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Research article

Subgraph induced by nonzero idempotent elements of clean graphs of some products of fields

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Abstract: The clean graph $Cl(R)$ of a ring R with identity is a simple graph whose vertices are of the form (e, u), where e is an idempotent and u is a unit of R. Two distinct vertices (e_1, u_1) and (e_2, u_2) are adjacent if $e_1e_2 = 0 = e_2e_1$ or $u_1u_2 = 1 = u_2u_1$. The graph $Cl_2(R)$ is a subgraph of $Cl(R)$ induced by vertices (e, u), where e is a nonzero idempotent element. For any prime numbers p, q such that $p \leq q$ and positive integers n, m, two structures of $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ are introduced to determine the order, size, maximum degree, minimum degree, diameter, connectivity, and planarity of both $Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ and $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$. Additionally, the clique number, chromatic number, and independence number of $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ are investigated.

Keywords: clean graph; idempotent element; unit
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1. Introduction

Graphs associated with rings provide a combinatorial framework for studying algebraic properties of rings. Many graph constructions have been studied, such as zero-divisor graphs, total graphs, and idempotent graphs; see [1–3]. More recently, clean graphs have also been introduced; see [4].

Let R be a ring with identity $1 \neq 0$. An element $a \in R$ is said to be a clean element if a can be expressed as a sum of an idempotent element and a unit of R. A ring R is called a clean ring if every element of R is clean. The idea of clean elements and clean rings was first introduced in 1977 by Nicholson [5] and has been developed since then; see [6–8]. Later in 2013, Akbari et al. [1] defined the idempotent graph whose vertices are nontrivial idempotent elements of R. Moreover, two distinct vertices e and f are adjacent if $ef = fe = 0$.

In 2021, Habibi et al. [4] defined a graph motivated by both idempotent graphs and clean rings, called a clean graph. The clean graph of R, denoted by $Cl(R)$, is a simple graph with vertices of the

form (e, u), where e is an idempotent element and u is a unit of R. Moreover, two distinct vertices (e_1, u_1) and (e_2, u_2) are adjacent if $e_1e_2 = 0 = e_2e_1$ or $u_1u_2 = 1 = u_2u_1$. Let $Cl_2(R)$ be the subgraph of $Cl(R)$ induced by $\{(e, u) \in R \times R \mid e$ is a nonzero idempotent element and u is a unit$\}$. The minimum degree, maximum degree, and diameter of $Cl_2(R)$ are also studied in [4]. Moreover, the clique number, chromatic number, and independence number of $Cl_2(R)$ are investigated when R is a commutative Artinian ring with identity.

In this paper, we focus on the clean graph of the finite ring $\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}$ of order p^nq^m where p and q are prime numbers such that $p \leq q$ and n and m are positive integers. While [4] studied $Cl_2(R)$ for general commutative Artinian rings and established an upper bound on the diameter, we further determine the exact diameter, vertex-connectivity, and edge-connectivity of both $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ and $Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$, as well as characterizing when these graphs are Eulerian or planar, extending significantly beyond [4]. In order to analyze $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$, we introduce two structures. The first structure is a partition of the vertex set according to the types of idempotents and units, which enables explicit computation of degrees and graph size. The second structure consists of triangles and 2-triangles, which is useful for proving connectivity and planarity results.

2. Preliminaries

In this section, we introduce our notation from ring theory and graph theory which can be found in [9] and [10], respectively.

Let R be a ring. The set of all idempotent elements of R is denoted by $Id(R)$, while $Id^*(R) = Id(R) \setminus \{0\}$. The set of all units of R is denoted by $U(R)$, which can be separated into two types: $U_1(R) = \{u \in U(R) \mid u^{-1} = u\}$ and $U_2(R) = \{u \in U(R) \mid u^{-1} \neq u\}$. Then $U(R)$ is a disjoint union of $U_1(R)$ and $U_2(R)$. Observe that if R_1 and R_2 are rings, then $U(R_1 \times R_2) = \{(u_1, u_2) \mid u_1 \in U(R_1)$ and $u_2 \in U(R_2)\}$.

Let G be a simple graph with the vertex set $V(G)$ and edge set $E(G)$. Any two adjacent vertices u and v in G are denoted by $u \sim v$. The join of two disjoint graphs G and H is denoted by $G \vee H$. The diameter, minimum degree, maximum degree, vertex-connectivity, edge-connectivity, clique number, chromatic number, and independence number of G are denoted by $\mathrm{diam}(G), \delta(G)$, $\Delta(G), \kappa(G), \lambda(G), \omega(G), \chi(G)$, and $\alpha(G)$, respectively.

