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Research article

Second-order stochastic differential equations with Lévy noise and mean boundary conditions

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Abstract: Stochastic boundary value problems of second order arise naturally in physical modeling, financial mathematics, and stochastic control theory. This paper focuses on establishing existence results for second-order stochastic differential equations with Lévy noise and mean boundary conditions. We combine deterministic and stochastic Picard iterations with fundamental techniques of stochastic calculus to obtain sufficient criteria for the existence of $\mathcal{L}^2(\mathbf{P}, \mathbb{R}^d)$-valued solutions. Moreover, we establish explicit sufficient conditions that ensure the solutions satisfy adaptedness and the càdlàg property. Several concrete examples are presented to demonstrate the applicability of our theoretical conclusions for such stochastic equations with mean-type boundary constraints.

Keywords: second-order stochastic differential equations; Lévy noise; boundary value problems; adapted; càdlàg
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1. Introduction

In the fields of mechanics, natural science, and engineering, Newtonian systems play an important role. By Newton's second law of motion, the acceleration of any moving body is governed by classical second-order ordinary differential equations. We present several representative examples as follows.

(1) Simple harmonic motion:

$$\ddot{x}(t) + a_1 x(t) = a_2 \cos \frac{2\pi t}{T}, \tag{1.1}$$

where $T > 0$, a_1 and a_2 denote constant parameters. Simple harmonic motion represents the most fundamental periodic vibration model found in natural systems and engineering practice. It is widely applied in fields such as engineering design, scientific instruments, and measurement.

(2) Pendulum equation:

$$\ddot{q}(t) + \lambda \sin q(t) = r(t), \tag{1.2}$$

where λ and Γ are positive constants, and r satisfies the periodicity condition $r(t + \Gamma) = r(t)$. Equation (1.2) models a thin wire swinging about a fixed pivot in a vertical plane and a rigid body oscillating about a fixed horizontal axis; see [1–3].

(3) Hill equation:

$$\ddot{p}(t) + \phi(t)p(t) = \rho(t, p(t)), \tag{1.3}$$

where ϕ and ρ are continuous and satisfy

$$\phi(t + \Gamma) = \phi(t), \qquad \rho(t + \Gamma, p) = \rho(t, p) \quad \text{for all } (t, p) \in \mathbb{R}^2.$$

This equation describes the motion of the moon, the theory of electric circuits, and cyclotron dynamics; see [4].

The examples presented above are special cases of the second-order ordinary differential system

$$\frac{d^2X_t}{dt^2} = f(t, X_t), \tag{1.4}$$

where the mapping $f : [a, b] \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous.

However, in actual mechanical processes, system (1.4) is often disturbed by stochastic noise. When the stochastic noise is continuous, the system can be modeled by the following second-order stochastic system:

$$\ddot{X}(t) = f(t, X(t)) + g(t, X(t))\dot{B}(t), \qquad t \in [a, b]. \tag{1.5}$$

Here B is an $\mathbb{R}^r$-valued Brownian motion and X is an $\mathbb{R}^d$-valued continuous-time stochastic process. The coefficients $f : [a, b] \times \mathbb{R}^d \to \mathbb{R}^d$ and $g : [a, b] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are continuous.

Unlike deterministic models incapable of capturing random fluctuations, stochastic differential equations (SDEs) yield more realistic descriptions of dynamical systems by quantifying the impact of stochastic perturbations on their evolution. SDEs play a fundamental role across a wide array of fields when uncertainties or random influences need to be taken into account. SDEs have found extensive applications in disciplines such as physics, biology, economics, population dynamics, finance, fluid flows in random media, and genetics. For example, studies on stochastic wave propagation and high-dimensional stochastic wave equations are available in [5, 6]. Stochastic Newtonian systems and periodic solutions in distribution are investigated in [7]. General background and numerical schemes for stochastic differential equations are provided in [8, 9]. SDEs provide suitable mathematical tools to model realistic problems with uncertainties that may be caused by many factors such as initial and boundary value conditions, and the geometric representation of the domain, among others.

In recent years, stochastic boundary value problems (SBVPs), particularly second-order SBVPs, have been widely applied across various fields. They arise from the application of Pontryagin's maximum principle to stochastic optimal control problems with a finite time horizon T, where the boundary condition corresponds to the transversality condition [10]. SBVPs are naturally employed to

model harmonic oscillators in physics [11, 12], and also appear in other domains such as wave motion in random media [13], maximum a posteriori estimation of diffusion trajectories [14], and smoothing techniques [15], among others.

SBVPs have attracted sustained attention; see [16] and the references therein. Ocone and Pardoux [17] investigated SBVPs with affine two-point boundary conditions. Their findings include a necessary and sufficient existence-uniqueness criterion, a sufficient criterion for probability densities of solutions, and sufficient conditions for the standard Markov process or Markov field property of solutions. Non-adaptive SBVPs were considered by Dembo and Zeitouni [18], who combined a generalized change of variables formula with optimal control methods to establish the existence of solutions. Nualart and Pardoux [19] investigated SDEs with two-point boundary conditions. They established existence and uniqueness results for solutions, and analyzed the Markov and Markov field properties. In the one-dimensional case, they showed that the solution process becomes a Markov field exactly when the drift term is affine, whereas higher-dimensional equations generally lead to more complicated Markovian structures. In [20], Nualart and Pardoux analyzed a second-order SDE with Dirichlet boundary conditions. Their main contribution was to prove pathwise existence and uniqueness by imposing certain regularity and monotonicity conditions on the coefficients. Using a generalized Girsanov theorem established by Kusuoka, they showed that the bivariate process $(X_t, \dot{X}_t)$ is Markov only for an affine drift function; for a nonlinear drift term, the process fails to be a germ-Markov field. In [21], Capietto and Priola considered the SDE

$$\frac{d^2X(t)}{dt^2} + f(t, X(t), \frac{dX(t)}{dt}) = \frac{dW(t)}{dt}, \qquad t \in [0, 1], \tag{1.6}$$

with $X(0) = X(1) = 0$. Using functional analytic tools and Malliavin calculus, they obtained sufficient conditions for uniqueness in distribution. Alabert and Nualart [22] studied second-order Stratonovich boundary value problems and showed that a linear drift is both necessary and sufficient for the solution to be a 2-Markov field. For linear nth-order stochastic differential equations over the interval $[0, 1]$, Alabert and Ferrante [23] introduced linear boundary restrictions imposed on a finite point set and explored the essential properties of their solutions. In all the above studies, the associated boundary conditions are deterministic. Nevertheless, deterministic boundary constraints cause $\{X_t\}$ to lack adaptedness with respect to the Brownian filtration. This raises a natural question: What forms of boundary restrictions guarantee adaptedness for solutions of stochastic boundary value problems? Motivated by the fact that noise makes terminal values uncertain, we adopt the class of mean-type terminal boundary constraints first proposed in [24]. Our work concerns stochastic Newtonian systems driven by white noise under these mean boundary constraints, and we show that the constructed solution process is measurable with respect to the filtration generated by Brownian motion.

However, it should be noted that discontinuous and drastic fluctuations frequently arise in practical scenarios, such as abrupt environmental events (earthquakes, hurricanes, etc.) and unpredictable changes. Such phenomena cannot be captured by Brownian motion. Applebaum [25] demonstrated that non-Gaussian Lévy noise is well-suited for characterizing jump-type stochastic perturbations. A Lévy process is a stochastic process featuring stationary and independent increments, with Brownian motion and Poisson processes as its basic examples. As an important class of semimartingales and Markov processes, Lévy processes play a central role across theoretical studies and practical applications. Building on the white noise framework of [24], we extend the analysis to stochastic

systems driven by Lévy noise. Indeed, research interest in SDEs driven by Lévy noise has been rising in recent years; see [26–28].

For system (1.4) perturbed by Lévy noise, we focus our analysis on the small-jump case $|y| < c$. The associated second-order stochastic system is as follows:

$$\frac{d^2X(t)}{dt^2} = f(t, X(t)) + g(t, X(t))\dot{B}_t + \int_{|y|<c} F(t, X(t), y)\dot{\tilde{N}}(t, dy). \tag{1.7}$$

Here $t \in [a, b]$ and $x, y \in \mathbb{R}^d$. The coefficients are continuous maps

$$f : [a, b] \times \mathbb{R}^d \to \mathbb{R}^d, \qquad g : [a, b] \times \mathbb{R}^d \to \mathbb{R}^{d\times r}, \qquad F : [a, b] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d.$$

The parameter $c \in (0, \infty]$ is the largest admissible jump size. Let N be an $\mathcal{F}_t$-adapted Poisson random measure on $[a, b] \times (\mathbb{R}^d \setminus \{0\})$. We also define the compensated Poisson random measure by

$$\tilde{N}(dt, dy) = N(dt, dy) - \nu(dy)dt,$$

where ν is a Lévy measure. The compensated measure $\tilde{N}$ is a martingale. We take B and N to be independent of $\mathcal{F}_0$, and call (B, N) a Lévy noise. Throughout the paper, we only consider stochastic systems driven by Lévy noise with small jumps, and do not cover general Lévy processes involving large jumps.

Taking into account the uncertainty and randomness of the boundary conditions, we introduce a class of mean boundary conditions at the terminal time $t = b$. These boundary conditions take the form:

$$X(a) = 0, \ \ \mathbf{E}X(b) = 0. \tag{1.8}$$

Nevertheless, Lévy noise brings additional challenges for the analysis concerning solution existence. The present work considers the second-order SBVP given below:

$$\begin{cases} \dfrac{d^2X(t)}{dt^2} = f(t, X(t)) + g(t, X(t))\dot{B}_t + \displaystyle\int_{|y|<c} F(t, X(t), y)\dot{\tilde{N}}(t, dy), \\ X(a) = 0, \ \ \mathbf{E}X(b) = 0. \end{cases} \tag{1.9}$$

Our goal is to determine conditions under which the second-order SBVP (1.9) has a solution. Any such solution is an $\mathbb{R}^d$-valued process $(X(t))_{t\in[a,b]}$, written in components as $X(t) = (X_1(t), \ldots, X_d(t))$. Imposing appropriate assumptions on coefficient functions, we derive existence results for such solutions. We further verify that the resulting process $X(t)$ satisfies adaptedness and possesses the càdlàg property.

The outline of this paper is given below. Section 2 collects necessary preliminaries, presents essential stochastic inequalities, and formulates the target second-order SBVP that we will investigate. Section 3 proves the main existence theorem for the second-order SBVP (1.9). Our proof relies on separate constructions of deterministic and stochastic Picard sequences. Section 4 deduces the adaptedness and càdlàg property of the constructed solutions, and Section 5 presents several examples to illustrate our theoretical conclusions.

