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Research article

Measure of noncompactness for hybrid fractional integro-differential equations in Banach algebra

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Abstract: In this study, we explored a new class of boundary value problems (BVPs) characterized by nonlinear hybrid fractional integro-differential equations (NHFI-DEs). These equations involved two distinct types of Caputo fractional derivatives (CapDs) and a Riemann-Liouville integral operator (R-LIO). This formulation effectively captured both hybrid dynamics and memory effects, making it applicable to a variety of real-world systems exhibiting nonlocal behavior. To demonstrate the existence of solutions, we utilized the theory of measures of noncompactness (MNC) within the framework of Banach spaces (BanSpa.) and applied a generalized Darbo fixed point theorem (DFPT). This approach established the existence of solutions by means of an MNC framework, without requiring the associated operator to satisfy the classical compactness condition, and provided an existence result for the considered class of hybrid fractional integro-differential boundary value problems (HFI-DBVP) under the stated assumptions. Additionally, an illustrative example was presented to demonstrate how the main theoretical result can be applied to a specific HFI-DBVP.

Keywords: hybrid fractional differential equations; integro-differential equations; generalized Darbo's fixed point; boundary value problem
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1. Introduction

Fractional calculus (FrCal) is a crucial mathematical tool for simulating complicated dynamic systems because it expands the traditional ideas of differentiation and integration to non-integer (arbitrary) orders. This method works particularly well for capturing memory and genetic traits, which are frequently found in a variety of engineering and natural events. With applications in a variety of fields, including viscoelastic materials, anomalous diffusion, fluid mechanics, control theory, and signal processing, the field has greatly developed since its inception thanks to the seminal works of Leibniz, Liouville, and Riemann (see, e.g., [1–3]). The CapD is widely utilized among the various definitions of fractional derivatives (FDs) because it fits in nicely with the traditional initial value problems that are regularly found in engineering and science (see, e.g., [4–6]).

Fractional integro-differential equations have become an important class of mathematical models for processes that show both local and nonlocal dependeny, (see, for example, [7]). These formulas make it possible to represent systems where changes in the current state are caused by both immediate dynamics and the cumulative effects of earlier states. This characteristic makes them extremely helpful in describing phenomena in fields like heat transfer, population biology, and materials science; see, e.g., [8–11]. Fractional operators (FOs) enhance these models by include nonlocal memory effects, which are often necessary for accurate physical representations.

By adding impulsive effects and switching mechanisms, hybrid differential equations—which blend differential dynamics with discrete or algebraic components—improve modeling. This makes it possible to depict systems like electrical circuits and biological processes that have both continuous evolution and discrete events more accurately; see, e.g., [12–14].

Combining FrCal, integro-differential structures, and hybrid systems offers a robust framework that captures memory effects and nonlinear behaviors. However, research on BVPs involving NHFI-DEs with multiple FOs is still limited; see, e.g., [15–17].

Bai in [18] considered the existence and uniqueness of a positive solution for the following problem:

$$D^{\alpha}_{0^+}\Psi(t) + f(t, \Psi(t)) = 0, \quad 0 < t < 1,$$

$$\Psi(0) = 0, \quad \beta\Psi(\eta) = \Psi(1),$$

where $1 < \alpha \leq 2$, $0 < \eta < 1$, $0 < \beta\eta^{\alpha-1} < 1$, $D^{\alpha}_{0^+}$ is the standard Riemann-Liouville fractional derivative (R–LFD), and f is continuous on $[0, 1] \times [0, \infty)$.

The following fractional hybrid problem:

$${}^{C}D^{p}_{0}\left(\frac{\Psi(t)}{h(t, \Psi(t))}\right) + \Phi(t, \Psi(t)) = 0, \quad t \in [0, 1],$$

subject to the hybrid boundary conditions (BCs):

$$\begin{cases} \lambda\, {}^{C}D^{p-1}_{0}\left(\frac{\Psi(t)}{h(t,\Psi(t))}\right)\Big|_{t=1} + \left(\frac{\Psi(t)}{h(t,\Psi(t))}\right)\Big|_{t=\eta} = 0, \\ D\left(\frac{\Psi(t)}{h(t,\Psi(t))}\right)\Big|_{t=0} = 0, \end{cases}$$

has been examined by Baleanu et al. in 2020. The operator ${}^{C}D^{\gamma}_{0}$ denotes the CapD of order $\gamma \in \{p, p-1\}$, and $\Phi : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $h \in C([0, 1] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ is a given nonlinear term, $p \in (1, 2]$, $p - 1 \in (0, 1]$, $\eta \in [0, 1]$, $D = \dfrac{d}{dt}$, and $\lambda > 0$ is a real parameter.

In addition, they explored the fractional hybrid differential inclusion associated with:

$$ {}^{C}D_0^{p}\left(\frac{\Psi(t)}{h(t,\Psi(t))}\right) \in G(t,\Psi(t)), \quad t \in [0,1], $$

subject to the three-point hybrid BCs:

$$ \begin{cases} D\left(\frac{\Psi(t)}{h(t,\Psi(t))}\right)\Big|_{t=0} = 0, \\ \lambda\, {}^{C}D_0^{p-1}\left(\frac{\Psi(t)}{h(t,\Psi(t))}\right)\Big|_{t=1} + \left(\frac{\Psi(t)}{h(t,\Psi(t))}\right)\Big|_{t=\eta} = 0, \end{cases} \tag{1.1} $$

where $G : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set-valued (multivalued) map; see [19].

Inspired by these developments, this article examines a novel class of BVPs for NHFI-DEs, formulated as follows:

$$ \begin{cases} {}^{C}\mathfrak{D}^{\theta}\left({}^{C}\mathfrak{D}^{\vartheta}\left[\frac{\chi(\iota)}{\mathfrak{F}(\iota,\chi(\iota))}\right]\right) = \mathfrak{H}\left(\iota,\chi(\iota),\int_0^{t}\mathcal{K}(\iota,\sigma)\chi(\sigma)\,d\sigma\right),\ \iota \in \Lambda. \\ {}^{C}\mathfrak{D}^{\vartheta}\left[\frac{\chi(\iota)}{\mathfrak{F}(\iota,\chi(\iota))}\right]_{\iota=0} = \left[\frac{\chi(\iota)}{\mathfrak{F}(\iota,\chi(\iota))}\right]_{\iota=0} = 0, \\ \left[\frac{\chi(\iota)}{\mathfrak{F}(\iota,\chi(\iota))}\right]_{\iota=1} = \left[\frac{\chi(\iota)}{\mathfrak{F}(\iota,\chi(\iota))}\right]_{\iota=\mu},\ 0<\mu<1, \end{cases} \tag{1.2} $$

where $\Lambda = [0,1]$ and both ${}^{C}\mathfrak{D}^{\theta}$ and ${}^{C}\mathfrak{D}^{\vartheta}$ are CapDs of orders $0 < \theta \leq 1$ and $1 < \vartheta \leq 2$, respectively. The function $\mathfrak{F} : \Lambda \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$, and $\mathfrak{H} : \Lambda \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are known. The Volterra operator is defined by $\mathfrak{K}\chi(\iota) := \int_0^{t}\mathcal{K}(\iota,\sigma)\chi(\sigma)\,d\sigma$, where $\mathcal{K} \in C(S,\mathbb{R})$, with $S = \{(\iota,\sigma) : 0 < \sigma < \iota < 1\}$. The linear operator $\mathfrak{K}$ is bounded on $C([0,1])$, with norm given by:

$$ K^{*} := \|\mathfrak{K}\|_{\mathcal{L}(X)} = \sup_{t\in[0,1]}\int_0^{t}|\mathcal{K}(t,\sigma)|d\sigma < \infty. $$

Consequently, for any $\chi \in X$, we have $\|\mathfrak{K}\chi\| \leq K^{*}\|\chi\|$ and $\|\chi\| + \|\mathfrak{K}\chi\| \leq (1 + K^{*})\|\chi\|$. An existence result is established via a generalization of DFPT (check, e.g., [19–22]).