The clean graph of a ring R with identity $1 \neq 0$, denoted by $Cl(R)$, is a simple graph with vertices of the form (e, u), where $e \in Id(R)$ and $u \in U(R)$. Any two distinct vertices (e_1, u_1) and (e_2, u_2) in $Cl(R)$ are adjacent if $e_1e_2 = 0 = e_2e_1$ or $u_1u_2 = 1 = u_2u_1$. We define $Cl_1(R)$ to be a subgraph of $Cl(R)$ induced by $\{(0, u) \mid u \in U(R)\}$ and $Cl_2(R)$ to be a subgraph of $Cl(R)$ induced by $\{(e, u) \mid e \in Id^*(R)$ and $u \in U(R)\}$. It is easy to see that $Cl_1(R)$ is a complete subgraph of $Cl(R)$. Indeed, $Cl(R) = Cl_1(R) \vee Cl_2(R)$, so the main purpose of this paper is to analyze $Cl_2(R)$.

3. Properties of clean graphs

Throughout this paper, we let p and q be prime numbers such that $p \leq q$ and n, m be positive integers, and the finite field of order p^n is denoted by $\mathbb{F}_{p^n}$. In this section, we investigate various properties of $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ and $Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ via two explicit structures. This section is organized as follows: In Section 3.1, we introduce the first structure, which is a partition of the vertex set of clean

graphs. We use this structure to determine the degree of vertices and to characterize when clean graphs are Eulerian. The order and size of clean graphs are studied in Section 3.2. In Section 3.3, we determine the minimum degree and maximum degree of clean graphs, as well as their diameter. In Section 3.4, we define another structure consisting of triangles and 2-triangles, and discuss the vertex-connectivity and edge-connectivity of clean graphs. Finally, we explore miscellaneous properties of clean graphs such as planarity, clique number, chromatic number, and independence number in Section 3.5.

3.1. Degree

Clearly, $\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}$ has $(0, 1)$ and $(1, 0)$ as nontrivial idempotents. The next lemma is directly obtained from [4, Theorem 2.2].

Lemma 3.1. *$Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ is connected.*

Lemma 3.2.

$$|U_1(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})| = \begin{cases} 1 & \text{if } p = q = 2, \\ 2 & \text{if } p = 2 \text{ and } q \geq 3, \\ 4 & \text{if } p \geq 3 \text{ and } q \geq 3, \end{cases}$$

and

$$|U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})| = \begin{cases} (2^n - 1)(2^m - 1) - 1 & \text{if } p = q = 2, \\ (2^n - 1)(q^m - 1) - 2 & \text{if } p = 2 \text{ and } q \geq 3, \\ (p^n - 1)(q^m - 1) - 4 & \text{if } p \geq 3 \text{ and } q \geq 3. \end{cases}$$

Proof. Note that $U_1(\mathbb{F}_{p^n}) = \{1, -1\}$. Therefore, $U_1(\mathbb{F}_{2^n}) = \{1\}$. Hence, we obtain the first equality. Since every nonzero element in a field is a unit, we have $|U(\mathbb{F}_{p^n})| = p^n - 1$, so $|U(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})| = (p^n - 1)(q^m - 1)$. As $U(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ is a disjoint union of $U_1(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ and $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$, we obtain the second equality. □

Next, we determine an explicit structure of $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ via $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$. As $U_1(\mathbb{F}_{p^n}) = \mathbb{F}_{p^n} \setminus \{0\}$ if and only if $p^n \in \{2, 3\}$, we deduce that $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) = \emptyset$ if and only if $p^n, q^m \in \{2, 3\}$. In fact, when $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) = \emptyset$, the graphs $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ are depicted in Figure 1.

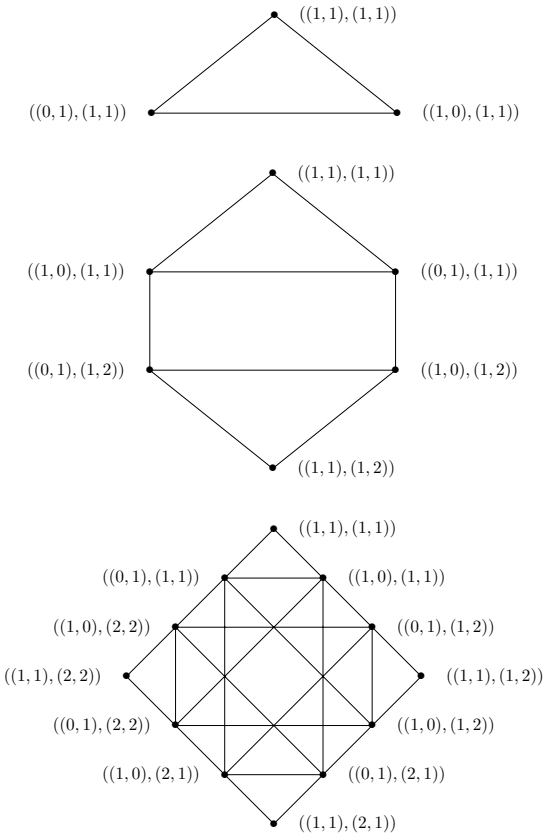


Figure 1. $Cl_2(\mathbb{F}_2 \times \mathbb{F}_2)$, $Cl_2(\mathbb{F}_2 \times \mathbb{F}_3)$, and $Cl_2(\mathbb{F}_3 \times \mathbb{F}_3)$, respectively.