2. Preliminaries

First, we introduce basic notations. Second, we present several definitions and key theorems concerning Lévy processes and stochastic calculus. These preliminary concepts form the basis for investigating SDEs with Lévy noise within our core results; see [25]. Finally, we introduce the second-order SBVP to be investigated and divide it into deterministic and stochastic components.

2.1. Notations

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space, and let $\{\mathcal{F}_t\}_{a\leq t\leq b}(0 \leq a < b)$ be a filtration that fulfills the usual conditions. For any random variable X on this probability space, its expectation $\mathbf{E}X$ is defined as the Lebesgue integral

$$\mathbf{E}X = \int_{\Omega} X d\mathbf{P},$$

whenever the integral is well-defined.

We say that an $\mathbb{R}^d$-valued random variable X belongs to $\mathcal{L}^2(\mathbf{P}, \mathbb{R}^d)$ if

$$\mathbf{E}|X|^2 = \int_{\Omega} |X|^2 d\mathbf{P} < \infty.$$

For each such X, define

$$\|X\|_2 := \left(\int_{\Omega} |X|^2 d\mathbf{P}\right)^{1/2}.$$

Endowed with the norm $\|\cdot\|_2$, $\mathcal{L}^2(\mathbf{P}, \mathbb{R}^d)$ forms a Hilbert space.

2.2. Lévy process

The following standard notions are recalled from [25].

Definition 2.1. *The stochastic process $X = (X(t), t \geq 0)$ is called a Lévy process when the three requirements below hold:*
(1) $X(0) = 0$ almost surely.
(2) X possesses independent, stationary increments.
(3) X is stochastically continuous. That is, for arbitrary $\alpha > 0$ and $t_0 \geq 0$,

$$\lim_{t\to t_0} \mathbf{P}[|X(t) - X(t_0)| > \alpha] = 0.$$

Definition 2.2. *Given a Lévy process X, we write $\Delta X = (\Delta X(t), t \geq 0)$ for its jump process. We define $\Delta X(t) = X(t) - X(t-)$, where $X(t-)$ stands for the left-hand limit of X at time t for all $t \geq 0$.*

Definition 2.3. *A function $f : [a, b] \to \mathbb{R}^d$ is càdlàg provided that it admits a left-hand limit at $t \in [a, b]$ and satisfies right continuity at t.*

Definition 2.4. *Suppose X is a Lévy process. Given $t \in [0, \infty)$ and a Borel set $E \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$, we set*

$$N(t, E) = \#\{0 \leq s \leq t; \Delta X(s) \in E\} = \sum_{0\leq s\leq t} 1_E(\Delta X(s)).$$

For fixed $\omega \in \Omega$ and $t \geq 0$, the mapping $E \mapsto N(t, E)(\omega)$ is a counting measure on $\mathcal{B}(\mathbb{R}^d \setminus \{0\})$. The intensity measure of X is denoted by $\nu(\cdot) = \mathbf{E}(N(1, \cdot))$.

Definition 2.5. *A Borel set $E \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ is bounded below whenever the origin does not belong to its closure, i.e., $0 \notin \bar{E}$.*

Lemma 2.6. *For any set E that is bounded below, $N(t, E)$ is almost surely finite for every $t \geq 0$.*

Definition 2.7. *For $t \geq 0$ and every bounded-below Borel set E, the compensated Poisson random measure $\tilde{N}(t, E)$ is*

$$\tilde{N}(t, E) = N(t, E) - t\nu(E).$$

We summarize the main properties of N which will be used extensively later.
(1) For all $t > 0$ and $\omega \in \Omega$, $N(t, \cdot)(\omega)$ constitutes a counting measure over $\mathcal{B}(\mathbb{R}^d \setminus \{0\})$.
(2) If E is bounded below, then $(N(t, E), t \geq 0)$ is a Poisson process, and its intensity measure is $\nu(E) = \mathbf{E}(N(1, E))$.
(3) $\tilde{N}$ is a σ-finite independently scattered martingale-valued measure.

Definition 2.8. *Given $f \in L^1(E, \nu(E))$ and $t \geq 0$, the compensated Poisson integral is defined by*

$$\int_E f(x)\tilde{N}(t, dx) = \int_E f(x)N(t, dx) - t\int_E f(x)\nu(dx).$$

Theorem 2.9. ***(Doob's martingale inequality)*** *Let $X = (X(t))_{t\geq 0}$ be a positive submartingale. For every $p > 1$ and $t > 0$,*

$$\mathbf{E}\Big(\sup_{0\leq s\leq t} X(s)^p\Big) \leq q^p \mathbf{E}[X(t)^p],$$

with $1/p + 1/q = 1$.

As a special case, consider an $\mathbb{R}^d$-valued submartingale $X = (X(t))_{t\geq 0}$. For each index $1 \leq i \leq d$, $\{X_i(t)^2, t \geq 0\}$ is a real-valued submartingale. Applying Theorem 2.9 with $p = q = 2$, we obtain for all $t \geq 0$,

$$\mathbf{E}\Big(\sup_{0\leq u\leq t} |X(u)|^2\Big) \leq \sum_{i=1}^{d} \mathbf{E}\Big(\sup_{0\leq u\leq t} X_i(u)^2\Big) \leq \sum_{i=1}^{d} 4\mathbf{E}(X_i(t)^2) = 4\mathbf{E}|X(t)|^2. \tag{2.1}$$

Theorem 2.10. ***(Chebyshev-Markov inequality)*** *Let X be a real random variable with $u = \mathbf{E}X$. For any $\alpha > 0$, $\lambda \in \mathbb{R}$, and $k \in \mathbb{N}$,*

$$\mathbf{P}(|X - \lambda u| \geq \alpha) \leq \frac{\mathbf{E}[|X - \lambda u|^k]}{\alpha^k}. \tag{2.2}$$

Lemma 2.11. ***(Borel-Cantelli lemma)*** *Let $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of events satisfying*

$$\sum_{n=1}^{\infty} \mathbf{P}(E_n) < \infty.$$

Then

$$\mathbf{P}\left(\limsup_{n\to\infty} E_n\right) = 0.$$

Theorem 2.12. ***(The Lévy-Itô decomposition)*** *The Lévy process X admits the representation*

$$X(t) = at + B_\Upsilon(t) + \int_{|u|<1} u\tilde{N}(t, du) + \int_{|u|\geq 1} uN(t, du), \qquad t \geq 0, \tag{2.3}$$

where $a \in \mathbb{R}^d$, B_Υ is a Brownian motion with covariance matrix Υ, and N is an independent Poisson random measure on $\mathbb{R}^+ \times (\mathbb{R}^d \setminus \{0\})$. The intensity measure ν satisfies

$$\int_{\mathbb{R}^d \setminus \{0\}} (|u|^2 \wedge 1)\nu(du) < \infty,$$

and $\tilde{N}$ denotes the associated compensated Poisson random measure.

2.3. Second-order SDE with Lévy noise and mean boundary conditions

We start our discussion from the deterministic BVP on $[0, t]$:

$$\begin{cases} \ddot{X}(t) = f(t, X(t)), \\ X(a) = 0, \quad X(b) = 0. \end{cases} \tag{2.4}$$

Denote the solution of (2.4) by $u(t)$. Then

$$\ddot{u}(t) = f(t, u(t)), \ t \in [a, b] \tag{2.5}$$

with $u(a) = 0$ and $u(b) = 0$. Given a solution $X(t)$ of (1.9), decompose it as

$$X(t) := u(t) + v(t).$$

Then $v(t)$ satisfies

$$\ddot{v}(t) = f(t, v(t) + u(t)) - f(t, u(t)) + g(t, v(t) + u(t))\dot{B}_t + \int_{|y|<c} F(t, v(t) + u(t), y)\dot{\tilde{N}}(t, dy), \tag{2.6}$$

with

$$v(a) = 0, \quad \mathbf{E}v(b) = 0. \tag{2.7}$$

As a consequence, (1.9) splits into the deterministic BVP (2.5) and the SBVP (2.6) subject to the mean boundary conditions (2.7). The existence of solutions to SBVP (1.9) can then be derived from the corresponding existence results of these two components.

3. Existence of the solution for a second-order SBVP

This section is devoted to establishing the main existence theorem of our work. With appropriate sufficient hypotheses, we establish the existence of at least one solution to the second-order SBVP (1.9) equipped with mean boundary constraints.

Theorem 3.1. *Let L, K be positive constants. Assume that the following conditions hold:*

(C1) Lipschitz conditions: *for every $t \in [a, b]$ and all $u_1, u_2 \in \mathbb{R}^d$,*

$$|f(t, u_1) - f(t, u_2)|^2 \vee \|g(t, u_1) - g(t, u_2)\|^2 \vee \int_{|y|<c} |F(t, u_1, y) - F(t, u_2, y)|^2 \nu(dy) \leq L|u_1 - u_2|^2.$$

(C2) Growth conditions: *for every $t \in [a, b]$ and $u \in \mathbb{R}^d$,*

$$|f(t,u)|^2 \vee \|g(t,u)\|^2 \vee \int_{|y|<c} |F(t,u,y)|^2 \nu(dy) \leq K(1+|u|^2).$$

If the constant L satisfies

$$8L(b-a)^3(2b-2a+1) + \frac{L(b-a)^4}{64} < 1, \tag{3.1}$$

then the second-order SBVP (1.9) *admits at least one solution $X_t(\omega)$ belonging to $\mathcal{L}^2(\mathbf{P}, \mathbb{R}^d)$.*

Proof. We first note a basic fact. If $f(t,0) = g(t,0) = 0$ and $\int_{|y|<c} |F(t,0,y)|^2 \nu(dy) = 0$ for every $t \in [a,b]$, then $X_t \equiv 0$ solves (1.9). If one of these three quantities is not identically zero, the zero process is not a solution. The proof proceeds in three steps.