Remark. The Volterra integral operator $\mathfrak{K}\chi(\iota) = \int_0^{t}K(\iota,\sigma)\chi(\sigma)\,d\sigma$ represents a generalized memory term. When the kernel is selected as $K(\iota,\sigma) = \frac{(\iota-\sigma)^{\theta-1}}{\Gamma(\theta)}$ for $\theta > 0$, the operator $\mathfrak{K}\chi(\iota)$ reduces precisely to the standard R–L fractional integral (FI) $I_{0+}^{\theta}\chi(\iota)$.

We summarize the principal contributions as follows:

1. We investigate a novel class of NHFI-DEs. Two nested CapDs with various fractional orders, a nonlinear hybrid transformation, and a Volterra-type memory term define these equations. To the best of our knowledge, the current literature has not looked at this specific formulation.
2. We extend and generalize earlier findings about fractional models by using the idea of MNC within BanSpa. to show the presence of solutions without requiring compactness criteria.
3. The proposed system incorporates a nonlinear transformation controlled by an auxiliary function, denoted as $\mathfrak{F}$. This introduces a novel type of coupling between the state function and its FDs, enhancing both the structural complexity and modeling flexibility of the problem.

We organize the rest as follows: In Section 2, we review essential definitions and preliminary results related to FrCal, including the CapDs of various orders and the R-LIO. Section 3 focuses on formulating the nonlinear hybrid fractional integro-differential BVP under consideration, and we establish the existence of solutions using the theory of MNC and a generalized version of DFPT in BanSpa. Finally, Section 4 presents an illustrative example that supports the theoretical results.

2. Preliminaries

To establish the groundwork for our main analysis, we introduce several key concepts and foundational results from the theory of FrCal and MNC.

Definition 2.1. *[2] For $k-1<\alpha\leq k$, define the Caputo domain by*

$$\mathcal{D}_C^{\alpha}:=\left\{Q:\ Q^{(k-1)}\in AC([0,1]),\quad Q^{(k)}\in L^1(0,1)\right\}.$$

For $Q\in\mathcal{D}_C^{\alpha}$, the CapDs of order α is

$${}^{C}D_{0+}^{\alpha}Q(\iota)=\frac{1}{\Gamma(k-\alpha)}\int_0^{\iota}(\iota-\varsigma)^{k-\alpha-1}Q^{(k)}(\varsigma)\,d\varsigma,\qquad 0\leq\iota\leq 1,$$

with the usual limiting interpretation when $\alpha=k$.

Definition 2.2. *[2] For a continuous function Q defined on $[0,\infty)$, the R-L FI of order $\delta>0$ is given by*

$$\mathfrak{I}_{0+}^{\vartheta}Q(\iota)=\frac{1}{\Gamma(\vartheta)}\int_0^{\iota}(\iota-\varsigma)^{\vartheta-1}Q(\varsigma)\,d\varsigma.$$

Lemma 2.1. *[2, 26] Let $\vartheta,\theta>0$ and let $Q\in C([0,1])$. Then the R-L integrals satisfy*

$$I_{0+}^{\vartheta}I_{0+}^{\theta}Q(\iota)=I_{0+}^{\vartheta+\theta}Q(\iota),\qquad 0\leq\iota\leq 1.$$

Moreover, if $I_{0+}^{\theta}Q\in\mathcal{D}_C^{\theta}$, then

$${}^{C}D_{0+}^{\theta}I_{0+}^{\theta}Q(\iota)=Q(\iota),\qquad 0\leq\iota\leq 1,$$

where the equality is understood pointwise whenever the CFD is continuous, and otherwise almost everywhere with its continuous representative.

Lemma 2.2. *[2] If $\vartheta>0$ and $m-1<\vartheta\leq m$, then for any sufficiently smooth function Q, we have:*

$$\mathfrak{I}_{0+}^{\vartheta}\,{}^{C}\mathfrak{D}_{0+}^{\vartheta}Q(\iota)=Q(\iota)-\sum_{i=0}^{m-1}b_i\iota^i,\quad \text{with } b_i\in\mathbb{R}.$$

Let the closed ball centered at $\chi\in X$ with radius $\mathcal{R}$ be denoted by $\mathfrak{B}(\chi,\mathcal{R})$ in a BanSpa. X. When $\chi=0$, we write $\mathfrak{B}_{\mathcal{R}}:=\mathfrak{B}(0,\mathcal{R})$. Let $\emptyset\neq Y\subset X$; denote by $\overline{Y}$ its closure and by $\mathrm{Conv}(Y)$ its convex hull.

We use $\mathcal{M}_X:=\{\phi\neq\mathcal{M}\subset X\}$, $\mathcal{N}_X$, its subfamily consisting of relatively compact subsets.

Definition 2.3. *[23] A mapping $F_{MNC}:\mathcal{M}_X\to[0,\infty)$ is called an* MNC *on X if it satisfies:*

1. $F_{MNC}(Y) = 0$ *implies* $Y \in \mathcal{N}_X$, *and* $\ker F_{MNC} := \{Y \in \mathcal{M}_X : F_{MNC}(Y) = 0\} \neq \emptyset$;
2. *If* $Z \subseteq Y$, *then* $F_{MNC}(Z) \leq F_{MNC}(Y)$;
3. $F_{MNC}(Y) = F_{MNC}(\overline{Y}) = F_{MNC}(\text{Conv}(Y))$;
4. $F_{MNC}(\delta_1 Y + \delta_2 Z) \leq \delta_1 F_{MNC}(Y) + \delta_2 F_{MNC}(Z)$, *for all* $\delta_1, \delta_2 \geq 0$ *with* $\delta_1 + \delta_2 = 1$;
5. *If* (Z_n) *is a decreasing sequence of closed subsets in* $\mathcal{M}_X$ *with* $F_{MNC}(Z_n) \to 0$, *then* $\bigcap_{n=1}^{\infty} Z_n \neq \emptyset$.

Definition 2.4. *[23] Let* $Z \subset X(\Lambda)$ *be a bounded set. The modulus of continuity of a function* $\mathfrak{f} \in Z$, *denoted* $\Omega(\mathfrak{f}, \varepsilon)$, *is defined by*

$$\Omega(\mathfrak{f}, \varepsilon) = \sup\{|\mathfrak{f}(\iota) - \mathfrak{f}(\varsigma)| : \iota, \varsigma \in \Lambda, |\iota - \varsigma| \leq \varepsilon\}, \quad \varepsilon > 0.$$

Further, we define

$$\Omega(Z, \varepsilon) = \sup\{\Omega(\mathfrak{f}, \varepsilon) : \mathfrak{f} \in Z\}, \quad \Omega_0(Z) = \lim_{\varepsilon \to 0} \Omega(Z, \varepsilon).$$

Definition 2.5. *[24] If* $X(\Lambda)$ *is a Banach algebra, then an MNC* F_{MNC} *satisfies property (m) if*

$$F_{MNC}(\mathfrak{f}\mathfrak{g}) \leq \|\mathfrak{f}\| F_{MNC}(\mathfrak{g}) + \|\mathfrak{g}\| F_{MNC}(\mathfrak{f}), \quad \forall \mathfrak{f}, \mathfrak{g} \in \mathcal{M}_C(\Lambda).$$

Lemma 2.3. *[25] The modulus* Ω_0 *introduced above satisfies the multiplicative property (m).*

Let $\mathcal{S}$ denote the class of functions $\Phi : (0, \infty) \to (b, \infty)$, with $b > 0$, and

$$\forall (\chi_n) \subset (0, \infty), \ \lim_{n \to \infty} \Phi(\chi_n) = b \quad \text{if and only if} \quad \lim_{n \to \infty} \chi_n = 0.$$

We now state two versions of DFPT, which will play a central role in our existence theory.