When $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \neq \emptyset$, we notice that the adjacency relation in $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ depends on two independent algebraic components: the idempotent part and the unit part. Since every nonzero idempotent of $\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}$ is one of $(0, 1), (1, 0)$, or $(1, 1)$, and every unit belongs either to $U_1(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ or $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$, the vertices naturally separate according to these two classifications. This yields six vertex classes as follows:

$$
\begin{aligned}
A_1 &= \left\{((0, 1), (x, y)) \mid (x, y)^{-1} = (x, y)\right\}, \\
A_2 &= \left\{((1, 0), (x, y)) \mid (x, y)^{-1} = (x, y)\right\}, \\
A_3 &= \left\{((1, 1), (x, y)) \mid (x, y)^{-1} = (x, y)\right\}, \\
A_4 &= \left\{((0, 1), (x, y)) \mid (x, y)^{-1} \neq (x, y)\right\}, \\
A_5 &= \left\{((1, 0), (x, y)) \mid (x, y)^{-1} \neq (x, y)\right\}, \\
A_6 &= \left\{((1, 1), (x, y)) \mid (x, y)^{-1} \neq (x, y)\right\}.
\end{aligned}
$$

The global structure of $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ can be determined by the above six vertex partitions as shown in the next proposition.

Proposition 3.3. *Suppose that $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \neq \emptyset$. Then the following statements hold:*

1) Every vertex in A_i is adjacent to every vertex in A_j, where $(i, j) \in \{(1, 2), (1, 5), (2, 4), (4, 5)\}$.

2) Each vertex in A_i is not adjacent to any vertex in A_j, where $(i, j) \in \{(1,4),(1,6),(2,5),(2,6),(3,4),(3,5),(3,6)\}$.
3) For each vertex in A_i, there exists a unique vertex in A_j such that they are adjacent, where $(i, j) \in \{(1,3),(2,3),(4,6),(5,6)\}$.
4) No two distinct vertices in A_i are adjacent, where $i \in \{1,2,3\}$.
5) For each vertex in A_i, there exists a unique distinct vertex in A_i such that they are adjacent, where $i \in \{4,5,6\}$.

Note that $|A_1| = |A_2| = |A_3| = |U_1(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})|$ and $|A_4| = |A_5| = |A_6| = |U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})|$. Thus, the size of each partition can be determined explicitly by Lemma 3.2.

Proposition 3.4. *Suppose that $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \neq \emptyset$. Then the following hold:*

1) For $p = q = 2$, $|A_1| = |A_2| = |A_3| = 1$ and $|A_4| = |A_5| = |A_6| = (2^n - 1)(2^m - 1) - 1$.
2) For $p = 2$ and $q \geq 3$, $|A_1| = |A_2| = |A_3| = 2$ and $|A_4| = |A_5| = |A_6| = (2^n - 1)(q^m - 1) - 2$.
3) For $p \geq 3$ and $q \geq 3$, $|A_1| = |A_2| = |A_3| = 4$ and $|A_4| = |A_5| = |A_6| = (p^n - 1)(q^m - 1) - 4$.

The degree of vertices in $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ depends on the partition they belong to, as stated in the next theorem.

Theorem 3.5. *Let $((e_1, e_2), (u_1, u_2))$ be a vertex in $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$, where $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \neq \emptyset$. Then*

$$\deg((e_1, e_2), (u_1, u_2)) = \begin{cases} 1 + (p^n - 1)(q^m - 1) & \text{if } ((e_1, e_2), (u_1, u_2)) \in A_1, \\ 1 + (p^n - 1)(q^m - 1) & \text{if } ((e_1, e_2), (u_1, u_2)) \in A_2, \\ 2 & \text{if } ((e_1, e_2), (u_1, u_2)) \in A_3, \\ 2 + (p^n - 1)(q^m - 1) & \text{if } ((e_1, e_2), (u_1, u_2)) \in A_4, \\ 2 + (p^n - 1)(q^m - 1) & \text{if } ((e_1, e_2), (u_1, u_2)) \in A_5, \\ 3 & \text{if } ((e_1, e_2), (u_1, u_2)) \in A_6. \end{cases}$$

Proof. By Proposition 3.3, we have

$$\deg((e_1, e_2), (u_1, u_2)) = \begin{cases} |A_2| + |A_5| + 1 & \text{if } ((e_1, e_2), (u_1, u_2)) \in A_1, \\ |A_1| + |A_4| + 1 & \text{if } ((e_1, e_2), (u_1, u_2)) \in A_2, \\ 1 + 1 & \text{if } ((e_1, e_2), (u_1, u_2)) \in A_3, \\ |A_2| + |A_5| + 1 + 1 & \text{if } ((e_1, e_2), (u_1, u_2)) \in A_4, \\ |A_1| + |A_4| + 1 + 1 & \text{if } ((e_1, e_2), (u_1, u_2)) \in A_5, \\ 1 + 1 + 1 & \text{if } ((e_1, e_2), (u_1, u_2)) \in A_6. \end{cases}$$

Hence, the result follows. □

Recall that a connected graph is Eulerian if and only if every vertex has even degree. By Theorem 3.5, any vertices in A_6 always have odd degree. Utilizing these facts, we investigate the condition for $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ to be an Eulerian graph.

