings of traditional entropy measures. Table 1 compares the proposed model with the existing models in terms of the features each one supports.

1.1. Motivation and contributions

Traditional mathematical models, including fuzzy and soft computing, are typically inadequate for modeling the combined effects of indecision and uncertainty, as well as for modeling complex interdependencies between attributes in systems such as evaluation and dialogue agent systems. Our primary motivation for this research is to develop a framework capable of modeling uncertainty flexibly. An objective method for evaluating and maximizing parameters and a data-driven technique for solving real-world problems will also be provided. The proposed research will be beneficial in addressing interdependence, inconsistency, and uncertainty to enhance decision-making.

The methodological contributions of this manuscript are summarized as follows to highlight the constraints of existing multi criteria decision-making (MCDM) approaches under complex uncertainty:

- A novel mathematical model, PPFHSS is developed to address decision-making problems in which each attribute is subdivided into multiple attribute values and each alternative is evaluated against these values, while incorporating the contradiction degree to represent the extent of disagreement among attribute values
- The conventional entropy weighting method is extended to be used in a PPFHSS environment to deal with multi-layered sub-attributes, which represent complex overlapping uncertainties.
- The multi-argument decision matrices are directly utilized. Each selected tuple is represented by a component of the candidate weight vector of particles, where the search space of PSO can be dynamically optimized for the decision ranks.
- Seamless workflow between PPFHSS assessments, entropy weight determinations, and PSO vector optimization is established for robust multi-criteria decision analysis.

1.2. Novelty

The novelty of this work lies in combining ideas that have so far been studied separately. Pythagorean fuzzy hypersoft sets can represent membership, non-membership, and hesitation over several attributes, but they do not say how strongly the values of an attribute conflict with each other. Plithogenic sets handle this through the degree of contradiction, though they cannot express falsity or hesitation and do not use a hypersoft structure with multiple arguments. We bring these two lines of work together in a single model, the proposed PPFHSS, in which attribute values are aggregated according to their contradiction degrees. A second difference is how the attribute weights are obtained. Approaches usually rely on weights assigned by experts, which adds subjective bias. The entropy-derived weights are refined through a regularized optimization procedure, reducing the direct dependence on manually assigned weights. As Table 1 shows, none of the models supports all these features together. The proposed approach therefore handles uncertainty, hesitation, contradiction, and complex attribute interactions within one consistent framework.

Table 1. Comparative analysis of existing models and the proposed approach.

Feature	**FS** [1]	**FSS** [8]	**PFS** [4]	**PFSS** [11]	**PFHSS** [10]	**PS** [13]	**Proposed**
Uncertainty	✓	✓	✓	✓	✓	✓	✓
Falsity	✗	✗	✓	✓	✓	✗	✓
Hesitation	✗	✗	✓	✓	✓	✗	✓
Hypersoft Structure	✗	✗	✗	✗	✓	✗	✓
Multi-Argument	✗	✗	✗	✗	✓	✗	✓
Contradiction Degree	✗	✗	✗	✗	✗	✓	✓
Plithogenic Aggregation	✗	✗	✗	✗	✗	✓	✓
Entropy Weighting	✗	✗	✗	✗	✗	✗	✓
Particle Swarm Optimization	✗	✗	✗	✗	✗	✗	✓
Complex Interaction	✗	✗	✗	✗	✗	✓	✓

✓ : supported ✗ : not supported
FS: Fuzzy Set; FSS: Fuzzy Soft Set; PFS: Pythagorean Fuzzy Set; PFSS: Pythagorean Fuzzy Soft Set; PFHSS: Pythagorean Fuzzy Hypersoft Set; PS: Plithogenic Set; Proposed: PPFHSS + Entropy-PSO.

1.3. Organization of the study

Fuzzy set, fuzzy soft set, fuzzy hypersoft set, Pythagorean fuzzy set, hypersoft set, Pythagorean fuzzy hypersoft set, and plithogenic set are defined in Section 2. In Section 3, the concept of the plithogenic Pythagorean fuzzy hypersoft set is introduced. In this section, we also present the concept's entropy measure. The PSO-Based entropy weight optimization for chatbot evaluation is established in Section 4. In Section 5, the conclusions are discussed.

2. Preliminaries

2.1. Fuzzy set

Zadeh [1] introduced a remarkable idea of FS, which can manage partial truth and uncertainty in the data. A membership function specifies a FS $\tilde{A}$ in a universe of discourse U.

$$\mathscr{F}_{\tilde{A}} : U \to [0, 1],$$

where the degree of membership of an element $x \in U$ in the set $\tilde{A}$ is represented by $\mathscr{F}_{\tilde{A}}(x)$. FS thus generalizes classical (crisp) sets by enabling gradual transitions between membership and non-membership.

2.2. Fuzzy soft set

Maji [8] put forward the idea of a FSS, a hybrid model that combines the notions of FS and soft sets, to better address uncertainty in parameterized decision-making situations. Let U be a universal set and E a set of parameters. Let $A \subseteq E$, A FSS $\mathcal{F}_A$ over U is defined as a set of ordered pairs

$$\mathcal{F}_A = \{(e, F(e)) \,|\, e \in A,\ F(e) \in \mathcal{F}(U)\},$$

where $\mathcal{F}(U)$ denotes the collection of all fuzzy subsets of U. For each parameter $e \in A$, $F(e)$ is a fuzzy subset of U characterized by the membership function $\lambda_{F(e)} : U \to [0, 1]$, representing the degree to which each element $x \in U$ satisfies the parameter e. FSS provides a parameterized framework to control uncertainty by associating fuzzy information with each parameter, thereby generalizing both fuzzy and soft set theories.

2.3. Fuzzy hypersoft set

The *fuzzy hypersoft set* (FHSS) [9] extends the structure of FSS by proposing multi-argument parameterization, where every parameter is associated with a collection of sub-parameters. This concept really helps in showing uncertainties more clearly, especially when we deal with complicated decision-making situations; It is like having a more detailed map when trying to navigate through complex choices. Consider U as a universal set and let $E = \{e_1, e_2, \ldots, e_n\}$ be a collection of parameters, and each parameter e_i has a corresponding set of attribute values $E_i = \{e_{i1}, e_{i2}, \ldots, e_{im_i}\}$. Let $\mathbb{E} = E_1 \times E_2 \times \cdots \times E_n$ be the Cartesian product of all attribute sets. A FHSS is defined by

$$\mathfrak{F} = \{(\varepsilon, \mathfrak{F}(\varepsilon)) \,|\, \varepsilon \in \mathbb{E},\ \mathfrak{F}(\varepsilon) \in \mathfrak{F}(U)\},$$

where $\mathfrak{F}(U)$ represents the family of all fuzzy subsets of U. For each multi-parameter $\varepsilon = (e_{1j_1}, e_{2j_2}, \ldots, e_{nj_n}) \in \mathbb{E}$, the associated fuzzy subset $F(\varepsilon)$ is defined by the function

$$\lambda_{\mathfrak{F}(\varepsilon)} : U \to [0, 1],$$

which expresses the degree to which each element $x \in U$ satisfies the combination of sub-parameters in ε.

