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Research article

Common fixed points of enriched Reich–Suzuki-type nonexpansive pair in Banach spaces via modified Picard-S iteration

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Abstract: In this article, the notion of an enriched γ-Reich–Suzuki-type nonexpansive pair (V_1, V_2) is proposed in the setting of Banach spaces, and the question of when two distinct such mappings share a fixed point is studied. It is shown that the pair admits a unique common fixed point and that the common fixed point problem for (V_1, V_2) is well-posed. To approximate this point, the modified Picard-S iteration is restructured to act on the π-Krasnoselskii maps of V_1 and V_2, and weak and strong convergence theorems are obtained under geometric assumptions on the space. A numerical example compares this scheme with modified Ishikawa, Abbas–Nazir, Noor, Agarwal, and Thakur iterations for several choices of control sequences; in every case, the modified Picard-S scheme attains the tolerance 10^{-12} in the fewest iterations. Finally, the theory is applied to a system of Fredholm integral equations in a uniformly convex Banach space $L^2([0, 1])$, yielding a unique, well-posed common solution to which the modified Picard-S sequence converges strongly.

Keywords: common fixed point; enriched Reich–Suzuki-type nonexpansive pair; weak convergence; strong convergence; Banach space; Fredholm integral equation
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1. Introduction

The study of fixed point theory for nonexpansive mappings, together with the convergence analysis of iterative methods for approximating these fixed points, has attracted sustained attention in functional analysis and its applications. A significant advance in this direction was made by Berinde [2], who introduced the class of enriched nonexpansive mappings in 2019 by incorporating an enrichment parameter $b \geq 0$ into the nonexpansiveness condition. This generalization strictly extends the classical framework and enables fixed point results in settings where standard nonexpansive mappings even fail. Motivated by this contribution, several enriched classes have since been studied: enriched Kannan mappings [5], enriched Suzuki nonexpansive mappings [16], and enriched Reich–Suzuki nonexpansive

type mappings [4]. More recently, enriched Suzuki nonexpansive mappings have been studied in Banach spaces via faster iterative schemes [18], and enriched contractions have been extended to enrichment parameters $b \in \mathbb{R} \setminus \{-1\}$ [19].

While the above works focus on single mappings, many problems in analysis and applied mathematics require the existence of a common fixed point for a pair of mappings. Common fixed point theory for a pair of nonexpansive type mappings has been investigated in [1, 7, 17], and recent studies have also begun to address enriched pair mappings in broader generalized frameworks. A related direction is pursued by Pitea and Ricinschi [12], who devise a three-step iterative scheme for common fixed points of L_2-operators in uniformly convex Banach spaces and use it to generate Julia and Mandelbrot sets, thereby connecting common fixed point iterations to the study of complex dynamics and fractal geometry.

In this paper, we introduce the class of enriched γ-Reich–Suzuki-type nonexpansive pairs (V_1, V_2) in Banach spaces. While the enriched single-mapping framework of [4] guarantees a fixed point for a single mapping V, it does not address the substantially different question of when two distinct mappings V_1 and V_2 share a common fixed point; this is precisely the gap our definition is designed to fill. Passing from a single mapping to a pair raises difficulties at every stage that are absent in [4]. First, the classical iterative schemes are designed for a single mapping, so they cannot be applied directly to a pair. We are forced to modify the Ishikawa, Abbas–Nazir, Noor, Agarwal, and Thakur iterations for a pair of mappings by deciding at each step whether V_1 or V_2 acts. The Picard-S iteration was introduced in [9] and modified for a pair of mappings by Khatoon et al. [15]; we apply this modified Picard-S iteration to V_{1_π} and V_{2_π} and compare it with the other five modified schemes. Moreover, the enriched condition (3.1) gives information only on the π-Krasnoselskii maps V_{1_π} and V_{2_π}. It controls only the mixed quantity $\|V_{1_\pi}\iota - V_{2_\pi}\kappa\|$ and never V_{1_π} or V_{2_π} separately. For this reason the modified schemes, and in particular the restructured modified Picard-S scheme (4.7), act on V_{1_π} and V_{2_π} rather than on V_1 and V_2.

Second, a numerical example cannot be built from two arbitrary self-mappings. Taking $\iota = \kappa$ equal to a fixed point of V_1 or V_2 in (3.1), for which the premise of Definition 11 holds trivially, shows that every enriched γ-Reich–Suzuki-type nonexpansive pair satisfies $F(V_1) = F(V_2)$, as proved in Remark 2. We are therefore forced to construct two distinct self-mappings with the same common fixed point and to verify the pair inequality on each region of their definition. Only then can the modified Picard-S scheme be compared with the other modified schemes. The example of Section 5 shows that it reaches the prescribed tolerance in fewer iterations.

Third, to apply our results to nonlinear integral equations, we define $V_i = -bI + (1 + b)W_i$. Then V_1 and V_2 are expansive, while their π-Krasnoselskii maps are exactly the Fredholm operators W_1 and W_2. Each W_i is a contraction, but the pair inequality compares W_1 at ι with W_2 at κ. We are therefore forced to control $W_1\kappa - W_2\kappa$ through a Lipschitz condition on $f_1 - f_2$, given in (H_4), whose constant is bounded in terms of the contraction constants of W_1 and W_2, as in (H_5). Moreover, since the pair inequality forces both equations to have the same solution set, the forcing terms g_1 and g_2 must satisfy the compatibility condition (H_6), which cannot be dropped by Remark 5.

Our first contribution is thus to extend the enriched single-mapping framework of [4] to a genuine pair of mappings, and to prove the existence, uniqueness, and well-posedness of common fixed points for such a pair. Our second contribution is the modified Picard-S iterative scheme, a restructured version of the Picard-S iteration tailored specifically to enriched γ-Reich–Suzuki-type nonexpansive pairs, for which we establish weak and strong convergence under standard geometric conditions on

the underlying Banach space. We modified the Ishikawa, Noor, Agarwal, Abbas–Nazir, and Thakur iteration schemes for a pair of mappings. Together with the modified Picard-S scheme of Khatoon et al. [15], we implemented each of the six schemes on the π-Krasnoselskii maps V_{1_π} and V_{2_π}, and compared them under an enriched γ-Reich–Suzuki-type nonexpansive pair. This comparison shows that the modified Picard-S iteration scheme attains the smallest error $|\iota_\eta - \iota^*|$ at every iterate. Our third contribution is an application of these results to a system of nonlinear Fredholm integral equations in the uniformly convex Banach space $L^2([0,1])$: We show that the corresponding pair of integral operators forms an enriched γ-Reich–Suzuki-type nonexpansive pair, from which the existence, uniqueness, and well-posedness of a common solution follow directly from the preceding fixed point theorems, and the modified Picard-S iteration scheme converges strongly to this solution in $L^2([0,1])$.

The remainder of the paper is organized as follows. Section 2 recalls the preliminary definitions and results needed in the sequel. Section 3 introduces the enriched γ-Reich–Suzuki-type nonexpansive pair and establishes the existence, uniqueness, and well-posedness of common fixed points. Section 4 develops the restructured modified Picard-S iterative scheme and proves the associated weak and strong convergence results. Section 5 presents a numerical example comparing the convergence behavior of the modified Picard-S scheme with that of existing iterative schemes. Section 6 applies the main results to a system of nonlinear Fredholm integral equations, and Section 7 concludes the paper.

2. Preliminaries

In this article, $\mathbb{Z}^+$ denotes the set of positive integers, $\mathbb{N}_0 := \mathbb{Z}^+ \cup \{0\}$, and $\mathbb{R}$ represents the set of real numbers. Moreover, $\mathcal{G}$ is a Banach space, and Y is a nonempty subset of $\mathcal{G}$. For a mapping V, $F(V)$ denotes its fixed point set. The following concepts and results will be employed later.

Initially, a Banach space $\mathcal{G}$ is said to be smooth if for each $\iota \in S_{\mathcal{G}}$, the limit

$$\lim_{t\to 0} \frac{\|\iota + t\kappa\| - \|\iota\|}{t} \tag{2.1}$$

exists for all $\kappa \in S_{\mathcal{G}}$, where $S_{\mathcal{G}} = \{\iota \in \mathcal{G} : \|\iota\| = 1\}$ denotes the unit sphere of $\mathcal{G}$.

The norm of $\mathcal{G}$ is said to be Fréchet differentiable if for each $\iota \in S_{\mathcal{G}}$, the limit in (2.1) exists uniformly for $\kappa \in S_{\mathcal{G}}$. In this case, for all $\iota, a \in \mathcal{G}$,

$$\frac{1}{2}\|\iota\|^2 + \langle a, J(\iota)\rangle \le \frac{1}{2}\|\iota + a\|^2 \le \frac{1}{2}\|\iota\|^2 + \langle a, J(\iota)\rangle + g\left(\|a\|\right), \tag{2.2}$$

where $J(\iota)$ denotes the Fréchet derivative of the functional $\frac{1}{2}\|\cdot\|^2$ at $\iota \in \mathcal{G}$, and $g : [0,\infty) \to [0,\infty)$ is an increasing function satisfying

$$\lim_{t\to 0} \frac{g(t)}{t} = 0.$$

For each $\epsilon \in [0,2]$, the modulus of convexity of $\mathcal{G}$ is defined by

$$\delta_{\mathcal{G}}(\epsilon) := \inf\left\{1 - \frac{\|\iota + \kappa\|}{2} : \|\iota\| \le 1,\ \|\kappa\| \le 1,\ \|\iota - \kappa\| \ge \epsilon\right\}.$$

The space $\mathcal{G}$ is called uniformly convex if $\delta_{\mathcal{G}}(\epsilon) > 0$ for every $\epsilon > 0$.

Definition 1. *Let $V_1, V_2, \ldots, V_m : Y \to Y$ be self-mappings. A point $\iota \in Y$ is called a common fixed point of $V_1, \ldots, V_m$ if $V_i\iota = \iota$ for all $i = 1, 2, \ldots, m$.*

Definition 2. *[6] A mapping $V : Y \to Y$ is called asymptotically regular on Y if*

$$\lim_{\eta\to\infty} \|V^{\eta+1}\iota - V^{\eta}\iota\| = 0, \quad \text{for every } \iota \in Y.$$

Definition 3. *Let $V : \mathcal{G} \to \mathcal{G}$ and $k \in \mathbb{Z}^+$.*

(1) V is called k-continuous if $V^k\iota_\eta \to V\varsigma$ whenever $\{\iota_\eta\} \subseteq \mathcal{G}$ satisfies $V^{k-1}\iota_\eta \to \varsigma$.

(2) V is called orbitally continuous if, for every $\iota \in \mathcal{G}$, $V^{\eta_j}\iota \to \varsigma$ for some subsequence $\{\eta_j\}$ implies $V^{\eta_j+1}\iota \to V\varsigma$.

Definition 4. *[10] The space $\mathcal{G}$ is said to satisfy the Opial property if, for every sequence $\{\iota_\eta\}$ in $\mathcal{G}$ converging weakly to some $\varsigma \in \mathcal{G}$,*

$$\liminf_{\eta\to\infty} \|\iota_\eta - \varsigma\| < \liminf_{\eta\to\infty} \|\iota_\eta - \varsigma^*\| \quad \text{for all } \varsigma^* \in \mathcal{G} \text{ with } \varsigma^* \neq \varsigma.$$

Definition 5. *[8] Suppose $\{\iota_\eta\}$ is a bounded sequence in $\mathcal{G}$.*

(1) For all $\iota \in \mathcal{G}$,

$$r(\iota, \{\iota_\eta\}) := \limsup_{\eta\to\infty} \|\iota_\eta - \iota\|$$

defines the asymptotic radius of $\{\iota_\eta\}$ at ι.

(2) For all $\iota \in \mathcal{G}$,

$$r(Y, \{\iota_\eta\}) := \inf\{r(\iota, \{\iota_\eta\}) : \iota \in Y\}$$

defines the asymptotic radius of $\{\iota_\eta\}$ with respect to Y.

(3) For all $\iota \in \mathcal{G}$,

$$A(Y, \{\iota_\eta\}) := \{\iota \in Y : r(\iota, \{\iota_\eta\}) = r(Y, \{\iota_\eta\})\}$$

defines the asymptotic center of $\{\iota_\eta\}$ with respect to Y.

In any uniformly convex Banach space, $A(Y, \{\iota_\eta\})$ consists of exactly one point.

Definition 6. *A mapping $V : Y \subseteq \mathcal{G} \to \mathcal{G}$ is called demiclosed at κ if, whenever ι_η converges weakly to ι in Y and $V\iota_\eta \to \kappa$ strongly in $\mathcal{G}$, it follows that $V\iota = \kappa$.*

Definition 7. *[7] Let $V_1, V_2 : Y \to Y$ be mappings with $F(V) = F(V_1) \cap F(V_2) \neq \emptyset$. The pair (V_1, V_2) satisfies Condition (B) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, such that*

$$\max\{\|V_1\iota - \iota\|, \|V_2\iota - \iota\|\} \geq f(d(\iota, F)) \quad \text{for all } \iota \in Y, \tag{2.3}$$

where $d(\iota, F) = \inf_{\lambda\in F} \|\iota - \lambda\|$.

Definition 8. *[11] A mapping $V : Y \to Y$ is called a γ-Reich–Suzuki-type nonexpansive mapping if there exists $\gamma \in [0, 1)$ such that, for all $\iota, \kappa \in Y$,*

$$\tfrac{1}{2}\|\iota - V\iota\| \leq \|\iota - \kappa\| \quad \textit{implies}$$

$$\|V\iota - V\kappa\| \leq \gamma\|V\iota - \iota\| + \gamma\|V\kappa - \kappa\| + (1 - 2\gamma)\|\iota - \kappa\|. \tag{2.4}$$

Definition 9. *[3] For $\pi \in (0, 1]$, the Krasnoselskii map associated with a self-mapping $V : Y \to Y$, where Y is convex, is defined by $V_\pi\iota = (1 - \pi)\iota + \pi V\iota,\ \iota \in Y$.*

Remark 1. *The fixed point sets of V and V_π coincide: $F(V_\pi) = F(V)$.*

Definition 10. *[4] A mapping $V : Y \to Y$ is called an enriched γ-Reich–Suzuki-type nonexpansive mapping if there exist $b \in [0, \infty)$ and $\gamma \in [0, 1)$ such that, for all $\iota, \kappa \in Y$,*

$$\tfrac{1}{2}\|\iota - V\iota\| \leq (b + 1)\|\iota - \kappa\| \quad \textit{implies}$$

$$\|b(\iota - \kappa) + V\iota - V\kappa\| \leq \gamma\|V\iota - \iota\| + \gamma\|V\kappa - \kappa\| + (b + 1)(1 - 2\gamma)\|\iota - \kappa\|. \tag{2.5}$$

Lemma 1. *[13] Let $\mathcal{G}$ be a uniformly convex Banach space, and let $\{b_\eta\}$ be a real sequence with $0 < \theta_1 \leq b_\eta \leq \theta_2 < 1$ for all $\eta \in \mathbb{Z}^+$. Suppose two sequences $\{\iota_\eta\}$ and $\{\kappa_\eta\}$ in $\mathcal{G}$ satisfy*

$$\limsup_{\eta\to\infty} \|\iota_\eta\| \leq \epsilon, \qquad \limsup_{\eta\to\infty} \|\kappa_\eta\| \leq \epsilon, \qquad \lim_{\eta\to\infty} \|(1 - b_\eta)\kappa_\eta + b_\eta\iota_\eta\| = \epsilon,$$

for some $\epsilon \geq 0$. Then $\lim_{\eta\to\infty} \|\kappa_\eta - \iota_\eta\| = 0$.