Theorem 3.6. *$Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ is an Eulerian graph if and only if $p^n = q^m = 2$.*

Proof. By Figure 1, it follows that $Cl_2(\mathbb{F}_2 \times \mathbb{F}_2)$ is Eulerian, while $Cl_2(\mathbb{F}_2 \times \mathbb{F}_3)$ and $Cl_2(\mathbb{F}_3 \times \mathbb{F}_3)$ are not Eulerian since they have a vertex of odd degree. For the case $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \neq \emptyset$, every vertex in A_6 has degree 3. □

Corollary 3.7. *$Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ is not an Eulerian graph.*

Proof. It is obvious that $Cl(\mathbb{F}_2 \times \mathbb{F}_2), Cl(\mathbb{F}_2 \times \mathbb{F}_3)$, and $Cl(\mathbb{F}_3 \times \mathbb{F}_3)$ have a vertex of odd degree by Figure 1. For the case $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \neq \emptyset$, choose $v_1 \in A_3$ and $v_2 \in A_6$. Then $\deg_{Cl_2}(v_1) = 2$ and $\deg_{Cl_2}(v_2) = 3$ by Proposition 3.3. Because $Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) = Cl_1(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \vee Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$, we have $\deg_{Cl}(v_1) = (p^n - 1)(q^m - 1) + 2$ and $\deg_{Cl}(v_2) = (p^n - 1)(q^m - 1) + 3$. Hence, either v_1 or v_2 has odd degree. □

3.2. Order and size

We begin this section by investigating the order and size of $Cl_1(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$. Since it is a complete graph, it is straightforward to prove that

$$|V(Cl_1(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}))| = (p^n - 1)(q^m - 1),$$
$$|E(Cl_1(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}))| = \frac{[(p^n - 1)(q^m - 1)]^2 - (p^n - 1)(q^m - 1)}{2}.$$

The orders of $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ and $Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ are determined as follows.

Proposition 3.8. $|V(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}))| = 3(p^n - 1)(q^m - 1)$.

Proof. Because $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ is induced by $\{(e, u) \mid e \in Id^*(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ and $u \in U(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})\}$, we have

$$\begin{aligned} |V(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}))| &= |\{(e, u) \mid e \in Id^*(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \text{ and } u \in U(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})\}| \\ &= |Id^*(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})| \cdot |U(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})| \\ &= 3(p^n - 1)(q^m - 1) \end{aligned}$$

as desired. □

Corollary 3.9. $|V(Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}))| = 4(p^n - 1)(q^m - 1)$.

The sizes of $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ and $Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ depend on $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$. By Figure 1, we obtain the following results when $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) = \emptyset$:

$$|E(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}))| = \begin{cases} 3 & \text{if } p^n = q^m = 2, \\ 8 & \text{if } p^n = 2 \text{ and } q^m = 3, \\ 24 & \text{if } p^n = q^m = 3, \end{cases}$$

and

$$|E(Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}))| = \begin{cases} 6 & \text{if } p^n = q^m = 2, \\ 21 & \text{if } p^n = 2 \text{ and } q^m = 3, \\ 78 & \text{if } p^n = q^m = 3. \end{cases}$$

Theorem 3.10. *Suppose that $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \neq \emptyset$. Then*

$$|E(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}))| = \begin{cases} [(2^n-1)(2^m-1)]^2 + \frac{7}{2}(2^n-1)(2^m-1) - \frac{3}{2} & \text{if } p = q = 2, \\ [(2^n-1)(q^m-1)]^2 + \frac{7}{2}(2^n-1)(q^m-1) - 3 & \text{if } p = 2 \text{ and } q \geq 3, \\ [(p^n-1)(q^m-1)]^2 + \frac{7}{2}(p^n-1)(q^m-1) - 6 & \text{if } p \geq 3 \text{ and } q \geq 3. \end{cases}$$

Proof. By Theorem 3.5, we obtain

$$\begin{aligned} \sum_{v \in V(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}))} \deg(v) &= [1 + (p^n-1)(q^m-1)]|A_1| + [1 + (p^n-1)(q^m-1)]|A_2| + 2|A_3| \\ &\quad + [2 + (p^n-1)(q^m-1)]|A_4| + [2 + (p^n-1)(q^m-1)]|A_5| + 3|A_6|. \end{aligned}$$