Step 1. Consider the deterministic second-order boundary value problem on $[a,b]$:

$$\begin{cases} \ddot{u}(t) = f(t,u(t)), \\ u(a) = 0,\ u(b) = 0. \end{cases} \tag{3.2}$$

The Green function representation of its solution is

$$u(t) = -\int_a^b G(t,s) f(s,u(s)) ds, \tag{3.3}$$

where G denotes the Green's function defined by

$$G(t,s) = \begin{cases} \frac{(b-t)(s-a)}{b-a}, & a \leq s \leq t \leq b, \\ \frac{(b-s)(t-a)}{b-a}, & a \leq t \leq s \leq b. \end{cases} \tag{3.4}$$

Observe that $G(t,s)$ is nonnegative and

$$0 \leq G(t,s) \leq \frac{b-a}{4}.$$

Set $u_0(t) \equiv 0$, and define the deterministic Picard iterative sequence by

$$u_n(t) = -\int_a^b G(t,s) f(s,u_{n-1}(s))\, ds, \qquad n \in \mathbb{N}_+, \quad t \in [a,b]. \tag{3.5}$$

Each $u_n(t)$ belongs to $C([a,b];\mathbb{R}^d)$. Since the sequence $\{u_n(t)\}$ is uniformly bounded and f satisfies the growth condition (C2), choose $M > 0$ such that

$$|f(t,u_n(t))| \leq M, \qquad t \in [a,b].$$

The remaining part of the proof follows the strategy proposed in [24]. Note that our Green's function is nonnegative. Consequently, $\{u_n(t)\}$ converges in $C([a,b];\mathbb{R}^d)$ to a limit function $u(t)$ satisfying (3.3), and $u(t)$ solves the deterministic second-order boundary value problem (3.2).

For later use, we state the quantitative bound derived from the above reasoning:

$$|u_{n+1}(t) - u_n(t)| \leq \frac{M(b-a)^2}{8} q^n, \tag{3.6}$$

where $q = \frac{\sqrt{L}(b-a)^2}{8}$.

Step 2. We consider the second-order stochastic boundary value problem on $[a, b]$:

$$\ddot{v}(t) = f(t, v(t) + u(t)) - f(t, u(t)) + g(t, v(t) + u(t))\dot{B}_t + \int_{|y|<c} F(t, v(t) + u(t), y)\dot{\tilde{N}}(t, dy)$$

with $v(a) = 0$ and $\mathbf{E}v(b) = 0$.

Letting $w(t) = \dot{v}(t)$, we convert the stochastic system (2.6) into an equivalent first-order system

$$dv(t) = w(t)dt, \tag{3.7}$$

$$\begin{aligned} dw(t) = {} & [f(t, v(t) + u(t)) - f(t, u(t))]dt + g(t, v(t) + u(t))dB(t) \\ & + \int_{|y|<c} F(t, v(t) + u(t), y)\tilde{N}(dt, dy). \end{aligned} \tag{3.8}$$

Integrating (3.7) and (3.8) over $[a, t]$ gives

$$v(t) = \int_a^t w(s)ds, \tag{3.9}$$

$$\begin{aligned} w(t) = {} & w(a) + \int_a^t \int_{|y|<c} F(s, v(s) + u(s), y)\tilde{N}(ds, dy) \\ & + \int_a^t [f(s, v(s) + u(s)) - f(s, u(s))]ds + \int_a^t g(s, v(s) + u(s))dB(s). \end{aligned} \tag{3.10}$$

Taking expectations in (3.9) gives

$$\begin{aligned} \mathbf{E}v(t) = \mathbf{E} \int_a^t \Big[& w(a) + \int_a^s \int_{|y|<c} F(z, v(z) + u(z), y)\tilde{N}(dz, dy) \\ & + \int_a^s (f(z, v(z) + u(z)) - f(z, u(z)))dz + \int_a^s g(z, v(z) + u(z))dB(z)\Big]ds. \end{aligned}$$

Using the boundary condition $\mathbf{E}v(b) = 0$, we obtain

$$0 = (b-a)\mathbf{E}w(a) + \mathbf{E}\Big[\int_a^b \int_a^s (f(z, v(z) + u(z)) - f(z, u(z)))dzds\Big]. \tag{3.11}$$

Equation (3.11) admits infinitely many random variables $w(a)$. To simplify the construction of a solution, we select a particular $\mathcal{F}_a$-measurable random variable defined by the conditional expectation

$$w(a) = -\frac{1}{b-a}\mathbf{E}\Big[\int_a^b \int_a^s (f(z, v(z) + u(z)) - f(z, u(z)))dzds\Big|\mathcal{F}_a\Big]. \tag{3.12}$$

Under this selection, a solution $v(t)$ to the second-order SBVP (2.6) with mean boundary conditions (2.7) admits the following integral representation:

$$
\begin{aligned}
v(t) &= \int_a^t w(s)ds \qquad (3.13)\\
&= \int_a^t w(a)ds + \int_a^t \int_a^s \int_{|y|<c} F(z, v(z) + u(z), y)\tilde{N}(dz, dy)ds\\
&\quad + \int_a^t \int_a^s [f(z, v(z) + u(z)) - f(z, u(z))]dzds + \int_a^t \int_a^s g(z, v(z) + u(z))dB(z)ds\\
&= (t-a)w(a) + \int_a^t \int_a^s \int_{|y|<c} F(z, v(z) + u(z), y)\tilde{N}(dz, dy)ds\\
&\quad + \int_a^t \int_a^s [f(z, v(z) + u(z)) - f(z, u(z))]dzds + \int_a^t \int_a^s g(z, v(z) + u(z))dB(z)ds.
\end{aligned}
$$

We start the stochastic Picard iterative sequence $\{v_n(t)\}_{n\in\mathbb{N}}$ with the term $v_0(t) \equiv 0$. To simplify the notation, set

$$\Phi_n(z) := f(z, v_n(z) + u_n(z)) - f(z, u_n(z)), \qquad n \geq 0,$$

and

$$w_n(a) = -\frac{1}{b-a}\mathbf{E}\left[\int_a^b \int_a^s \Phi_n(z)\, dz\, ds \,\middle|\, \mathcal{F}_a\right].$$

For $n \geq 1$, the stochastic Picard iterations are defined by

$$
\begin{aligned}
v_n(t) &= (t-a)w_{n-1}(a) + \int_a^t \int_a^s \int_{|y|<c} F(z, v_{n-1}(z) + u_{n-1}(z), y)\, \tilde{N}(dz, dy)\, ds \qquad (3.14)\\
&\quad + \int_a^t \int_a^s \Phi_{n-1}(z)\, dz\, ds + \int_a^t \int_a^s g(z, v_{n-1}(z) + u_{n-1}(z))\, dB(z)\, ds.
\end{aligned}
$$

Since $u_0 = v_0 = 0$, we have $\Phi_0(z) = 0$, and hence

$$v_1(t) - v_0(t) = \int_a^t \int_a^s g(z, 0)\, dB(z)\, ds + \int_a^t \int_a^s \int_{|y|<c} F(z, 0, y)\, \tilde{N}(dz, dy)\, ds.$$

Applying the elementary inequality $|x + y|^2 \leq 2|x|^2 + 2|y|^2$, we obtain

$$\mathbf{E}|v_1(t) - v_0(t)|^2 \leq 2\mathbf{E}\left|\int_a^t \int_a^s g(z, 0)\, dB(z)\, ds\right|^2 + 2\mathbf{E}\left|\int_a^t \int_a^s \int_{|y|<c} F(z, 0, y)\, \tilde{N}(dz, dy)\, ds\right|^2.$$

By the Cauchy-Schwarz inequality and Itô's isometry,

$$\mathbf{E}|v_1(t) - v_0(t)|^2 \leq 2(t-a)\int_a^t \int_a^s \mathbf{E}|g(z, 0)|^2\, dz\, ds + 2(t-a)\int_a^t \int_a^s \int_{|y|<c} \mathbf{E}|F(z, 0, y)|^2 \nu(dy)\, dz\, ds.$$

Condition (C2) yields

$$|g(z, 0)|^2 \vee \int_{|y|<c} |F(z, 0, y)|^2 \nu(dy) \leq K.$$

Therefore,

$$\mathbf{E}|v_1(t) - v_0(t)|^2 \leq 2K(b-a)^3.$$

For $n \geq 1$, define

$$\Delta g_n(z) := g(z, v_n(z) + u_n(z)) - g(z, v_{n-1}(z) + u_{n-1}(z)),$$
$$\Delta F_n(z, y) := F(z, v_n(z) + u_n(z), y) - F(z, v_{n-1}(z) + u_{n-1}(z), y),$$

and

$$\Delta w_n(a) := w_n(a) - w_{n-1}(a).$$

Then

$$\begin{aligned} & v_{n+1}(t) - v_n(t) \\ &= (t-a)\Delta w_n(a) + \int_a^t \int_a^s \int_{|y|<c} \Delta F_n(z, y)\, \tilde{N}(dz, dy)\, ds \\ &\quad + \int_a^t \int_a^s [\Phi_n(z) - \Phi_{n-1}(z)]\, dz\, ds + \int_a^t \int_a^s \Delta g_n(z)\, dB(z)\, ds. \end{aligned} \tag{3.15}$$

Using $|x_1 + \cdots + x_4|^2 \leq 4 \sum_{j=1}^{4} |x_j|^2$, we obtain

$$\begin{aligned} & |v_{n+1}(t) - v_n(t)|^2 \\ &\leq 4|(t-a)\Delta w_n(a)|^2 + 4\left| \int_a^t \int_a^s \int_{|y|<c} \Delta F_n(z, y)\, \tilde{N}(dz, dy)\, ds \right|^2 \\ &\quad + 4\left| \int_a^t \int_a^s [\Phi_n(z) - \Phi_{n-1}(z)]\, dz\, ds \right|^2 + 4\left| \int_a^t \int_a^s \Delta g_n(z)\, dB(z)\, ds \right|^2. \end{aligned} \tag{3.16}$$

Hence,

$$\begin{aligned} & \mathbf{E}|v_{n+1}(t) - v_n(t)|^2 \\ &\leq 4(t-a)^2 \mathbf{E}|\Delta w_n(a)|^2 + 4\mathbf{E}\left| \int_a^t \int_a^s \int_{|y|<c} \Delta F_n(z, y)\, \tilde{N}(dz, dy)\, ds \right|^2 \\ &\quad + 4\mathbf{E}\left| \int_a^t \int_a^s [\Phi_n(z) - \Phi_{n-1}(z)]\, dz\, ds \right|^2 + 4\mathbf{E}\left| \int_a^t \int_a^s \Delta g_n(z)\, dB(z)\, ds \right|^2. \end{aligned} \tag{3.17}$$

For the first term in (3.17), the definition of $\Delta w_n(a)$ gives

$$\Delta w_n(a) = -\frac{1}{b-a} \mathbf{E}\left[\int_a^b \int_a^s (\Phi_n(z) - \Phi_{n-1}(z))\, dz\, ds \,\middle|\, \mathcal{F}_a \right].$$

By the conditional Jensen inequality

$$\mathbf{E}\left[|\mathbf{E}[X \mid \mathcal{F}_a]|^2 \right] \leq \mathbf{E}|X|^2,$$

we have

$$\mathbf{E}|\Delta w_n(a)|^2 \leq \frac{1}{(b-a)^2} \mathbf{E}\left| \int_a^b \int_a^s (\Phi_n(z) - \Phi_{n-1}(z))\, dz\, ds \right|^2.$$