Lemma 2. *[14] Consider two sequences of non-negative real numbers, $\{\iota_\eta\}$ and $\{\kappa_\eta\}$ such that*

$$\iota_{\eta+1} \leq \iota_\eta + \kappa_\eta,$$

for all $\eta \geq 1$. If $\sum_{\eta=1}^{\infty} \kappa_\eta$ converges, then $\lim_{\eta\to\infty} \iota_\eta$ exists.

3. Results on common fixed points for an enriched pair of mappings

We now present the enriched Reich–Suzuki-type nonexpansive pair and derive some related results.

Definition 11. *Let Y be a nonempty subset of $\mathcal{G}$. A pair of mappings $V_1, V_2 : Y \to Y$, with $V_1 \neq V_2$, is known as an enriched γ-Reich–Suzuki-type nonexpansive pair if there exist $b \in [0, \infty)$ and $\gamma \in (0, 1)$ such that, for all $\iota, \kappa \in Y$,*

$$\tfrac{1}{2}\min\{\|\iota - V_1\iota\|, \|\kappa - V_2\kappa\|\} \leq (b + 1)\|\iota - \kappa\| \quad \textit{implies}$$

$$\|b(\iota - \kappa) + V_1\iota - V_2\kappa\| \leq \gamma\|V_1\iota - \iota\| + \gamma\|V_2\kappa - \kappa\| + (b + 1)(1 - 2\gamma)\|\iota - \kappa\|. \tag{3.1}$$

Example 1. *Let $\mathcal{G} = \mathbb{R}$ be endowed with the usual norm, and define $V_1, V_2 : \mathcal{G} \to \mathcal{G}$ by*

$$V_1\iota = -2\iota, \qquad V_2\iota = -3\iota, \qquad \iota \in \mathcal{G}.$$

Clearly $V_1 \neq V_2$ and $F(V_1) \cap F(V_2) = \{0\}$. We show that (V_1, V_2) is an enriched γ-Reich–Suzuki-type nonexpansive pair with $b = \frac{5}{2}$ and $\gamma = \frac{1}{6}$.

For all $\iota, \kappa \in \mathcal{G}$,

$$\|V_1\iota - \iota\| = 3\|\iota\| \quad \textit{and} \quad \|V_2\kappa - \kappa\| = 4\|\kappa\|.$$

Moreover,

$$b(\iota - \kappa) + V_1\iota - V_2\kappa = \tfrac{5}{2}\iota - \tfrac{5}{2}\kappa - 2\iota + 3\kappa = \tfrac{1}{2}\iota + \tfrac{1}{2}\kappa,$$

so that

$$\|b(\iota - \kappa) + V_1\iota - V_2\kappa\| \leq \tfrac{1}{2}\|\iota\| + \tfrac{1}{2}\|\kappa\|.$$

On the other hand, since $b + 1 = \frac{7}{2}$ and $1 - 2\gamma = \frac{2}{3}$,

$$\gamma\|V_1\iota - \iota\| + \gamma\|V_2\kappa - \kappa\| + (b+1)(1-2\gamma)\|\iota - \kappa\| = \tfrac{1}{6}\cdot 3\|\iota\| + \tfrac{1}{6}\cdot 4\|\kappa\| + \tfrac{7}{3}\|\iota - \kappa\| = \tfrac{1}{2}\|\iota\| + \tfrac{2}{3}\|\kappa\| + \tfrac{7}{3}\|\iota - \kappa\|.$$

Combining the two estimates, we see that (3.1) *holds for all $\iota, \kappa \in \mathcal{G}$, in particular whenever the premise of Definition 11 is satisfied, with $b = \frac{5}{2}$ and $\gamma = \frac{1}{6}$. Hence (V_1, V_2) is an enriched γ-Reich–Suzuki-type nonexpansive pair.*

Remark 2. *Let (V_1, V_2) be an enriched γ-Reich–Suzuki-type nonexpansive pair on Y and $\pi = \frac{1}{b+1}$. Since $\iota - V_{i_\pi}\iota = \pi(\iota - V_i\iota)$ for $i = 1, 2$ and $b(\iota - \kappa) + V_1\iota - V_2\kappa = (b + 1)(V_{1_\pi}\iota - V_{2_\pi}\kappa)$, Definition 11 is equivalent to the following: for all $\iota, \kappa \in Y$ satisfying*

$$P(\iota, \kappa): \quad \tfrac{1}{2}\min\{\|\iota - V_{1_\pi}\iota\|, \|\kappa - V_{2_\pi}\kappa\|\} \leq \|\iota - \kappa\|, \tag{3.2}$$

we have

$$\|V_{1_\pi}\iota - V_{2_\pi}\kappa\| \leq \gamma\|V_{1_\pi}\iota - \iota\| + \gamma\|V_{2_\pi}\kappa - \kappa\| + (1 - 2\gamma)\|\iota - \kappa\|. \tag{3.3}$$

In particular, by Remark 1, $P(\iota, \kappa)$ holds whenever $\iota \in F(V_1)$ or $\kappa \in F(V_2)$.

(i) Both sides of (3.1) *are symmetric in (ι, κ) when $V_1 = V_2$. Hence, if $V_1 = V_2 = V$, the defining condition of Definition 11 reduces to that of Definition 10, and for $b = 0$ to that of Definition 8. The requirement $V_1 \neq V_2$ excludes this case, since the purpose of the pair setting is to find a common fixed point of two distinct mappings.*

(ii) The two mappings of an enriched γ-Reich–Suzuki-type nonexpansive pair have the same fixed points. Let $V_2\varsigma = \varsigma$. Then $V_{2_\pi}\varsigma = \varsigma$ by Remark 1, so $P(\varsigma, \varsigma)$ holds since $\|\varsigma - V_{2_\pi}\varsigma\| = 0$, and (3.3) *with $\iota = \kappa = \varsigma$ gives $\|V_{1_\pi}\varsigma - \varsigma\| \leq \gamma\|V_{1_\pi}\varsigma - \varsigma\|$, that is, $(1 - \gamma)\|V_{1_\pi}\varsigma - \varsigma\| \leq 0$. Since $1 - \gamma > 0$, we get $V_{1_\pi}\varsigma = \varsigma$, and hence $V_1\varsigma = \varsigma$ by Remark 1. Thus $F(V_2) \subseteq F(V_1)$. Conversely, let $V_1\varsigma = \varsigma$. Then $P(\varsigma, \varsigma)$ holds since $\|\varsigma - V_{1_\pi}\varsigma\| = 0$, and* (3.3) *with $\iota = \kappa = \varsigma$ gives $(1 - \gamma)\|V_{2_\pi}\varsigma - \varsigma\| \leq 0$, so $V_2\varsigma = \varsigma$. Consequently $F(V_1) = F(V_2)$, and a fixed point of either mapping is a common fixed point of the pair.*

Lemma 3. *Let $V_1, V_2 : Y \to Y$ be an enriched γ-Reich–Suzuki-type nonexpansive pair, let $\pi = \frac{1}{b+1}$, and let $\varsigma \in F(V_1) \cap F(V_2)$. Then both π-Krasnoselskii maps V_{1_π} and V_{2_π} are quasinonexpansive with respect to ς; that is,*

$$\|V_{1_\pi}\iota - \varsigma\| \leq \|\iota - \varsigma\| \quad \textit{and} \quad \|V_{2_\pi}\kappa - \varsigma\| \leq \|\kappa - \varsigma\| \quad \forall\, \iota, \kappa \in Y.$$

Proof. By (3.1), we have

$$\|b(\iota-\kappa)+V_1\iota-V_2\kappa\| \leq \gamma\|V_1\iota-\iota\|+\gamma\|V_2\kappa-\kappa\|+(b+1)(1-2\gamma)\,\|\iota-\kappa\|\,,$$

whenever $P(\iota,\kappa)$, for all $\iota,\kappa\in Y$. By assuming $\pi=\frac{1}{b+1}$, we get

$$\|V_{1_\pi}\iota-V_{2_\pi}\kappa\| \leq \gamma\|V_{1_\pi}\iota-\iota\|+\gamma\|V_{2_\pi}\kappa-\kappa\|+(1-2\gamma)\|\iota-\kappa\|. \tag{3.4}$$

Since $V_2\varsigma=\varsigma$, we have $V_{2_\pi}\varsigma=\varsigma$ by Remark 1, so $P(\iota,\varsigma)$ holds for every $\iota\in Y$. Hence (3.3) with $\kappa=\varsigma$ gives

$$\|V_{1_\pi}\iota-\varsigma\| \leq \gamma\|V_{1_\pi}\iota-\iota\|+(1-2\gamma)\|\iota-\varsigma\| \quad \forall\ \iota\in Y.$$

By the triangle inequality, $\|V_{1_\pi}\iota-\iota\| \leq \|V_{1_\pi}\iota-\varsigma\|+\|\varsigma-\iota\|$, so

$$\|V_{1_\pi}\iota-\varsigma\| \leq \gamma\|V_{1_\pi}\iota-\varsigma\|+(1-\gamma)\|\iota-\varsigma\|,$$

that is, $(1-\gamma)\|V_{1_\pi}\iota-\varsigma\| \leq (1-\gamma)\|\iota-\varsigma\|$. As $1-\gamma>0$, we obtain $\|V_{1_\pi}\iota-\varsigma\| \leq \|\iota-\varsigma\|$ for all $\iota\in Y$.

Setting $\iota=\varsigma$ in (3.4), since $V_{1_\pi}\varsigma=\varsigma$, $P(\varsigma,\kappa)$ holds for every $\kappa\in Y$, and an argument identical to that in first case by using the triangle inequality $\|V_{2_\pi}\kappa-\kappa\| \leq \|V_{2_\pi}\kappa-\varsigma\|+\|\varsigma-\kappa\|$ and the fact that $1-\gamma>0$, yields

$$\|V_{2_\pi}\kappa-\varsigma\| \leq \|\kappa-\varsigma\|,$$

for all $\kappa\in Y$. This completes the proof. □

Lemma 4. *Suppose (V_1,V_2) is an enriched γ-Reich–Suzuki-type nonexpansive pair on a nonempty subset Y of $\mathcal{G}$, and let $\pi=\frac{1}{b+1}$. Then*

$$\|\iota-V_{2_\pi}\kappa\| \leq \left(\frac{1+\gamma}{1-\gamma}\right)\|\iota-V_{1_\pi}\iota\|+\|\iota-\kappa\|$$

for all $\iota,\kappa\in Y$ satisfying $P(\iota,\kappa)$.

Proof. Let $\iota,\kappa\in Y$ satisfy $P(\iota,\kappa)$. By Remark 2,

$$\|V_{1_\pi}\iota-V_{2_\pi}\kappa\| \leq \gamma\|V_{1_\pi}\iota-\iota\|+\gamma\|V_{2_\pi}\kappa-\kappa\|+(1-2\gamma)\|\iota-\kappa\|.$$

Consider,

$$\begin{aligned}\|\iota-V_{2_\pi}\kappa\| &\leq \|\iota-V_{1_\pi}\iota\|+\|V_{1_\pi}\iota-V_{2_\pi}\kappa\|\\ &\leq (1+\gamma)\|\iota-V_{1_\pi}\iota\|+\gamma\|\kappa-V_{2_\pi}\kappa\|+(1-2\gamma)\|\iota-\kappa\|\\ &\leq (1+\gamma)\|\iota-V_{1_\pi}\iota\|+\gamma\|\kappa-\iota\|+\gamma\|\iota-V_{2_\pi}\kappa\|+(1-2\gamma)\|\iota-\kappa\|\\ &= (1+\gamma)\|\iota-V_{1_\pi}\iota\|+\gamma\|\iota-V_{2_\pi}\kappa\|+(1-\gamma)\|\iota-\kappa\|.\end{aligned}$$

Since $1-\gamma>0$, we get

$$\|\iota-V_{2_\pi}\kappa\| \leq \left(\frac{1+\gamma}{1-\gamma}\right)\|\iota-V_{1_\pi}\iota\|+\|\iota-\kappa\|.$$

□

Theorem 1. *Let (V_1, V_2) be an enriched γ-Reich–Suzuki type nonexpansive pair on a nonempty subset Y of $\mathcal{G}$, where $\gamma \in (0,1)$ and $\pi = \frac{1}{b+1}$. Suppose that V_{1_π} and V_{2_π} are asymptotically regular on Y, and that V_{1_π} and V_{2_π} are either k-continuous for some $k \in \mathbb{Z}^+$ or orbitally continuous. Then V_1 and V_2 have a unique common fixed point ς, and*

$$\lim_{\eta\to\infty} V_{1_\pi}^{\eta}\iota = \varsigma = \lim_{\eta\to\infty} V_{2_\pi}^{\eta}\iota \quad \textit{for any } \iota \in \mathcal{G}.$$

Proof. We show this through the following cases:

Case 1: $b > 0$. This gives $0 < \pi < 1$. Applying Definition 11 with $\pi = \frac{1}{b+1}$, the enriched γ-Reich–Suzuki condition becomes

$$\|V_{1_\pi}\iota - V_{2_\pi}\kappa\| \le \gamma\|V_{1_\pi}\iota - \iota\| + \gamma\|V_{2_\pi}\kappa - \kappa\| + (1-2\gamma)\|\iota - \kappa\|, \tag{3.5}$$

whenever $P(\iota, \kappa)$, for all $\iota, \kappa \in \mathcal{G}$. We first show that $\lim_{\eta\to\infty} \|V_{1_\pi}^{\eta}\iota - V_{2_\pi}^{\eta}\iota\| = 0$ for any $\iota \in \mathcal{G}$.

Subcase 1: $1 - 2\gamma = 0$. Then (3.5) gives

$$\|V_{1_\pi}\iota - V_{2_\pi}\kappa\| \le \frac{1}{2}(\|V_{1_\pi}\iota - \iota\| + \|V_{2_\pi}\kappa - \kappa\|), \tag{3.6}$$

Substituting $\iota = V_{1_\pi}^{\eta-1}\iota$ and $\kappa = V_{2_\pi}^{\eta-1}\iota$ in (3.6),

$$\|V_{1_\pi}^{\eta}\iota - V_{2_\pi}^{\eta}\iota\| \le \frac{1}{2}(\|V_{1_\pi}^{\eta}\iota - V_{1_\pi}^{\eta-1}\iota\| + \|V_{2_\pi}^{\eta}\iota - V_{2_\pi}^{\eta-1}\iota\|).$$

Taking the limit and using the asymptotic regularity of V_{1_π} and V_{2_π},

$$\lim_{\eta\to\infty} \|V_{1_\pi}^{\eta}\iota - V_{2_\pi}^{\eta}\iota\| = 0. \tag{3.7}$$

Subcase 2: $1 - 2\gamma < 0$, that is, $\gamma \in (\frac{1}{2}, 1)$. Substituting $\iota = V_{1_\pi}^{\eta-1}\iota$ and $\kappa = V_{2_\pi}^{\eta-1}\iota$ in (3.5),

$$\|V_{1_\pi}^{\eta}\iota - V_{2_\pi}^{\eta}\iota\| \le \gamma\|V_{1_\pi}^{\eta}\iota - V_{1_\pi}^{\eta-1}\iota\| + \gamma\|V_{2_\pi}^{\eta}\iota - V_{2_\pi}^{\eta-1}\iota\| + (1-2\gamma)\|V_{1_\pi}^{\eta-1}\iota - V_{2_\pi}^{\eta-1}\iota\|.$$

Since $1 - 2\gamma < 0$, the last term is nonpositive, and hence

$$\|V_{1_\pi}^{\eta}\iota - V_{2_\pi}^{\eta}\iota\| \le \gamma\|V_{1_\pi}^{\eta}\iota - V_{1_\pi}^{\eta-1}\iota\| + \gamma\|V_{2_\pi}^{\eta}\iota - V_{2_\pi}^{\eta-1}\iota\|.$$

By the asymptotic regularity of V_{1_π} and V_{2_π}, we get $\lim_{\eta\to\infty} \|V_{1_\pi}^{\eta}\iota - V_{2_\pi}^{\eta}\iota\| = 0$.