2.4. Pythagorean fuzzy set

Yager [4] presented an expansion of the IFS named PFS, which provides more flexibility in expressing uncertainty. Let U be a discourse universe. The definition of a PFS $\tilde{P}$ in U is

$$\tilde{P} = \{\langle x, \lambda_{\tilde{P}}(x), \gamma_{\tilde{P}}(x)\rangle \,|\, x \in U\},$$

where $\lambda_{\tilde{P}}(x) : U \to [0, 1]$ is the degree of membership and $\gamma_{\tilde{P}}(x) : U \to [0, 1]$ is the degree of non-membership. Furthermore, it satisfies the following condition:

$$\lambda^2_{\tilde{P}}(x) + \gamma^2_{\tilde{P}}(x) \leq 1, \quad \forall x \in U.$$

The hesitant degree $\pi_{\tilde{P}}(x)$, is given by

$$\pi_{\tilde{P}}(x) = \sqrt{1 - \lambda^2_{\tilde{P}}(x) - \gamma^2_{\tilde{P}}(x)}.$$

2.5. Pythagorean fuzzy hypersoft set

Let U be a universal set and $E = \{e_1, e_2, \ldots, e_n\}$ a set of parameters, where each parameter e_i has a corresponding set of attribute values $E_i = \{e_{i1}, e_{i2}, \ldots, e_{im_i}\}$. Let $\mathbb{E} = E_1 \times E_2 \times \cdots \times E_n$ be the Cartesian product of all attribute sets. A PFHSS [10] $\mathbb{G}$ over U is defined as a set of ordered pairs

$$\mathbb{G} = \{(\varepsilon, \hat{F}(\varepsilon)) \,|\, \varepsilon \in \mathbb{E},\ \hat{F}(\varepsilon) \in \hat{F}(U)\},$$

where $\hat{F}(U)$ denotes the family of all Pythagorean fuzzy subsets of U. For the sake of convenance, let $\hat{F}(\varepsilon) = \hat{A}$, which defined by mathematical expression

$$\hat{A} = \{(x, \lambda_{\hat{A}}(x), \gamma_{\hat{A}}(x)) \mid x \in X\},$$

In this regard, $\lambda_{\hat{A}}(x)$ and $\beta_{\hat{A}}(x)$ are membership and non-membership functions, respectively, defined by

$$\lambda_{\hat{A}}(x), \gamma_{\hat{A}}(x) : X \to [0, 1],$$

in addition to the condition

$$0 \leq (\lambda_{\hat{A}}(x))^2 + (\gamma_{\hat{A}}(x))^2 \leq 1.$$

The degree of indeterminacy for any PFHSS is given by

$$\pi_{\hat{A}}(x) = \sqrt{1 - ((\lambda_{\hat{A}}(x))^2 + (\gamma_{\hat{A}}(x))^2)}.$$

2.6. Plithogenic set

Smarandache [13] introduced the enhanced framework of a PS, which is the extension of intuitionistic fuzzy, fuzzy, and classical neutrosophic sets. A PS $\mathbb{P}$ is an extended structure, where each element is described by one or more attributes, each of which may have numerous possible values. For each attribute value, the degree of appurtenance is allocated. This framework incorporates a degree of contradiction (or dissimilarity) between attribute values and dominant attribute value, to improve the expressiveness and accuracy of this concept.

3. Entropy measures of the plithogenic Pythagorean fuzzy hypersoft set

The idea of a PPFHSS and its associated entropy measure is presented in this section not just as a mathematical construct, but also as a reflection of the fundamental complexity of real-world intelligence.

3.1. Plithogenic Pythagorean fuzzy hypersoft set

Let $P \subseteq U$ be a non-empty set of objects, and let U be a universe of discourse. Let $K = \{K_1, K_2, K_3, \ldots, K_n\}$ be the set of the corresponding sub-attributes of K_i, respectively with $K_i \cap K_j = \Phi$ for $n \geq 1$ for each $i, j \in \{1, 2, 3, \ldots n\}$ and $i \neq j$. Each v' is an n-tuple, and let $v' = \{v_1, v_2, \ldots, v_m\}$ be the collection of all attribute values. The five tuple (p', a, v', d', c) is known as a PPFHSS, where

- a is the given attribute (the compound attribute defined by the Cartesian product v).
- $d' : p' \times v' \to P([0, 1]^2)$ is the Pythagorean fuzzy degree function.
- $c : K_i \times K_i \to P[0, 1], \forall i = 1, 2, 3, \ldots, n$, is the fuzzy value contradiction (dissimilarity) degree function.

PPFHSS has another alternate representation; For an n-tuple $t \in v'$, where $t = (t_1, t_2, t_3, \ldots, t_n)$, $t_i \in K_i$, $\forall i = 1, 2, 3, \ldots, n$.

$$\mathbb{P}_f(t_1, t_2, t_3, \ldots, t_n) = \{x'((d'(t_1), d'(t_2), \ldots, d'(t_n)), x' \in U\},$$

$$\mathbb{P}_f(t_i) = \{x'(\lambda(t_i), \gamma(t_i))\rangle), \forall i = 1, 2, 3, \ldots, n\}.$$

In this context, $\lambda(t_i)$, are the membership degrees and $\gamma(t_i)$ are non-membership degrees, meeting the condition $(\lambda(t_i)^2 + \gamma(t_i)^2) \leq 1$, for $i = 1, 2, 3, \ldots, n$.

3.2. Entropy measure

Entropy is a quantitative measurement for the amount of disorder within a system. An absolute certainty has zero entropy, and the highest disorder is total uncertainty (or total disorder). For example, in the case of a coin flip, it gives maximum two outcomes: one lands on the head and the other lands on the tail; then their entropy is $\frac{1}{2}$, as shown in Figure 1.

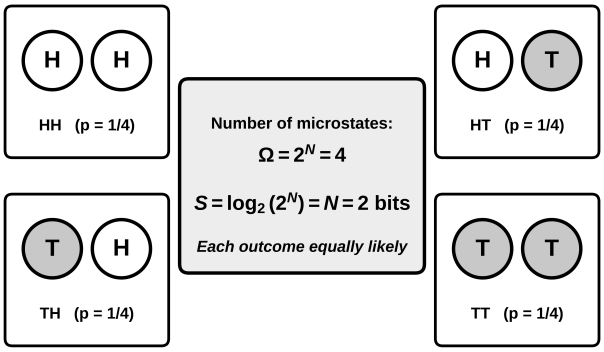


Figure 1. Coin entropy.

Similarly, the entropy of a broken egg is an example of disorder: a fresh egg breaks easily, but the already broken egg does not spontaneously reassemble itself. The reason is that there are many ways to break the egg, as shown in Figure 2, as the broken egg has a higher entropy.

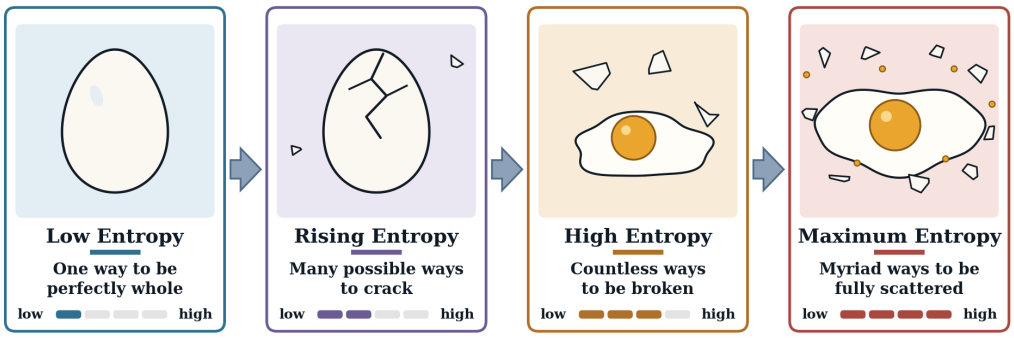


Figure 2. Entropy increases as an egg moves from a whole state to a fully scattered state.

3.3. PPFHSS entropy measure

An entropy measure is a mapping E : PPFHSS$(P) \to [0,1]$, which quantifies the uncertainty associated with a plithogenic Pythagorean fuzzy hypersoft set. The proposed measure is required to satisfy the following axioms.