Theorem 2.4. *[22] Let* $\phi \neq U \subset X(\Lambda)$ *be a closed, convex, and bounded subset of a BanSpa., and suppose* $L : U \to U$ *is continuous. If a constant* $q \in [0, 1)$ *exists:*

$$F_{MNC}(LV) \leq q F_{MNC}(V), \quad \text{for all nonempty } V \subseteq U,$$

then L admits at least one FP in U.

Theorem 2.5. *[22][Generalized Darbo] Let* $\phi \neq W \subset X(\Lambda)$ *be a bounded, closed, and convex subset of a BanSpa., and let* $L : W \to W$ *be continuous. Suppose a function* $F \in \mathcal{S}$ *and a constant* $q \in [0, 1)$ *exist: For all nonempty subsets* $D \subseteq W$ *with* $F_{MNC}(LD) > 0$, *the inequality*

$$F(F_{MNC}(LD)) \leq (F(F_{MNC}(D)))^q$$

holds. Then L possesses at least one FP in W.

3. Main results

To establish our main existence results, let us first introduce the underlying function space. Let $(X, \|\cdot\|)$ denote the BanSpa. of all real-valued, continuous functions defined on the closed unit interval $\Lambda = [0, 1]$, equipped with the standard supremum norm:

$$\|\Upsilon\| = \sup_{\iota \in \Lambda} |\Upsilon(\iota)|, \quad \Upsilon \in X.$$

Lemma 3.1. *Suppose $\iota \in \Lambda$ and $\mathfrak{h} \in X$. Then, the unique solution to the BVP*

$$\begin{cases} {}^{C}\mathfrak{D}^{\theta}({}^{C}\mathfrak{D}^{\vartheta}\Upsilon(\iota)) = \mathfrak{h}(\iota), \\ \\ {}^{C}\mathfrak{D}^{\vartheta}\Upsilon(0) = \Upsilon(0) = 0, \quad \Upsilon(1) = \Upsilon(\mu),\ 0 < \mu < 1, \end{cases} \tag{3.1}$$

is given explicitly by

$$\begin{aligned} \Upsilon(\iota) = \frac{1}{\Gamma(\vartheta+\theta)}\int_0^{\iota}(\iota-\varsigma)^{\vartheta+\theta-1}\mathfrak{h}(\varsigma)\,d\varsigma + \frac{\iota}{1-\mu}\Bigg[\frac{1}{\Gamma(\vartheta+\theta)}\int_0^{\mu}(\mu-\varsigma)^{\vartheta+\theta-1}\mathfrak{h}(\varsigma)\,d\varsigma \\ - \frac{1}{\Gamma(\vartheta+\theta)}\int_0^{1}(1-\varsigma)^{\vartheta+\theta-1}\mathfrak{h}(\varsigma)\,d\varsigma\Bigg]. \end{aligned} \tag{3.2}$$

Proof. To derive the solution, we begin by applying the FI operator $\mathfrak{I}_{0+}^{\theta}$ to both sides of (3.1). Utilizing Lemma 2.2, the expression for ${}^{C}\mathfrak{D}^{\vartheta}\Upsilon(\iota)$ becomes:

$${}^{C}\mathfrak{D}^{\vartheta}\Upsilon(\iota) = \mathfrak{I}_{0+}^{\theta}\mathfrak{h}(\iota) + d_0,$$

where d_0 is an integration constant to be determined. By evaluating at $\iota = 0$ and using the initial condition ${}^{C}\mathfrak{D}^{\vartheta}\Upsilon(0) = 0$, we find $d_0 = 0$. This simplifies the expression to:

$${}^{C}\mathfrak{D}^{\vartheta}\Upsilon(\iota) = \mathfrak{I}_{0+}^{\theta}\mathfrak{h}(\iota). \tag{3.3}$$

We now apply another FI operator, namely, $\mathfrak{I}_{0+}^{\vartheta}$, to both sides of (3.3). By again using Lemma 2.2, the general solution to the original problem can be expressed as:

$$\Upsilon(\iota) = \mathfrak{I}_{0+}^{\vartheta+\theta}\mathfrak{h}(\iota) + d_1\iota + d_2, \tag{3.4}$$

where d_1 and d_2 are constants. Substituting $\iota = 0$ into (3.4) and applying the condition $\Upsilon(0) = 0$, we immediately get $d_2 = 0$, which simplifies the formula to:

$$\Upsilon(\iota) = \mathfrak{I}_{0+}^{\vartheta+\theta}\mathfrak{h}(\iota) + d_1\iota. \tag{3.5}$$

To evaluate d_1, we make use of the condition $\Upsilon(1) = \Upsilon(\mu)$, which upon substitution into (3.5) gives:

$$d_1 = \frac{\mathfrak{I}_{0+}^{\vartheta+\theta}\mathfrak{h}(\mu) - \mathfrak{I}_{0+}^{\vartheta+\theta}\mathfrak{h}(1)}{1-\mu}.$$

Inserting this result back into Eq (3.5) yields the closed-form expression for $\Upsilon(\iota)$ as given in (3.2). Conversely, let $\Upsilon \in X$ be a continuous function satisfying the integral equation:

$$\Upsilon(\iota) = \mathfrak{I}_{0+}^{\theta}\left[\mathfrak{I}_{0+}^{\vartheta}h(\iota,\Upsilon(\iota))\right] + d_0 + d_1\iota + d_2\iota^{\vartheta-1} + d_3\iota^{\vartheta}, \quad \iota \in [0,1],$$

where the constants $d_0, d_1, d_2, d_3 \in \mathbb{R}$ are explicitly defined by the boundary and initial conditions as in (3.2)–(3.5).

Verification of the differential equation: Applying the CFD ${}^{c}\mathfrak{D}_{0+}^{\theta}$ of order $\theta \in (0,1]$ to both sides of the integral equation yields:

$${}^{c}\mathfrak{D}_{0+}^{\theta}\Upsilon(\iota) = {}^{c}\mathfrak{D}_{0+}^{\theta}\left(\mathfrak{I}_{0+}^{\theta}\left[\mathfrak{I}_{0+}^{\vartheta}h(\iota,\Upsilon(\iota))\right]\right) + {}^{c}\mathfrak{D}_{0+}^{\theta}\left(d_0 + d_1\iota + d_2\iota^{\vartheta-1} + d_3\iota^{\vartheta}\right).$$

Set $Q(\iota) = I_{0+}^{\vartheta} h(\iota, \Upsilon(\iota))$. Because $\vartheta > 1$ and $h(\cdot, \Upsilon(\cdot)) \in C([0,1])$, the function Q has the additional regularity required for the Caputo inversion identity; in particular, $Q \in \mathcal{D}_C^{\theta}$. Therefore, by Lemma 2.1,

$$^{C}\mathfrak{D}_{0+}^{\theta} I_{0+}^{\theta} Q(\iota) = Q(\iota) = I_{0+}^{\vartheta} h(\iota, \Upsilon(\iota)).$$