Subcase 3: $0 < 1 - 2\gamma < 1$, that is, $\gamma \in \left(0, \frac{1}{2}\right)$. Put $\theta := 1 - 2\gamma$, so that $0 < \theta < 1$, and set

$$d_\eta := \|V_{1_\pi}^{\eta}\iota - V_{2_\pi}^{\eta}\iota\| \quad (\eta \ge 0), \qquad r_\eta := \gamma\|V_{1_\pi}^{\eta}\iota - V_{1_\pi}^{\eta-1}\iota\| + \gamma\|V_{2_\pi}^{\eta}\iota - V_{2_\pi}^{\eta-1}\iota\| \quad (\eta \ge 1).$$

Note that $d_0 = \|\iota - \iota\| = 0$. Substituting $\iota = V_{1_\pi}^{\eta-1}\iota$ and $\kappa = V_{2_\pi}^{\eta-1}\iota$ in (3.5), we obtain

$$d_\eta \le r_\eta + \theta\, d_{\eta-1} \qquad \text{for all } \eta \ge 1. \tag{3.8}$$

We first show that the sequence $\{d_\eta\}$ is bounded. By the asymptotic regularity of V_{1_π} and V_{2_π}, we have $r_\eta \to 0$; hence $M := \sup_{\eta\geq 1} r_\eta < \infty$, and (3.8) gives

$$d_\eta \leq M + \theta\, d_{\eta-1} \qquad \text{for all } \eta \geq 1. \tag{3.9}$$

We claim that

$$d_\eta \leq M\left(1 + \theta + \theta^2 + \cdots + \theta^{\eta-1}\right) \qquad \text{for all } \eta \geq 1. \tag{3.10}$$

For $\eta = 1$, (3.9) and $d_0 = 0$ give $d_1 \leq M$. Assume (3.10) holds for some $\eta \geq 1$. Since $\theta > 0$, it follows from (3.9) and the induction hypothesis that

$$d_{\eta+1} \leq M + \theta\, d_\eta \leq M + \theta M\left(1 + \theta + \cdots + \theta^{\eta-1}\right) = M\left(1 + \theta + \cdots + \theta^{\eta}\right),$$

which is (3.10) for $\eta + 1$. This proves (3.10). Since $0 < \theta < 1$,

$$1 + \theta + \cdots + \theta^{\eta-1} = \frac{1-\theta^\eta}{1-\theta} \leq \frac{1}{1-\theta} = \frac{1}{2\gamma},$$

and therefore $0 \leq d_\eta \leq \dfrac{M}{2\gamma}$ for all $\eta \geq 1$. Consequently,

$$J := \limsup_{\eta\to\infty} d_\eta \quad \text{satisfies} \quad 0 \leq J \leq \frac{M}{2\gamma} < \infty.$$

We now show that $d_\eta \to 0$. Taking $\limsup_{\eta\to\infty}$ in (3.8), and using the subadditivity of lim sup, the fact that $\lim_{\eta\to\infty} r_\eta = 0$, the positivity of θ, and $\limsup_{\eta\to\infty} d_{\eta-1} = \limsup_{\eta\to\infty} d_\eta$, we obtain

$$J \leq \lim_{\eta\to\infty} r_\eta + \theta \limsup_{\eta\to\infty} d_{\eta-1} = \theta J = (1-2\gamma)J.$$

Since $J < \infty$, this gives $2\gamma J \leq 0$, and since $\gamma > 0$, we conclude $J = 0$. As $d_\eta \geq 0$, it follows that

$$\lim_{\eta\to\infty} \|V_{1_\pi}^\eta \iota - V_{2_\pi}^\eta \iota\| = 0. \tag{3.11}$$

In each of the above subcases, $\lim_{\eta\to\infty} \|V_{1_\pi}^\eta \iota - V_{2_\pi}^\eta \iota\| = 0$.

We now show that the sequence $\iota_\eta := V_{1_\pi}^\eta \iota$ is Cauchy. Suppose for contradiction that $\{\iota_\eta\}$ is not Cauchy. Then there exist $\epsilon > 0$ and sequences of integers $\{\upsilon(i)\}$, and $\{\eta(i)\}$ with $\upsilon(i) > \eta(i) \geq i$ such that

$$\|V_{1_\pi}^{\upsilon(i)} \iota - V_{1_\pi}^{\eta(i)} \iota\| \geq \epsilon, \qquad \text{for } i = 1, 2, \ldots. \tag{3.12}$$

Choosing $\upsilon(i)$ as the smallest integer exceeding $\eta(i)$ satisfying (3.12),

$$\|V_{1_\pi}^{\upsilon(i)-1} \iota - V_{1_\pi}^{\eta(i)} \iota\| < \epsilon.$$

By the triangle inequality and (3.12),

$$\epsilon \leq \|V_{1_\pi}^{\upsilon(i)} \iota - V_{1_\pi}^{\eta(i)} \iota\| \leq \|V_{1_\pi}^{\upsilon(i)} \iota - V_{1_\pi}^{\upsilon(i)-1} \iota\| + \|V_{1_\pi}^{\upsilon(i)-1} \iota - V_{1_\pi}^{\eta(i)} \iota\| < \|V_{1_\pi}^{\upsilon(i)} \iota - V_{1_\pi}^{\upsilon(i)-1} \iota\| + \epsilon.$$

Taking $i \to \infty$ and using asymptotic regularity,

$$\lim_{i\to\infty} \|V_{1_\pi}^{\upsilon(i)}\iota - V_{1_\pi}^{\eta(i)}\iota\| = \epsilon. \tag{3.13}$$

From the asymptotic regularity of V_{1_π} and the triangle inequality,

$$\|V_{1_\pi}^{\upsilon(i)-1}\iota - V_{1_\pi}^{\eta(i)-1}\iota\| \le \|V_{1_\pi}^{\upsilon(i)-1}\iota - V_{1_\pi}^{\upsilon(i)}\iota\| + \|V_{1_\pi}^{\upsilon(i)}\iota - V_{1_\pi}^{\eta(i)}\iota\| + \|V_{1_\pi}^{\eta(i)}\iota - V_{1_\pi}^{\eta(i)-1}\iota\|,$$

so

$$\lim_{i\to\infty} \|V_{1_\pi}^{\upsilon(i)-1}\iota - V_{1_\pi}^{\eta(i)-1}\iota\| = \epsilon. \tag{3.14}$$

By the triangle inequality,

$$\|V_{1_\pi}^{\upsilon(i)}\iota - V_{1_\pi}^{\eta(i)}\iota\| \le \|V_{1_\pi}^{\upsilon(i)}\iota - V_{2_\pi}^{\eta(i)}\iota\| + \|V_{2_\pi}^{\eta(i)}\iota - V_{1_\pi}^{\eta(i)}\iota\|.$$

Applying (3.5) with $\iota = V_{1_\pi}^{\upsilon(i)-1}\iota$ and $\kappa = V_{2_\pi}^{\eta(i)-1}\iota$,

$$\begin{aligned}\|V_{1_\pi}^{\upsilon(i)}\iota - V_{1_\pi}^{\eta(i)}\iota\| \le \|V_{2_\pi}^{\eta(i)}\iota - V_{1_\pi}^{\eta(i)}\iota\| + \gamma\|V_{1_\pi}^{\upsilon(i)}\iota - V_{1_\pi}^{\upsilon(i)-1}\iota\| \\ + \gamma\|V_{2_\pi}^{\eta(i)}\iota - V_{2_\pi}^{\eta(i)-1}\iota\| + (1-2\gamma)\|V_{1_\pi}^{\upsilon(i)-1}\iota - V_{2_\pi}^{\eta(i)-1}\iota\|.\end{aligned}$$

Letting $i \to \infty$ and applying (3.11), (3.13), (3.14), and the asymptotic regularity of V_{1_π} and V_{2_π},

$$\epsilon \le (1-2\gamma)\epsilon,$$

which gives $2\gamma\epsilon \le 0$. Since $\epsilon > 0$, this contradicts $\gamma > 0$. Hence $\{\iota_\eta\}$ is a Cauchy sequence. Since $\mathcal{G}$ is complete, $\iota_\eta \to \varsigma$ for some $\varsigma \in \mathcal{G}$.

By the triangle inequality and (3.11),

$$\|V_{2_\pi}^{\eta}\iota - \varsigma\| \le \|V_{2_\pi}^{\eta}\iota - V_{1_\pi}^{\eta}\iota\| + \|V_{1_\pi}^{\eta}\iota - \varsigma\| \to 0,$$

so $V_{2_\pi}^{\eta}\iota \to \varsigma$ as well.

Suppose V_{1_π} is k-continuous. Since $V_{1_\pi}^{k-1}\iota_\eta = \iota_{\eta+k-1} \to \varsigma$, we obtain $V_{1_\pi}^{k}\iota_\eta = \iota_{\eta+k} \to V_{1_\pi}\varsigma$, so $V_{1_\pi}\varsigma = \varsigma$. If instead V_{1_π} is orbitally continuous, then $\iota_\eta \to \varsigma$ and orbital continuity give $V_{1_\pi}\iota_\eta \to V_{1_\pi}\varsigma$, so again $V_{1_\pi}\varsigma = \varsigma$. In either case, $V_1\varsigma = \varsigma$. An identical argument gives $V_2\varsigma = \varsigma$.

For uniqueness, suppose $\varpi \in \mathcal{G}$ satisfies $V_{1_\pi}\varpi = \varpi = V_{2_\pi}\varpi$ with $\varpi \neq \varsigma$. Setting $\iota = \varsigma$ and $\kappa = \varpi$ in (3.5),

$$\|\varsigma - \varpi\| \le (1-2\gamma)\|\varsigma - \varpi\|,$$

so $2\gamma\|\varsigma - \varpi\| \le 0$. Since $\gamma > 0$, this forces $\varsigma = \varpi$, a contradiction. Hence ς is the unique common fixed point of V_1 and V_2.

Case 2: $b = 0$. Then $\pi = 1$ and the Krasnoselskii maps reduce to $V_{1_\pi} = V_1$ and $V_{2_\pi} = V_2$. The conclusion follows by arguments identical to those in Case 1.

□

Definition 12. *The common fixed point problem for self-mappings V_1 and V_2 on $\mathcal{G}$ is called well-posed if the set of common fixed points of V_1 and V_2 is a singleton, i.e., $V_2\varsigma = \varsigma = V_1\varsigma$, and for any sequence $\{\iota_\eta\}$ in $\mathcal{G}$ satisfying $\lim_{\eta\to\infty} \|V_1\iota_\eta - \iota_\eta\| = 0$ or $\lim_{\eta\to\infty} \|V_2\iota_\eta - \iota_\eta\| = 0$ implies that $\lim_{\eta\to\infty} \iota_\eta = \varsigma$.*

Theorem 2. *Suppose (V_1, V_2) is an enriched γ-Reich–Suzuki-type nonexpansive pair as in Theorem 1. Then the common fixed point problem of the pair (V_1, V_2) is well-posed.*

Proof. By Theorem 1, ς is the unique common fixed point of V_1 and V_2. Suppose $\lim_{\eta\to\infty} \|V_{1_\pi}\iota_\eta - \iota_\eta\| = 0$. Using (3.5), we have,

$$\begin{aligned}\|\iota_\eta - \varsigma\| &\leq \|\iota_\eta - V_{1_\pi}\iota_\eta\| + \|V_{1_\pi}\iota_\eta - V_{2_\pi}\varsigma\| \\ &\leq \|\iota_\eta - V_{1_\pi}\iota_\eta\| + \gamma\|V_{1_\pi}\iota_\eta - \iota_\eta\| + \gamma\|V_{2_\pi}\varsigma - \varsigma\| + (1-2\gamma)\|\iota_\eta - \varsigma\|,\end{aligned}$$

that is,

$$2\gamma \left\|\iota_\eta - \varsigma\right\| \leq (1+\gamma) \left\|\iota_\eta - V_{1_\pi}\iota_\eta\right\|. \tag{3.15}$$

Similarly, if we assume $\lim_{\eta\to\infty} \left\|\iota_\eta - V_{2_\pi}\iota_\eta\right\| = 0$, we can apply arguments similar to those presented above to obtain the following:

$$2\gamma \left\|\iota_\eta - \varsigma\right\| \leq (1+\gamma) \left\|\iota_\eta - V_{2_\pi}\iota_\eta\right\|. \tag{3.16}$$

It can be concluded from Eqs (3.15) and (3.16) that

$$\lim_{\eta\to\infty} \iota_\eta = \varsigma \tag{3.17}$$

provided that $\lim_{\eta\to\infty} \|V_{1_\pi}\iota_\eta - \iota_\eta\| = 0$ or $\lim_{\eta\to\infty} \|V_{2_\pi}\iota_\eta - \iota_\eta\| = 0$. □

4. Convergence results via modified Picard-S iteration

This section presents several iterative schemes for two mappings $V_1, V_2 : Y \to Y$ and illustrates their convergence behavior towards a common fixed point.