(i) **Axiom E1 (Crispness Condition):** $E(\text{PPFHSS}) = 0$ if and only if, for every $x' \in P$, all appurtenance degrees are crisp and the following pairwise compatibility conditions hold:

$$\begin{cases} d'(x', v_1) = d'(x', v_2), & c(v_1, v_2) < 0.5, \\ d'(x', v_1) \neq d'(x', v_2), & c(v_1, v_2) > 0.5, \\ d'(x', v_1), d'(x', v_2) \in \{0, 1\}, & c(v_1, v_2) = 0.5. \end{cases}$$

(ii) **Axiom E2 (Maximum Uncertainty):** The entropy measure attains its maximum value, namely $E(\text{PPFHSS}) = 1$, if and only if $d'(x', v') = \frac{1}{\sqrt{2}}$ for every $x' \in P$ and $v' \in V_k$.

(iii) **Axiom E3 (Complement Invariance):** $E(\overline{\text{PPFHSS}}) = E(\text{PPFHSS})$, where the Pythagorean complement is defined by $d'^c(x', v') = \sqrt{1 - d'^2(x', v')}$.

3.3.1. Appurtenance uncertainty function

Let $\delta : [0,1] \to [0,1]$ be an appurtenance uncertainty function satisfying:

(i) $\delta(0) = \delta(1) = 0$.

(ii) $\delta(1/\sqrt{2}) = 1$.

(iii) $\delta(\lambda) = \delta(\sqrt{1-\lambda^2})$ for all $\lambda \in [0,1]$.

(iv) δ is strictly increasing on $[0, 1/\sqrt{2}]$.

Lemma 3.1. *The function $\delta(\lambda) = 1 - |2\lambda^2 - 1|$ satisfies all the above conditions.*

Proof. First, $\delta(0) = 1 - |-1| = 0$, $\delta(1) = 1 - |1| = 0$. Moreover, $\delta\left(\frac{1}{\sqrt{2}}\right) = 1 - |1-1| = 1$. For complement symmetry,

$$\delta(\sqrt{1-\lambda^2}) = 1 - |2(1-\lambda^2) - 1| = 1 - |1 - 2\lambda^2| = \delta(\lambda).$$

Finally, for $0 \le \lambda \le 1/\sqrt{2}$, $\delta(\lambda) = 2\lambda^2$, which is strictly increasing on the specified interval. Thus, all conditions are satisfied. □

3.3.2. Contradiction activation function

Let $\Omega : [0,1] \to [0,1]$ be a contradiction activation function defined by $\Omega(\gamma) = \max\{0, 2\gamma - 1\}$. It satisfies the following properties:

(i) $\Omega(\gamma) = 0$ for $\gamma \in [0, 0.5]$.

(ii) $\Omega(1) = 1$.

(iii) Ω is strictly increasing on $[0.5, 1]$.

Lemma 3.2. *The function $\Omega(\gamma) = \max\{0, 2\gamma - 1\}$ satisfies all the above properties.*

Proof. For $\gamma \in [0, 0.5]$, we have $2\gamma - 1 \le 0$, and hence $\Omega(\gamma) = 0$. Additionally, $\Omega(1) = \max\{0, 1\} = 1$. On $[0.5, 1]$, the function reduces to $\Omega(\gamma) = 2\gamma - 1$, which is strictly increasing. Therefore, all properties hold. □

3.3.3. Entropy components

The appurtenance entropy component is defined as $\varepsilon(x', v') = \delta(d'(x', v'))$. For two sub-attributes $v_1, v_2 \in V_k$, the pairwise contradiction entropy component is defined by

$$\vartheta(x', v_1, v_2) = \Omega(1 - c(v_1, v_2)) \left| d'^2(x', v_1) - d'^2(x', v_2) \right| + \Omega(c(v_1, v_2)) \left| d'^2(x', v_1) + d'^2(x', v_2) - 1 \right|.$$

For each $x' \in P$ and $v' \in V_k$, define $\vartheta_{\max}(x', v') = \max_{\substack{u' \in V_k \\ u' \neq v'}} \vartheta(x', v', u')$. If V_k contains only one element, the empty maximum is defined as zero.

Lemma 3.3 (Contradiction Component Bounds)**.** *For any $x' \in P$ and $v_1, v_2 \in V_k$, $0 \le \vartheta(x', v_1, v_2) \le 1$. Consequently, $0 \le \vartheta_{\max}(x', v') \le 1$.*

Proof. Since all appurtenance degrees belong to $[0, 1]$, $|d_1'^2 - d_2'^2| \leq 1$ and $|d_1'^2 + d_2'^2 - 1| \leq 1$. Furthermore, $0 \leq \Omega(\gamma) \leq 1$. The first activation term can be positive only when $c(v_1, v_2) < 0.5$, whereas the second can be positive only when $c(v_1, v_2) > 0.5$. Thus, the two terms cannot be positive simultaneously. Therefore, $0 \leq \vartheta(x', v_1, v_2) \leq 1$. Taking the maximum over the admissible pairs gives the corresponding bound for $\vartheta_{\max}$. □

Theorem 3.4 (Canonical Entropy Realization)**.** *Let* PPFHSS *be defined over a finite parameter set P, with r attributes. Let V_k denote the set of sub-attributes associated with the k-th attribute, and let $q_k > 0$ be its normalized weight, satisfying $\sum_{k=1}^{r} q_k = 1$. Then the proposed entropy measure is defined as*

$$E(\text{PPFHSS}) = \frac{1}{|P|} \sum_{x' \in P} \sum_{k=1}^{r} q_k \left[\frac{1}{|V_k|} \sum_{v' \in V_k} H(x', v') \right],$$

where $H(x', v') = \delta(d'(x', v')) + (1 - \delta(d'(x', v')))\,\rho\,\vartheta_{\max}(x', v')$, with $0 < \rho < 1$. Here, q_k represents the weight of the k-th attribute, and V_k denotes its set of sub-attributes.

Proof. For convenience, define the local entropy contribution as

$$H(x', v') = \delta(d'(x', v')) + (1 - \delta(d'(x', v')))\rho\vartheta_{\max}(x', v').$$

(i) **Axiom E1 (Crispness).**

Suppose that all appurtenance degrees are crisp and satisfy the pairwise compatibility conditions. Then $d'(x', v') \in \{0, 1\}$, which gives $\delta(d'(x', v')) = 0$. Consider any pair $v_1, v_2 \in V_k$. If $c(v_1, v_2)<0.5$, the crispness condition requires $d'(x', v_1) = d'(x', v_2)$. Hence, $|d_1'^2 - d_2'^2| = 0$. The remaining term is multiplied by $\Omega(c(v_1, v_2)) = 0$. Therefore, $\vartheta(x', v_1, v_2) = 0$. If $c(v_1, v_2) > 0.5$, the crispness condition requires unequal crisp degrees. Thus, $|d_1'^2 + d_2'^2 - 1| = 0$, while $\Omega(1 - c(v_1, v_2)) = 0$. Consequently, $\vartheta(x', v_1, v_2) = 0$. When $c(v_1, v_2) = 0.5$, both activation functions vanish. Hence, the contradiction component is also zero, irrespective of the crisp pair. It follows that $\vartheta_{\max}(x', v') = 0$. Therefore, $H(x', v') = 0$ for every $x' \in P$ and $v' \in V_k$, and $E(\text{PPFHSS}) = 0$. Conversely, suppose that $E(\text{PPFHSS}) = 0$. Since the entropy is an weighted average with positive weights of nonnegative local contributions, every local contribution must be zero. In particular, $\delta(d'(x', v')) = 0$. By the definition of δ, $d'(x', v') \in \{0, 1\}$. Furthermore, since $0 < \rho < 1$, the vanishing of every local contribution also requires $\vartheta_{\max}(x', v') = 0$. The definition of Ω then enforces equal crisp grades when $c(v_1, v_2) < 0.5$, unequal crisp grades when $c(v_1, v_2) > 0.5$, and permits either crisp configuration when $c(v_1, v_2) = 0.5$. Thus, the stated crispness conditions are necessary and sufficient.