By substituting the size of each partition in Proposition 3.4, we get

$$\sum_{v \in V(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}))} \deg(v) = \begin{cases} 2[(2^n-1)(2^m-1)]^2 + 7(2^n-1)(2^m-1) - 3 & \text{if } p = q = 2, \\ 2[(2^n-1)(q^m-1)]^2 + 7(2^n-1)(q^m-1) - 6 & \text{if } p = 2 \text{ and } q \geq 3, \\ 2[(p^n-1)(q^m-1)]^2 + 7(p^n-1)(q^m-1) - 12 & \text{if } p \geq 3 \text{ and } q \geq 3, \end{cases}$$

and the result follows. □

Theorem 3.11. *Suppose that $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \neq \emptyset$. Then*

$$|E(Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}))| = \begin{cases} \frac{9}{2}[(2^n-1)(2^m-1)]^2 + 3(2^n-1)(2^m-1) - \frac{3}{2} & \text{if } p = q = 2, \\ \frac{9}{2}[(2^n-1)(q^m-1)]^2 + 3(2^n-1)(q^m-1) - 3 & \text{if } p = 2 \text{ and } q \geq 3, \\ \frac{9}{2}[(p^n-1)(q^m-1)]^2 + 3(p^n-1)(q^m-1) - 6 & \text{if } p \geq 3 \text{ and } q \geq 3. \end{cases}$$

Proof. Since $Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) = Cl_1(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \vee Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$, the number of edges between $Cl_1(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ and $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ is

$$\begin{aligned} |V(Cl_1(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}))| \cdot |V(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}))| &= (p^n-1)(q^m-1) \cdot 3(p^n-1)(q^m-1) \\ &= 3[(p^n-1)(q^m-1)]^2 \end{aligned}$$

by Proposition 3.8. Thus, the total number of edges of $Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ is

$$|E(Cl_1(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}))| + 3[(p^n-1)(q^m-1)]^2 + |E(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}))|.$$

By substituting the above values using Theorem 3.10, we obtain our result. □

3.3. Maximum degree, minimum degree, and diameter

We start this section by considering the minimum and maximum degree of both $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ and $Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$.

Theorem 3.12. *$\delta(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = 2$, and*

$$\Delta(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = \begin{cases} 2 & \text{if } p^n = q^m = 2, \\ 3 & \text{if } p^n = 2 \text{ and } q^m = 3, \\ 5 & \text{if } p^n = q^m = 3, \\ 2 + (p^n-1)(q^m-1) & \text{otherwise.} \end{cases}$$

Proof. For the minimum degree of $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$, it follows that $\delta(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = 2$ when $p^n, q^m \in \{2, 3\}$ by Figure 1. On the other hand, Theorem 3.5 yields $\delta(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = 2$ when $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \neq \emptyset$.

For the maximum degree of $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$, the first three cases are clear by Figure 1. Assume that $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \neq \emptyset$. Then $A_i \neq \emptyset$ for all $i = 4, 5, 6$. Because $(p^n - 1)(q^m - 1) \geq 1$, we get $2 + (p^n - 1)(q^m - 1) \geq 3$, so $\Delta(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = 2 + (p^n - 1)(q^m - 1)$ by Theorem 3.5. □

Theorem 3.13. $\delta(Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = 2 + (p^n - 1)(q^m - 1)$ *and*

$$\Delta(Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = 4(p^n - 1)(q^m - 1) - 1.$$

Proof. By Theorem 3.12, we have $\delta(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = 2$. Since $Cl_1(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ is a complete graph of order $(p^n - 1)(q^m - 1)$, we get $\delta(Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = 2 + (p^n - 1)(q^m - 1)$.

For the maximum degree of $Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$, we notice that $((0, 0), (u_1, u_2)) \in Cl_1(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$. Because this vertex is adjacent to every vertex except itself, we have $\deg((0, 0), (u_1, u_2)) = |V(Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}))| - 1 = 4(p^n - 1)(q^m - 1) - 1$ by Theorem 3.12. However, $\Delta(Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) \leq |V(Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}))| - 1$, so the result follows. This completes the proof. □

For the diameter, we notice $Cl_1(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ is complete. However, $Cl_1(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \cong K_1$ if and only if $p^n = q^m = 2$. Consequently, we obtain

$$\operatorname{diam}(Cl_1(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = \begin{cases} 0 & \text{if } p^n = q^m = 2, \\ 1 & \text{otherwise.} \end{cases}$$

By a similar argument, $\operatorname{diam}(Cl_2(\mathbb{F}_2 \times \mathbb{F}_2)) = 1 = \operatorname{diam}(Cl(\mathbb{F}_2 \times \mathbb{F}_2))$. Thus, we will determine the diameter of $Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ and $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ when $p^n \neq 2$ or $q^m \neq 2$.

Theorem 3.14. *Suppose that $p^n \neq 2$ or $q^m \neq 2$. Then* $\operatorname{diam}(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = 3$.