A variety of iterative algorithms have been developed to approximate fixed points of nonlinear mappings in metric spaces. In problems involving common fixed points, it is natural to consider paired self-mappings, and many classical iterative schemes have consequently been adapted to operate with two mappings. For two mappings $V_1, V_2 : Y \to Y$ and an initial point $\iota_0 \in Y$, several such modified schemes are listed below, where the sequence $\{\iota_\eta\}$ is generated as follows:

(1) Modified Ishikawa iteration.

$$\begin{cases}\kappa_\eta = (1-b_\eta)\iota_\eta + b_\eta V_1\iota_\eta, \\ \iota_{\eta+1} = (1-a_\eta)\iota_\eta + a_\eta V_2\kappa_\eta,\end{cases} \qquad a_\eta, b_\eta \in (0,1),\ \eta \in \mathbb{N}_0. \tag{4.1}$$

(2) Modified Abbas-Nazir iteration.

$$\begin{cases}\lambda_\eta = (1-c_\eta)\iota_\eta + c_\eta V_1\iota_\eta, \\ \kappa_\eta = (1-b_\eta)V_1\iota_\eta + b_\eta V_2\lambda_\eta, \\ \iota_{\eta+1} = (1-a_\eta)V_1\kappa_\eta + a_\eta V_2\lambda_\eta,\end{cases} \qquad a_\eta, b_\eta, c_\eta \in [0,1),\ \eta \in \mathbb{N}_0. \tag{4.2}$$

(3) Modified Noor iteration.

$$\begin{cases} \lambda_\eta = (1-c_\eta)\iota_\eta + c_\eta V_1 \iota_\eta, \\ \kappa_\eta = (1-b_\eta)\iota_\eta + b_\eta V_2 \lambda_\eta, \\ \iota_{\eta+1} = (1-a_\eta)\iota_\eta + a_\eta V_1 \kappa_\eta, \end{cases} \qquad a_\eta, b_\eta, c_\eta \in [0,1),\ \eta \in \mathbb{N}_0. \tag{4.3}$$

(4) Modified Agarwal iteration.

$$\begin{cases} \kappa_\eta = (1-b_\eta)\iota_\eta + b_\eta V_1 \iota_\eta, \\ \iota_{\eta+1} = (1-a_\eta)V_1 \iota_\eta + a_\eta V_2 \kappa_\eta, \end{cases} \qquad a_\eta, b_\eta \in [0,1),\ \eta \in \mathbb{N}_0. \tag{4.4}$$

(5) Modified Thakur iteration.

$$\begin{cases} \lambda_\eta = (1-c_\eta)\iota_\eta + c_\eta V_2(\iota_\eta), \\ \kappa_\eta = (1-b_\eta)\lambda_\eta + b_\eta V_1(\lambda_\eta), \\ \iota_{\eta+1} = (1-a_\eta)V_1(\lambda_\eta) + a_\eta V_2(\kappa_\eta), \end{cases} \qquad a_\eta, b_\eta, c_\eta \in [0,1),\ \eta \in \mathbb{N}_0. \tag{4.5}$$

(6) Modified Picard-S iteration.

Later, Khatoon et al. [15] modified the Picard-S iteration process [9] for a pair of mappings.

$$\begin{cases} \lambda_\eta = (1-b_\eta)\iota_\eta + b_\eta V_2 \iota_\eta, \\ \kappa_\eta = (1-a_\eta)V_1 \iota_\eta + a_\eta V_2 \lambda_\eta, \\ \iota_{\eta+1} = V_1 \kappa_\eta, \end{cases} \qquad a_\eta, b_\eta \in [0,1),\ \eta \in \mathbb{N}_0. \tag{4.6}$$

Furthermore, we consider a restructured version of the modified Picard-S iteration and employ it to establish common fixed point results for enriched γ-Reich–Suzuki-type nonexpansive pair. The proposed iteration is defined by

$$\begin{cases} \lambda_\eta = (1-b_\eta)\iota_\eta + b_\eta V_{2_\pi} \iota_\eta, \\ \kappa_\eta = (1-a_\eta)V_{1_\pi} \iota_\eta + a_\eta V_{2_\pi} \lambda_\eta, \\ \iota_{\eta+1} = V_{1_\pi} \kappa_\eta, \end{cases} \qquad a_\eta, b_\eta \in (0,1),\ \eta \in \mathbb{N}_0. \tag{4.7}$$

Initially, we establish the following lemma using the iterative method (4.7) for enriched γ-Reich–Suzuki-type nonexpansive pairs.

Lemma 5. *Suppose Y is a closed convex subset of $\mathcal{G}$. Let $V_1, V_2 : Y \to Y$ be an enriched γ-Reich–Suzuki-type nonexpansive pair with $F(V) = F(V_1) \cap F(V_2) \neq \emptyset$. If $\varsigma \in F(V)$ and the sequence $\{\iota_\eta\}$ is defined by (4.7), then $\lim_{\eta\to\infty} \left\|\iota_\eta - \varsigma\right\|$ exists for all $\varsigma \in F(V)$.*

Proof. Since (V_1, V_2) is an enriched γ-Reich–Suzuki-type nonexpansive pair with $F(V) = F(V_1) \cap F(V_2) \neq \emptyset$ and $\pi = \frac{1}{b+1}$, Lemma 3 shows that the π-Krasnoselskii maps V_{1_π}, and V_{2_π} are quasinonexpansive, that is,

$$\|V_{1_\pi}\iota - \varsigma\| \leq \|\iota - \varsigma\| \quad \text{and} \quad \|V_{2_\pi}\iota - \varsigma\| \leq \|\iota - \varsigma\| \quad \forall\ \iota \in Y, \tag{4.8}$$

for each $\varsigma \in F(V)$. Fix $\varsigma \in F(V)$. By (4.7), the triangle inequality, and (4.8),

$$\begin{aligned} \|\lambda_\eta - \varsigma\| &= \|(1-b_\eta)(\iota_\eta - \varsigma) + b_\eta(V_{2_\pi}\iota_\eta - \varsigma)\| \\ &\leq (1-b_\eta)\|\iota_\eta - \varsigma\| + b_\eta\|V_{2_\pi}\iota_\eta - \varsigma\| \\ &\leq (1-b_\eta)\|\iota_\eta - \varsigma\| + b_\eta\|\iota_\eta - \varsigma\| = \|\iota_\eta - \varsigma\|. \end{aligned} \tag{4.9}$$

Similarly, using (4.8) and then (4.9),

$$\begin{aligned} \|\kappa_\eta - \varsigma\| &= \|(1-a_\eta)(V_{1_\pi}\iota_\eta - \varsigma) + a_\eta(V_{2_\pi}\lambda_\eta - \varsigma)\| \\ &\leq (1-a_\eta)\|V_{1_\pi}\iota_\eta - \varsigma\| + a_\eta\|V_{2_\pi}\lambda_\eta - \varsigma\| \\ &\leq (1-a_\eta)\|\iota_\eta - \varsigma\| + a_\eta\|\lambda_\eta - \varsigma\| \\ &\leq (1-a_\eta)\|\iota_\eta - \varsigma\| + a_\eta\|\iota_\eta - \varsigma\| = \|\iota_\eta - \varsigma\|. \end{aligned} \tag{4.10}$$

Finally, by (4.8) and (4.10),

$$\|\iota_{\eta+1} - \varsigma\| = \|V_{1_\pi}\kappa_\eta - \varsigma\| \leq \|\kappa_\eta - \varsigma\| \leq \|\iota_\eta - \varsigma\|. \tag{4.11}$$

The preceding argument implies that $\left\{\|\iota_\eta - \varsigma\|\right\}$ is nonincreasing and bounded below for each $\varsigma \in F(V_1) \cap F(V_2)$. Thus, $\lim_{\eta\to\infty}\|\iota_\eta - \varsigma\|$ exists. □

Based on the lemma mentioned earlier, we establish an equivalent condition to demonstrate the presence of common fixed points in enriched γ-Reich–Suzuki-type nonexpansive pairs.

Theorem 3. *Consider a nonempty closed convex subset Y of a uniformly convex Banach space $\mathcal{G}$. Let $V_1, V_2 : Y \to Y$ be an enriched γ-Reich–Suzuki-type nonexpansive pair. For any $\iota_0 \in Y$, let $\{\iota_\eta\}$ be the sequence defined by the iteration scheme given in (4.7), where the control sequences satisfy $0 < \theta_1 \leq a_\eta, b_\eta \leq \theta_2 < 1$ for all $\eta \in \mathbb{N}_0$. Then $F(V) = F(V_1) \cap F(V_2) \neq \emptyset$ if and only if $\{\iota_\eta\}$ is bounded and $\lim_{\eta\to\infty}\|\iota_\eta - V_1\iota_\eta\| = 0 = \lim_{\eta\to\infty}\|\iota_\eta - V_2\iota_\eta\|$.*

Proof. Since Y is convex and $V_i(Y) \subseteq Y$, we have $V_{i_\pi}\iota = (1-\pi)\iota + \pi V_i\iota \in Y$ for all $\iota \in Y$ and $i = 1, 2$; hence, by (4.7), all ι_η, λ_η, and κ_η lie in Y. Fix $\varsigma \in F(V)$. By Lemma 5, the sequence $\left\{\|\iota_\eta - \varsigma\|\right\}$ is nonincreasing. In particular, $\|\iota_\eta\| \leq \|\varsigma\| + \|\iota_0 - \varsigma\|$ for all $\eta \in \mathbb{N}_0$, so $\{\iota_\eta\}$ is bounded, and $\{\|\iota_\eta - \varsigma\|\}$ converges. We denote its limit by

$$m := \lim_{\eta\to\infty}\|\iota_\eta - \varsigma\|. \tag{4.12}$$

By Lemma 3,

$$\|V_{1_\pi}\iota - \varsigma\| \leq \|\iota - \varsigma\| \quad \text{and} \quad \|V_{2_\pi}\iota - \varsigma\| \leq \|\iota - \varsigma\|, \qquad \forall\, \iota \in Y, \tag{4.13}$$

and hence, by (4.12),

$$\limsup_{\eta\to\infty}\|V_{1_\pi}\iota_\eta - \varsigma\| \leq m, \tag{4.14}$$

and

$$\limsup_{\eta\to\infty}\|V_{2_\pi}\iota_\eta - \varsigma\| \leq m. \tag{4.15}$$

From (4.7),

$$\begin{aligned}\|\lambda_\eta - \varsigma\| &= \|(1-b_\eta)(\iota_\eta - \varsigma) + b_\eta(V_{2_\pi}\iota_\eta - \varsigma)\| \\ &\leq (1-b_\eta)\|\iota_\eta - \varsigma\| + b_\eta\|V_{2_\pi}\iota_\eta - \varsigma\| \\ &\leq \|\iota_\eta - \varsigma\|. \end{aligned} \tag{4.16}$$

Applying (4.16) and Lemma 3,

$$\limsup_{\eta\to\infty} \|V_{1_\pi}\lambda_\eta - \varsigma\| \leq \limsup_{\eta\to\infty} \|\lambda_\eta - \varsigma\| \leq m. \tag{4.17}$$

From (4.7),

$$\begin{aligned}\|\kappa_\eta - \varsigma\| &= \|(1-a_\eta)(V_{1_\pi}\iota_\eta - \varsigma) + a_\eta(V_{2_\pi}\lambda_\eta - \varsigma)\| \\ &\leq (1-a_\eta)\|V_{1_\pi}\iota_\eta - \varsigma\| + a_\eta\|V_{2_\pi}\lambda_\eta - \varsigma\| \\ &\leq (1-a_\eta)\|\iota_\eta - \varsigma\| + a_\eta\|\lambda_\eta - \varsigma\| \\ &\leq \|\iota_\eta - \varsigma\|. \end{aligned} \tag{4.18}$$

By Lemma 3 and (4.18),

$$\|\iota_{\eta+1} - \varsigma\| = \|V_{1_\pi}\kappa_\eta - \varsigma\| \leq \|\kappa_\eta - \varsigma\| \leq \|\iota_\eta - \varsigma\|. \tag{4.19}$$

Since $\lim_{\eta\to\infty} \|\iota_\eta - \varsigma\| = \lim_{\eta\to\infty} \|\iota_{\eta+1} - \varsigma\| = m$, the squeeze theorem applied to (4.19) yields

$$\lim_{\eta\to\infty} \|\kappa_\eta - \varsigma\| = m. \tag{4.20}$$

From (4.18) we have the pointwise bound

$$\|\kappa_\eta - \varsigma\| \leq (1-a_\eta)\|\iota_\eta - \varsigma\| + a_\eta\|\lambda_\eta - \varsigma\| = \|\iota_\eta - \varsigma\| - a_\eta(\|\iota_\eta - \varsigma\| - \|\lambda_\eta - \varsigma\|),$$

so that

$$0 \leq a_\eta(\|\iota_\eta - \varsigma\| - \|\lambda_\eta - \varsigma\|) \leq \|\iota_\eta - \varsigma\| - \|\kappa_\eta - \varsigma\|.$$

Since $a_\eta \geq \theta_1 > 0$ and $\|\lambda_\eta - \varsigma\| \leq \|\iota_\eta - \varsigma\|$ by (4.16),

$$0 \leq \|\iota_\eta - \varsigma\| - \|\lambda_\eta - \varsigma\| \leq \frac{1}{\theta_1}(\|\iota_\eta - \varsigma\| - \|\kappa_\eta - \varsigma\|) \to 0,$$

and hence

$$\lim_{\eta\to\infty} \|\lambda_\eta - \varsigma\| = m. \tag{4.21}$$

From (4.7),

$$\lambda_\eta - \varsigma = (1-b_\eta)(\iota_\eta - \varsigma) + b_\eta(V_{2_\pi}\iota_\eta - \varsigma),$$

so (4.21) gives

$$\lim_{\eta\to\infty} \|(1-b_\eta)(\iota_\eta - \varsigma) + b_\eta(V_{2_\pi}\iota_\eta - \varsigma)\| = m. \tag{4.22}$$

We now apply Lemma 1 with the sequences $\{\iota_\eta\}$ and $\{\kappa_\eta\}$ of Lemma 1 replaced by $\{V_{2_\pi}\iota_\eta - \varsigma\}$ and $\{\iota_\eta - \varsigma\}$, respectively, with $\epsilon = m$, and we verify its hypotheses. The space $\mathcal{G}$ is uniformly convex by assumption, and $0 < \theta_1 \leq b_\eta \leq \theta_2 < 1$ for all η. By (4.12), $\limsup_{\eta\to\infty} \|\iota_\eta - \varsigma\| = m$. Since $\varsigma \in F(V_1) \cap F(V_2)$ and $\iota_\eta \in Y$, Lemma 3 gives $\|V_{2_\pi}\iota_\eta - \varsigma\| \leq \|\iota_\eta - \varsigma\|$, and hence $\limsup_{\eta\to\infty} \|V_{2_\pi}\iota_\eta - \varsigma\| \leq m$. Finally, by (4.22),

$$\lim_{\eta\to\infty} \|(1 - b_\eta)(\iota_\eta - \varsigma) + b_\eta(V_{2_\pi}\iota_\eta - \varsigma)\| = m,$$

and therefore Lemma 1 yields

$$\lim_{\eta\to\infty} \|\iota_\eta - V_{2_\pi}\iota_\eta\| = 0. \tag{4.23}$$

Consequently, from (4.7) and (4.23),

$$\lim_{\eta\to\infty} \|\lambda_\eta - \iota_\eta\| = \lim_{\eta\to\infty} b_\eta \|V_{2_\pi}\iota_\eta - \iota_\eta\| = 0. \tag{4.24}$$

By the triangle inequality,

$$\|\iota_\eta - V_{1_\pi}\iota_\eta\| \leq \|V_{1_\pi}\iota_\eta - V_{2_\pi}\iota_\eta\| + \|V_{2_\pi}\iota_\eta - \iota_\eta\|.$$

Since (V_1, V_2) is an enriched γ-Reich–Suzuki-type nonexpansive pair,

$$\|V_{1_\pi}\iota_\eta - V_{2_\pi}\iota_\eta\| \leq \gamma\|V_{1_\pi}\iota_\eta - \iota_\eta\| + \gamma\|V_{2_\pi}\iota_\eta - \iota_\eta\|.$$