(ii) **Axiom E2 (Maximum Uncertainty).** Suppose that $d'(x', v') = \frac{1}{\sqrt{2}}$ for every $x' \in P$ and $v' \in V_k$. Then $\delta\left(\frac{1}{\sqrt{2}}\right) = 1$. Consequently, $H(x', v') = 1 + (1 - 1)\rho\vartheta_{\max}(x', v') = 1$. Thus, $E(\text{PPFHSS}) = 1$. Conversely, suppose that $E(\text{PPFHSS}) = 1$. Since the entropy is the weighted average with positive weights of finitely many local contributions bounded above by one, every local contribution must equal one. However,

$$H = \delta + (1 - \delta)\rho\vartheta_{\max} \leq \delta + (1 - \delta)\rho = 1 - (1 - \delta)(1 - \rho).$$

Because $0 < \rho < 1$, the equality $H = 1$ is possible only when $\delta = 1$. By the definition of δ, this is equivalent to $d'(x', v') = \frac{1}{\sqrt{2}}$. Hence, the maximum-uncertainty condition is satisfied if and only if all appurtenance degrees equal $1/\sqrt{2}$.

(iii) **Axiom E3 (Complement Invariance).** Let $d'^c(x', v') = \sqrt{1 - d'^2(x', v')}$. For the appurtenance component,

$$\delta(d'^c) = 1 - \left|2(d'^c)^2 - 1\right| = 1 - \left|2(1 - d'^2) - 1\right| = 1 - \left|1 - 2d'^2\right| = \delta(d').$$

For the contradiction component, let $d_1'^c = \sqrt{1 - d_1'^2}$ and $d_2'^c = \sqrt{1 - d_2'^2}$. Then

$$|(d_1'^c)^2 - (d_2'^c)^2| = |d_1'^2 - d_2'^2|, \text{ and } |(d_1'^c)^2 + (d_2'^c)^2 - 1| = |d_1'^2 + d_2'^2 - 1|.$$

Since the contradiction degree remains unchanged under complementation, $\vartheta^c(x', v_1, v_2) = \vartheta(x', v_1, v_2)$. Therefore, both entropy components remain invariant, which gives $E(\overline{\text{PPFHSS}}) = E(\text{PPFHSS})$. This completes the proof.

□

Theorem 3.5 (Normalization Property)**.** *For any PPFHSS, the proposed entropy measure satisfies* $0 \leq E(\text{PPFHSS}) \leq 1$.

Proof. By construction, $0 \leq \delta(d'(x', v')) \leq 1$ and $0 \leq \vartheta_{\max}(x', v') \leq 1$. Since $0 < \rho < 1$, we obtain $0 \leq H(x', v') \leq \delta + (1 - \delta)\rho \leq 1$. The entropy measure is the convex combination of these bounded local contributions. Hence, $0 \leq E(\text{PPFHSS}) \leq 1$. □

Theorem 3.6 (Monotonicity of Local Entropy)**.** *The local entropy aggregation function*

$$H(\delta, \vartheta_{\max}) = \delta + (1 - \delta)\rho\vartheta_{\max}$$

is not decreasing in both arguments on $[0, 1] \times [0, 1]$.

Proof. The partial derivatives are $\frac{\partial H}{\partial \delta} = 1 - \rho\vartheta_{\max} \geq 1 - \rho > 0$ and $\frac{\partial H}{\partial \vartheta_{\max}} = \rho(1 - \delta) \geq 0$. Therefore, the local entropy aggregation function is not decreasing in both arguments. □

Lemma 3.7 (Pythagorean Boundary Behavior)**.** *The appurtenance entropy component satisfies:*

- $\varepsilon(x', v') = 0$ *if and only if* $d'(x', v') \in \{0, 1\}$*;*
- $\varepsilon(x', v') = 1$ *if and only if* $d'(x', v') = 1/\sqrt{2}$*.*

Proof. Both statements follow directly from $\varepsilon(x', v') = 1 - \left|2d'^2(x', v') - 1\right|$. □

Corollary 3.8 (Crisp Set Entropy)**.** *If a PPFHSS satisfies the crispness conditions specified in Axiom E1, then* $E(\text{PPFHSS}) = 0$.

Corollary 3.9 (Maximum Entropy Configuration)**.** *The proposed entropy measure attains its maximum value if and only if* $d'(x', v') = \frac{1}{\sqrt{2}}$ *for all* $x' \in P$ *and* $v' \in V_k$.

Corollary 3.10 (Complement Invariance)**.** *The entropy of a PPFHSS remains unchanged under Pythagorean complementation:* $E(\overline{\text{PPFHSS}}) = E(\text{PPFHSS})$.

The above results establish the normalization, crispness, maximum-uncertainty, complement-invariance, and monotonicity properties of the proposed entropy measure.

4. PSO-based entropy weight optimization for chatbot evaluation

The proposed framework integrates PPFHSS-based decision modeling, entropy-based weighting, and particle swarm optimization (PSO) for chatbot evaluation. Its main methodological features are:

- The PPFHSS structure represents evaluations through combinations of sub-attributes, enabling the decision model to retain information associated with multi-attribute parameter tuples.
- In this framework, tuple-based evaluations map each selected tuple to a specific component of the candidate weight vector. This mechanism establishes a systematic bridge between the hypersoft decision structure and the weight-optimization process.

4.1. Hypersoft decision structure

Let $C = \{C_1, C_2, \ldots, C_m\}$ denote the set of chatbot alternatives, and let $A = \{A_1, A_2, \ldots, A_n\}$ represent the primary evaluation attributes. Each primary attribute is decomposed into a set of sub-attributes: $A_k = \{a_{k1}, a_{k2}, \ldots, a_{kq_k}\}, \quad k = 1, \ldots, n$. The hypersoft parameter space is defined by $\Pi = A_1 \times A_2 \times \cdots \times A_n$. Each element $\pi_j = (a_{1l_1}, a_{2l_2}, \ldots, a_{nl_n}) \in \Pi$ represents a combination of sub-attributes. The evaluation of chatbot C_i with respect to tuple π_j is represented by a Pythagorean fuzzy number $(\lambda_{ij}, \gamma_{ij})$, where λ_{ij} and γ_{ij} denote the membership and non-membership degrees, respectively. They satisfy $\lambda_{ij}^2 + \gamma_{ij}^2 \leq 1$. Thus, the decision matrix is expressed as $\mathcal{D} = [(\lambda_{ij}, \gamma_{ij})]_{m\times|\Pi|}$.

4.2. Illustrative sub-attribute structure and decision matrix

For the chatbot evaluation problem, four primary attributes and their corresponding sub-attributes are considered:

$$A_1 = \{\text{precision, recall}\}, \quad A_2 = \{\text{adaptability, training speed}\},$$
$$A_3 = \{\text{memory, relevance}\}, \quad A_4 = \{\text{user experience, response quality}\}.$$

The complete parameter space contains $2^4 = 16$ possible sub-attribute tuples. For the numerical illustration, four representative tuples are selected:

$$\pi_1 = (0,0,0,0), \quad \pi_2 = (0,1,0,1), \quad \pi_3 = (1,0,1,0), \quad \pi_4 = (1,1,1,1).$$

Here, 0 denotes the first sub-attribute and 1 denotes the second sub-attribute within each primary attribute. These tuples represent a selected subset of the complete parameter space.

The corresponding decision matrix is

$$\mathcal{D} = \begin{bmatrix} (0.85, 0.25) & (0.82, 0.30) & (0.88, 0.22) & (0.86, 0.24) \\ (0.78, 0.35) & (0.74, 0.40) & (0.76, 0.36) & (0.72, 0.38) \\ (0.80, 0.32) & (0.77, 0.35) & (0.79, 0.33) & (0.75, 0.36) \\ (0.83, 0.28) & (0.80, 0.30) & (0.82, 0.29) & (0.81, 0.27) \end{bmatrix}.$$

Table 2 presents the PPFHSS decision table constructed based on the selected sub-attribute tuples.