Thus,

$$^{c}\mathfrak{D}_{0+}^{\theta} \Upsilon(\iota) = \mathfrak{I}_{0+}^{\vartheta} h(\iota, \Upsilon(\iota)) + {}^{c}\mathfrak{D}_{0+}^{\theta} \left(d_0 + d_1 \iota + d_2 \iota^{\vartheta - 1} + d_3 \iota^{\vartheta} \right).$$

Next, applying the CFD ${}^{c}\mathfrak{D}_{0+}^{\vartheta}$ of order $\vartheta \in (1, 2]$ to both sides gives:

$$^{c}\mathfrak{D}_{0+}^{\vartheta} \left({}^{c}\mathfrak{D}_{0+}^{\theta} \Upsilon(\iota) \right) = {}^{c}\mathfrak{D}_{0+}^{\vartheta} \left(\mathfrak{I}_{0+}^{\vartheta} h(\iota, \Upsilon(\iota)) \right) + {}^{c}\mathfrak{D}_{0+}^{\vartheta} \left({}^{c}\mathfrak{D}_{0+}^{\theta} (d_0 + d_1 \iota + d_2 \iota^{\vartheta - 1} + d_3 \iota^{\vartheta}) \right).$$

Using ${}^{c}\mathfrak{D}_{0+}^{\vartheta} \mathfrak{I}_{0+}^{\vartheta} h(\iota, \Upsilon(\iota)) = h(\iota, \Upsilon(\iota))$ and noting that the CFD of polynomials of degree less than or equal to $\theta + \vartheta$ under ${}^{c}\mathfrak{D}_{0+}^{\vartheta}\, {}^{c}\mathfrak{D}_{0+}^{\theta}$ vanishes identically, we obtain:

$$^{c}\mathfrak{D}_{0+}^{\vartheta} \left({}^{c}\mathfrak{D}_{0+}^{\theta} \Upsilon(\iota) \right) = h(\iota, \Upsilon(\iota)), \quad \text{for all } \iota \in [0,1].$$

Verification of boundary and initial conditions

At $\iota = 0$: Evaluating $\Upsilon(\iota)$ at $\iota = 0$, we observe that all Riemann-Liouville integral terms $\mathfrak{I}_{0+}^{\theta}[\cdot]$ vanish, i.e., $\mathfrak{I}_{0+}^{\theta}[\cdot](0) = 0$. With $d_0 = 0$, it immediately follows that $\Upsilon(0) = 0$.

Derivatives at $\iota = 0$: Taking ${}^{c}\mathfrak{D}_{0+}^{\vartheta} \Upsilon(\iota)$ at $\iota = 0$, the integral term vanishes at the lower limit $\iota = 0$, which gives ${}^{c}\mathfrak{D}_{0+}^{\vartheta} \Upsilon(0) = 0$.

At $\iota = 1$: Substituting $\iota = 1$ into the expression for $\Upsilon(\iota)$ and using the explicit forms of d_1, d_2, d_3 derived from solving the system of BCs, we obtain:

$$\Upsilon(1) = \mathfrak{I}_{0+}^{\theta+\vartheta} h(1, \Upsilon(1)) + d_1 + d_2 + d_3 = \Upsilon(\mu).$$

Hence, all BCs specified in (3.1) are identically satisfied.Therefore, $\Upsilon(\iota)$ is indeed a solution to the BVP (3.1).

The integral equation

$$\begin{aligned} \Upsilon(\iota) = \mathfrak{F}(\iota, \Upsilon(\iota)) \Bigg(& \frac{1}{\Gamma(\vartheta + \theta)} \int_0^{\iota} (\iota - \varsigma)^{\vartheta + \theta - 1} \mathfrak{H}\left(\varsigma, \Upsilon(\varsigma), \aleph\Upsilon(\varsigma)\right) d\varsigma \\ & + \frac{\iota}{1 - \mu} \Bigg[\frac{1}{\Gamma(\vartheta + \theta)} \int_0^{\mu} (\mu - \varsigma)^{\vartheta + \theta - 1} \mathfrak{H}\left(\varsigma, \Upsilon(\varsigma), \aleph\Upsilon(\varsigma)\right) d\varsigma \\ & - \frac{1}{\Gamma(\vartheta + \theta)} \int_0^{1} (1 - \varsigma)^{\vartheta + \theta - 1} \mathfrak{H}\left(\varsigma, \Upsilon(\varsigma), \aleph\Upsilon(\varsigma)\right) d\varsigma \Bigg] \Bigg), \end{aligned} \tag{3.6}$$

is the mild formulation of the BVP (1.2). Hence, every fixed point of the operator L in $X = C([0,1])$ is a mild solution of (1.2). If, in addition, the nonlinear term and the fixed point have sufficient regularity so that the relevant CFD belong to the domains $\mathcal{D}_C^{\theta}$ and $\mathcal{D}_C^{\vartheta}$, then the mild solution satisfies (1.2) in the classical Caputo sense. This integral formulation allows us to define an appropriate operator $L : X \to X$ by

$$L\Upsilon(\iota) = \mathfrak{F}(\iota, \Upsilon(\iota)) \Bigg(\frac{1}{\Gamma(\vartheta + \theta)} \int_0^{\iota} (\iota - \varsigma)^{\vartheta + \theta - 1} \mathfrak{H}\left(\varsigma, \Upsilon(\varsigma), \aleph\Upsilon(\varsigma)\right) d\varsigma$$

$$+\frac{\iota}{1-\mu}\Bigg[\frac{1}{\Gamma(\vartheta+\theta)}\int_0^{\mu}(\mu-\varsigma)^{\vartheta+\theta-1}\mathfrak{H}\left(\varsigma,\Upsilon(\varsigma),\aleph\Upsilon(\varsigma)\right)\,d\varsigma$$
$$-\frac{1}{\Gamma(\vartheta+\theta)}\int_0^{1}(1-\varsigma)^{\vartheta+\theta-1}\mathfrak{H}\left(\varsigma,\Upsilon(\varsigma),\aleph\Upsilon(\varsigma)\right)\,d\varsigma\Bigg]\Bigg),$$

such that any fixed point of L corresponds to a solution of the BVP (1.2).

Before presenting the main results, we will examine (3.6) based on the following assumptions:

($\mathcal{H}_1$): The function $\mathfrak{F}\in C([0,1]\times\mathbb{R},\mathbb{R}\setminus\{0\})$,

($\mathcal{H}_2$): A real number $p\in(0,1)$ exists:

$$|\mathfrak{F}(\iota,\Upsilon)-\mathfrak{F}(\iota,\zeta)|\leq(|\Upsilon-\zeta|+b)^p-b^p,\quad\forall\iota\in[0,1],\ \Upsilon,\zeta\in\mathbb{R},\ b\in\mathbb{R}^+.$$

($\mathcal{H}_3$): $\mathfrak{H}:\Lambda\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfies the following:

(i) $\mathfrak{H}(\iota,.,.):\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous for $\iota\in\Lambda$.

(ii) A continuous function $\ell_{\mathfrak{H}}\in L^1(\Lambda,(0,\infty))$ exists, and a continuous nondecreasing function $\Phi:\mathbb{R}^+\to\mathbb{R}^+$ with $\Phi(0)=0$:

$$|\mathfrak{H}(\iota,\Upsilon,\zeta)|\leq\|\ell_{\mathfrak{H}}\|_{L^1}\,\Phi\left(|\Upsilon|+|\zeta|\right),\quad\text{for a.e. }\iota\in\Lambda\text{ and each }\Upsilon,\zeta\in\mathbb{R}.$$

($\mathcal{H}_4$): A positive real number exists:

$$((\mathcal{R}_0+b)^p-b^p+\mathfrak{F}_0)\,\frac{\|\ell_{\mathfrak{H}}\|_{L^1}\Phi\left((1+K^*)\mathcal{R}_0\right)}{\Gamma(\vartheta+\theta+1)}\left[1+\frac{1+\mu^{\vartheta+\theta}}{1-\mu}\right]\leq\mathcal{R}_0,$$

where $\mathfrak{F}_0=\sup_{t\in[0,1]}|\mathfrak{F}(t,0)|$.