Substituting,

$$\|\iota_\eta - V_{1_\pi}\iota_\eta\| \leq \gamma\|V_{1_\pi}\iota_\eta - \iota_\eta\| + (1 + \gamma)\|V_{2_\pi}\iota_\eta - \iota_\eta\|.$$

Rearranging,

$$(1 - \gamma)\|\iota_\eta - V_{1_\pi}\iota_\eta\| \leq (1 + \gamma)\|V_{2_\pi}\iota_\eta - \iota_\eta\|,$$

hence

$$\|\iota_\eta - V_{1_\pi}\iota_\eta\| \leq \left(\frac{1 + \gamma}{1 - \gamma}\right)\|\iota_\eta - V_{2_\pi}\iota_\eta\|. \tag{4.25}$$

Taking the limit and applying (4.23),

$$\lim_{\eta\to\infty} \|\iota_\eta - V_{1_\pi}\iota_\eta\| = 0. \tag{4.26}$$

Since $\iota_\eta - V_{i_\pi}\iota_\eta = \pi(\iota_\eta - V_i\iota_\eta)$ for $i = 1, 2$, (4.23) and (4.26) give

$$\lim_{\eta\to\infty} \|\iota_\eta - V_2\iota_\eta\| = 0 \quad \text{and} \quad \lim_{\eta\to\infty} \|\iota_\eta - V_1\iota_\eta\| = 0.$$

Conversely, suppose that $\{\iota_\eta\}$ is bounded and that

$$\lim_{\eta\to\infty} \|\iota_\eta - V_1\iota_\eta\| = 0 = \lim_{\eta\to\infty} \|\iota_\eta - V_2\iota_\eta\|.$$

Since $\iota_\eta - V_{i_\pi}\iota_\eta = \pi(\iota_\eta - V_i\iota_\eta)$ for $i = 1, 2$, we obtain

$$\lim_{\eta\to\infty} \|\iota_\eta - V_{1_\pi}\iota_\eta\| = 0 = \lim_{\eta\to\infty} \|\iota_\eta - V_{2_\pi}\iota_\eta\|. \tag{4.27}$$

As $\mathcal{G}$ is uniformly convex and $Y \subseteq \mathcal{G}$ is a nonempty closed convex set, the bounded sequence $\{\iota_\eta\}$ has an asymptotic center $A(Y, \{\iota_\eta\})$ consisting of a single point; call it $\varsigma \in Y$. Thus,

$$r(\varsigma, \{\iota_\eta\}) = r(Y, \{\iota_\eta\}). \tag{4.28}$$

By Lemma 4 with $\iota = \iota_\eta$ and $\kappa = \varsigma$,

$$\|\iota_\eta - V_{2_\pi}\varsigma\| \leq \left(\frac{1+\gamma}{1-\gamma}\right)\|\iota_\eta - V_{1_\pi}\iota_\eta\| + \|\iota_\eta - \varsigma\| \qquad \text{for all } \eta \in \mathbb{N}_0.$$

Taking $\limsup_{\eta\to\infty}$ and applying (4.27) and (4.28), we get

$$r(V_{2_\pi}\varsigma, \{\iota_\eta\}) \leq r(\varsigma, \{\iota_\eta\}) = r(Y, \{\iota_\eta\}).$$

Since $\varsigma \in Y$, $V_2\varsigma \in Y$, and Y is convex, we have $V_{2_\pi}\varsigma = (1-\pi)\varsigma + \pi V_2\varsigma \in Y$, and hence

$$r(V_{2_\pi}\varsigma, \{\iota_\eta\}) \geq r(Y, \{\iota_\eta\}),$$

by the definition of $r(Y, \{\iota_\eta\})$. Therefore, $r(V_{2_\pi}\varsigma, \{\iota_\eta\}) = r(Y, \{\iota_\eta\})$, that is, $V_{2_\pi}\varsigma \in A(Y, \{\iota_\eta\})$. Since $A(Y, \{\iota_\eta\}) = \{\varsigma\}$, it follows that $V_{2_\pi}\varsigma = \varsigma$. Similarly, we can show that $V_{1_\pi}\varsigma = \varsigma$. Consequently, $\varsigma \in F(V_{1_\pi}) \cap F(V_{2_\pi})$, and by Remark 1, $\varsigma \in F(V)$. Thus, $F(V) \neq \emptyset$. □

The following lemma discusses the demiclosedness principle of enriched γ-Reich–Suzuki-type nonexpansive pairs at zero.

Lemma 6. *Let Y be a nonempty closed convex subset of $\mathcal{G}$, where $\mathcal{G}$ has the Opial property. Let $V_1, V_2 : Y \to Y$ be an enriched γ-Reich–Suzuki-type nonexpansive pair. Then $I - V_1$ and $I - V_2$ are demiclosed at zero.*

Proof. Let $\{\iota_\eta\} \subseteq Y$ converge weakly to $\varsigma \in Y$, and let $\pi = \frac{1}{b+1}$.

First assume that $\lim_{\eta\to\infty} \|\iota_\eta - V_1\iota_\eta\| = 0$. Since $\|\iota_\eta - V_{1_\pi}\iota_\eta\| = \pi\|\iota_\eta - V_1\iota_\eta\|$, we have $\lim_{\eta\to\infty} \|\iota_\eta - V_{1_\pi}\iota_\eta\| = 0$. By Lemma 4 with $\iota = \iota_\eta$ and $\kappa = \varsigma$,

$$\left\|\iota_\eta - V_{2_\pi}\varsigma\right\| \leq \left(\frac{1+\gamma}{1-\gamma}\right)\left\|\iota_\eta - V_{1_\pi}\iota_\eta\right\| + \left\|\iota_\eta - \varsigma\right\|. \tag{4.29}$$

Applying $\liminf_{\eta\to\infty}$ to both sides of (4.29), we get

$$\liminf_{\eta\to\infty} \left\|\iota_\eta - V_{2_\pi}\varsigma\right\| \leq \liminf_{\eta\to\infty} \left\|\iota_\eta - \varsigma\right\|. \tag{4.30}$$

If $V_{2_\pi}\varsigma \neq \varsigma$, then the Opial property of $\mathcal{G}$ gives

$$\liminf_{\eta\to\infty} \|\iota_\eta - \varsigma\| < \liminf_{\eta\to\infty} \|\iota_\eta - V_{2_\pi}\varsigma\|,$$

which contradicts (4.30). Hence $V_{2_\pi}\varsigma = \varsigma$, so $V_2\varsigma = \varsigma$ by Remark 1, and $V_1\varsigma = \varsigma$ by Remark 2(ii). Therefore, $I - V_1$ is demiclosed at zero.

Now assume that $\lim_{\eta\to\infty} \|\iota_\eta - V_2\iota_\eta\| = 0$, so that $\lim_{\eta\to\infty} \|\iota_\eta - V_{2_\pi}\iota_\eta\| = 0$. For every $\iota \in Y$, the triangle inequality and (3.3) give

$$\|\iota - V_{1_\pi}\iota\| \leq \|\iota - V_{2_\pi}\iota\| + \|V_{2_\pi}\iota - V_{1_\pi}\iota\| \leq (1+\gamma)\|\iota - V_{2_\pi}\iota\| + \gamma\|\iota - V_{1_\pi}\iota\|,$$

that is, $\|\iota - V_{1_\pi}\iota\| \leq (\frac{1+\gamma}{1-\gamma})\|\iota - V_{2_\pi}\iota\|$. Hence $\lim_{\eta\to\infty} \|\iota_\eta - V_{1_\pi}\iota_\eta\| = 0$, and the argument above gives $V_2\varsigma = \varsigma$. Therefore, $I - V_2$ is demiclosed at zero. □

Theorem 4. *Let $\mathcal{G}$ be a uniformly convex Banach space with Opial's condition, and Y be a nonempty closed convex subset of $\mathcal{G}$. Consider a pair of enriched γ-Reich–Suzuki-type nonexpansive mappings $V_1, V_2 : Y \to Y$ such that the intersection of their fixed point sets is nonempty, i.e., $F(V) = F(V_1) \cap F(V_2) \neq \emptyset$. For any initial point $\iota_0 \in Y$, the sequence $\{\iota_\eta\}$ generated by the iteration scheme (4.7) with control sequences satisfying $0 < \theta_1 \leq a_\eta, b_\eta \leq \theta_2 < 1$, for all $\eta \in \mathbb{N}_0$, converges weakly to a common fixed point of V_1 and V_2.*

Proof. Since $F(V_1) \cap F(V_2) \neq \emptyset$ and $V_1, V_2 : Y \to Y$ form an enriched γ-Reich–Suzuki-type nonexpansive pair, using Theorem 3, we obtain

$$\lim_{\eta\to\infty} \left\|\iota_\eta - V_1\iota_\eta\right\| = 0 = \lim_{\eta\to\infty} \left\|\iota_\eta - V_2\iota_\eta\right\|.$$

By Theorem 3, $\{\iota_\eta\}$ is bounded. Since $\mathcal{G}$ is uniformly convex, it is reflexive, so $\{\iota_\eta\}$ has weakly convergent subsequences. Let α_1 and α_2 be two weak subsequential limits of $\{\iota_\eta\}$. Since Y is closed and convex, it is weakly closed, and hence $\alpha_1, \alpha_2 \in Y$. We aim to demonstrate that $\alpha_1, \alpha_2 \in F(V)$. By Lemma 6, $I - V_1$ and $I - V_2$ are demiclosed at zero. Then we have $(I - V_1)\alpha_1 = 0$. This implies that $V_1\alpha_1 = \alpha_1$. Following similar arguments, we can conclude that $V_2\alpha_1 = \alpha_1$. Therefore, $\alpha_1 \in F(V)$. In a similar manner, it can be established that $\alpha_2 \in F(V)$. Now, we aim to prove that $\alpha_1 = \alpha_2$. Suppose, for contradiction, that $\alpha_1 \neq \alpha_2$. Since $\lim_{\eta\to\infty} \|\iota_\eta - \alpha_k\|$ exists for $k = 1, 2$ by Lemma 5, the Opial property gives

$$\begin{aligned}
\lim_{\eta\to\infty} \|\iota_\eta - \alpha_1\| &= \liminf_{i\to\infty} \|\iota_{\eta_i} - \alpha_1\| \\
&< \liminf_{i\to\infty} \|\iota_{\eta_i} - \alpha_2\| \\
&= \lim_{\eta\to\infty} \|\iota_\eta - \alpha_2\| \\
&= \liminf_{j\to\infty} \|\iota_{\eta_j} - \alpha_2\| \\
&< \liminf_{j\to\infty} \|\iota_{\eta_j} - \alpha_1\| \\
&= \lim_{\eta\to\infty} \|\iota_\eta - \alpha_1\|,
\end{aligned}$$

which leads to a contradiction. Therefore, $\alpha_1 = \alpha_2$, which indicates that the sequence $\{\iota_\eta\}$ converges weakly to an element of $F(V)$. □

It should be noted that not every uniformly convex Banach space satisfies Opial's property. Therefore, our previous result cannot be applied to such spaces. To overcome this limitation, we assume a Fréchet differentiable norm in the following result.

Theorem 5. *Let $\mathcal{G}$ be a uniformly convex Banach space with a Fréchet differentiable norm, and let Y be a nonempty closed convex subset of $\mathcal{G}$. Let $V_1, V_2 : Y \to Y$ be an enriched γ-Reich–Suzuki-type nonexpansive pair with $F(V) = F(V_1) \cap F(V_2) \neq \emptyset$, and let $\{\iota_\eta\}$ be generated by* (4.7) *with arbitrary $\iota_0 \in Y$ and control sequences satisfying $0 < \theta_1 \leq a_\eta, b_\eta \leq \theta_2 < 1$, for all $\eta \in \mathbb{N}_0$. Assume that*

$$\lim_{\eta\to\infty} \|t\iota_\eta + (1-t)a - b\|$$

exists for all $t \in [0,1]$ and all $a, b \in F(V)$. If $I - V_1$ and $I - V_2$ are demiclosed at zero, then $\{\iota_\eta\}$ converges weakly to a common fixed point of V_1 and V_2.

Proof. We will show that the sequence $\{\iota_\eta\}$ has a unique limit point. Assume $\{\iota_{\eta_i}\}$ converges weakly to α_1 and $\{\iota_{\eta_j}\}$ converges weakly to α_2. Since $F(V) \neq \emptyset$, by Theorem 3, we have

$$\lim_{\eta\to\infty} \left\|\iota_\eta - V_1\iota_\eta\right\| = 0 = \lim_{\eta\to\infty} \left\|\iota_\eta - V_2\iota_\eta\right\|.$$

Given that $I - V_1$ and $I - V_2$ are demiclosed at zero, it follows that $\alpha_1, \alpha_2 \in F(V)$. Substituting $a - b$ for ι and $t(\iota_\eta - a)$ for a in (2.2), we have

$$\begin{aligned}\frac{1}{2}\|a-b\|^2 + t\langle \iota_\eta - a, J(a-b)\rangle &\leq \frac{1}{2}\left\|t\iota_\eta + (1-t)a - b\right\|^2 \\ &\leq \frac{1}{2}\|a-b\|^2 + t\langle \iota_\eta - a, J(a-b)\rangle + g(t\left\|\iota_\eta - a\right\|).\end{aligned}$$

Since $\{\iota_\eta\}$ is bounded by Theorem 3, $R := \sup_\eta \|\iota_\eta - a\| < \infty$, and since g is increasing, $g(t\|\iota_\eta - a\|) \leq g(tR)$. Using the given condition, we get

$$\begin{aligned}\frac{1}{2}\|a-b\|^2 + t\limsup_{\eta\to\infty}\langle \iota_\eta - a, J(a-b)\rangle &\leq \frac{1}{2}\lim_{\eta\to\infty}\left\|t\iota_\eta + (1-t)a - b\right\|^2 \\ &\leq \frac{1}{2}\|a-b\|^2 + t\liminf_{\eta\to\infty}\langle \iota_\eta - a, J(a-b)\rangle + g(tR).\end{aligned}$$

Thus,

$$\limsup_{\eta\to\infty}\langle \iota_\eta - a, J(a-b)\rangle \leq \liminf_{\eta\to\infty}\langle \iota_\eta - a, J(a-b)\rangle + \frac{g(tR)}{t}.$$

Since $\frac{g(tR)}{t} \to 0$ as $t \to 0^+$, it follows that $\lim_{\eta\to\infty}\langle \iota_\eta - a, J(a-b)\rangle$ exists. Now, we have

$$\langle \alpha_1 - a, J(a-b)\rangle = d \text{ (say)},$$

and

$$\langle \alpha_2 - a, J(a-b)\rangle = d.$$

By the above equalities, we get

$$\langle \alpha_1 - \alpha_2, J(a-b)\rangle = 0,$$

for all $a, b \in F(V)$. From this

$$\|\alpha_1 - \alpha_2\|^2 = \langle \alpha_1 - \alpha_2, J(\alpha_1 - \alpha_2)\rangle = 0,$$

hence $\alpha_1 = \alpha_2$. Thus, $\{\iota_\eta\}$ converges weakly to a common fixed point of V_1 and V_2. □

Remark 3. *Theorems 4 and 5 both assume uniform convexity, but within uniformly convex spaces Opial's condition and Fréchet differentiability of the norm are independent and not mutually exclusive. For instance, Hilbert spaces satisfy both, $L^p([0,1])$ with $1 < p < \infty$, $p \neq 2$, satisfies only Fréchet differentiability, and a finite-dimensional uniformly convex space whose unit ball has corners satisfies only Opial's condition. Hence each theorem covers spaces the other does not.*

By Lemma 2 and Condition (B), we prove strong convergence via the iterative scheme (4.7).