Combining this estimate with (3.17), and applying Cauchy-Schwarz and Itô's isometry to the remaining terms, we obtain

$$\begin{aligned}\mathbf{E}|v_{n+1}(t)-v_n(t)|^2 \le{}& \frac{4(t-a)^2}{b-a}\int_a^b (s-a)\int_a^s \mathbf{E}|\Phi_n(z)-\Phi_{n-1}(z)|^2\,dz\,ds\\ &+4(t-a)\int_a^t\int_a^s\int_{|y|<c}\mathbf{E}|\Delta F_n(z,y)|^2\nu(dy)\,dz\,ds\\ &+4(t-a)\int_a^t (s-a)\int_a^s \mathbf{E}|\Phi_n(z)-\Phi_{n-1}(z)|^2\,dz\,ds\\ &+4(t-a)\int_a^t\int_a^s \mathbf{E}|\Delta g_n(z)|^2\,dz\,ds, \end{aligned}\tag{3.18}$$

where

$$E|\Phi_n(z)-\Phi_{n-1}(z)|^2 \le\ 2\mathbf{E}|f(z,v_n(z)+u_n(z))-f(z,v_{n-1}(z)+u_{n-1}(z))|^2+2|f(z,u_n(z))-f(z,u_{n-1}(z))|^2.$$

For $n\ge 1$, set

$$U_n(z):=u_n(z)-u_{n-1}(z),\qquad V_n(z):=v_n(z)-v_{n-1}(z).$$

Applying the Lipschitz condition (C1) and $|x+y|^2\le 2|x|^2+2|y|^2$, we obtain

$$\begin{aligned}&\mathbf{E}|v_{n+1}(t)-v_n(t)|^2\\ &\le \frac{8L(t-a)^2}{b-a}\int_a^b (s-a)\int_a^s [\mathbf{E}|V_n(z)+U_n(z)|^2+|U_n(z)|^2]\,dz\,ds\\ &\quad+4L(t-a)\int_a^t\int_a^s \mathbf{E}|V_n(z)+U_n(z)|^2\,dz\,ds\\ &\quad+8L(t-a)\int_a^t (s-a)\int_a^s [\mathbf{E}|V_n(z)+U_n(z)|^2+|U_n(z)|^2]\,dz\,ds\\ &\quad+4L(t-a)\int_a^t\int_a^s \mathbf{E}|V_n(z)+U_n(z)|^2\,dz\,ds\\ &\le 8L(t-a)\int_a^b (s-a)\int_a^s [2\mathbf{E}|V_n(z)|^2+3|U_n(z)|^2]\,dz\,ds\\ &\quad+8L(t-a)\int_a^t\int_a^s [E|V_n(z)|^2+|U_n(z)|^2]\,dz\,ds\\ &\quad+8L(t-a)\int_a^t (s-a)\int_a^s [2\mathbf{E}|V_n(z)|^2+3|U_n(z)|^2]\,dz\,ds\\ &\quad+8L(t-a)\int_a^t\int_a^s [E|V_n(z)|^2+|U_n(z)|^2]\,dz\,ds\\ &\le 48L(t-a)\int_a^b (s-a)\int_a^s |U_n(z)|^2\,dz\,ds\\ &\quad+32L(t-a)\int_a^b (s-a)\int_a^s E|V_n(z)|^2\,dz\,ds\\ &\quad+16L(t-a)\int_a^t\int_a^s |U_n(z)|^2\,dz\,ds+16L(t-a)\int_a^b\int_a^b E|V_n(z)|^2\,dz\,ds. \end{aligned}\tag{3.19}$$

Notice that

$$|u_n(t) - u_{n-1}(t)| \leq \frac{M(b-a)^2}{8} q^{n-1}.$$

Then

$$\begin{aligned} & 48L(t-a)\int_a^b (s-a)\int_a^s |U_n(z)|^2 dzds \\ & \leq \frac{3LM^2}{4}(b-a)^4(t-a)q^{2n-2}\int_a^b (s-a)^2 ds \\ & \leq \frac{LM^2}{4}(b-a)^8 q^{2n-2}, \end{aligned} \tag{3.20}$$

$$16L(t-a)\int_a^t \int_a^s |U_n(z)|^2 dzds \leq \frac{LM^2}{8}(b-a)^7 q^{2n-2}. \tag{3.21}$$

Substituting the integral bounds (3.20) and (3.21) into inequality (3.19), we obtain the following estimate:

$$\begin{aligned} & \mathbf{E}|v_{n+1}(t) - v_n(t)|^2 \\ & \leq \frac{LM^2}{4}(b-a)^8 q^{2n-2} + \frac{LM^2}{8}(b-a)^7 q^{2n-2} \\ & \quad + 32L(t-a)\int_a^b (s-a)\int_a^b \mathbf{E}|v_n(z) - v_{n-1}(z)|^2 dzds \\ & \quad + 16L(t-a)\int_a^b \int_a^b \mathbf{E}|v_n(z) - v_{n-1}(z)|^2 dzds \\ & \leq \frac{LM^2}{8}(b-a)^7(2b-2a+1)q^{2n-2} \\ & \quad + 16L(b-a)(2b-2a+1)(t-a)\int_a^b \mathbf{E}|v_n(z) - v_{n-1}(z)|^2 dz. \end{aligned} \tag{3.22}$$

Substituting $q = \frac{\sqrt{L}(b-a)^2}{8}$ into the preceding estimate yields

$$\begin{aligned} & \mathbf{E}|v_{n+1}(t) - v_n(t)|^2 \\ & = \frac{LM^2}{8}(b-a)^7(2b-2a+1)\Big[\frac{\sqrt{L}(b-a)^2}{8}\Big]^{2n-2} \\ & \quad + 16L(b-a)(2b-2a+1)(t-a)\int_a^b \mathbf{E}|v_n(z) - v_{n-1}(z)|^2 dz \\ & = 8M^2(b-a)^3(2b-2a+1)\Big[\frac{\sqrt{L}(b-a)^2}{8}\Big]^{2n} \\ & \quad + 16L(b-a)(2b-2a+1)(t-a)\int_a^b \mathbf{E}|v_n(z) - v_{n-1}(z)|^2 dz. \end{aligned} \tag{3.23}$$

Now we introduce two positive constants to simplify the subsequent recursive estimate. Let A_1, A_2 be positive constants determined by

$$A_1 = 8M^2(b-a)^3(2b-2a+1), \;\; A_2 = 16L(b-a)(2b-2a+1). \tag{3.24}$$

Then the above estimate (3.23) can be rewritten as

$$\mathbf{E}|v_{n+1}(t) - v_n(t)|^2 \leq A_1 q^{2n} + A_2(t-a)\int_a^b \mathbf{E}|v_n(z) - v_{n-1}(z)|^2 dz, \tag{3.25}$$

for every $t \in [a, b]$. By applying the estimate (3.25), we proceed by mathematical induction. To illustrate the recursive pattern, we derive the corresponding estimates successively as follows.
We first consider the case $n = 1$:

$$\begin{aligned}
&\mathbf{E}|v_2(t) - v_1(t)|^2 \\
&\leq A_1 q^2 + A_2(t-a)\int_a^b \mathbf{E}|v_1(z) - v_0(z)|^2 dz \\
&\leq A_1 q^2 + 2KA_2(b-a)^4(t-a).
\end{aligned}$$

When $n = 2$, the estimate becomes

$$\begin{aligned}
&\mathbf{E}|v_3(t) - v_2(t)|^2 \\
&\leq A_1 q^4 + A_2(t-a)\int_a^b \mathbf{E}|v_2(z) - v_1(z)|^2 dz \\
&\leq A_1 q^4 + A_2(t-a)\int_a^b [A_1 q^2 + 2KA_2(b-a)^4(z-a)]dz \\
&= A_1 q^4 + A_2[A_1 q^2(b-a) + KA_2(b-a)^6](t-a).
\end{aligned}$$

When $n = 3$, we obtain

$$\begin{aligned}
&\mathbf{E}|v_4(t) - v_3(t)|^2 \\
&\leq A_1 q^6 + A_2(t-a)\int_a^b \mathbf{E}|v_3(z) - v_2(z)|^2 dz \\
&\leq A_1 q^6 + A_2(t-a)\int_a^b \{A_1 q^4 + A_2(z-a)[A_1 q^2(b-a) + KA_2(b-a)^6]\}dz \\
&= A_1 q^6 + A_2[A_1 q^4(b-a) + \frac{1}{2}KA_2^2(b-a)^8 + \frac{1}{2}A_1 A_2 q^2(b-a)^3](t-a).
\end{aligned}$$

When $n = 4$, it follows that

$$\begin{aligned}
&\mathbf{E}|v_5(t) - v_4(t)|^2 \\
&\leq A_1 q^8 + A_2(t-a)\int_a^b \mathbf{E}|v_4(z) - v_3(z)|^2 dz \\
&\leq A_1 q^8 + A_2(t-a)\int_a^b \{A_1 q^6 + A_2(z-a)[A_1 q^4(b-a) + \frac{1}{2}KA_2^2(b-a)^8 + \frac{1}{2}A_1 A_2 q^2(b-a)^3]\}dz \\
&= A_1 q^8 + A_2[A_1 q^6(b-a) + \frac{1}{2^2}KA_2^3(b-a)^{10} + \frac{1}{2}A_1 A_2 q^4(b-a)^3 + \frac{1}{2^2}A_1 A_2^2 q^2(b-a)^5](t-a).
\end{aligned}$$

Finally, when $n = 5$, we get

$$
\begin{aligned}
&\mathbf{E}|v_6(t) - v_5(t)|^2 \\
&\leq A_1 q^{10} + A_2(t-a)\int_a^b \mathbf{E}|v_5(z) - v_4(z)|^2 dz \\
&\leq A_1 q^{10} + A_2(t-a)\int_a^b \{A_1 q^8 + A_2[A_1 q^6(b-a) + \frac{1}{2^2}KA_2^3(b-a)^{10} \\
&\quad + \frac{1}{2}A_1A_2q^4(b-a)^3 + \frac{1}{2^2}A_1A_2^2q^2(b-a)^5](z-a)\}dz \\
&\leq A_1 q^{10} + A_2\{A_1 q^8(b-a) + \frac{1}{2^3}KA_2^4(b-a)^{12} + \frac{1}{2}A_1A_2q^6(b-a)^3 \\
&\quad + \frac{1}{2^2}A_1A_2^2q^4(b-a)^5 + \frac{1}{2^3}A_1A_2^3q^2(b-a)^7](t-a)\}.
\end{aligned}
$$