Proof. If $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) = \emptyset$, then $\operatorname{diam}(Cl_2(\mathbb{F}_2 \times \mathbb{F}_3)) = 3 = \operatorname{diam}(Cl_2(\mathbb{F}_3 \times \mathbb{F}_3))$ by Figure 1. Assume that $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \neq \emptyset$. Let $((1, 1), (x, y)) \in A_3$ and $((1, 1), (a, b)) \in A_6$. Then $(x, y)^{-1} = (x, y)$ while $(a, b)^{-1} \neq (a, b)$. Moreover, we have a path

$$((1, 1), (x, y)) \sim ((0, 1), (x, y)) \sim ((1, 0), (a, b)^{-1}) \sim ((1, 1), (a, b)).$$

We will claim that the distance between $((1, 1), (x, y))$ and $((1, 1), (a, b))$ is 3. By Proposition 3.3, each vertex in A_3 is not adjacent to any vertex in A_6. As a result, $d(((1, 1), (x, y)), ((1, 1), (a, b))) \neq 1$. Suppose $d(((1, 1), (x, y)), ((1, 1), (a, b))) = 2$. There exists $((e_1, e_2), (u_1, u_2)) \in V(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}))$ such that $((1, 1), (x, y)) \sim ((e_1, e_2), (u_1, u_2)) \sim ((1, 1), (a, b))$. Each possibility $((e_1, e_2), (u_1, u_2)) \in A_i$ for all $i = 1, 2, \ldots, 6$ leads to a contradiction by Proposition 3.3. As a result, we have $d(((1, 1), (x, y)), ((1, 1), (a, b))) = 3$. By [4, Theorem 2.2], we get $\operatorname{diam}(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) \leq 3$. Hence, $\operatorname{diam}(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = 3$ as desired. □

Proposition 3.15. *Suppose that $p^n \neq 2$ or $q^m \neq 2$. Then* $\operatorname{diam}(Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = 2$.

Proof. First, we claim that if $p^n \neq 2$ or $q^m \neq 2$, then $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ is not complete. If $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) = \emptyset$, then $Cl_2(\mathbb{F}_2 \times \mathbb{F}_3)$ and $Cl_2(\mathbb{F}_3 \times \mathbb{F}_3)$ are not complete by Figure 1. If $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \neq \emptyset$, then $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ is not complete by Proposition 3.3. Thus, we obtain our claim. As $Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) = Cl_1(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \vee Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$, we have that $Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ is not complete and $\operatorname{diam}(Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = 2$. □

3.4. Connectivity

Although the partition $\{A_1, A_2, \ldots, A_6\}$ is useful for studying local properties such as vertex degrees, it does not clearly reveal the cycle structure of the graph. For connectivity arguments, it is more convenient to group vertices according to a fixed unit and its inverse. This observation naturally leads to the notions of triangles and 2-triangles as follows:

For any $(x, y) \in U(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$, define $T(x, y) \subseteq Id^*(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \times U(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ by

$$T(x,y) = \begin{cases} \{((0,1),(x,y)),((1,0),(x,y)),((1,1),(x,y))\} & \text{if } (x,y) \in U_1(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}), \\ \{((0,1),(x,y)),((1,0),(x,y)),((1,1),(x,y)) & \\ ((0,1),(x^{-1},y^{-1})),((1,0),(x^{-1},y^{-1})),((1,1),(x^{-1},y^{-1}))\}, & \text{if } (x,y) \in U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}). \end{cases}$$

Indeed, a subgraph of $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ induced by $T(x, y)$ is a triangle for every $(x, y) \in U_1(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$. Meanwhile, for any $(x, y) \in U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$, a subgraph of $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ induced by $T(x, y)$ is shown in Figure 2 and is called a 2-triangle.

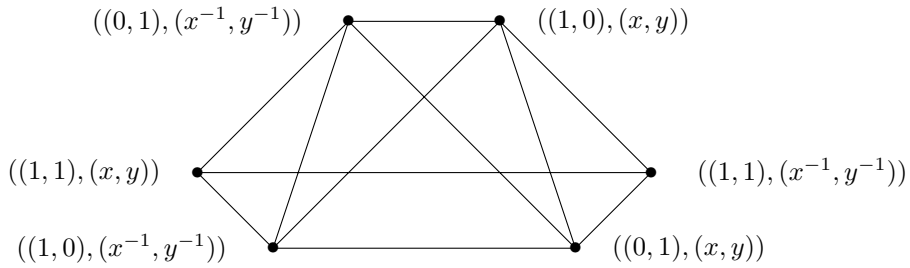


Figure 2. 2-triangle graph.

Hence, there are three cases of subgraphs induced by $T(x, y)$ and $T(a, b)$ as shown in Figures 3–5.