Theorem 6. *Let $\mathcal{G}$ be a uniformly convex Banach space and $Y \subseteq \mathcal{G}$ nonempty, closed, and convex. Let $V_1, V_2 : Y \to Y$ be an enriched γ-Reich–Suzuki-type nonexpansive pair with $F(V) = F(V_1) \cap F(V_2) \neq \emptyset$. Let $\{\iota_\eta\}$ be generated by* (4.7) *with control sequences satisfying $0 < \theta_1 \leq a_\eta,\ b_\eta \leq \theta_2 < 1$, for all $\eta \in \mathbb{N}_0$. If (V_1, V_2) satisfies Condition (B), then $\{\iota_\eta\}$ converges strongly to a common fixed point of V_1 and V_2.*

Proof. Write $F := F(V)$. By (4.11),

$$\|\iota_{\eta+1} - \varsigma\| \leq \|\iota_\eta - \varsigma\| \qquad \forall\, \eta \in \mathbb{N}_0,\ \forall\, \varsigma \in F. \tag{4.31}$$

Since $d(\iota_{\eta+1}, F)$ is a lower bound of $\{\|\iota_{\eta+1} - \lambda\| : \lambda \in F\}$, (4.31) gives $d(\iota_{\eta+1}, F) \leq \|\iota_{\eta+1} - \varsigma\| \leq \|\iota_\eta - \varsigma\|$ for every $\varsigma \in F$; taking the infimum over $\varsigma \in F$ yields $d(\iota_{\eta+1}, F) \leq d(\iota_\eta, F)$. Thus, $\{d(\iota_\eta, F)\}$ is nonincreasing and bounded below, so by the monotone convergence theorem $L := \lim_{\eta\to\infty} d(\iota_\eta, F) = \inf_{\eta\in\mathbb{N}_0} d(\iota_\eta, F)$ exists.

By Theorem 3 and Condition (B), since $f : [0,\infty) \to [0,\infty)$,

$$0 \leq f(d(\iota_\eta, F)) \leq \max\{\|V_1\iota_\eta - \iota_\eta\|, \|V_2\iota_\eta - \iota_\eta\|\} \to 0,$$

so $\lim_{\eta\to\infty} f(d(\iota_\eta, F)) = 0$ by the squeeze theorem. Since $d(\iota_\eta, F) \geq L$ for all η and f is nondecreasing, $0 \leq f(L) \leq f(d(\iota_\eta, F))$ for all η; letting $\eta \to \infty$ gives $f(L) = 0$, and hence $L = 0$.

Consequently, we can choose a strictly increasing sequence $\{\eta_i\} \subseteq \mathbb{N}_0$ with $d(\iota_{\eta_i}, F) < \frac{1}{2^i}$, and, by the definition of the infimum, $\mu_i \in F$ with $\|\iota_{\eta_i} - \mu_i\| < \frac{1}{2^i}$ for all $i \in \mathbb{Z}^+$. Applying (4.31) repeatedly with $\varsigma = \mu_i$ gives $\|\iota_m - \mu_i\| \leq \|\iota_{\eta_i} - \mu_i\|$ for all $m \geq \eta_i$. Hence,

$$\|\iota_m - \iota_n\| \leq \|\iota_m - \mu_i\| + \|\iota_n - \mu_i\| \leq 2\|\iota_{\eta_i} - \mu_i\| < \frac{2}{2^i}, \qquad \forall\, m, n \geq \eta_i.$$

Given $\epsilon > 0$, choosing i with $\frac{2}{2^i} < \epsilon$ shows that $\{\iota_\eta\}$ is a Cauchy sequence. Since Y is convex and $V_{k_\pi}(Y) \subseteq Y$ for $k = 1, 2$, all iterates of (4.7) lie in Y; since $\mathcal{G}$ is complete and Y is closed, $\iota_\eta \to p$ for some $p \in Y$.

It remains to show $p \in F$. First, F is closed: If $\{q_m\} \subseteq F$ and $q_m \to q \in Y$, then Lemma 3 with $\varsigma = q_m$ and $\iota = q$ gives $\|V_{k_\pi} q - q_m\| \leq \|q - q_m\| \to 0$, so $V_{k_\pi} q = q$ for $k = 1, 2$, and $q \in F$ by Remark 1. Moreover,

$$\|\mu_i - p\| \leq \|\mu_i - \iota_{\eta_i}\| + \|\iota_{\eta_i} - p\| < \frac{1}{2^i} + \|\iota_{\eta_i} - p\| \to 0 \qquad (i \to \infty),$$

so $\mu_i \to p$ with $\{\mu_i\} \subseteq F$. Since F is closed, $p \in F$. Therefore, $\{\iota_\eta\}$ converges strongly to the common fixed point p of V_1 and V_2. □

5. Numerical example

Example 2. *Let $\mathcal{G} = \mathbb{R}$ with the usual norm and let $Y = [0, \infty) \subseteq \mathcal{G}$. Define two self-mappings $V_1, V_2 : Y \to Y$ by*

$$V_1(\iota) = \begin{cases} \frac{2+\iota}{17}, & \left[0, \frac{199}{200}\right], \\ \frac{3+\iota}{25}, & \left(\frac{199}{200}, \infty\right), \end{cases} \qquad V_2(\iota) = \begin{cases} \frac{7+\iota}{57}, & \left[0, \frac{199}{200}\right], \\ \frac{1+\iota}{9}, & \left(\frac{199}{200}, \infty\right). \end{cases}$$

(1) (V_1, V_2) is an enriched γ-Reich–Suzuki-type nonexpansive pair.

(2) (V_1, V_2) has a unique common fixed point.

Solving $V_1(\iota) = \iota$ and $V_2(\iota) = \iota$,

$$\frac{2+\iota}{17} = \iota \implies \iota = \frac{1}{8}, \qquad \frac{7+\iota}{57} = \iota \implies \iota = \frac{1}{8},$$

so V_1 and V_2 share the same fixed point $\iota = \frac{1}{8} \in \left[0, \frac{199}{200}\right]$. Thus, the pair (V_1, V_2) has a unique common fixed point given by

$$F(V_1) \cap F(V_2) = \left\{\frac{1}{8}\right\}.$$

We now show that (V_1, V_2) is an enriched γ-Reich–Suzuki-type nonexpansive pair.

Case 1: When $\iota \in \left[0, \frac{199}{200}\right]$ and $\kappa \in \left(\frac{199}{200}, \infty\right)$, we can find a $b \in [0, \infty)$ such that

$$\begin{aligned} \|b(\iota - \kappa) + V_1\iota - V_2\kappa\| &= \left\| b(\iota - \kappa) + \frac{2+\iota}{17} - \frac{1+\kappa}{9} \right\| \\ &= \left\| b(\iota - \kappa) + \frac{1 + 9\iota - 17\kappa}{153} \right\| \\ &\leq b\|\iota - \kappa\| + \left\| \frac{9\iota - 9\kappa}{153} \right\| + \left\| \frac{1 - 8\kappa}{153} \right\| \\ &= \left(b + \frac{1}{17} \right) \|\iota - \kappa\| + \frac{1}{17} \left\| \frac{1 - 8\kappa}{9} \right\| \\ &\leq \left(\frac{b+1}{3} \right) \|\iota - \kappa\| + \frac{1}{17} \left\| \frac{1 - 8\kappa}{9} \right\| \\ &\leq \left(\frac{b+1}{3} \right) \|\iota - \kappa\| + \frac{1}{3} \left\| \frac{1 - 8\kappa}{9} \right\| + \frac{1}{3} \left\| \frac{2 - 16\iota}{17} \right\|. \end{aligned}$$

Assuming $\gamma = \frac{1}{3} \in (0, 1)$, we can find a constant $b \in \left[0, \frac{7}{17}\right] \subseteq [0, \infty)$. Thus, (V_1, V_2) forms an enriched γ-Reich–Suzuki-type nonexpansive pair.

Case 2: When $\iota \in \left(\frac{199}{200}, \infty\right)$ and $\kappa \in \left[0, \frac{199}{200}\right]$, we can find a $b \in [0, \infty)$ such that

$$\begin{aligned}
\|b(\iota-\kappa)+V_1\iota-V_2\kappa\| &= \left\|b(\iota-\kappa)+\frac{3+\iota}{25}-\frac{7+\kappa}{57}\right\| \\
&= \left\|b(\iota-\kappa)+\frac{57\iota-25\kappa-4}{1425}\right\| \\
&\leq b\,\|\iota-\kappa\|+\left\|\frac{25\iota-25\kappa}{1425}\right\|+\left\|\frac{32\iota-4}{1425}\right\| \\
&= \left(b+\frac{1}{57}\right)\|\iota-\kappa\|+\left\|\frac{32\iota-4}{1425}\right\| \\
&\leq \left(\frac{b+1}{3}\right)\|\iota-\kappa\|+\left\|\frac{32\iota-4}{1425}\right\| \\
&= \left(\frac{b+1}{3}\right)\|\iota-\kappa\|+\frac{4}{57}\left\|\frac{8\iota-1}{25}\right\| \\
&\leq \left(\frac{b+1}{3}\right)\|\iota-\kappa\|+\frac{1}{3}\left\|\frac{3-24\iota}{25}\right\|+\frac{1}{3}\left\|\frac{7-56\kappa}{57}\right\|.
\end{aligned}$$

Assuming $\gamma = \frac{1}{3} \in (0, 1)$, we can find a constant $b \in \left[0, \frac{9}{19}\right] \subseteq [0, \infty)$. Thus, (V_1, V_2) forms an enriched γ-Reich–Suzuki-type nonexpansive pair.

Case 3: When $\iota, \kappa \in \left[0, \frac{199}{200}\right]$, we can find a $b \in [0, \infty)$ such that

$$\begin{aligned}
\|b(\iota-\kappa)+V_1\iota-V_2\kappa\| &= \left\|b(\iota-\kappa)+\frac{2+\iota}{17}-\frac{7+\kappa}{57}\right\| \\
&= \left\|b(\iota-\kappa)+\frac{57\iota-17\kappa-5}{969}\right\| \\
&\leq b\,\|\iota-\kappa\|+\left\|\frac{17\iota-17\kappa}{969}\right\|+\left\|\frac{40\iota-5}{969}\right\| \\
&= \left(b+\frac{1}{57}\right)\|\iota-\kappa\|+\left\|\frac{40\iota-5}{969}\right\| \\
&\leq \left(\frac{b+1}{3}\right)\|\iota-\kappa\|+\left\|\frac{40\iota-5}{969}\right\| \\
&\leq \left(\frac{b+1}{3}\right)\|\iota-\kappa\|+\frac{1}{3}\left\|\frac{2-16\iota}{17}\right\|+\frac{1}{3}\left\|\frac{7-56\kappa}{57}\right\|.
\end{aligned}$$

Assuming $\gamma = \frac{1}{3} \in (0, 1)$, we can find a constant $b \in \left[0, \frac{9}{19}\right] \subseteq [0, \infty)$. Thus, (V_1, V_2) forms an enriched γ-Reich–Suzuki-type nonexpansive pair.

Case 4: When $\iota, \kappa \in \left(\frac{199}{200}, \infty\right)$, we can find a $b \in [0, \infty)$ such that

$$\begin{aligned}
\|b(\iota-\kappa)+V_1\iota-V_2\kappa\| &= \left\|b(\iota-\kappa)+\frac{3+\iota}{25}-\frac{1+\kappa}{9}\right\| \\
&= \left\|b(\iota-\kappa)+\frac{2+9\iota-25\kappa}{225}\right\|
\end{aligned}$$

$$\begin{aligned}
&\leq b\,\|\iota-\kappa\| + \left\|\frac{9\iota-9\kappa}{225}\right\| + \left\|\frac{2-16\kappa}{225}\right\| \\
&= \left(b+\frac{1}{25}\right)\|\iota-\kappa\| + \left\|\frac{2-16\kappa}{225}\right\| \\
&\leq \left(\frac{b+1}{3}\right)\|\iota-\kappa\| + \frac{1}{3}\left\|\frac{1-8\kappa}{9}\right\| \\
&\leq \left(\frac{b+1}{3}\right)\|\iota-\kappa\| + \frac{1}{3}\left\|\frac{1-8\kappa}{9}\right\| + \frac{1}{3}\left\|\frac{3-24\iota}{25}\right\|.
\end{aligned}$$

Assuming $\gamma = \frac{1}{3} \in (0,1)$, we can find a constant $b \in \left[0, \frac{11}{25}\right] \subseteq [0,\infty)$. Thus, (V_1, V_2) forms an enriched γ-Reich–Suzuki-type nonexpansive pair.

Combining all four cases, taking

$$\gamma = \frac{1}{3}, \qquad b \in \left[0,\ \min\left\{\frac{7}{17},\ \frac{9}{19},\ \frac{11}{25}\right\}\right] = \left[0,\ \frac{7}{17}\right] \subseteq [0,\infty),$$

satisfies inequality (3.1) simultaneously for all $\iota, \kappa \in Y$, so (V_1, V_2) is an enriched γ-Reich–Suzuki-type nonexpansive pair with $\gamma = \frac{1}{3}$ and any $b \in \left[0, \frac{7}{17}\right]$.

Accordingly, we take $b = \frac{7}{17}$, the right endpoint of this interval, so that $\pi = \frac{1}{b+1} = \frac{17}{24}$. Since Definition 11 and Theorems 1, and 3–6 are formulated in terms of the π-Krasnoselskii maps, the iteration schemes (4.1)–(4.6) are applied not to V_1 and V_2 directly, but to their associated π-Krasnoselskii maps

$$V_{i_\pi}\iota = (1-\pi)\iota + \pi V_i\iota = \tfrac{7}{24}\,\iota + \tfrac{17}{24}\,V_i\iota, \qquad i = 1,2.$$

Explicitly,

$$V_{1_\pi}(\iota) = \begin{cases} \frac{1+4\iota}{12}, & \iota \in [0, \frac{199}{200}], \\ \frac{17+64\iota}{200}, & \iota \in (\frac{199}{200}, \infty), \end{cases} \qquad V_{2_\pi}(\iota) = \begin{cases} \frac{119+416\iota}{1368}, & \iota \in [0, \frac{199}{200}], \\ \frac{17+80\iota}{216}, & \iota \in (\frac{199}{200}, \infty). \end{cases}$$

Since $\frac{1}{8} \in [0, \frac{199}{200}]$, $V_{1_\pi}(\frac{1}{8}) = \frac{1+\frac{1}{2}}{12} = \frac{1}{8}$ and $V_{2_\pi}(\frac{1}{8}) = \frac{119+52}{1368} = \frac{171}{1368} = \frac{1}{8}$, so $\iota^* = \frac{1}{8}$ is a common fixed point of V_{1_π} and V_{2_π}.