The main result is presented in the following theorem.

Theorem 3.2. *Assuming that conditions* ($\mathcal{H}_1$) *through* ($\mathcal{H}_4$) *are met, Eq* (1.2) *has a solution in the Banach algebra* $X(\Lambda)$ *if*

$$R:=\frac{\|\ell_{\mathfrak{H}}\|_{L^1}\Phi\left(1+K^*\right)}{\Gamma(\vartheta+\theta+1)}\left[1+\frac{1+\mu^{\vartheta+\theta}}{1-\mu}\right]\leq 1.$$

Proof. Consider the operator $L:X(\Lambda)\to X(\Lambda)$ defined as follows:

$$(L\Upsilon)(\iota)=(M\Upsilon)(\iota)(N\Upsilon)(\iota),\tag{3.7}$$

where

$$(M\Upsilon)(\iota)=\mathfrak{F}(\iota,\Upsilon(\iota)),$$

$$(N\Upsilon)(\iota)=\frac{1}{\Gamma(\vartheta+\theta)}\int_0^{\iota}(\iota-\varsigma)^{\vartheta+\theta-1}\mathfrak{H}\left(\varsigma,\Upsilon(\varsigma),\aleph\Upsilon(\varsigma)\right)\,d\varsigma$$
$$+\frac{\iota}{1-\mu}\Bigg[\frac{1}{\Gamma(\vartheta+\theta)}\int_0^{\mu}(\mu-\varsigma)^{\vartheta+\theta-1}\mathfrak{H}\left(\varsigma,\Upsilon(\varsigma),\aleph\Upsilon(\varsigma)\right)\,d\varsigma$$
$$-\frac{1}{\Gamma(\vartheta+\theta)}\int_0^{1}(1-\varsigma)^{\vartheta+\theta-1}\mathfrak{H}\left(\varsigma,\Upsilon(\varsigma),\aleph\Upsilon(\varsigma)\right)\,d\varsigma\Bigg].$$

First, we will demonstrate that for every $\Upsilon \in X(\Lambda)$, the operator $(L\Upsilon)$ also belongs to $X(\Lambda)$, meaning that $(M\Upsilon)(N\Upsilon) \in X(\Lambda)$. Based on assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$(i), we can conclude that $(M\Upsilon) \in X(\Lambda)$. Next, we will verify that $(N\Upsilon) \in X(\Lambda)$ as well.
Let $\Upsilon \in X(\Lambda)$ and choose arbitrary $\iota_2, \iota_1 \in \Lambda$ such that $\iota_2 > \iota_1$. By assumption $(\mathcal{H}_3)$(ii), we have:

$$
\begin{aligned}
&|N\Upsilon(\iota_2) - N\Upsilon(\iota_1)| \\
&\leq \frac{1}{\Gamma(\vartheta+\theta)} \left| \int_0^{\iota_1} [(\iota_2-\varsigma)^{\vartheta+\theta-1} - (\iota_1-\varsigma)^{\vartheta+\theta-1}] \mathfrak{H}\left(\varsigma, \Upsilon(\varsigma), \aleph\Upsilon(\varsigma)\right), d\varsigma \right. \\
&\quad \left. + \int_{\iota_1}^{\iota_2} (\iota_2-\varsigma)^{\vartheta+\theta-1}, \mathfrak{H}\left(\varsigma, \Upsilon(\varsigma), \aleph\Upsilon(\varsigma)\right), d\varsigma \right| \\
&+ \frac{|\iota_2-\iota_1|}{1-\mu} \left| \frac{1}{\Gamma(\vartheta+\theta)} \int_0^{\mu} (\mu-\varsigma)^{\vartheta+\theta-1} \mathfrak{H}\left(\varsigma, \Upsilon(\varsigma), \aleph\Upsilon(\varsigma)\right), d\varsigma \right. \\
&- \left. \frac{1}{\Gamma(\vartheta+\theta)} \int_0^{1} (1-\varsigma)^{\vartheta+\theta-1} \mathfrak{H}\left(\varsigma, \Upsilon(\varsigma), \aleph\Upsilon(\varsigma)\right), d\varsigma \right| \\
&\leq |\ell_{\mathfrak{H}}|L^1, \Phi\left((1+K^*)|\Upsilon|\right) \frac{1}{\Gamma(\vartheta+\theta)} \left[\int_0^{\iota_1} [(\iota_2-\varsigma)^{\vartheta+\theta-1} - (\iota_1-\varsigma)^{\vartheta+\theta-1}] d\varsigma + \int \iota_1{}^{\iota_2} (\iota_2-\varsigma)^{\vartheta+\theta-1} d\varsigma \right] \\
&\quad + |\ell_{\mathfrak{H}}|L^1, \Phi\left((1+K^*)|\Upsilon|\right) \frac{|\iota_2-\iota_1|}{1-\mu} \left| \frac{1}{\Gamma(\vartheta+\theta)} \int_0^{\mu} (\mu-\varsigma)^{\vartheta+\theta-1} d\varsigma - \frac{1}{\Gamma(\vartheta+\theta)} \int_0^{1} (1-\varsigma)^{\vartheta+\theta-1} d\varsigma \right| \\
&\leq |\ell\mathfrak{H}|_{L^1}, \Phi\left((1+K^*)|\Upsilon|\right) \left(\frac{\iota_2^{\vartheta+\theta} - \iota_1^{\vartheta+\theta}}{\Gamma(\vartheta+\theta+1)} + \frac{|\iota_2-\iota_1|}{1-\mu} \frac{1-\mu^{\vartheta+\theta}}{\Gamma(\vartheta+\theta+1)} \right).
\end{aligned}
$$

As $\iota_2 \to \iota_1$, the above expression tends to zero uniformly, confirming that $N\Upsilon \in X(\Lambda)$. Now, for each $\Upsilon \in X(\Lambda)$ and $\iota \in \Lambda$, we can estimate $|(N\Upsilon)(\iota)|$:

$$
\begin{aligned}
|(N\Upsilon)(\iota)| &= \left| \int_0^{\iota} \frac{(\iota-\varsigma)^{\vartheta+\theta-1}}{\Gamma(\vartheta+\theta)} \mathfrak{H}\left(\varsigma, \Upsilon(\varsigma), \aleph\Upsilon(\varsigma)\right) d\varsigma \right. \\
&\quad + \frac{\iota}{1-\mu} \left[\frac{1}{\Gamma(\vartheta+\theta)} \int_0^{\mu} (\mu-\varsigma)^{\vartheta+\theta-1} \mathfrak{H}\left(\varsigma, \Upsilon(\varsigma), \aleph\Upsilon(\varsigma)\right) d\varsigma \right. \\
&\quad \left. \left. + \frac{1}{\Gamma(\vartheta+\theta)} \int_0^{1} (1-\varsigma)^{\vartheta+\theta-1} \mathfrak{H}\left(\varsigma, \Upsilon(\varsigma), \aleph\Upsilon(\varsigma)\right) d\varsigma \right| \right. \\
&\quad \leq \int_0^{\iota} \frac{(\iota-\varsigma)^{\vartheta+\theta-1}}{\Gamma(\vartheta+\theta)} \left| \mathfrak{H}\left(\varsigma, \Upsilon(\varsigma), \aleph\Upsilon(\varsigma)\right) \right| d\varsigma \\
&\quad + \frac{\iota}{1-\mu} \left[\frac{1}{\Gamma(\vartheta+\theta)} \int_0^{\mu} (\mu-\varsigma)^{\vartheta+\theta-1} \left| \mathfrak{H}\left(\varsigma, \Upsilon(\varsigma), \aleph\Upsilon(\varsigma)\right) \right| d\varsigma \right. \\
&\quad \left. \left. + \frac{1}{\Gamma(\vartheta+\theta)} \int_0^{1} (1-\varsigma)^{\vartheta+\theta-1} \left| \mathfrak{H}\left(\varsigma, \Upsilon(\varsigma), \aleph\Upsilon(\varsigma)\right) \right| d\varsigma \right] \right. \\
&\quad \leq \int_0^{\iota} \frac{(\iota-\varsigma)^{\vartheta+\theta-1}}{\Gamma(\vartheta+\theta)} d\varsigma \|\ell_{\mathfrak{H}}\|_{L^1} \Phi\left((1+K^*)\|\Upsilon\|\right) \\
&\quad + \frac{\iota}{1-\mu} \left[\frac{1+\mu^{\vartheta+\theta}}{\Gamma(\vartheta+\theta+1)} \right] \|\ell_{\mathfrak{H}}\|_{L^1} \Phi\left((1+K^*)\|\Upsilon\|\right)
\end{aligned}
$$

$$\leq \left[1 + \frac{1+\mu^{\vartheta+\theta}}{1-\mu}\right] \frac{\|\ell_{\mathfrak{H}}\|_{L^1}\, \Phi\left((1+K^*)\|\Upsilon\|\right)}{\Gamma(\vartheta+\theta+1)}.$$

Using the estimation of $|(N\Upsilon)(t)|$, the complete majoration for $\|L\Upsilon\| = \sup_{t\in[0,1]} |(M\Upsilon)(t)(N\Upsilon)(t)|$ becomes:

$$\begin{aligned}
|(L\Upsilon)(\iota)| &= |(M\Upsilon)(\iota)(N\Upsilon)(\iota)| \\
&\leq (|\mathfrak{F}(\iota,\Upsilon(\iota)) - \mathfrak{F}(\iota,0)| + |\mathfrak{F}(\iota,0)|) \cdot \left[1 + \frac{1+\mu^{\vartheta+\theta}}{1-\mu}\right] \frac{\|\ell_{\mathfrak{H}}\|_{L^1}\, \Phi\left((1+K^*)\|\Upsilon\|\right)}{\Gamma(\vartheta+\theta+1)} \\
&\leq ((\|\Upsilon\| + b)^p - b^p + \mathfrak{F}_0) \cdot \left[1 + \frac{1+\mu^{\vartheta+\theta}}{1-\mu}\right] \frac{\|\ell_{\mathfrak{H}}\|_{L^1}\, \Phi\left((1+K^*)\|\Upsilon\|\right)}{\Gamma(\vartheta+\theta+1)}.
\end{aligned}$$

After estimating all the necessary terms and constants, we arrive at the following result:

$$\|L\Upsilon\| \leq ((\|\Upsilon\| + b)^p - b^p + \mathfrak{F}_0) \cdot \left[1 + \frac{1+\mu^{\vartheta+\theta}}{1-\mu}\right] \frac{\|\ell_{\mathfrak{H}}\|_{L^1}\, \Phi\left((1+K^*)\|\Upsilon\|\right)}{\Gamma(\vartheta+\theta+1)}.$$

Define the closed ball:

$$\mathfrak{B}_{\mathcal{R}_0} = \{\Upsilon \in X(\Lambda) : \|\Upsilon\| \leq \mathcal{R}_0\}, \tag{3.8}$$

where $\mathcal{R}_0$ is a positive constant that satisfies assumption $(\mathcal{H}_4)$. Then, it follows that $L(\mathfrak{B}_{\mathcal{R}_0}) \subseteq \mathfrak{B}_{\mathcal{R}_0}$. In the next step, we will demonstrate that L defined in (3.7) is continuous on $\mathfrak{B}_{\mathcal{R}_0}$. This will be established by proving the continuity of M and N separately.

For M, let $\Upsilon, \zeta \in \mathfrak{B}_{\mathcal{R}_0}$ and $\iota \in \Lambda$, and choose an $\varepsilon > 0$. There exists a $\varrho > 0$: If $\|\Upsilon - \zeta\| < \varrho$, then:

$$|M(\Upsilon)(\iota) - M(\zeta)(\iota)| = |\mathfrak{F}(\iota,\Upsilon(\iota)) - \mathfrak{F}(\iota,\zeta(\iota))| \leq (\|\Upsilon - \zeta\| + b)^p - b^p \leq (\varrho + b)^p - b^p < \varepsilon. \tag{3.9}$$

The operator M meets the continuity criteria on the set $\mathfrak{B}_{\mathcal{R}_0}$. To examine the continuity of the operator N on $\mathfrak{B}_{\mathcal{R}_0}$, let $\{\Upsilon_n\} \subset \mathfrak{B}_{\mathcal{R}_0}$ and suppose that $\|\Upsilon_n - \Upsilon\|_\infty \longrightarrow 0$. Since $\mathfrak{K}$ is a bounded linear Volterra operator,

$$\|\mathfrak{K}\Upsilon_n - \mathfrak{K}\Upsilon\|_\infty \leq K^* \|\Upsilon_n - \Upsilon\|_\infty \longrightarrow 0.$$

Consequently,

$$(\Upsilon_n(\iota), \mathfrak{K}\Upsilon_n(\iota)) \longrightarrow (\Upsilon(\iota), \mathfrak{K}\Upsilon(\iota))$$

uniformly on Λ. All these arguments belong to the compact set

$$[-R_0, R_0] \times [-K^* R_0, K^* R_0],$$

since, (H3)(i) is stated in the uniform form that $\mathfrak{H}$ is jointly continuous on the relevant compact product (equivalently, uniformly continuous in its last two variables uniformly with respect to ι). Thus, $\mathfrak{H}$ is uniformly continuous on that compact product with Λ. Define

$$g_n(\iota) = \mathfrak{H}(\iota, \Upsilon_n(\iota), \mathfrak{K}\Upsilon_n(\iota)), \qquad g(\iota) = \mathfrak{H}(\iota, \Upsilon(\iota), \mathfrak{K}\Upsilon(\iota)).$$

Then

$$\|g_n - g\|_\infty \longrightarrow 0.$$

Writing $\alpha = \vartheta + \theta$, the operator N in (3.7) is a linear combination of fractional integrals of order α. Therefore,

$$\|I^{\alpha}_{0+}(g_n - g)\|_{\infty} \leq \frac{\|g_n - g\|_{\infty}}{\Gamma(\alpha + 1)}.$$

It follows directly from the three terms defining N that

$$\|N\Upsilon_n - N\Upsilon\|_{\infty} \leq \frac{1 + 2/(1-\mu)}{\Gamma(\alpha + 1)}\|g_n - g\|_{\infty} \longrightarrow 0.$$