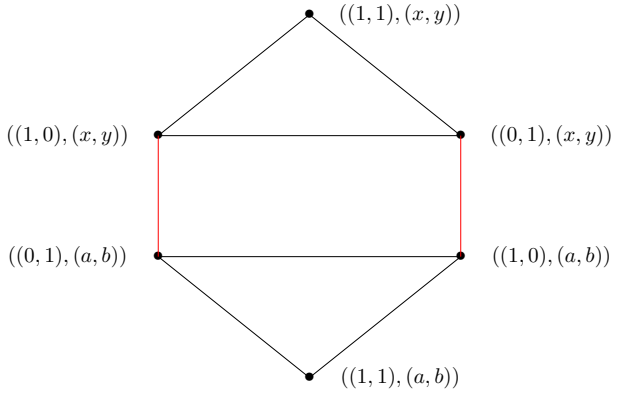


Figure 3. Subgraph induced by two triangles $T(x, y)$ and $T(a, b)$.

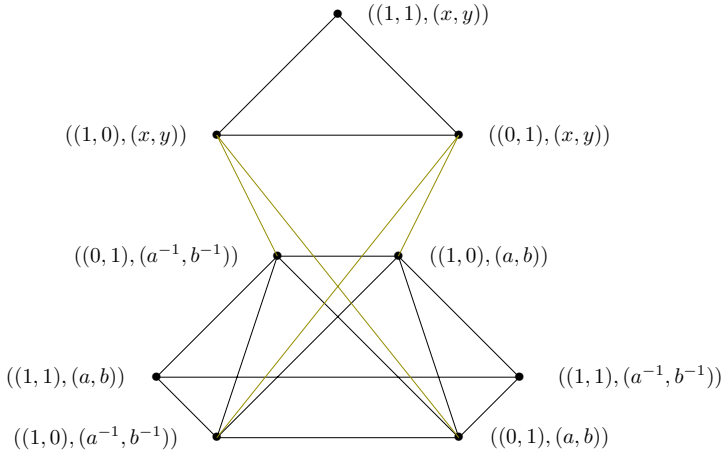


Figure 4. Subgraph induced by a triangle $T(x,y)$ and a 2-triangle $T(a,b)$.

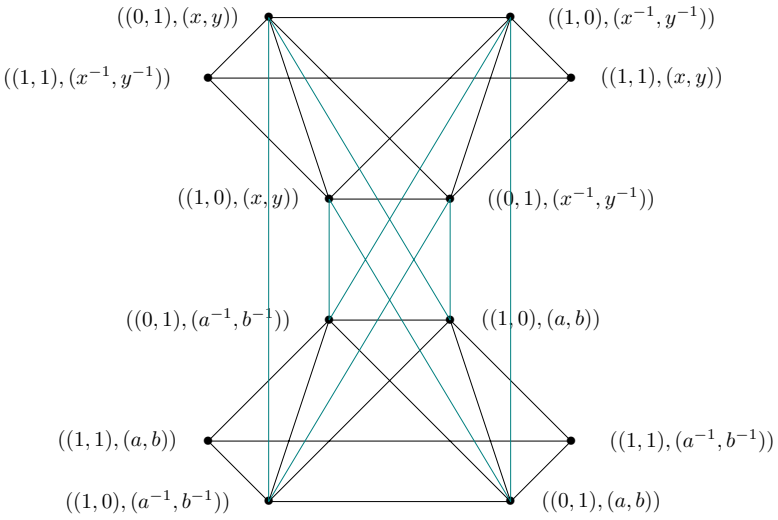


Figure 5. Subgraph induced by two 2-triangles $T(x,y)$ and $T(a,b)$.

The number of triangles and 2-triangles can be directly determined by Lemma 3.2.

1) If $p = q = 2$, then $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ has a triangle and $\dfrac{(2^n-1)(2^m-1)-1}{2}$ 2-triangles.
2) If $p = 2$ and $q \geq 3$, then $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ has two triangles and $\dfrac{(2^n-1)(q^m-1)-2}{2}$ 2-triangles.
3) If $p \geq 3$ and $q \geq 3$, then $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ has four triangles and $\dfrac{(p^n-1)(q^m-1)-4}{2}$ 2-triangles.

We are now ready to analyze the edge-connectivity of $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$. Recall that an edge of a graph G is a bridge if and only if it is not contained in any cycle of G.

Theorem 3.16. $\lambda(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = 2$.

Proof. For $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) = \emptyset$, it is easy to see that $\lambda(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = 2$ by Figure 1. Next, we suppose that $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \neq \emptyset$. Notice that

$$0 \leq \kappa(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) \leq \lambda(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) \leq \delta(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = 2$$

by Theorem 3.12. Hence, $\lambda(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = 0, 1$, or 2. By Lemma 3.1, $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ is connected, so $\lambda(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) \neq 0$.