From the above iterative estimates for $n = 1, 2, \ldots, 5$, we identify the general recursive structure and state the following estimate. We now claim that for $n > 3$,

$$
\begin{aligned}
&\mathbf{E}|v_n(t) - v_{n-1}(t)|^2 \\
&\leq A_1 q^{2n-2} + A_2\Big[A_1 q^{2n-4}(b-a) + \frac{1}{2^{n-3}}KA_2^{n-2}(b-a)^{2n} \\
&\quad + \sum_{k=1}^{n-3}\frac{1}{2^k}A_1A_2^k q^{2n-2k-4}(b-a)^{2k+1}\Big](t-a).
\end{aligned} \tag{3.26}
$$

Assume (3.26) holds for a fixed integer $n > 3$. We next establish the corresponding estimate with n replaced by $n+1$, which gives (3.27):

$$
\begin{aligned}
&\mathbf{E}|v_{n+1}(t) - v_n(t)|^2 \\
&\leq A_1 q^{2n} + A_2(t-a)\int_a^b \mathbf{E}|v_n(z) - v_{n-1}(z)|^2 dz \\
&\leq A_1 q^{2n} + A_2\Big[A_1 q^{2n-2}(b-a) + \frac{1}{2^{n-2}}KA_2^{n-1}(b-a)^{2n+2} \\
&\quad + \frac{1}{2}A_1A_2q^{2n-4}(b-a)^3 + \sum_{k=1}^{n-3}\frac{1}{2^{k+1}}A_1A_2^{k+1}q^{2n-2k-4}(b-a)^{2k+3}(t-a)\Big] \\
&\leq A_1 q^{2n} + A_2\Big[A_1 q^{2n-2}(b-a) + \frac{1}{2^{n-2}}KA_2^{n-1}(b-a)^{2n+2} + A_1q^2(b-a)\Big(\frac{A_2(b-a)^2}{2} + q^2\Big)^{n-2}\Big](t-a).
\end{aligned} \tag{3.27}
$$

This matches the form of estimate (3.26) with n replaced by $n+1$.

Furthermore, according to inequality (3.16), we have

$$
\begin{aligned}
&\sup_{a\leq t\leq b}|v_{n+1}(t) - v_n(t)|^2 \\
&\leq 4(b-a)^2|\Delta w_n(a)|^2 + 4\left|\int_a^b\int_a^s\int_{|y|<c}\Delta F_n(z,y)\,\tilde{N}(dz,dy)\,ds\right|^2 \\
&\quad + 4\left|\int_a^b\int_a^s(\Phi_n(z) - \Phi_{n-1}(z))\,dz\,ds\right|^2 + 4\left|\int_a^b\int_a^s\Delta g_n(z)\,dB(z)\,ds\right|^2.
\end{aligned} \tag{3.28}
$$

Taking expectations on both sides of (3.28) and applying the conditional Jensen inequality gives

$$\begin{aligned}
&\mathbf{E}\left(\sup_{a\le t\le b}|v_{n+1}(t)-v_n(t)|^2\right) \qquad (3.29)\\
&\le 4(b-a)^2\mathbf{E}|\Delta w_n(a)|^2+4\mathbf{E}\left|\int_a^b\int_a^s\int_{|y|<c}\Delta F_n(z,y)\,\tilde{N}(dz,dy)\,ds\right|^2\\
&\quad+4\mathbf{E}\left|\int_a^b\int_a^s(\Phi_n(z)-\Phi_{n-1}(z))\,dz\,ds\right|^2+4\mathbf{E}\left|\int_a^b\int_a^s\Delta g_n(z)\,dB(z)\,ds\right|^2\\
&\le 8\mathbf{E}\left|\int_a^b\int_a^s(\Phi_n(z)-\Phi_{n-1}(z))\,dz\,ds\right|^2+4\mathbf{E}\left|\int_a^b\int_a^s\Delta g_n(z)\,dB(z)\,ds\right|^2\\
&\quad+4\mathbf{E}\left|\int_a^b\int_a^s\int_{|y|<c}\Delta F_n(z,y)\,\tilde{N}(dz,dy)\,ds\right|^2.
\end{aligned}$$

By Cauchy-Schwarz and Itô's isometry, we have

$$\begin{aligned}
&\mathbf{E}\left(\sup_{a\le t\le b}|v_{n+1}(t)-v_n(t)|^2\right) \qquad (3.30)\\
&\le 8(b-a)^2\int_a^b\int_a^s\mathbf{E}|\Phi_n(z)-\Phi_{n-1}(z)|^2\,dz\,ds+4(b-a)\int_a^b\int_a^s\mathbf{E}|\Delta g_n(z)|^2\,dz\,ds\\
&\quad+4(b-a)\int_a^b\int_a^s\int_{|y|<c}\mathbf{E}|\Delta F_n(z,y)|^2\nu(dy)\,dz\,ds.
\end{aligned}$$

Using $|x+y|^2\le 2|x|^2+2|y|^2$ and Lipschitz conditions (C1), we obtain

$$\begin{aligned}
&\mathbf{E}\left(\sup_{a\le t\le b}|v_{n+1}(t)-v_n(t)|^2\right) \qquad (3.31)\\
&\le 16L(b-a)^2\int_a^b\int_a^s[\mathbf{E}|V_n(z)+U_n(z)|^2+|U_n(z)|^2]\,dz\,ds\\
&\quad+4L(b-a)\int_a^b\int_a^s\mathbf{E}|V_n(z)+U_n(z)|^2\,dz\,ds+4L(b-a)\int_a^b\int_a^s\mathbf{E}|V_n(z)+U_n(z)|^2\,dz\,ds\\
&\le 48L(b-a)^2\int_a^b\int_a^s|U_n(z)|^2\,dz\,ds+32L(b-a)^2\int_a^b\int_a^s\mathbf{E}|V_n(z)|^2\,dz\,ds\\
&\quad+16L(b-a)\int_a^b\int_a^s|U_n(z)|^2\,dz\,ds+16L(b-a)\int_a^b\int_a^s\mathbf{E}|V_n(z)|^2\,dz\,ds.
\end{aligned}$$

Using the deterministic estimate (3.6)

$$|U_n(t)|^2=|u_n(t)-u_{n-1}(t)|^2\le\frac{M^2(b-a)^4}{64}q^{2n-2},$$

and

$$\begin{aligned}\mathbf{E}|V_n(t)|^2 &= \mathbf{E}|v_n(t) - v_{n-1}(t)|^2\\ &\le A_1q^{2n-2} + A_2\Big[A_1q^{2n-4}(b-a) + \frac{1}{2^{n-3}}KA_2^{n-2}(b-a)^{2n}\\ &\quad + A_1q^2(b-a)\Big(\frac{A_2(b-a)^2}{2} + q^2\Big)^{n-3}\Big](t-a),\end{aligned}$$

we deduce that

$$\begin{aligned}&\mathbf{E}\Big(\sup_{a\le t\le b}|v_{n+1}(t) - v_n(t)|^2\Big) \qquad (3.32)\\ &\le 16L(b-a)(3b-3a+1)\int_a^b\int_a^b \frac{M^2(b-a)^4}{64}q^{2n-2}dzds\\ &\quad + 16L(b-a)(2b-2a+1)\int_a^b\int_a^b \Big\{A_1q^{2n-2} + A_2\Big[A_1(b-a)q^{2n-4}\\ &\quad + \frac{1}{2^{n-3}}KA_2^{n-2}(b-a)^{2n} + A_1q^2(b-a)\Big(\frac{A_2(b-a)^2}{2} + q^2\Big)^{n-3}\Big](z-a)\Big\}dzds\\ &= \frac{1}{4}LM^2(b-a)^7(3b-3a+1)q^{2n-2} + 16L(b-a)(2b-2a+1)\Big\{A_1q^{2n-2}(b-a)^2\\ &\quad + \frac{1}{2}A_2\Big[A_1(b-a)q^{2n-4} + \frac{1}{2^{n-3}}KA_2^{n-2}(b-a)^{2n} + A_1q^2(b-a)\Big(\frac{A_2(b-a)^2}{2} + q^2\Big)^{n-3}\Big](b-a)^3\Big\}.\end{aligned}$$

We can conclude that $\{v_n(t)\}$ converges in $\mathcal{L}^2(\mathbf{P},\mathbb{R}^d)$ for each $t \in [a,b]$. Indeed, denote by $\|\cdot\|_2 = (\mathbf{E}|\cdot|^2)^{\frac{1}{2}}$ the standard $\mathcal{L}^2$-norm. For integers $n > m > 3$, we have

$$\begin{aligned}&\|v_n(t) - v_m(t)\|_2 \qquad (3.33)\\ &\le \sum_{k=m}^{n-1}\|v_{k+1}(t) - v_k(t)\|_2 = \sum_{k=m}^{n-1}\Big(\mathbf{E}|v_{k+1}(t) - v_k(t)|^2\Big)^{\frac{1}{2}}\\ &\le \sum_{k=m}^{n-1}\Big[\mathbf{E}\Big(\sup_{a\le t\le b}|v_{k+1}(t) - v_k(t)|^2\Big)\Big]^{\frac{1}{2}}\\ &\le \sum_{k=m}^{n-1}\Big\{\frac{\sqrt{L(b-a)(3b-3a+1)}}{2}M(b-a)^3q^{k-1} + 4\sqrt{L(b-a)(2b-2a+1)}\\ &\quad \Big[A_1q^{2k-2}(b-a)^2 + \frac{1}{2}A_1A_2(b-a)^4q^{2k-4} + \frac{1}{2^{k-2}}KA_2^{k-1}(b-a)^{2k+3}\\ &\quad + \frac{1}{2}A_1A_2q^2(b-a)^4\Big(\frac{A_2(b-a)^2}{2} + q^2\Big)^{k-3}\Big]^{\frac{1}{2}}\Big\}\\ &\le C_1M(b-a)^3\sum_{k=m}^{\infty}q^{k-1} + C_2\Big[\sqrt{A_1}(b-a)\sum_{k=m}^{\infty}q^{k-1} + \sqrt{\frac{A_1A_2}{2}}(b-a)^2\sum_{k=m}^{\infty}q^{k-2}\\ &\quad + 2\sqrt{\frac{K}{A_2}}(b-a)^{\frac{3}{2}}\sum_{k=m}^{\infty}\Big(\sqrt{\frac{A_2}{2}}(b-a)\Big)^k + C_3\sum_{k=m}^{\infty}\Big(\sqrt{\frac{A_2(b-a)^2}{2} + q^2}\Big)^k\Big]\Big\},\end{aligned}$$

where $C_1 = \frac{\sqrt{L(b-a)(3b-3a+1)}}{2}$, $C_2 = 4\sqrt{L(b-a)(2b-2a+1)}$, and $C_3 = \frac{2q\sqrt{A_1A_2}(b-a)^2}{[A_2(b-a)^2+2q^2]^{\frac{3}{2}}}$.