Table 2. PPFHSS decision table based on selected sub-attribute tuples.

Chatbot	π_1	π_2	π_3	π_4
C_1	(0.85,0.25)	(0.82,0.30)	(0.88,0.22)	(0.86,0.24)
C_2	(0.78,0.35)	(0.74,0.40)	(0.76,0.36)	(0.72,0.38)
C_3	(0.80,0.32)	(0.77,0.35)	(0.79,0.33)	(0.75,0.36)
C_4	(0.83,0.28)	(0.80,0.30)	(0.82,0.29)	(0.81,0.27)

4.3. Application-specific entropy index

Let Π_{sel} denote the selected four-tuple subset. The intrinsic uncertainty of each tuple is evaluated by $U(\pi_j) = \frac{1}{m}\sum_{i=1}^{m}\left(1 - \left|\lambda_{ij}^2 - \gamma_{ij}^2\right|\right)$. The reference tuple is selected as $\pi_d = \pi_1 = (0, 0, 0, 0)$. The tuple-level contradiction degree is calculated using the normalized Hamming distance: $c_j = c(\pi_j, \pi_d) = \frac{1}{n}\sum_{k=1}^{n}\mathbb{I}(\pi_{j,k} \neq \pi_{d,k})$, where $n = 4$ and $\mathbb{I}(\cdot)$ is the indicator function. For the selected tuples, $(c_1, c_2, c_3, c_4) = (0, 0.5, 0.5, 1)$. The application-specific entropy index is defined as

$$E^*(\pi_j) = U(\pi_j) + (1 - U(\pi_j))\rho c_j,$$

where $0 < \rho < 1$ controls the contribution of tuple-level contradiction. In the numerical illustration, $\rho = 0.5$.

The calculated intrinsic uncertainty values are $\mathbf{U} = (0.426500,\ 0.502400,\ 0.430625,\ 0.481975)$. Consequently, the application-specific entropy values are

$$\mathbf{E}^* = (0.426500,\ 0.626800,\ 0.572969,\ 0.740988).$$

The overall application-specific entropy index is $E^*_{\text{app}} = \frac{1}{|\Pi_{\text{sel}}|}\sum_{\pi_j \in \Pi_{\text{sel}}} E^*(\pi_j) = 0.591814$. The index is bounded because $0 \leq U(\pi_j), c_j \leq 1$ and $0 < \rho < 1$. Its monotonicity follows from $\frac{\partial E^*}{\partial U} = 1 - \rho c_j > 0$ and $\frac{\partial E^*}{\partial c_j} = \rho(1 - U) \geq 0$. The index is an application-specific adaptation motivated by the canonical entropy measure in Theorem 3.4. It uses tuple-level contradiction in place of the canonical pairwise contradiction component. Therefore, it is not a direct numerical evaluation of the canonical entropy measure.

4.4. Initial entropy-based tuple weights

The initial weight assigned to each selected tuple is calculated by

$$w_j^{(0)} = \frac{1 - E^*(\pi_j)}{\sum_{\ell=1}^{4}(1 - E^*(\pi_\ell))}.$$

The resulting initial weight vector is

$$\mathbf{w}^{(0)} = (0.351249,\ 0.228572,\ 0.261542,\ 0.158636).$$

These values represent initial tuple weights rather than primary-attribute weights.

4.5. PSO-based weight refinement

Let $\mathbf{w} = (w_1, w_2, w_3, w_4)$ denote a candidate tuple-weight vector. The feasible weight domain is the probability simplex $\mathcal{W} = \left\{\mathbf{w} \in \mathbb{R}^4 : w_j \geq 0,\ \sum_{j=1}^4 w_j = 1\right\}$. For each tuple, define the information score $a_j = 1 - E^*(\pi_j)$. The fitness function is formulated as

$$\max_{\mathbf{w}\in\mathcal{W}} F(\mathbf{w}) = \sum_{j=1}^{4} w_j a_j - \tau \sum_{j=1}^{4} (w_j - w_j^{(0)})^2,$$

where $\tau > 0$ is a regularization parameter. The first term rewards weights assigned to tuples with lower application-specific entropy, while the second term controls deviation from the initial weights. In the numerical illustration, $\tau = 10$. Each particle represents a candidate weight vector. Its velocity is updated according to

$$\mathbf{v}_i^{t+1} = \omega \mathbf{v}_i^t + c_1 r_1 (\mathbf{p}_i^t - \mathbf{w}_i^t) + c_2 r_2 (\mathbf{g}^t - \mathbf{w}_i^t),$$

where ω is the inertia coefficient, c_1, c_2 are acceleration coefficients, and r_1, r_2 are random values in $[0, 1]$. The position update is $\mathbf{w}_i^{t+1} = \Pi_{\mathcal{W}}(\mathbf{w}_i^t + \mathbf{v}_i^{t+1})$, where $\Pi_{\mathcal{W}}$ denotes Euclidean projection onto the feasible simplex. This projection ensures nonnegative weights whose sum is one.

4.5.1. Analytical reference solution

The fitness function is strictly concave because $\nabla^2 F(\mathbf{w}) = -2\tau I$. Therefore, it has a unique global maximizer over the convex feasible simplex. Using the Lagrange multiplier condition, the unconstrained stationary solution is

$$w_j^{\text{an}} = w_j^{(0)} + \frac{a_j - \bar{a}}{2\tau}, \quad \bar{a} = \frac{1}{4}\sum_{j=1}^{4} a_j.$$

For the present data, all resulting weights are nonnegative. Hence, the stationary solution is feasible and is the unique global maximizer:

$$\mathbf{w}^{\text{an}} = (0.359515,\ 0.226823,\ 0.262484,\ 0.151178).$$

The corresponding maximum fitness value is $F(\mathbf{w}^{\text{an}}) = 0.440799$. The analytical solution provides a reference against which the numerical PSO solution can be evaluated. PSO is used to approximate this solution; it is not claimed to outperform the analytical optimizer.

4.5.2. PSO configuration

For the numerical implementation, the following settings are specified: $N = 40$, $T_{\max} = 300$, $\omega = 0.7$, $c_1 = c_2 = 1.5$, and $\tau = 10$. The algorithm is intended to be executed over 30 independent runs. The mean optimized weights, standard deviations, and convergence statistics must be obtained from the actual rerun using the revised application-specific entropy index and fitness function.

Due to the stochastic nature of particle swarm optimization, the proposed optimization procedure is evaluated through 30 independent runs. Table 3 reports the mean optimized weights, their corresponding standard deviations, and the convergence statistics obtained across all executions.

Table 3. PSO configuration and statistical performance over 30 independent runs.

Category	Parameter / Measure	Value
PSO Configuration	Swarm size (N)	40
	Maximum iterations ($T_{\max}$)	300
	Inertia weight (ω)	0.7
	Cognitive coefficient (c_1)	1.5
	Social coefficient (c_2)	1.5
	Regularization parameter (τ)	10
	Independent runs	30
Optimized Weights	Mean optimized weight (π_1)	0.3595149626
	Mean optimized weight (π_2)	0.2268229624
	Mean optimized weight (π_3)	0.2624845125
	Mean optimized weight (π_4)	0.1511775624
Optimization Statistics	SD of π_1	6.2183×10^{-10}
	SD of π_2	6.0829×10^{-10}
	SD of π_3	6.0503×10^{-10}
	SD of π_4	5.7315×10^{-10}
Convergence Performance	Mean best fitness	0.4407989684
	Fitness SD	1.0308×10^{-17}
	Mean convergence iteration	48.17
	Convergence iteration SD	4.53

Note: SD denotes standard deviation. All statistical measures are calculated from 30 independent PSO executions. The optimized weights satisfy the normalization constraint $\sum_{j=1}^{4} \pi_j = 1$.