The numerical experiment is carried out with $\gamma = \frac{1}{3}$ and $b = \frac{7}{17}$, so that $\pi = \frac{1}{1+b} = \frac{17}{24}$. Each scheme is applied to the π-Krasnoselskii maps V_{1_π} and V_{2_π} with initial point $\iota_0 = 1$ and constant control sequences a_η, b_η, and c_η; the modified Ishikawa, Agarwal, and Picard-S schemes use only a_η and b_η. For each scheme, the reported number of iterations is the smallest $\eta \geq 1$ such that $|\iota_\eta - \iota^*| < 10^{-12}$, where $\iota^* = \frac{1}{8}$, at which point the scheme is terminated. All computations were performed in MATLAB R2026a.

Table 1 and Figure 1 show the iterates ι_η of the six schemes for $(a_\eta, b_\eta, c_\eta) = (0.8, 0.4, 0.6)$. At $\eta = 1$, the distances $|\iota_1 - \iota^*|$ are approximately 7.2×10^{-2}, 1.3×10^{-1}, 1.4×10^{-1}, 2.1×10^{-1}, 3.3×10^{-1}, and 3.3×10^{-1} for the modified Picard-S, Thakur, Abbas–Nazir, Agarwal, Ishikawa, and Noor iterations, respectively, so the modified Picard-S iterate is the closest to ι^* from the first iteration.

Table 1. Iterates of each scheme for $\eta = 0, 1, \ldots, 15$, with $\iota_0 = 1$ and $(a_\eta, b_\eta, c_\eta) = (0.8, 0.4, 0.6)$.

η	Modified Ishikawa	Modified Abbas–Nazir	Modified Noor	Modified Agarwal	Modified Thakur	Modified Picard-S
0	1.000000000000	1.000000000000	1.000000000000	1.000000000000	1.000000000000	1.000000000000
1	0.454966081871	0.266416920078	0.456802183236	0.335966081871	0.258426034221	0.196751642481
2	0.249859679947	0.148215530880	0.250906316174	0.176701083456	0.144045431344	0.130793297122
3	0.172247097605	0.128811148438	0.172776661076	0.137670292811	0.127718573307	0.125467756422
4	0.142878375413	0.125625652391	0.143129426806	0.128105086184	0.125388053213	0.125037767107
5	0.131765205137	0.125102709438	0.131879428343	0.125760957963	0.125055391295	0.125003049353
6	0.127559964174	0.125016861166	0.127610481557	0.125186486618	0.125007906636	0.125000246208
7	0.125968694436	0.125002767992	0.125990578522	0.125045701945	0.125001128605	0.125000019879
8	0.125366555485	0.125000454404	0.125375886896	0.125011200095	0.125000161099	0.125000001605
9	0.125138705167	0.125000074597	0.125142634789	0.125002744787	0.125000022995	0.125000000130
10	0.125052486252	0.125000012246	0.125054124481	0.125000672660	0.125000003282	0.125000000010
11	0.125019860879	0.125000002010	0.125020538183	0.125000164848	0.125000000469	0.125000000001
12	0.125007515388	0.125000000330	0.125007793460	0.125000040399	0.125000000067	0.125000000000
13	0.125002843834	0.125000000054	0.125002957322	0.125000009900	0.125000000010	0.125000000000
14	0.125001076111	0.125000000009	0.125001122191	0.125000002426	0.125000000001	0.125000000000
15	0.125000407202	0.125000000001	0.125000425829	0.125000000595	0.125000000000	0.125000000000

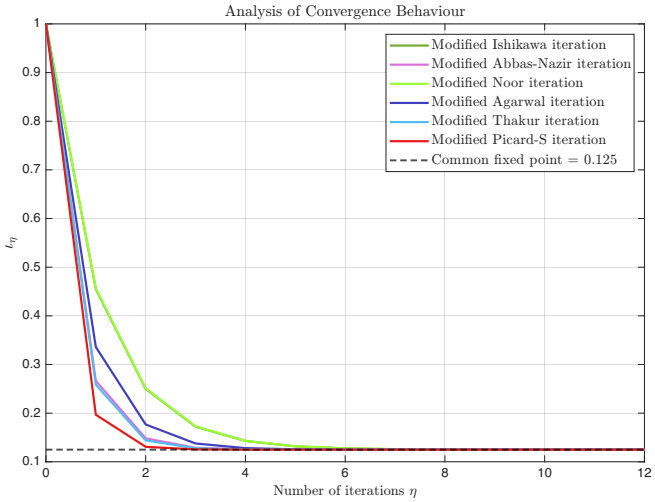


Figure 1. Iterates ι_η of the six schemes with $\iota_0 = 1$ and $(a_\eta, b_\eta, c_\eta) = (0.8, 0.4, 0.6)$, converging to the common fixed point $\iota^* = \frac{1}{8}$.

To show that this behavior does not depend on a particular choice of the control sequences, Table 2 reports the number of iterations required by each scheme for four further choices of (a_η, b_η, c_η), and Figure 2 shows the corresponding errors $|\iota_\eta - \iota^*|$ on a logarithmic scale. In every case, the modified Picard-S scheme requires the fewest iterations and has the smallest error at every step.

Table 2. Number of iterations required by each scheme to satisfy $|\iota_\eta - \iota^*| < 10^{-12}$ for different control sequences, with $\iota_0 = 1$.

(a_η, b_η, c_η)	Modified Ishikawa	Modified Abbas–Nazir	Modified Noor	Modified Agarwal	Modified Thakur	Modified Picard-S
(0.6, 0.3, 0.7)	46	14	46	22	15	12
(0.7, 0.5, 0.3)	34	18	34	20	17	11
(0.5, 0.8, 0.5)	50	14	48	20	15	11
(0.7, 0.4, 0.6)	36	15	36	21	15	12

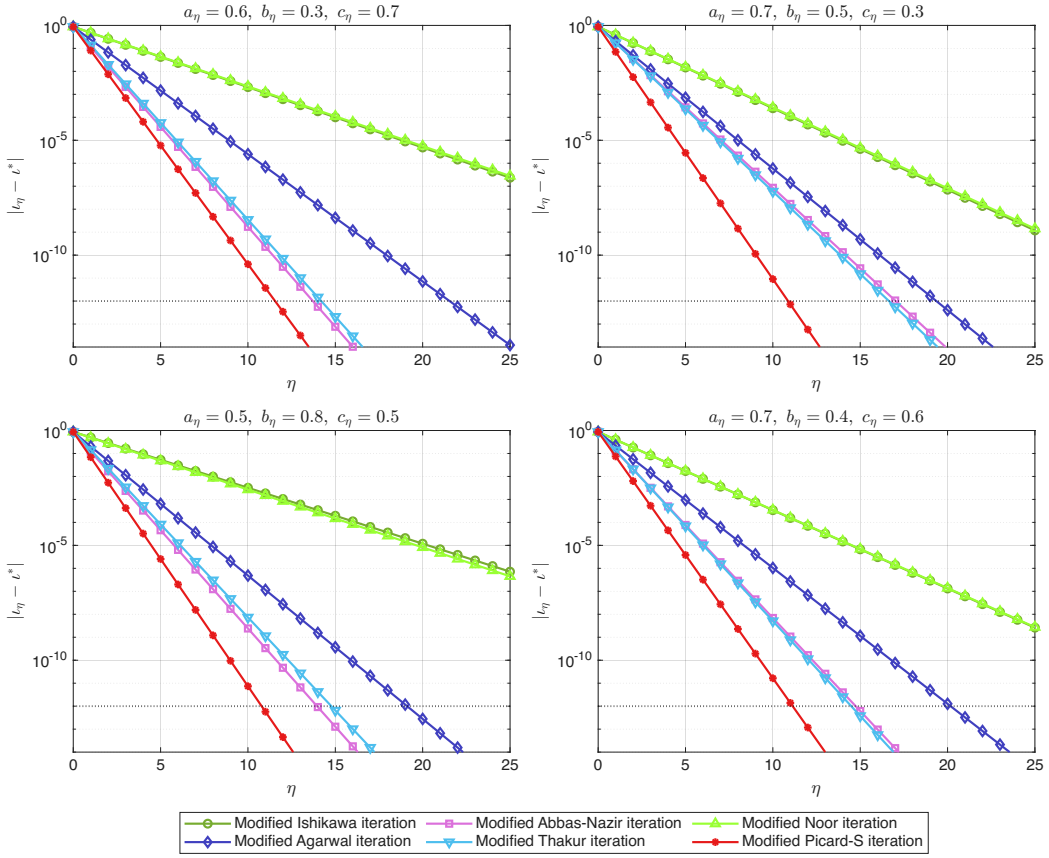


Figure 2. Errors $|\iota_\eta - \iota^*|$ (logarithmic scale) of the six schemes with $\iota_0 = 1$ for four combinations of the control sequences. The dotted line marks the level 10^{-12}.

6. Application to a system of nonlinear Fredholm integral equations

In this section, we apply Theorems 1, 2, and 6 to establish uniqueness, well-posedness, and strong convergence to a common solution of a system of nonlinear Fredholm integral equations in a Banach space, whose existence is guaranteed by the compatibility condition (H_6).

Let $\mathcal{G} = L^2([0, 1])$ denote the Banach space of all Lebesgue square-integrable real-valued functions on $[0, 1]$, equipped with the norm

$$\|\iota\| := \left(\int_0^1 |\iota(t)|^2 \, dt \right)^{1/2}, \qquad \iota \in \mathcal{G}.$$

Note that $\mathcal{G} = L^2([0,1])$ is a uniformly convex Banach space. We seek a single function $\iota \in \mathcal{G}$ satisfying simultaneously

$$\iota(t) = g_i(t) + \int_0^1 K(t,s) f_i(s, \iota(s)) \, ds, \qquad i = 1,2, \quad t \in [0,1], \tag{6.1}$$

where $K : [0,1] \times [0,1] \to \mathbb{R}$ is a measurable kernel, $g_1, g_2 \in \mathcal{G}$ are forcing terms, and $f_1, f_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$ are nonlinear functions with $f_1 \neq f_2$. Such a function is called a common solution of (6.1), that is, a single function solving both equations.

For $i = 1, 2$, define the Fredholm operators $W_i : \mathcal{G} \to \mathcal{G}$ by

$$(W_i \iota)(t) := g_i(t) + \int_0^1 K(t,s) f_i\big(s, \iota(s)\big) \, ds, \qquad t \in [0,1], \tag{6.2}$$

and, for a fixed enrichment parameter $b > 0$, the operators $V_1, V_2 : \mathcal{G} \to \mathcal{G}$ by

$$V_i \iota := -b\, \iota + (1+b)\, W_i \iota, \qquad i = 1,2. \tag{6.3}$$

Since $V_i \iota = \iota$ if and only if $(1+b)(W_i \iota - \iota) = 0$, we have $F(V_i) = F(W_i)$, so $V_i \iota = \iota$ if and only if ι solves the i^{th} equation of (6.1). Hence $\iota^* \in \mathcal{G}$ is a common fixed point of V_1 and V_2 if and only if ι^* is a common solution of (6.1). We assume throughout that $W_1 \neq W_2$. Since $V_1 - V_2 = (1+b)(W_1 - W_2)$ by (6.3), this gives $V_1 \neq V_2$, so (V_1, V_2) is a genuine pair of mappings rather than a single mapping paired with itself. Moreover, with $\pi := \frac{1}{b+1}$,

$$V_{i_\pi} \iota = (1-\pi)\iota + \pi V_i \iota = \frac{b}{b+1}\iota + \frac{1}{b+1}\big(-b\iota + (1+b) W_i \iota\big) = W_i \iota, \qquad i = 1,2, \tag{6.4}$$

that is, the π-Krasnoselskii maps of V_1, and V_2 are exactly the Fredholm operators W_1, and W_2.

Assume the following conditions:

(H_1) $K \in L^2([0,1] \times [0,1])$ and

$$0 < M := \left(\int_0^1 \int_0^1 |K(t,s)|^2 \, ds \, dt \right)^{\frac{1}{2}} < \infty.$$

(H_2) For each $i \in \{1,2\}$, $f_i(\cdot, u)$ is measurable for every $u \in \mathbb{R}$, $f_i(\cdot, 0) \in L^2([0,1])$, and there exists a constant $L_i > 0$ such that

$$|f_i(s,u) - f_i(s,v)| \leq L_i \, |u - v| \qquad \forall\, s \in [0,1],\ u, v \in \mathbb{R}.$$

(H_3) $\alpha_i := M L_i < 1$ for $i = 1, 2$.

(H_4) The function $h := f_1 - f_2$ satisfies, for some constant $\delta > 0$,

$$|h(s,u) - h(s,v)| \leq \delta \, |u - v| \qquad \forall\, s \in [0,1],\ u, v \in \mathbb{R}.$$

(H_5) $2M\delta \le (1-\alpha_1)(1-\alpha_2)$.

(H_6) There exists $\varphi \in \mathcal{G}$ such that

$$g_i(t) = \varphi(t) - \int_0^1 K(t,s) f_i(s,\varphi(s))\, ds, \qquad i = 1,2, \quad \text{for a.e. } t \in [0,1].$$

Condition (H_6) is equivalent to $W_1\varphi = \varphi = W_2\varphi$. Indeed, substituting the expression for g_i from (H_6) into (6.2) gives

$$(W_i\varphi)(t) = \varphi(t) - \int_0^1 K(t,s) f_i(s,\varphi(s))\, ds + \int_0^1 K(t,s) f_i(s,\varphi(s))\, ds = \varphi(t), \qquad i = 1,2.$$

Remark 4. *Under (H_1)–(H_5), define*

$$\gamma := \frac{M\delta}{1-\alpha_2}. \tag{6.5}$$

Since $\delta > 0$, $M > 0$, by (H_1) and $1-\alpha_2 > 0$ by (H_3), we have $\gamma > 0$. Condition (H_5) is equivalent to $2\gamma \le 1-\alpha_1$, so $\gamma \le \frac{1-\alpha_1}{2} < \frac{1}{2}$ and hence $\gamma \in (0, \frac{1}{2}) \subset (0,1)$.