This proves the continuity of N in the supremum norm. The continuity of M follows from (H1) and (H2), and hence $L\Upsilon = (M\Upsilon)(N\Upsilon)$ is continuous on $\mathfrak{B}_{\mathcal{R}_0}$.
Next, to analyze the modulus of continuity of L, consider a nonempty subset $U \subseteq \mathfrak{B}_{\mathcal{R}_0}$.
For any $\Upsilon \in U$ and $\iota_1, \iota_2 \in \Lambda$ with $|\iota_2 - \iota_1| < \varrho$, by using the hypotheses $(\mathcal{H}_2)$ we have

$$\begin{aligned}
|(Mu)(\iota_2) - (Mu)(\iota_1)| &= |\mathfrak{F}(\iota_2, \Upsilon(\iota_2)) - \mathfrak{F}(\iota_1, \Upsilon(\iota_1))| \\
&\leq |\mathfrak{F}(\iota_2, \Upsilon(\iota_2)) - \mathfrak{F}(\iota_2, \Upsilon(\iota_1))| + |\mathfrak{F}(\iota_2, \Upsilon(\iota_1)) - \mathfrak{F}(\iota_1, \Upsilon(\iota_1))| \\
&\leq [(|\Upsilon(\iota_2) - \Upsilon(\iota_1)| + b)^p - b^p] + \Omega(\mathfrak{F}, \varepsilon) \\
&\leq [(\mathfrak{F}(U, \varepsilon) + b)^p - b^p] + \Omega(\mathfrak{F}, \varepsilon), \qquad (3.10)
\end{aligned}$$

where

$$\Omega(\mathfrak{F}, \varepsilon) := \sup\{|\mathfrak{F}(\iota_2, \Upsilon) - \mathfrak{F}(\iota_1, \Upsilon)| : \iota_1, \iota_2 \in \Lambda, |\iota_2 - \iota_1| < \varepsilon, \Upsilon \in [-\mathcal{R}_0, \mathcal{R}_0]\}.$$

The equation presented in (3.10) can be expressed as follows:

$$\Omega(MU, \varepsilon) \leq (\Omega(U, \varepsilon) + b)^p - b^p + \Omega(\mathfrak{F}, \varepsilon), \qquad (3.11)$$

Since $\mathfrak{F}$ is uniformly continuous on $\Lambda \times [-\mathcal{R}_0, \mathcal{R}_0]$, then $\Omega(\mathfrak{F}, \varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, Eq (3.11) transforms into the desired form

$$\Omega_0(MU) \leq (\Omega_0(U) + b)^p - b^p. \qquad (3.12)$$

Similarly, there exists $\varrho = \varrho(\varepsilon)$ for which $|\iota_2 - \iota_1| < \varrho$. Set

$$\alpha = \vartheta + \theta > 1, \qquad C_{\mathfrak{H}} = \|\ell_{\mathfrak{H}}\|_{L^1}\Phi((1 + K^*)R_0).$$

For every $\Upsilon \in U \subset B_{R_0}$, (H3) gives $|\mathfrak{H}(\iota, \Upsilon(\iota), K\Upsilon(\iota))| \leq C_{\mathfrak{H}}$. If $\iota_2 > \iota_1$, the fractional-integral term in $N\Upsilon$ satisfies, using $\alpha > 1$,

$$\begin{aligned}
|I^{\alpha}_{0+}g_{\Upsilon}(\iota_2) - I^{\alpha}_{0+}g_{\Upsilon}(\iota_1)| &\leq \frac{C_{\mathfrak{H}}}{\Gamma(\alpha + 1)}|\iota_2^{\alpha} - \iota_1^{\alpha}| \\
&\leq \frac{\alpha C_{\mathfrak{H}}}{\Gamma(\alpha + 1)}|\iota_2 - \iota_1|,
\end{aligned}$$

where $g_{\Upsilon}(\iota) = \mathfrak{H}(\iota, \Upsilon(\iota), K\Upsilon(\iota))$. The remaining boundary terms in $N\Upsilon$ are linear in ι; hence their difference is bounded by a constant multiple of $|\iota_2 - \iota_1|$. Thus, there is a constant $C_N > 0$, independent of $\Upsilon \in U$: $|(N\Upsilon)(\iota_2) - (N\Upsilon)(\iota_1)| \leq C_N|\iota_2 - \iota_1|$. In particular,

$$\Omega(NU, \delta) \leq C_N\delta \longrightarrow 0 \qquad (\delta \to 0),$$

and therefore

$$\Omega_0(N\Omega) = 0. \tag{3.13}$$

This estimate is uniform over the entire bounded set U, so it supplies precisely the equicontinuity required in the measure-of-noncompactness argument. No subdivision into the three cases in the previous version is necessary. The estimation of $\Omega_0(L\Omega)$ can be achieved using Lemma 2.3 along with (3.8), (3.12), and (3.13).

$$\begin{aligned}
\Omega_0(LU) &= \Omega_0(MU, NU) \\
&\leq \|MU\|\,\Omega_0(NU) + \|NU\|\,\Omega_0(MU) \\
&\leq \|M(\mathcal{B}_{\mathcal{R}_0})\|\,\Omega_0(NU) + \|N(\mathcal{B}_{\mathcal{R}_0})\|\,\Omega_0(MU) \\
&\leq [(\Omega_0(U) + b)^p - b^p]\left(\frac{\|\ell_{\mathfrak{F}}\|_{L^1}\,\Phi(\mathcal{R}_0)}{\Gamma(\vartheta+\theta+1)}\left[1 + \frac{1+\mu^{\vartheta+\theta}}{1-\mu}\right]\right) \\
&= [(\Omega_0(U) + b)^p - b^p]\,R.
\end{aligned}$$

By $(\mathcal{H}_4)$ and noting that $R \leq 1$, we can conclude that:

$$\Omega_0(LU) + b^p \leq (\Omega_0(U) + b)^p.$$

Due to Theorem 2.4, the contractive condition is clearly satisfied with $\Phi(\Upsilon) = \Upsilon + b$, where Φ is a member of $\mathcal{S}$. By applying Theorem 2.5, we decisively conclude that L possesses at least one FP within $\mathcal{B}_{\mathcal{R}_0}$. Thus, (1.2) has at least one solution in $\mathcal{B}_{\mathcal{R}_0}$.

4. Example

Consider the following NHFI-DBVP:

$$\begin{cases}
{}^{C}\mathfrak{D}^{1/2}\left({}^{C}\mathfrak{D}^{3/2}\left[\dfrac{\Upsilon(\iota)}{\sqrt{|\Upsilon(\iota)|+9}}\right]\right) = \dfrac{\iota}{20}\left(\cos(\Upsilon(\iota)) + \displaystyle\int_0^{\iota}(\iota-\sigma)^{-2/3}\Upsilon(\sigma)\,d\sigma\right), \quad \iota \in [0,1], \\[2ex]
{}^{C}\mathfrak{D}^{3/2}\left[\dfrac{\Upsilon(\iota)}{\sqrt{|\Upsilon(\iota)|+9}}\right]_{\iota=0} = 0, \quad \dfrac{\Upsilon(0)}{\sqrt{|\Upsilon(0)|+9}} = 0, \\[2ex]
\dfrac{\Upsilon(1)}{\sqrt{|\Upsilon(1)|+9}} = \dfrac{\Upsilon(1/10)}{\sqrt{|\Upsilon(1/10)|+9}}.
\end{cases} \tag{4.1}$$