4.6. Algorithm: PSO-based entropy weight optimization

- Define chatbot alternatives $C = \{C_1, \ldots, C_m\}$.
- Define the primary attributes and their sub-attribute sets.
- Construct the hypersoft parameter space

$$\Pi = A_1 \times A_2 \times \cdots \times A_n.$$

- Select the tuple subset Π_{sel} and construct the Pythagorean fuzzy decision matrix.
- Calculate the intrinsic uncertainty $U(\pi_j)$ for each selected tuple.
- Calculate the tuple-level contradiction degree c_j using the normalized Hamming distance.
- Compute the application-specific entropy

$$E^*(\pi_j) = U(\pi_j) + (1 - U(\pi_j))\rho c_j.$$

- Determine the initial tuple weights

$$w_j^{(0)} = \frac{1 - E^*(\pi_j)}{\sum_\ell (1 - E^*(\pi_\ell))}.$$

- Initialize particles within the feasible simplex $\mathcal{W}$.

- Evaluate the fitness function

$$F(\mathbf{w}) = \sum_j w_j (1 - E^*(\pi_j)) - \tau \sum_j (w_j - w_j^{(0)})^2.$$

- Update particle velocities and positions using the PSO equations.

- Project each updated position onto $\mathcal{W}$.

- Update the personal-best and global-best solutions according to the fitness values.

- Repeat the updates until the stopping condition or maximum iteration count is reached.

- Obtain the optimized tuple-weight vector $\mathbf{w}^*$.

- Calculate the quadratic decision score for each chatbot:

$$S_Q(C_i) = \sum_{j=1}^{4} w_j^* (\lambda_{ij}^2 - \gamma_{ij}^2).$$

- Rank the chatbot alternatives in descending order of their decision scores.

Figure 3 demonstrates the conceptual flow of the proposed methodology combining Pythagorean fuzzy hypersoft decision modeling, entropy measures, and PSO optimization to determine optimal attribute weights and rank chatbot alternatives.

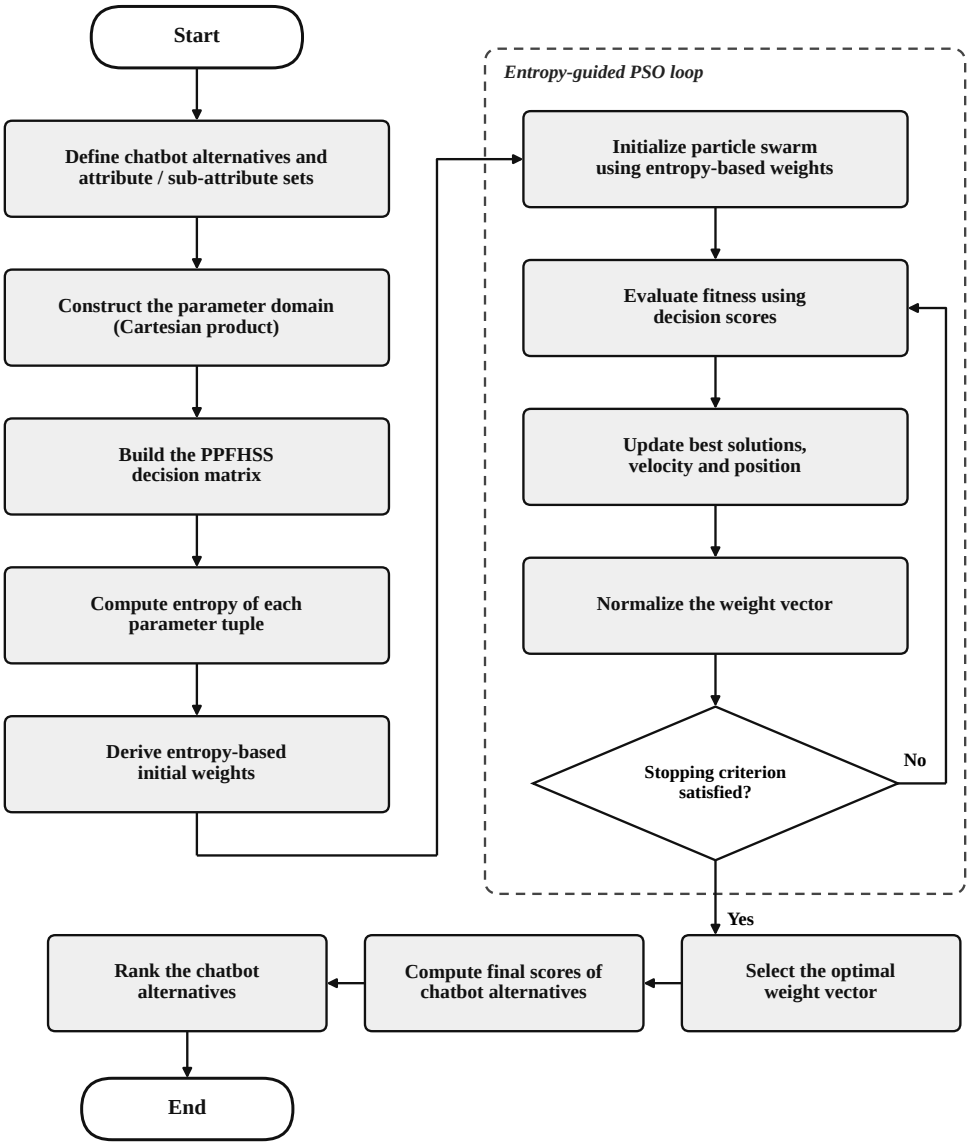


Figure 3. PSO-driven entropy optimization process for decision-making in chatbot evaluation using hypersoft fuzzy structures.

4.7. Ranking of chatbots and decision scores

The quadratic score of a Pythagorean fuzzy number is defined as $S(\lambda_{ij}, \gamma_{ij}) = \lambda_{ij}^2 - \gamma_{ij}^2$. Using the analytical reference weights, the aggregated decision score of chatbot C_i is calculated by

$$S_Q(C_i) = \sum_{j=1}^{4} w_j^{\text{an}} (\lambda_{ij}^2 - \gamma_{ij}^2).$$

The resulting decision scores and corresponding rankings are presented in Table 4.

Table 4. Final chatbot ranking based on the quadratic decision scores.

Chatbot	Decision Score	Rank
C_1	0.663048	1
C_4	0.586823	2
C_3	0.500650	3
C_2	0.436738	4

The results show that C_1 obtains the highest decision score of 0.663048, followed by C_4, C_3, and C_2. Thus, the resulting preference order is $C_1 \succ C_4 \succ C_3 \succ C_2$. The complete theoretical hypersoft parameter space contains $2^4 = 16$ possible tuples. However, this numerical illustration considers four representative tuples. Therefore, the reported scores and ranking correspond to the selected four-tuple implementation rather than an exhaustive evaluation of all possible sub-attribute combinations.

Figure 4 illustrates the aggregated quadratic decision scores of the four chatbot alternatives. It confirms that C_1 achieves the highest score, whereas C_2 obtains the lowest score.