Remark 5. *Condition (H_6) cannot be dropped. Indeed, if (V_1, V_2) satisfies* (6.11) *below, then taking $\iota = \kappa$ gives $\|W_1\iota - W_2\iota\| \le \gamma(\|W_1\iota - \iota\| + \|W_2\iota - \iota\|)$, so every solution of one equation in* (6.1) *is automatically a solution of the other. Thus, the enriched pair condition forces the two equations to have the same solution set, and a compatibility condition of the type (H_6) is necessary.*

Theorem 7. *Let $\mathcal{G} = L^2([0,1])$, suppose (H_1)–(H_6) hold, let γ be as in* (6.5)*, and let $b > 0$ and $\pi := \frac{1}{b+1}$. Then the following hold:*

(i) *The pair (V_1, V_2) defined by* (6.3) *is an enriched γ-Reich–Suzuki-type nonexpansive pair on $\mathcal{G}$ with enrichment parameter b and $\gamma \in (0, \frac{1}{2})$.*

(ii) *V_1 and V_2 possess a unique common fixed point $\iota^* = \varphi$, which is the unique common solution of* (6.1)*, and*

$$\lim_{\eta\to\infty} V_{1_\pi}^\eta \iota = \iota^* = \lim_{\eta\to\infty} V_{2_\pi}^\eta \iota \qquad \text{for any } \iota \in \mathcal{G}.$$

Proof. Let $\iota, \kappa \in \mathcal{G}$ be arbitrary. Since the forcing term g_i cancels, the Cauchy–Schwarz inequality and (H_2) give, for each $i \in \{1,2\}$,

$$\begin{aligned} \|W_i\iota - W_i\kappa\|^2 &= \int_0^1 \left| \int_0^1 K(t,s)[f_i(s,\iota(s)) - f_i(s,\kappa(s))] ds \right|^2 dt \\ &\le \int_0^1 \left[\int_0^1 |K(t,s)|^2\, ds \int_0^1 |f_i(s,\iota(s)) - f_i(s,\kappa(s))|^2\, ds \right] dt \\ &\le L_i^2 \|\iota - \kappa\|^2 \int_0^1 \int_0^1 |K(t,s)|^2\, ds\, dt = M^2 L_i^2 \|\iota - \kappa\|^2. \end{aligned} \tag{6.6}$$

Taking square roots,

$$\|W_i\iota - W_i\kappa\| \le \alpha_i \|\iota - \kappa\|, \qquad i = 1,2. \tag{6.7}$$

By (H_6),

$$g_1 - g_2 = -\int_0^1 K(\cdot, s)h(s, \varphi(s))\, ds,$$

hence

$$W_1\kappa - W_2\kappa = \int_0^1 K(\cdot, s)[h(s, \kappa(s)) - h(s, \varphi(s))]ds.$$

Applying the Cauchy–Schwarz inequality and (H_4) exactly as in (6.6),

$$\|W_1\kappa - W_2\kappa\| \leq M\delta\, \|\kappa - \varphi\|. \tag{6.8}$$

Since $W_2\varphi = \varphi$ by (H_6), the triangle inequality and (6.7) give

$$\|\kappa - \varphi\| \leq \|\kappa - W_2\kappa\| + \|W_2\kappa - W_2\varphi\| \leq \|W_2\kappa - \kappa\| + \alpha_2\|\kappa - \varphi\|,$$

and therefore, as $\alpha_2 < 1$,

$$\|\kappa - \varphi\| \leq \frac{\|W_2\kappa - \kappa\|}{1 - \alpha_2}. \tag{6.9}$$

Combining (6.8) and (6.9),

$$\|W_1\kappa - W_2\kappa\| \leq \frac{M\delta}{1 - \alpha_2}\|W_2\kappa - \kappa\| = \gamma\, \|W_2\kappa - \kappa\|. \tag{6.10}$$

By the triangle inequality, (6.7), (6.10), and $\alpha_1 \leq 1 - 2\gamma$ by Remark 4,

$$\begin{aligned}\|W_1\iota - W_2\kappa\| &\leq \|W_1\iota - W_1\kappa\| + \|W_1\kappa - W_2\kappa\| \\ &\leq \alpha_1\|\iota - \kappa\| + \gamma\|W_2\kappa - \kappa\| \\ &\leq \gamma\|W_1\iota - \iota\| + \gamma\|W_2\kappa - \kappa\| + (1 - 2\gamma)\|\iota - \kappa\|. \end{aligned} \tag{6.11}$$

By (6.3),

$$b(\iota - \kappa) + V_1\iota - V_2\kappa = (1 + b)(W_1\iota - W_2\kappa), \qquad V_i\iota - \iota = (1 + b)(W_i\iota - \iota), \quad i = 1, 2.$$

Multiplying (6.11) by $1 + b$ therefore gives

$$\|b(\iota - \kappa) + V_1\iota - V_2\kappa\| \leq \gamma\|V_1\iota - \iota\| + \gamma\|V_2\kappa - \kappa\| + (b + 1)(1 - 2\gamma)\|\iota - \kappa\|, \tag{6.12}$$

for all $\iota, \kappa \in \mathcal{G}$. Together with Remark 4, this establishes (i). Now, we prove the (ii). By (6.4) and (6.7), $V_{i_\pi} = W_i$ is a contraction with constant $\alpha_i \in [0, 1)$:

$$\|V_{i_\pi}\iota - V_{i_\pi}\kappa\| \leq \alpha_i\|\iota - \kappa\|, \qquad i = 1, 2. \tag{6.13}$$

Applying (6.13) with ι replaced by $V_{i_\pi}^{\eta}\iota$ and κ replaced by $V_{i_\pi}^{\eta-1}\iota$, and then inductively,

$$\|V_{i_\pi}^{\eta+1}\iota - V_{i_\pi}^{\eta}\iota\| \leq \alpha_i\|V_{i_\pi}^{\eta}\iota - V_{i_\pi}^{\eta-1}\iota\| \leq \cdots \leq \alpha_i^{\eta}\|V_{i_\pi}\iota - \iota\| \to 0 \text{ as } \eta \to \infty, \tag{6.14}$$

so V_{i_π} is asymptotically regular on $\mathcal{G}$. Moreover, V_{i_π} is orbitally continuous: If $V_{i_\pi}^{\eta_x}\iota \to z$ as $x \to \infty$ for some $z \in \mathcal{G}$ and some subsequence $\{\eta_x\}$, then by (6.13)

$$\|V_{i_\pi}^{\eta_x+1}\iota - V_{i_\pi}z\| \leq \alpha_i\|V_{i_\pi}^{\eta_x}\iota - z\| \to 0 \qquad (x \to \infty).$$

By (i), $\gamma \in (0, 1)$, and the above, all hypotheses of Theorem 1 are satisfied. Hence V_1 and V_2 have a unique common fixed point ι^*, and $V_{1_\pi}^{\eta}\iota \to \iota^*$, and $V_{2_\pi}^{\eta}\iota \to \iota^*$ for every $\iota \in \mathcal{G}$. Since $W_1\varphi = \varphi = W_2\varphi$ by (H_6) and $F(V_i) = F(W_i)$, φ is a common fixed point of V_1 and V_2; by uniqueness, $\iota^* = \varphi$, which is the unique common solution of (6.1). □

Theorem 8. *Under the assumptions of Theorem 7, the common fixed point problem for* (V_1, V_2) *is well-posed.*

Proof. By Theorem 7, all assumptions of Theorem 2 are satisfied, and the conclusion follows directly. □

Corollary 1. *Under* (H_1)–(H_6)*, let* $b > 0$ *and let* $\{\iota_\eta\}$ *be the sequence generated by the modified Picard-S iteration* (4.7) *with arbitrary* $\iota_0 \in \mathcal{G}$*, where the control sequences satisfy* $0 < \theta_1 \le a_\eta, b_\eta \le \theta_2 < 1$ *for all* $\eta \in \mathbb{N}_0$*. Then the pair* (V_1, V_2) *satisfies Condition (B) with* $f(r) := (1 + b)(1 - \alpha_1)r$*, and* $\{\iota_\eta\}$ *converges strongly to the unique common solution* $\iota^* = \varphi$ *of* (6.1) *in* $L^2([0, 1])$.

Proof. By Theorem 7, (V_1, V_2) is an enriched γ-Reich–Suzuki-type nonexpansive pair on the uniformly convex Banach space $\mathcal{G} = L^2([0, 1])$ with $F(V_1) \cap F(V_2) = \{\iota^*\}$, so $d(\iota, F) = \|\iota - \iota^*\|$. For any $\iota \in \mathcal{G}$, since $W_1\iota^* = \iota^*$, the triangle inequality and (6.7) give

$$\|\iota - \iota^*\| \le \|W_1\iota - \iota\| + \|W_1\iota - W_1\iota^*\| \le \|W_1\iota - \iota\| + \alpha_1\|\iota - \iota^*\|,$$

so $(1 - \alpha_1)\|\iota - \iota^*\| \le \|W_1\iota - \iota\|$. Since $V_1\iota - \iota = (1 + b)(W_1\iota - \iota)$,

$$\max\{\|V_1\iota - \iota\|, \|V_2\iota - \iota\|\} \ge \|V_1\iota - \iota\| \ge (1 + b)(1 - \alpha_1)\|\iota - \iota^*\| = f(d(\iota, F)),$$

where $f(r) := (1+b)(1-\alpha_1)r$ is nondecreasing with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$. Hence (V_1, V_2) satisfies Condition (B), and strong convergence of $\{\iota_\eta\}$ to ι^* follows from Theorem 6. □

Example 3. *Consider the kernel and nonlinear functions*

$$K(t, s) := \frac{s(1 - t)}{35}, \qquad f_1(s, u) := \frac{u}{1 + u^2}, \qquad f_2(s, u) := \frac{u}{2(1 + u^2)}, \tag{6.15}$$

the target function $\varphi(t) := t$*, and the forcing terms determined by* (H_6)*. With*

$$\vartheta := \int_0^1 \frac{s^2}{1 + s^2}\, ds = 1 - \arctan 1 \approx 0.2146,$$

we have $\int_0^1 K(t, s)f_1(s, s)\, ds = \frac{\vartheta(1-t)}{35}$ *and, since* $f_2 = \frac{1}{2}f_1$, $\int_0^1 K(t, s)f_2(s, s)\, ds = \frac{\vartheta(1-t)}{70}$*. Hence*

$$g_1(t) = t - \frac{\vartheta(1 - t)}{35}, \qquad g_2(t) = t - \frac{\vartheta(1 - t)}{70}, \tag{6.16}$$

and (6.1) *becomes*

$$\iota(t) = t - \frac{\vartheta(1-t)}{35} + \int_0^1 \frac{s(1-t)}{35} \cdot \frac{\iota(s)}{1+\iota(s)^2}\, ds, \qquad \iota(t) = t - \frac{\vartheta(1-t)}{70} + \int_0^1 \frac{s(1-t)}{35} \cdot \frac{\iota(s)}{2(1+\iota(s)^2)}\, ds.$$

We verify (H_1)–(H_6).
(H_1)*: By Fubini's theorem,*

$$M^2 = \frac{1}{35^2}\left(\int_0^1 s^2\, ds\right)\left(\int_0^1 (1 - t)^2\, dt\right) = \frac{1}{35^2 \cdot 9},$$

so $0 < M = \frac{1}{105} < \infty$.
(H_2): *Clearly* $f_i(\cdot, u)$ *is measurable and* $f_i(s, 0) = 0$, *so* $f_i(\cdot, 0) \in L^2([0, 1])$. *Moreover,*

$$|f_1(s, u) - f_1(s, v)| = \frac{|u - v|\,|1 - uv|}{(1 + u^2)(1 + v^2)}.$$

Since $|1 - uv| \le 1 + |uv| \le 1 + \frac{u^2+v^2}{2}$ *and* $(1 + u^2)(1 + v^2) \ge 1 + u^2 + v^2$, *setting* $\tau = u^2 + v^2 \ge 0$, *we obtain*

$$\frac{|1 - uv|}{(1 + u^2)(1 + v^2)} \le \frac{2 + \tau}{2(1 + \tau)} \le 1,$$

since $2 + \tau \le 2 + 2\tau$ *for all* $\tau \ge 0$. *Hence* $|f_1(s, u) - f_1(s, v)| \le |u - v|$, *so* $L_1 = 1$. *Since* $f_2 = \frac{1}{2} f_1$, *we get* $|f_2(s, u) - f_2(s, v)| \le \frac{1}{2}|u - v|$, *so* $L_2 = \frac{1}{2}$.

(H_3): $\alpha_1 = ML_1 = \frac{1}{105} < 1$ *and* $\alpha_2 = ML_2 = \frac{1}{210} < 1$.

(H_4): $h = f_1 - f_2 = \frac{1}{2} f_1$, *so* $|h(s, u) - h(s, v)| \le \frac{1}{2}|u - v|$ *and* $\delta = \frac{1}{2}$.

(H_5): $2M\delta = \frac{1}{105}$ *and* $(1 - \alpha_1)(1 - \alpha_2) = \frac{104}{105} \cdot \frac{209}{210} = \frac{21736}{22050}$, *so* $2M\delta \le (1 - \alpha_1)(1 - \alpha_2)$.

(H_6): *holds with* $\varphi(t) = t$ *by the construction* (6.16).

By (6.5),

$$\gamma = \frac{M\delta}{1 - \alpha_2} = \frac{\frac{1}{2} \cdot \frac{1}{105}}{1 - \frac{1}{210}} = \frac{\frac{1}{210}}{\frac{209}{210}} = \frac{1}{209} \in (0, \tfrac{1}{2}).$$

Take $b = 2$, *so* $\pi = \frac{1}{3}$ *and*

$$(V_i\iota)(t) = -2\iota(t) + 3g_i(t) + 3\int_0^1 K(t, s) f_i(s, \iota(s))\, ds, \qquad i = 1, 2.$$

The operators V_1, *and* V_2 *are not nonexpansive, and hence not contractions. Indeed, for* $\iota, \kappa \in \mathcal{G}$, *by* (6.3), *the reverse triangle inequality, and* (6.7),

$$\|V_i\iota - V_i\kappa\| = \| - 2(\iota - \kappa) + 3(W_i\iota - W_i\kappa)\| \ge 2\|\iota - \kappa\| - 3\alpha_i\|\iota - \kappa\|,$$

so that

$$\|V_1\iota - V_1\kappa\| \ge \frac{69}{35}\|\iota - \kappa\|, \qquad \|V_2\iota - V_2\kappa\| \ge \frac{139}{70}\|\iota - \kappa\|,$$

and $\|V_i\iota - V_i\kappa\| > \|\iota - \kappa\|$ *whenever* $\iota \ne \kappa$. *Hence the Banach contraction principle and results for nonexpansive mappings cannot be applied to* V_1, *and* V_2 *directly, whereas all hypotheses* (H_1)–(H_6) *hold. By Theorems 7 and 8 and Corollary 1, the system has the unique common solution* $\iota^*(t) = t$, *the problem is well-posed, and the modified Picard-S iteration* (4.7), *applied to* $V_{1_\pi} = W_1$ *and* $V_{2_\pi} = W_2$, *converges strongly to* ι^* *in* $L^2([0, 1])$ *from any starting point.*

7. Conclusions

This work introduced enriched γ-Reich–Suzuki-type nonexpansive pairs (V_1, V_2) on Banach spaces and showed that such a pair has a unique common fixed point, with a well-posed common fixed point

problem. A modified Picard-S scheme acting on the π-Krasnoselskii maps V_{1_π} and V_{2_π} was shown to converge weakly under Opial's condition or a Fréchet differentiable norm, and strongly under Condition (B), in uniformly convex spaces. In the numerical example, tested with five choices of control sequences, this scheme reached the tolerance 10^{-12} in the fewest iterations and had the smallest error at each step, compared with the modified Ishikawa, Abbas–Nazir, Noor, Agarwal, and Thakur schemes. Finally, for two nonlinear Fredholm-type integral equations in $L^2([0,1])$, the operators $V_i = -bI + (1+b)W_i$ are expansive, yet under (H_1)–(H_6) they form a pair of this class with $\gamma \in (0, \frac{1}{2})$. Hence the two equations share a unique, well-posed solution, which the modified Picard-S sequence approximates strongly.

Author contributions

The first author contributed to the conceptualization, methodology, investigation, software, visualization, and writing of the original draft. The second author contributed to the supervision, validation, and review and editing of the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

No potential conflict of interest was reported by the authors.

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