According to (4.1), we observe that

$$\theta = \frac{1}{2},\ \vartheta = \frac{3}{2},\ \mu = \frac{1}{10},\ b = 3.$$

$$\mathfrak{F}(\iota, \Upsilon) = \sqrt{|\Upsilon|+9} - 3,$$

$$\mathfrak{H}\left(\iota, \Upsilon(\iota), \int_0^{\iota}\mathcal{K}(\iota,\sigma)\Upsilon(\sigma)\,d\sigma\right) = \frac{\iota}{20}\left(\cos(\Upsilon(\iota)) + \int_0^{\iota}(\iota-\sigma)^{-2/3}\Upsilon(\sigma)\,d\sigma\right),$$

$$with \quad \Re\Upsilon(\iota) = \int_0^{\iota} (\iota - \sigma)^{-2/3} \Upsilon(\sigma)\, d\sigma,$$

which is bounded on $X = C([0,1])$, with operator norm:

$$K^* = \|\Re\|_{\mathcal{L}(X)} = \sup_{t\in[0,1]} \int_0^t (t-\sigma)^{-2/3} d\sigma = \sup_{t\in[0,1]} 3t^{1/3} = 3.$$

Both hypotheses $(\mathcal{H}_1)$ and $(\mathcal{H}_3 - (i))$ are satisfied. For hypothesis $(\mathcal{H}_2)$, we take $p = \frac{1}{2}$. Moreover, consider the function $y(\Upsilon) = \sqrt{|\Upsilon| + 9} - 3$. It is easy to verify that $y(0) = 0$ and that y is concave. Since $y(\Upsilon)$ is concave, we apply the subadditive property of concave functions to obtain:

$$\begin{aligned} |\mathfrak{F}(\iota, \Upsilon) - \mathfrak{F}(\iota, \zeta)| &= |y(\Upsilon) - y(\zeta)| \\ &\leq y(\Upsilon - \zeta) = \sqrt{|\Upsilon - \zeta| + 9} - 3 \\ &\leq (|\Upsilon - \zeta| + 3)^{\frac{1}{2}} - 3^{\frac{1}{2}}, \end{aligned}$$

which confirms that hypothesis $(\mathcal{H}_2)$ holds, and let $\ell_{\mathfrak{H}}(\iota) = \frac{\iota}{20} \in L^1([0,1])$, with $\|\ell_{\mathfrak{H}}\|_{L^1} = \frac{1}{40} = 0.025$. Define $\Phi(r) = r$, which is continuous, nondecreasing, and $\Phi(0) = 0$. Then,

$$|\mathfrak{H}(\iota, \Upsilon, \Re\Upsilon)| \leq \frac{\iota}{20} (|\cos(\Upsilon)| + |\Re\Upsilon|) \leq \frac{1}{20}(|\Upsilon| + |\Re\Upsilon|) \leq \|\ell_{\mathfrak{H}}\|_{L^1} \cdot \Phi(|\Upsilon| + |\Re\Upsilon|),$$

thus, verifying hypothesis $(\mathcal{H}_3)$.
Setting $\mathcal{R}_0 = 2$, we evaluate the terms in $(\mathcal{H}_4)$:

- $(\mathcal{R}_0 + b)^p - b^p + \delta_0 = (2+3)^{1/2} - 3^{1/2} + 0 = \sqrt{5} - \sqrt{3} \approx 2.236 - 1.732 = 0.504.$
- $(1 + K^*)\mathcal{R}_0 = (1+3)(2) = 8$, so $\Phi((1+K^*)\mathcal{R}_0) = 8.$
- $\vartheta + \theta = \frac{3}{2} + \frac{1}{2} = 2$, and $\Gamma(\vartheta + \theta + 1) = \Gamma(3) = 2! = 2.$
- $\mu^{\vartheta+\theta} = \left(\frac{1}{10}\right)^2 = 0.01.$

We now evaluate the left-hand side of the self-mapping inequality in $(\mathcal{H}_4)$ we obtain that:

$$(0.504) \cdot \frac{0.025 \cdot 8}{2} \cdot \left[1 + \frac{1 + 0.01}{1 - 0.1}\right] = 0.504 \cdot 0.1 \cdot \left[1 + \frac{1.01}{0.9}\right] \approx 0.0504 \cdot 2.122 \approx 0.107 \leq 2 = \mathcal{R}_0.$$

Thus, condition $(\mathcal{H}_4)$ holds and $L(\mathfrak{B}_{\mathcal{R}_0}) \subseteq \mathfrak{B}_{\mathcal{R}_0}$.
Finally, we compute the constant R for Theorem 3.2:

$$R = \frac{\|\ell_{\mathfrak{H}}\|_{L^1}(1 + K^*)}{\Gamma(\vartheta + \theta + 1)} \left[1 + \frac{1 + \mu^{\vartheta+\theta}}{1 - \mu}\right] = \frac{0.025 \cdot (1+3)}{2} \cdot [2.122] = 0.05 \cdot 2.122 = 0.1061 < 1.$$

All hypotheses $(\mathcal{H}_1)$–$(\mathcal{H}_4)$ are satisfied, and the radius condition $R < 1$ is met. Therefore, by Theorem 3.2, the BVP (4.1) has at least one solution in the Banach algebra $X(\Lambda) = C([0,1])$.

5. Conclusions

In this study, we explored a new class of BVPs that involve NHFI-DEs. This work incorporates two distinct CapDs along with an R-LIO. To demonstrate the existence of solutions, we utilized the theory of MNC within the framework of BanSpa. and applied a generalized version of DFPT. We included an illustrative example to validate the practical applicability of our theoretical findings. In future research, we aim to expand this framework to include systems that feature variable-order FDs, impulsive effects, or stochastic perturbations. Additionally, we plan to investigate the stability in the sense of Ulam, its generalizations, and controllability of

$$\begin{cases} {}^{C}\mathfrak{D}^{\theta}\left({}^{C}\mathfrak{D}^{\vartheta}\left[\dfrac{\Upsilon(\iota)}{\mathfrak{F}(\iota,\Upsilon(\iota))}\right]\right) = \displaystyle\sum_{i=1}^{n}\mathfrak{H}_{\mathrm{i}}\left(\iota,\Upsilon(\iota),\int_{0}^{\iota}\mathcal{K}(\iota,\sigma)\Upsilon(\sigma)\,d\sigma\right),\ \iota\in\Lambda, \\ {}^{C}\mathfrak{D}^{\vartheta}\left[\dfrac{\Upsilon(\iota)}{\mathfrak{F}(\iota,\Upsilon(\iota))}\right]_{\iota=0} = \left[\dfrac{\Upsilon(\iota)}{\mathfrak{F}(\iota,\Upsilon(\iota))}\right]_{\iota=0} = 0, \\ \left[\dfrac{\Upsilon(\iota)}{\mathfrak{F}(\iota,\Upsilon(\iota))}\right]_{\iota=1} = \left[\dfrac{\Upsilon(\iota)}{\mathfrak{F}(\iota,\Upsilon(\iota))}\right]_{\iota=\mu},\ 0<\mu<1, \end{cases}$$

where $\Lambda = [0,1]$ and both ${}^{C}\mathfrak{D}^{\theta}$ and ${}^{C}\mathfrak{D}^{\vartheta}$ are CapDs of orders $0 < \theta \leq 1$ and $1 < \vartheta \leq 2$, respectively. The function $\mathfrak{F} : \Lambda \times X \to X \setminus \{0\}$, and $\mathfrak{H}_{\mathrm{i}} : \Lambda \times X \times X \to X$ are known.

Author contributions

Entesar Aljarallah, Salman Alsaeed, Ymnah Alruwaily: Conceptualization, methodology, formal analysis, numerical examples, software, validation; El-sayed El-hady, Kulandhaivel Karthikeyan, Hamid Lmou, Asmaa Baihi: Investigation, supervision, visualization, guidance on theoretical results, writing original draft, review and editing. All authors have read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Data availability

The data used in this study are available from the corresponding author upon request.

Conflicts of interest

The authors declare that they have no conflicts of interest.

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