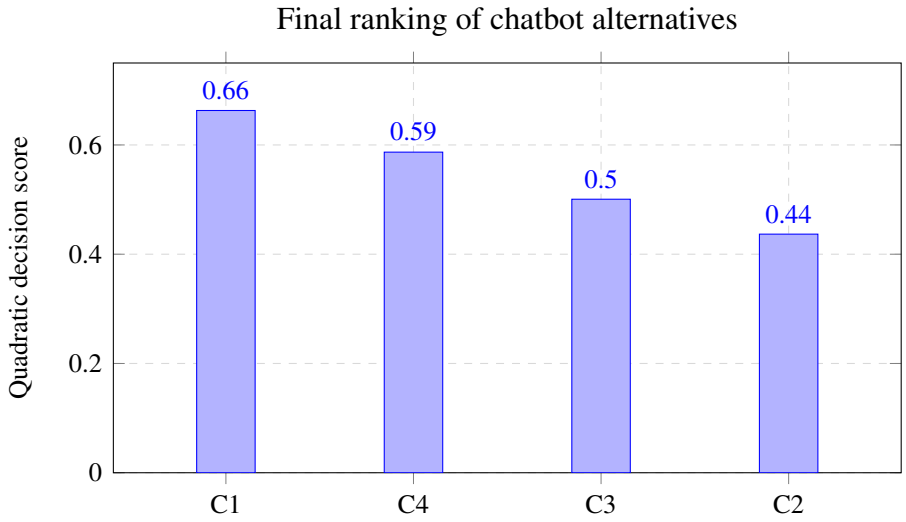


Figure 4. Final ranking of chatbot alternatives based on the aggregated quadratic decision scores.

The difference between the scores of the two highest-ranked alternatives, C_1 and C_4, is 0.076226. This result further indicates the separation between the two leading alternatives in the considered decision matrix.

4.8. Comparative analysis

To examine the effect of different weighting strategies, the analytical reference solution is compared with equal weighting, initial entropy-based weighting, and the numerical weights obtained from PSO and GA. All approaches use the same four-tuple decision matrix and quadratic scoring relation. For the PSO and GA approaches, the mean optimized weight vectors obtained over 30 independent runs are used to calculate the corresponding decision scores. The GA mean weight vector is $\mathbf{w}^{GA} =$

(0.35952342, 0.22681295, 0.26253372, 0.15112991), which agrees with the analytical weights to within 5×10^{-5}.

Table 5 shows that all five weighting strategies produce the same preference order, namely $C_1 \succ C_4 \succ C_3 \succ C_2$. The PSO-based scores closely match the analytical results, while the GA-based scores also remain very close to the analytical reference scores. The differences among the resulting scores are small and do not affect the ranking of the alternatives in this numerical example. These findings demonstrate the consistency of the obtained ranking across the considered weighting strategies. However, this agreement is specific to this decision matrix and does not establish the general superiority of the proposed weighting strategy for other datasets or evaluation settings. Table 5 shows comparison of chatbot decision scores under different weighting strategies.

Table 5. Comparison of chatbot decision scores under different weighting strategies.

Weighting Strategy	C_1	C_4	C_3	C_2	**Ranking**
Equal Weighting	0.662600	0.583000	0.489025	0.423875	$C_1 > C_4 > C_3 > C_2$
Initial Entropy Weights	0.663015	0.586534	0.499772	0.435767	$C_1 > C_4 > C_3 > C_2$
Analytical Optimized Weights	0.663048	0.586823	0.500650	0.436738	$C_1 > C_4 > C_3 > C_2$
PSO Mean Weights	0.663048	0.586823	0.500650	0.436738	$C_1 > C_4 > C_3 > C_2$
GA Mean Weights	0.663051	0.586824	0.500654	0.436743	$C_1 > C_4 > C_3 > C_2$

4.9. Sensitivity analysis

The influence of the regularization parameter τ on the optimized tuple weights is examined using five values: 0.01, 0.1, 1, 10, and 100. For each value, the constrained regularized optimization problem is solved over the feasible weight simplex.

Table 6 indicates that the regularization parameter controls the deviation of the optimized weights from the initial entropy-based weights. Smaller values of τ allow the information-based objective to exert a stronger influence on the solution and may produce boundary solutions with zero weights. In contrast, larger values impose a stronger penalty on deviations from the initial weights. Consequently, the optimized weights approach the initial entropy-based weight vector as τ increases.

Table 6. Sensitivity of analytically optimized tuple weights to the regularization parameter τ.

τ	w_1^*	w_2^*	w_3^*	w_4^*
0.01	1.000000	0.000000	0.000000	0.000000
0.10	0.911025	0.000000	0.088975	0.000000
1.00	0.433906	0.211079	0.270965	0.084050
10.00	0.359515	0.226823	0.262484	0.151178
100.00	0.352076	0.228397	0.261636	0.157890

Note: The reported values are obtained by solving the simplex-constrained optimization problem. Equivalently, they correspond to the Euclidean projection of $\mathbf{w}^{(0)} + \mathbf{a}/(2\tau)$ onto the feasible simplex $\mathcal{W}$.

For the numerical illustration, $\tau = 10$ is adopted as a compromise between the information-based objective and the initial weighting scheme. This choice is specific to the considered decision matrix

and should not be interpreted as a universally optimal parameter value.

4.9.1. PSO-based numerical validation

The Particle Swarm Optimization (PSO) algorithm is executed over 30 independent runs, with 40 particles and 300 iterations per run. The mean final fitness is 0.4407990905, with a standard deviation of 2.5563×10^{-16} (population standard deviation, ddof = 0). The best final fitness is also 0.4407990905. The means of the 30 final best weight vectors are $\mathbf{w}^{PSO} = (0.35951495, 0.22682301, 0.26248438, 0.15117767)$. Table 7 presents PSO performance over 30 independent runs.

Table 7. PSO performance over 30 independent runs.

Performance Measure	**Value**
Number of independent runs	30
Number of particles	40
Maximum iterations	300
Mean final fitness	0.4407990905
Standard deviation (ddof = 0)	2.5563×10^{-16}
Best final fitness	0.4407990905
Analytical optimal fitness	0.4407990905

Figure 5 illustrates the mean best-fitness progression of PSO over 300 iterations and 30 independent runs. The mean fitness increases from 0.326502 at initialization and approaches the analytical optimum by approximately iteration 40. Thereafter, the improvement becomes negligible, indicating that the algorithm maintains a stable solution.

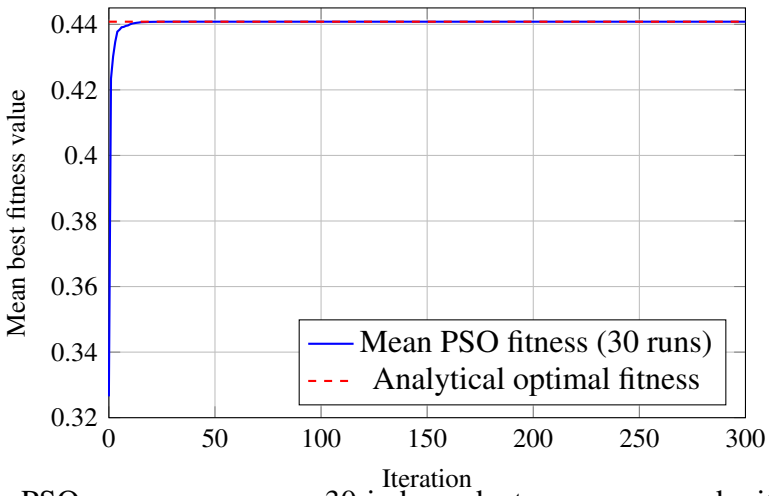


Figure 5. Mean PSO convergence over 30 independent runs compared with the analytical optimal fitness.

4.9.2. Genetic algorithm comparison

As an additional numerical comparison, a real-coded Genetic Algorithm (GA) is applied to the same regularized objective function used in the PSO experiment. The GA employs a population size of 40 individuals and is executed for 300 generations over 30 independent runs. Binary tournament selection with a tournament size of two is used to select parent individuals. The crossover operation employs BLX-0.5 with a crossover probability of $p_c = 0.9$. Gaussian mutation is applied with a mutation probability of $p_m = 0.2$ and a standard deviation of $\sigma = 0.05$. Elitism is maintained by preserving the best individual in each generation. The same Euclidean projection onto the feasible simplex is applied to ensure that all candidate weight vectors satisfy the optimization constraints.