A sufficient condition for the series in (3.33) to converge is

$$\sqrt{\frac{A_2(b-a)^2}{2}+q^2}<1.$$

Substituting $A_2 = 16L(b-a)(2b-2a+1)$ yields the following restriction on the Lipschitz constant L:

$$8L(b-a)^3(2b-2a+1)+\frac{L(b-a)^4}{64}<1. \tag{3.34}$$

Under this condition, each term of the above infinite series becomes arbitrarily small when m is taken large enough. This implies

$$\|v_n(t)-v_m(t)\|_2\to 0,\quad \text{as } n,m\to\infty,$$

uniformly in $t\in[a,b]$. Hence $\{v_n(t)\}$ is Cauchy in $\mathcal{L}^2(\mathbf{P},\mathbb{R}^d)$, and its limit is denoted by $v(t)$.

Now we verify that $v(t)$ indeed satisfies the SDE (2.6) with mean boundary conditions (2.7). Set

$$\Phi(z):=f(z,v(z)+u(z))-f(z,u(z)).$$

Comparing the integral representations of $v_{n+1}(t)$ and $v(t)$ and using the conditional Jensen inequality, the Cauchy-Schwarz inequality, and Itô's isometry, we obtain

$$\begin{aligned}
&\mathbf{E}\Big|(t-a)(w(a)-w_n(a))\Big|^2+\mathbf{E}\Big|\int_a^t\int_a^s(\Phi(z)-\Phi_n(z))dzds\Big|^2\\
&+\mathbf{E}\Big|\int_a^t\int_a^s[g(z,v(z)+u(z))-g(z,v_n(z)+u_n(z))]dB(z)ds\Big|^2\\
&+\mathbf{E}\Big|\int_a^t\int_a^s\int_{|y|<c}[F(z,v(z)+u(z),y)-F(z,v_n(z)+u_n(z),y)]\tilde{N}(dz,dy)ds\Big|^2\\
&\le\frac{(t-a)^2}{(b-a)^2}\mathbf{E}\Big|\int_a^b\int_a^s(\Phi(z)-\Phi_n(z))dzds\Big|^2+\mathbf{E}\Big|\int_a^t\int_a^s(\Phi(z)-\Phi_n(z))dzds\Big|^2\\
&+\mathbf{E}\Big|\int_a^t\int_a^s[g(z,v(z)+u(z))-g(z,v_n(z)+u_n(z))]dB(z)ds\Big|^2\\
&+\mathbf{E}\Big|\int_a^t\int_a^s\int_{|y|<c}[F(z,v(z)+u(z),y)-F(z,v_n(z)+u_n(z),y)]\tilde{N}(dz,dy)ds\Big|^2\\
&\le\frac{(t-a)^2}{b-a}\int_a^b(s-a)\int_a^s\mathbf{E}\Big|\Phi(z)-\Phi_n(z)\Big|^2dzds+(t-a)\int_a^t(s-a)\int_a^s\mathbf{E}\Big|\Phi(z)-\Phi_n(z)\Big|^2dzds\\
&\quad+(t-a)\int_a^t\int_a^s\mathbf{E}|g(z,v(z)+u(z))-g(z,v_n(z)+u_n(z))|^2dzds\\
&\quad+(t-a)\int_a^t\int_a^s\int_{|y|<c}\mathbf{E}|F(z,v(z)+u(z),y)-F(z,v_n(z)+u_n(z),y)|^2\nu(dy)dzds\\
&\le 2(b-a)^2\int_a^b\int_a^s\mathbf{E}\Big|\Phi(z)-\Phi_n(z)\Big|^2dzds\\
&\quad+(b-a)\int_a^b\int_a^s\mathbf{E}|g(z,v(z)+u(z))-g(z,v_n(z)+u_n(z))|^2dzds\\
&\quad+(b-a)\int_a^b\int_a^s\int_{|y|<c}\mathbf{E}|F(z,v(z)+u(z),y)-F(z,v_n(z)+u_n(z),y)|^2\nu(dy)dzds,
\end{aligned}$$

where

$$\begin{aligned}
&\mathbf{E}\Big|\Phi(z)-\Phi_n(z)\Big|^2\\
&=\mathbf{E}|f(z,v(z)+u(z))-f(z,u(z))-f(z,v_n(z)+u_n(z))+f(z,u_n(z))|^2\\
&\le 2\mathbf{E}|f(z,v(z)+u(z))-f(z,v_n(z)+u_n(z))|^2+2\mathbf{E}|f(z,u(z))-f(z,u_n(z))|^2.
\end{aligned}$$

According to the Lipschitz condition (C1), we have

$$\begin{aligned}
&\mathbf{E}\Big|(t-a)(w(a)-w_n(a))\Big|^2+\mathbf{E}\Big|\int_a^t\int_a^s(\Phi(z)-\Phi_n(z))dzds\Big|^2\\
&+\mathbf{E}\Big|\int_a^t\int_a^s[g(z,v(z)+u(z))-g(z,v_n(z)+u_n(z))]dB(z)ds\Big|^2\\
&+\mathbf{E}\Big|\int_a^t\int_a^s\int_{|y|<c}[F(z,v(z)+u(z),y)-F(z,v_n(z)+u_n(z),y)]\tilde{N}(dz,dy)ds\Big|^2\\
&\le 4L(b-a)^2\int_a^b\int_a^s[\mathbf{E}|v(z)-v_n(z)+u(z)-u_n(z)|^2+|u(z)-u_n(z)|^2]dzds\\
&\quad+2L(b-a)\int_a^b\int_a^s\mathbf{E}|v(z)-v_n(z)+u(z)-u_n(z)|^2dzds\\
&\le 4L(b-a)(3b-3a+1)\int_a^b\int_a^s|u(z)-u_n(z)|^2dzds\\
&\quad+4L(b-a)(2b-2a+1)\int_a^b\int_a^s\mathbf{E}|v(z)-v_n(z)|^2dzds\ \to 0,\ as\ n\to\infty.
\end{aligned}$$

Letting $n\to\infty$ in (3.14), we see that $v(t)$ satisfies (3.13). It follows that $v(t)$ solves the SDE (2.6) with mean boundary conditions (2.7).

Step 3. We consider the second-order SBVP (1.9).
Starting with $X_0(t)\equiv 0$, define

$$X_n(t):=u_n(t)+v_n(t),\qquad n\ge 1.$$

For $\forall\ m,n\in\mathbb{N}_+$ satisfying $m>n$ and $t\in[a,b]$, the following estimate holds:

$$\begin{aligned}
&\|X_m(t)-X_n(t)\|_2\\
&\le|u_m(t)-u_n(t)|+\|v_m(t)-v_n(t)\|_2\\
&\le\sum_{k=n}^{\infty}|u_{k+1}(t)-u_k(t)|+\|v_m(t)-v_n(t)\|_2\to 0,\ \ as\ \ n,m\to\infty.
\end{aligned}$$

This implies that $\{X_n,\ n\in\mathbb{N}\}$ is a Cauchy sequence in $\mathcal{L}^2(\mathbb{P},\mathbb{R}^d)$, and hence it admits a limit $X(t)\in\mathcal{L}^2(\mathbb{P},\mathbb{R}^d)$. Consequently,

$$X(t):=\lim_{n\to\infty}X_n(t)=u(t)+v(t)$$

solves the second-order SBVP (1.9), and admits the representation

$$X(t) = \int_a^b G(t,s)f(s,u(s))ds + (t-a)w(a) + \int_a^t \int_a^s \int_{|y|<c} F(z, v(z)+u(z), y)\tilde{N}(dz,dy)ds \tag{3.35}$$
$$+ \int_a^t \int_a^s [f(z, v(z)+u(z)) - f(z,u(z))]dzds + \int_a^t \int_a^s g(z, v(z)+u(z))dB(z)ds.$$

□

Remark 3.2. *Equation (3.11) admits infinitely many families of random variables $w(a)$. To simplify the construction of solutions, we focus on a special case with the conditional expectation form (3.12) of $w(a)$ to carry out the existence analysis. This $\mathcal{F}_a$-measurable selection provides a specific solution for the second-order SBVP (2.6) with mean boundary conditions (2.7).*

Remark 3.3. *Theorem 3.1 only guarantees existence of an $\mathcal{L}^2$-solution for the second-order stochastic differential equation with Lévy noise and mean boundary conditions. The mean boundary condition restricts the statistical expectation at terminal time instead of each sample path, which may allow multiple distinct $\mathcal{L}^2$-solutions. The hypotheses employed in our argument are adequate for proving existence, yet they are insufficient for pathwise or mean-square uniqueness. A rigorous uniqueness result would demand stronger hypotheses and different techniques, which are not explored in this paper.*

4. Adaptedness and càdlàg property of solutions

Theorem 4.1. *Assume that conditions (C1) and (C2) hold. If the Lipschitz constant L satisfies*

$$8L(b-a)^3(2b-2a+1) + \frac{L(b-a)^4}{64} < \frac{1}{4}, \tag{4.1}$$

then the solution (3.35) to the second-order SBVP (1.9) is adapted and càdlàg.

Notice that condition (4.1) in Theorem 4.1 imposes a stricter constraint on L than condition (3.1) in Theorem 3.1. Condition (3.1) suffices to guarantee $\mathcal{L}^2$-convergence of Picard iterates and the existence of an $\mathcal{L}^2$-solution. However, to establish almost sure uniform convergence via the Borel-Cantelli lemma for the adaptedness and càdlàg property of the solution, we require the stricter bound (4.1), which ensures summability of all geometric series appearing in the subsequent estimates.