Across the 30 independent runs, the GA achieves a mean final fitness of 0.4407960913, with a standard deviation of 2.13×10^{-6} (population standard deviation, ddof = 0), as reported in Table 8. The best final fitness is 0.4407988594, compared with the analytical optimum of 0.4407990905. The means of the 30 final best weight vectors are $\mathbf{w}^{GA} = (0.35952342,\ 0.22681295,\ 0.26253372,\ 0.15112991)$. The GA mean weights agree with the analytical weights to within 5×10^{-5}. Furthermore, the GA-based decision scores preserve the preference order $C_1 \succ C_4 \succ C_3 \succ C_2$, which is consistent with the analytical and PSO-based rankings. The comparison indicates that PSO exhibits lower run-to-run variation than GA in the selected experimental settings. Nevertheless, both metaheuristic approaches produce solutions close to the unique analytical optimum of the strictly concave objective function. Since the analytical solution is available, these experiments should be interpreted as numerical verification of the optimization procedure rather than evidence that either metaheuristic is superior to the analytical method.

Table 8. Performance of the genetic algorithm over 30 independent runs.

Performance Measure	**Value**
Number of independent runs	30
Population size	40
Number of generations	300
Mean final fitness	0.4407960913
Standard deviation (ddof = 0)	2.13×10^{-6}
Best final fitness	0.4407988594
Analytical optimal fitness	0.4407990905

5. Conclusions

In this article, we offer an advanced mathematical model that utilizes the latest mathematical modeling techniques, such as the plithogenic Pythagorean fuzzy hypersoft set, and determines the weight of each criterion through entropy and PSO to handle the natural ambiguity in multi-criteria decision making. Besides that, because overlapping attributes of plithogenic with respect to the nonlinear constraints of Pythagorean fuzzy logic is one of the properties that enable more flexibility than the classical crisp set, the plithogenic model will be a better way of understanding and simulating the complexities of human-machine interaction than ever before. The involvement of PSO inside the model makes chatbot evaluation a dynamic process that can be improved until the most optimal

solution is found. In this study, we demonstrate how advanced fuzzy logic and the development of intelligent algorithms provide an effective, scalable, and mathematically precise approach to measuring excellence in the rapidly evolving world of artificial intelligence. The proposed model can be extended by incorporating different intelligent optimization methods and comparing their performance with PSO. This technique may also be applied to real-world fields such as supplier selection, healthcare, risk assessment, and recommendation systems. Furthermore, hybrid models based on advanced fuzzy environments and machine learning techniques may provide additional improvements in decision-making accuracy and computational efficiency.

Authors contribution

Mahrukh Irfan, Usman Afzal and Shah Zeb contributed equally to conceptualization, data curation, formal analysis, investigation, methodology, visualization, and writing the original draft. Khalil Ur Rehman and Wasfi Shatanawi contributed equally to data curation, formal analysis, funding, and methodology. All authors reviewed and approved the final version.

Use of AI tools declaration

The authors confirm that this paper was not created using artificial intelligence (AI) methods. However, Quill-bot and Chat Gpt were utilized for rephrasing.

Acknowledgment

This work was supported by Prince Sultan University, Saudi Arabia, for technical support through the TAS research lab.

Conflict of interest

The authors confirm that they have no conflicts of interest.

References

1. L. A. Zadeh, Fuzzy sets, *Information and Control*, **8** (1965), 338–353. http://dx.doi.org/10.1016/S0019-9958(65)90241-X
2. A. Shahzad, A. Shoaib, N. Mlaiki, S. S. Aiadi, Results for fuzzy mappings and stability of fuzzy sets with applications, *Fractal Fract.*, **6** (2022), 556. http://dx.doi.org/10.3390/fractalfract6100556
3. K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Set. Syst.*, **20** (1986), 87–96. http://dx.doi.org/10.1016/S0165-0114(86)80034-3
4. R. R. Yager, Pythagorean fuzzy subsets, *Proceedings of the Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS)*, 2013, 57–61. http://dx.doi.org/10.1109/IFSA-NAFIPS.2013.6608375
5. P. A. Ejegwa, Modal operators on Pythagorean fuzzy sets and some of their properties, *Journal of Fuzzy Mathematics*, **27** (2019), 939–956.

6. D. Molodtsov, Soft set theory—first results, *Comput. Math. Appl.*, **37** (1999), 19–31. http://dx.doi.org/10.1016/S0898-1221(99)00056-5
7. P. Zhu, Q. Wen, Operations on soft sets revisited, *J. Appl. Math.*, **2013** (2013), 105752. http://dx.doi.org/10.1155/2013/105752
8. A. R. Roy, P. K. Maji, A fuzzy soft set theoretic approach to decision making problems, *J. Comput. Appl. Math.*, **203** (2007), 412–418. http://dx.doi.org/10.1016/j.cam.2006.04.008
9. F. Smarandache, Extension of soft set to hypersoft set, and then to plithogenic hypersoft set, *Neutrosophic Set. Syst.*, **22** (2018), 168–170. http://dx.doi.org/10.5281/zenodo.2159755
10. R. M. Zulqarnain, X. L. Xin, M. Saeed, A development of Pythagorean fuzzy hypersoft set with basic operations and decision-making approach based on the correlation coefficient, In: *Theory and application of hypersoft set*, Brussels: Pons Publishing House, 2021, 85–106. http://dx.doi.org/10.5281/zenodo.4788064
11. R. M. Zulqarnain, X. L. Xin, H. Garg, W. A. Khan, Aggregation operators of Pythagorean fuzzy soft sets with their application for green supplier chain management, *J. Intell. Fuzzy Syst.*, **40** (2021), 5545–5563. http://dx.doi.org/10.3233/JIFS-202781
12. S. Khan, M. I. Asjad, M. B. Riaz, A. Wahab, H. Ashaq, T. Muhammad, Innovative player evaluation: dual-possibility Pythagorean fuzzy hypersoft sets for accurate international football rankings, *Heliyon*, **10** (2024), e36993. http://dx.doi.org/10.1016/j.heliyon.2024.e36993
13. F. Smarandache, *Plithogeny, plithogenic set, logic, probability, and statistics*, Brussels: Pons Publishing House, 2017.
14. C. E. Shannon, A mathematical theory of communication, *Bell System Technical Journal*, **27** (1948), 379–423. http://dx.doi.org/10.1002/j.1538-7305.1948.tb01338.x
15. A. De Luca, S. Termini, A definition of a nonprobabilistic entropy in the setting of fuzzy sets theory, *Information and Control*, **20** (1972), 301–312. http://dx.doi.org/10.1016/S0019-9958(72)90199-4
16. S. G. Quek, G. Selvachandran, F. Smarandache, J. Vimala, S. H. Le, Q. T. Bui, et al., Entropy measures for plithogenic sets and applications in multi-attribute decision making, *Mathematics*, **8** (2020), 965. http://dx.doi.org/10.3390/math8060965
17. J. Kennedy, R. Eberhart, Particle swarm optimization, *Proceedings of ICNN'95—International Conference on Neural Networks*, 1995, 1942–1948. http://dx.doi.org/10.1109/ICNN.1995.488968
18. M. C. Z. Zambou, A. S. T. Kammogne, M. S. Siewe, A. T. Azar, S. Ahmed, I. A. Hameed, Optimized nonlinear PID control for maximum power point tracking in PV systems using particle swarm optimization, *Math. Comput. Appl.*, **29** (2024), 88. http://dx.doi.org/10.3390/mca29050088
19. M. Soleimani, A. Harooni, N. Erfani, A. R. Khan, T. Saba, S. A. Bahaj, Classification of cancer types based on microRNA expression using a hybrid radial basis function and particle swarm optimization algorithm, *Microsc. Res. Tech.*, **87** (2024), 1052–1062. http://dx.doi.org/10.1002/jemt.24492



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)