Proof. We first prove that $\{X_n(t)\}_{n\in\mathbb{N}}$ converges to $X(t)$ almost surely and uniformly over $t \in [a,b]$. Indeed,

$$\begin{aligned}
&\mathbf{E}\Big(\sup_{a\le t\le b}|X_{n+1}(t)-X_n(t)|^2\Big)\\
&\le 2\mathbf{E}\Big(\sup_{a\le t\le b}|v_{n+1}(t)-v_n(t)|^2\Big) + 2\sup_{a\le t\le b}|u_{n+1}(t)-u_n(t)|^2\\
&\le \frac{1}{32}M^2(b-a)^4q^{2n} + \frac{LM^2}{2}(b-a)^7(3b-3a+1)q^{2n-2}\\
&\quad + 32L(b-a)(2b-2a+1)\Big[A_1(b-a)^2q^{2n-2} + \frac{1}{2}A_1A_2(b-a)^4q^{2n-4}\\
&\quad + \frac{1}{2^{n-2}}KA_2^{n-1}(b-a)^{2n+3} + \frac{1}{2}A_1A_2q^2(b-a)^4\Big(\frac{1}{2}A_2(b-a)^2+q^2\Big)^{n-3}\Big].
\end{aligned}$$

Applying the Chebyshev–Markov inequality (2.2), we obtain

$$\begin{aligned}
&\mathbf{P}\Big(\sup_{a\le t\le b}|X_{n+1}(t)-X_n(t)|\ge\frac{1}{2^n}\Big)\\
&\le 4^n\mathbf{E}\Big(\sup_{a\le t\le b}|X_{n+1}(t)-X_n(t)|^2\Big)\\
&\le \frac{1}{32}M^2(b-a)^4(4q^2)^n+\frac{LM^2}{2q^2}(b-a)^7(3b-3a+1)(4q^2)^n\\
&\quad+32L(b-a)(2b-2a+1)\Big\{\frac{A_1}{q^2}(b-a)^2(4q^2)^n+\frac{A_1A_2}{2q^4}(b-a)^4(4q^2)^n\\
&\quad+\frac{4K}{A_2}(b-a)^3\Big[4\cdot\frac{A_2(b-a)^2}{2}\Big]^n+\frac{4A_1A_2q^2(b-a)^4}{[A_2(b-a)^2+2q^2]^3}\cdot\Big[4\Big(\frac{A_2(b-a)^2}{2}+q^2\Big)\Big]^n\Big\}.
\end{aligned}$$

Since the Lipschitz constant L satisfies

$$8L(2b-2a+1)(b-a)^3+\frac{L}{64}(b-a)^4<\frac{1}{4},$$

it follows that

$$4\Big[\frac{A_2(b-a)^2}{2}+q^2\Big]<1,$$

which implies that $4q^2<1$ and $2A_2(b-a)^2<1$. Hence the upper estimate decays geometrically with respect to n, and the sum over n of these probability bounds converges. Applying Lemma **2.11**, we have

$$\mathbf{P}\Big(\limsup_{n\to\infty}\sup_{a\le t\le b}|X_{n+1}(t)-X_n(t)|\ge\frac{1}{2^n}\Big)=0.$$

Thus, the sequence $\{X_n(t)\}_{n\in\mathbb{N}}$ converges almost surely uniformly on the finite interval $[a,b]$ to $X(t)$.

Each Picard iterate $X_n(t)$ is adapted and càdlàg. Almost sure uniform convergence implies pointwise almost sure convergence for every fixed $t\in[a,b]$. The pointwise almost sure limit of adapted processes remains adapted. Moreover, the almost sure uniform limit of càdlàg sample paths on a compact time interval admits a càdlàg modification. Therefore, the limiting process $X(t)$ is adapted and possesses a càdlàg modification. □

5. Examples

This section supplies concrete examples to demonstrate the theoretical conclusions derived above.

Example 5.1. Consider a scalar second-order SDE on $[0,1]$ with small-jump noise and mean boundary conditions:

$$\begin{aligned}
&\ddot{X}(t)=\int_{|y|<1}\lambda X(t)y\dot{\tilde{N}}(t,dy),\\
&X(0)=0,\quad \mathbf{E}X(1)=0,
\end{aligned}\tag{5.1}$$

where λ represents a constant parameter defined below, and $\tilde{N}$ corresponds to the compensated Poisson random measure associated with the Lévy measure $\nu(\cdot)$. We set

$$\nu(dy)=\frac{1}{2|y|^2}dy.$$

We now verify the conditions of Theorem 3.1 for this example. Here the jump coefficient is $F(t, X, y) = \lambda Xy$.

For condition (C1),

$$\begin{aligned}&\int_{|y|<1} |F(t, X_1, y) - F(t, X_2, y)|^2 \nu(dy)\\ =&\int_{|y|<1} \lambda^2 |X_1 - X_2|^2 y^2 \cdot \frac{1}{2|y|^2} dy\\ =&\lambda^2 |X_1 - X_2|^2.\end{aligned}$$

Since $f \equiv 0$ and $g \equiv 0$,

$$\begin{aligned}|f(t, X_1) - f(t, X_2)|^2 = 0,\\ |g(t, X_1) - g(t, X_2)|^2 = 0.\end{aligned}$$

Thus, condition (C1) holds with Lipschitz constant $L = \lambda^2$.

For condition (C2),

$$\int_{|y|<1} |F(t, X, y)|^2 \nu(dy) = \int_{|y|<1} \lambda^2 |X|^2 |y|^2 \cdot \frac{1}{2|y|^2} dy = \lambda^2 |X|^2.$$

Then

$$|f(t, X)|^2 \vee |g(t, X)|^2 \vee \int_{|y|<1} |F(t, X, y)|^2 \nu(dy) = \lambda^2 |X|^2 \leq \lambda^2 (1 + |X|^2).$$

Hence linear growth (C2) holds with $K = \lambda^2$.

For condition (3.1) of Theorem 3.1, $[a, b] = [0, 1]$, so $b - a = 1$. Substitute $L = \lambda^2$:

$$\begin{aligned}&8L(b-a)^3(2b - 2a + 1) + \frac{L(b-a)^4}{64}\\ =&8\lambda^2 \cdot 1 \cdot (2 + 1) + \frac{\lambda^2}{64}\\ =&\frac{1537}{64}\lambda^2.\end{aligned}$$

Condition (3.1) requires $\frac{1537}{64}\lambda^2 < 1$, i.e., $\lambda < \sqrt{\frac{64}{1537}} \approx 0.204$. When $\lambda \in (0, 1/5)$, this inequality is satisfied, so (3.1) holds. Equation (5.1) satisfies the conditions of Theorem 3.1.

We consider the equivalent first-order system for the stochastic differential Eq (5.1)

$$\begin{aligned}\dot{X}(t) = Y(t),\\ \dot{Y}(t) = \int_{|y|<1} \lambda X(t) y \dot{\tilde{N}}(t, dy).\end{aligned}$$

Integration gives

$$X(t) = \int_0^t Y(s) ds,$$

and

$$Y(t) = Y(0) + \int_0^t \int_{|y|<1} \lambda X(s) y \tilde{N}(ds, dy).$$

Then

$$\begin{aligned} X(t) &= \int_0^t Y(s) ds \\ &= Y(0)t + \int_0^t \int_0^s \int_{|y|<1} \lambda X(u) y \tilde{N}(du, dy) ds. \end{aligned}$$

From the mean boundary constraint $\mathbf{E}X(1) = 0$, we obtain

$$0 = \mathbf{E}X(1) = \mathbf{E}Y(0) + \mathbf{E}\Big(\int_0^1 \int_0^s \int_{|y|<1} \lambda X(u) y \tilde{N}(du, dy) ds \Big). \tag{5.2}$$

Then we choose

$$Y(0) = \mathbf{E}\left[- \int_0^1 \int_0^s \int_{|y|<1} \lambda X(u) y \, \tilde{N}(du, dy) \, ds \,\middle|\, \mathcal{F}_0 \right] = 0,$$

which is $\mathcal{F}_0$-measurable and certainly satisfies (5.2). Consequently, the integral representation becomes

$$X(t) = \lambda \int_0^t \int_0^s \int_{|y|<1} X(u) y \tilde{N}(du, dy) ds. \tag{5.3}$$

As $f \equiv g \equiv 0$ and $F(t, 0, y) \equiv 0$, the trivial process $X(t) \equiv 0$ solves (5.1).

For Theorem 4.1, substituting $L = \lambda^2$ into condition (4.1), we obtain

$$8L(b-a)^3(2b - 2a + 1) + \frac{L(b-a)^4}{64} < \frac{1}{4}.$$

We obtain

$$\frac{1537}{64}\lambda^2 < \frac{1}{4} \implies \lambda^2 < \frac{16}{1537} \implies \lambda < \sqrt{\frac{16}{1537}} \approx 0.102.$$

Therefore, the admissible range $\lambda \in (0, 1/10)$ satisfies inequality (4.1). Theorem 4.1 guarantees that the trivial solution is $\mathcal{F}_t$-adapted and càdlàg.

Example 5.1 corresponds to the trivial case discussed in the proof of Theorem 3.1.

Remark 5.1. *Equation (5.1) describes the random vibration of nanoparticles subjected to small-jump-size* ($|y| < 1$) *random collisions. The particles start from rest, and their mean position returns to the equilibrium state after a finite time interval. The second-order stochastic boundary value system (5.1) can describe the dynamic characteristics of microscopic particles in a thermal environment.*

Example 5.2. Consider the following one-dimensional second-order SDE:

$$\ddot{X}(t) + \lambda X(t) = \lambda \dot{W}(t) + \int_{|y|<2} \lambda X(t) \dot{\tilde{N}}(t, dy), \quad t \in [0, 1], \tag{5.4}$$

subject to the boundary constraints $X(0) = 0$ and $\mathbf{E}X(1) = 0$. Here λ is a constant, $W(t)$ represents a one-dimensional Wiener process, and $\tilde{N}$ stands for a compensated Poisson random measure. Choose the Lévy measure $\nu(\cdot)$ given by

$$\nu(dy) = e^{-|y|}dy.$$

Let

$$f(t, X) = -\lambda X, \quad g(t, X) = \lambda, \quad F(t, X, y) = \lambda X.$$

For condition (C1),

$$|f(t, X_1) - f(t, X_2)|^2 = \lambda^2 |X_1 - X_2|^2,$$

and

$$|g(t, X_1) - g(t, X_2)|^2 = 0.$$

Furthermore,

$$\int_{|y|<2} |F(t, X_1, y) - F(t, X_2, y)|^2 \nu(dy) = \lambda^2 |X_1 - X_2|^2 \int_{|y|<2} e^{-|y|}dy < 2\lambda^2 |X_1 - X_2|^2.$$

Combining these estimates gives

$$|f(t, X_1) - f(t, X_2)|^2 \vee |g(t, X_1) - g(t, X_2)|^2 \vee \int_{|y|<2} |F(t, X_1, y) - F(t, X_2, y)|^2 \nu(dy) < 2\lambda^2 |X_1 - X_2|^2.$$

Hence condition (C1) holds with Lipschitz constant $L = 2\lambda^2$.

For condition (C2),

$$\int_{|y|<2} |F(t, X, y)|^2 \nu(dy) = \int_{|y|<2} \lambda^2 |X|^2 e^{-|y|}dy \leq 2\lambda^2 (1 + |X|^2).$$

This confirms that condition (C2) is satisfied by taking $K = 2\lambda^2$.

For condition (3.1), substituting $L = 2\lambda^2$, we have

$$8L(b-a)^3(2b - 2a + 1) + \frac{L(b-a)^4}{64} = L\left(8 \cdot 1 \cdot 3 + \frac{1}{64}\right) = \frac{1537}{64}L.$$

Substitute $L = 2\lambda^2$:

$$\frac{1537}{64} \cdot 2\lambda^2 = \frac{1537}{32}\lambda^2.$$

Condition (3.1) requires

$$\frac{1537}{32}\lambda^2 < 1.$$

When $\lambda \in (0, 1/10)$, substituting $\lambda = 1/10$ gives $\frac{1537}{32}\lambda^2 = \frac{1537}{3200} < 1$. Hence, the conditions of Theorem 3.1 are satisfied.

We rewrite the second-order stochastic system (5.4) as a first-order system

$$\begin{aligned} \dot{X}(t) &= Y(t), \\ \dot{Y}(t) + \lambda X(t) &= \lambda \dot{W}(t) + \int_{|y|<2} \lambda X(t) \dot{\tilde{N}}(t, dy). \end{aligned} \tag{5.5}$$

Upon integrating system (5.5) over $[0, t]$, we obtain

$$X(t) = X(0) + \int_0^t Y(s)ds,$$
$$Y(t) = Y(0) - \int_0^t \lambda X(s)ds + \lambda W(t) + \int_0^t \int_{|y|<2} \lambda X(s)\tilde{N}(ds, dy).$$

Then

$$X(t) = Y(0)t - \int_0^t \int_0^s \lambda X(u)duds + \int_0^t \lambda W(s)ds + \int_0^t \int_0^s \int_{|y|<2} \lambda X(u)\tilde{N}(du, dy)ds.$$

Taking the expectation and applying the condition $\mathbf{E}X(1) = 0$, we get

$$\mathbf{E}X(1) = \mathbf{E}Y(0) - \mathbf{E}\int_0^1 \int_0^s \lambda X(u)duds = 0. \tag{5.6}$$

Choose

$$Y(0) = \lambda \mathbf{E}\Big[\int_0^1 \int_0^s X(u)duds\Big|\mathcal{F}_0\Big].$$

It follows that

$$Y(0) = 0,$$

and the explicit solution is as follows:

$$X(t) = -\lambda \int_0^t \int_0^s X(u)duds + \lambda \int_0^t W(s)ds + \lambda \int_0^t \int_0^s \int_{|y|<2} X(u)\tilde{N}(du, dy)ds. \tag{5.7}$$

For Theorem 4.1, substituting $L = 2\lambda^2$ into condition (4.1), we have

$$8L(b-a)^3(2b-2a+1) + \frac{L(b-a)^4}{64} < \frac{1}{4}.$$

We obtain

$$\frac{1537}{32}\lambda^2 < \frac{1}{4} \implies \lambda^2 < \frac{8}{1537}, \quad \lambda < \sqrt{\frac{8}{1537}} \approx 0.0722.$$

Therefore, the admissible range $\lambda \in (0, 7/100)$ satisfies inequality (4.1). Theorem 4.1 implies that the solution (5.7) is adapted and càdlàg.

Remark 5.2. *This model describes the random vibration behavior of microscopic particles constrained by linear restoring forces, under the influence of continuous Brownian noise and small-jump-size collisions. The stochastic system also captures the dynamical behavior of nanoparticles, colloidal particles, etc., in a thermal environment.*

Example 5.3. Consider a one-dimensional second-order stochastic boundary value problem on $[0, 1]$:

$$\ddot{X}(t) + \lambda X(t) = \sin \pi t + \lambda \dot{W}(t) + \int_{|y|<1} \lambda X(t) y \dot{\tilde{N}}(t, dy), \tag{5.8}$$
$$X(0) = 0, \quad \mathbf{E}X(1) = 0,$$

where λ denotes a constant parameter, $W(t)$ is a Wiener process, and $\tilde{N}$ represents a compensated Lévy random measure whose intensity measure ν is defined on $\{|y| < 1\}$. We assume that

$$\int_{|y|<1} \nu(dy) = \infty, \qquad \int_{|y|<1} |y|^2 \nu(dy) = 1.$$

We now verify the conditions of Theorem 3.1.

For condition (C1),

$$| - \lambda X_1 + \sin \pi t + \lambda X_2 - \sin \pi t|^2 = \lambda^2 |X_1 - X_2|^2.$$

For the jump term,

$$\begin{aligned}\int_{|y|<1} |\lambda X_1 y - \lambda X_2 y|^2 \nu(dy) &= \lambda^2 |X_1 - X_2|^2 \int_{|y|<1} |y|^2 \nu(dy) \\ &= \lambda^2 |X_1 - X_2|^2.\end{aligned}$$

Taking the maximum of the three terms, condition (C1) holds with Lipschitz constant $L = \lambda^2$.

For condition (C2),

$$|\sin \pi t - \lambda X|^2 \leq 2 + 2\lambda^2 |X|^2,$$

and

$$\int_{|y|<1} |F(t, X, y)|^2 \nu(dy) = \int_{|y|<1} |\lambda X y|^2 \nu(dy) = \lambda^2 |X|^2.$$

Hence condition (C2) holds with constant $K = 2\lambda^2$.

For condition (3.1), substituting $L = \lambda^2$, we have

$$8L(b-a)^3(2b - 2a + 1) + \frac{L(b-a)^4}{64} = L\left(8 \cdot 1 \cdot 3 + \frac{1}{64}\right) = \frac{1537}{64}\lambda^2.$$

Condition (3.1) requires

$$\frac{1537}{64}\lambda^2 < 1.$$

This holds for $\lambda \in (0, 1/5)$. Hence, all conditions of Theorem 3.1 are satisfied.

Consider the first-order stochastic system equivalent to (5.8):

$$\begin{aligned}\dot{X}(t) &= Y(t), \\ \dot{Y}(t) + \lambda X(t) &= \sin \pi t + \lambda \dot{W}(t) + \int_{|y|<1} \lambda X(t) y \dot{\tilde{N}}(t, dy).\end{aligned} \tag{5.9}$$

Integrating both equations in system (5.9) over the interval $[0, t]$, we obtain

$$X(t) = \int_0^t Y(s) ds,$$

and

$$Y(t) = Y(0) - \lambda \int_0^t X(s) ds + \frac{1}{\pi} - \frac{1}{\pi} \cos \pi t + \lambda W(t) + \lambda \int_0^t \int_{|y|<1} X(s) y \tilde{N}(ds, dy).$$

Then

$$X(t) = Y(0)t - \lambda \int_0^t \int_0^s X(u)duds + \frac{t}{\pi} - \frac{1}{\pi^2}\sin \pi t + \lambda \int_0^t W(s)ds + \lambda \int_0^t \int_0^s \int_{|y|<1} X(u)y\tilde{N}(du, dy)ds.$$

From the boundary condition $\mathbf{E}X(1) = 0$, we have

$$\mathbf{E}X(1) = \mathbf{E}Y(0) - \lambda \int_0^1 \int_0^s \mathbf{E}X(u)duds + \frac{1}{\pi} = 0. \tag{5.10}$$

Choose

$$Y(0) = \lambda \mathbf{E}\Big[\int_0^1 \int_0^s \mathbf{E}X(u)duds\Big|\mathcal{F}_0\Big] - \frac{1}{\pi}. \tag{5.11}$$

Then

$$Y(0) = -\frac{\pi}{\pi^2 - \lambda},$$

which satisfies (5.10), and $Y(0)$ is $\mathcal{F}_0$-measurable. We obtain a solution

$$\begin{aligned} X(t) = &- \frac{\pi t}{\pi^2 - \lambda} - \lambda \int_0^t \int_0^s X(u)duds + \frac{t}{\pi} - \frac{1}{\pi^2}\sin \pi t \\ &+ \lambda \int_0^t W(u)du + \lambda \int_0^t \int_0^s \int_{|y|<1} X(u)y\tilde{N}(du, dy)ds. \end{aligned} \tag{5.12}$$

Hence, a solution (5.12) for the second-order stochastic boundary value problem (5.8) is obtained.

For Theorem 4.1, substituting $L = \lambda^2$ into condition (4.1), then

$$8L(b-a)^3(2b-2a+1) + \frac{L(b-a)^4}{64} < \frac{1}{4}.$$

We obtain

$$\frac{1537}{64}\lambda^2 < \frac{1}{4} \implies \lambda^2 < \frac{16}{1537} \implies \lambda < \sqrt{\frac{16}{1537}} \approx 0.102.$$

Therefore, the admissible range $\lambda \in (0, 1/10)$ satisfies inequality (4.1) and the solution given in (5.12) is adapted and càdlàg.

Remark 5.3. *Equation (5.8) with mean boundary conditions describes the oscillations of a weakly damped elastic system under the action of a periodic external force and random impulsive disturbances. The mean boundary conditions characterize a type of statistical steady-state behavior: The system will not drift far from equilibrium after long-term oscillation, and its average position returns to the equilibrium state.*

6. Conclusions

In this paper, we studied second-order stochastic boundary value problems driven by Brownian motion and small-jump Lévy noise with mean boundary conditions. By combining a deterministic Picard iteration for the associated deterministic boundary value problem with a stochastic Picard iteration for the associated stochastic boundary value problem, we established sufficient conditions

for the existence of $\mathcal{L}^2(\mathbf{P}, \mathbb{R}^d)$-valued solutions. Under a contraction-type smallness condition on the Lipschitz constant, we further proved that the constructed solution is adapted to the filtration $\{\mathcal{F}_t\}_{a\le t\le b}$ and possesses the càdlàg property. Several concrete examples illustrated the applicability of the theoretical results.

Our analysis is restricted to small-jumps ($|y| < c$) and therefore does not cover general Lévy processes with large-jumps. The paper focuses on existence rather than uniqueness. A rigorous uniqueness theory would require stronger assumptions and different techniques, and is left for future work.

Author contributions

Yingjia Guo: Conceptualization, Formal analysis, Writing-original draft, Writing-editing; Xiaomeng Jiang: Methodology, Formal analysis. Both authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors have used Artificial Intelligence (AI) tools to check the English text and make sure the formulas are correct.

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Conflict of interest

All authors declare no financial or proprietary interests in any material discussed in this article.

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