To show $\lambda(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) \neq 1$, we will claim that any edge of $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ is not a bridge. Equivalently, any edge of $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ must be contained in a cycle. Let e be an edge in $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$. If e is an edge of a subgraph of $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ induced by a triangle or a 2-triangle, it follows that e is contained in a cycle. Suppose that e is not an edge of a subgraph of $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ induced by $T(x, y)$ for all $(x, y) \in U(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$. As a result, there exist two distinct $T(u_1, u_2)$ and $T(v_1, v_2)$ such that e is an edge joining a vertex of $T(u_1, u_2)$ and a vertex of $T(v_1, v_2)$. Observe that (u_1, u_2) and (v_1, v_2) are distinct.

If $(u_1, u_2), (v_1, v_2) \in U_1(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$, then $T(u_1, u_2)$ and $T(v_1, v_2)$ are triangles. We observe that $((1, 0), (u_1, u_2)) \sim ((0, 1), (u_1, u_2)) \sim ((1, 0), (v_1, v_2)) \sim ((0, 1), (v_1, v_2)) \sim ((1, 0), (u_1, u_2))$ is a cycle containing e. Next, suppose that $(u_1, u_2) \in U_1(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ and $(v_1, v_2) \in U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$. Then $T(u_1, u_2)$ is a triangle and $T(v_1, v_2)$ is a 2-triangle. Moreover, $(u_1, u_2), (v_1, v_2)$, and (v_1^{-1}, v_2^{-1}) are all distinct, so $((1, 0), (u_1, u_2)) \sim ((0, 1), (v_1^{-1}, v_2^{-1})) \sim ((1, 0), (v_1, v_2)) \sim ((0, 1), (u_1, u_2)) \sim ((1, 0), (v_1^{-1}, v_2^{-1})) \sim ((0, 1), (v_1, v_2)) \sim ((1, 0), (u_1, u_2))$ is a cycle containing e. Finally, if $(u_1, u_2), (v_1, v_2) \in U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$, then $T(u_1, u_2)$ and $T(v_1, v_2)$ are 2-triangles. In addition, $(u_1, u_2), (u_1^{-1}, u_2^{-1}), (v_1, v_2)$, and (v_1^{-1}, v_2^{-1}) are all distinct. We notice that either

$$((1, 0), (u_1, u_2)) \sim ((0, 1), (v_1^{-1}, v_2^{-1})) \sim ((1, 0), (u_1^{-1}, u_2^{-1})) \sim ((0, 1), (v_1, v_2)) \sim ((1, 0), (u_1, u_2)),$$

or

$$((0, 1), (u_1^{-1}, u_2^{-1})) \sim ((1, 0), (v_1, v_2)) \sim ((0, 1), (u_1, u_2)) \sim ((1, 0), (v_1^{-1}, v_2^{-1})) \sim ((0, 1), (u_1^{-1}, u_2^{-1}))$$

is a cycle containing e. By the three cases mentioned above, we conclude that $\lambda(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) \neq 1$. Consequently, $\lambda(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = 2$. □

The vertex-connectivity of $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ is stated in the following theorem.

Theorem 3.17. $\kappa(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = 2$.

Proof. By Figure 1, it follows that $\kappa(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = 2$ when $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) = \emptyset$. Assume that $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \neq \emptyset$. As in the proof of Theorem 3.16, $\kappa(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = 0, 1$, or 2. By Lemma 3.1, $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ is connected, so $\kappa(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) \neq 0$.

We show that $\kappa(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) \neq 1$ by contradiction. Suppose $\kappa(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = 1$. Let $v = ((e_1, e_2), (u_1, u_2))$ be a cut-vertex in $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$. By Proposition 3.8, $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ has at least six vertices. Thus, $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) - v$ is nontrivial and contains at least two components, say G_1 and G_2.

Let $x = ((e_1', e_2'), (u_1', u_2'))$ be a vertex in G_1 such that $x \sim v$. Then $(e_1, e_2) \cdot (e_1', e_2') = (0, 0)$ or $(u_1, u_2) \cdot (u_1', u_2') = (1, 1)$. Similarly, let $y = ((e_1'', e_2''), (u_1'', u_2''))$ be a vertex in G_2 such that $y \sim v$. Then $(e_1, e_2) \cdot (e_1'', e_2'') = (0, 0)$ or $(u_1, u_2) \cdot (u_1'', u_2'') = (1, 1)$. Next, we consider the following cases. If $(e_1, e_2) \cdot (e_1', e_2') = (0, 0)$ and $(e_1, e_2) \cdot (e_1'', e_2'') = (0, 0)$,

$$x = ((e_1', e_2'), (u_1', u_2')) \sim ((e_1, e_2), (u_1', u_2')) \sim ((e_1'', e_2''), (u_1'', u_2'')) = y$$

is a path in $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ not containing v. If $(e_1, e_2) \cdot (e_1', e_2') = (0, 0)$ and $(u_1, u_2) \cdot (u_1'', u_2'') = (1, 1)$,

$$x = ((e_1', e_2'), (u_1', u_2')) \sim ((e_1, e_2), (u_1'', u_2'')) \sim ((e_1'', e_2''), (u_1, u_2)) \sim ((e_1'', e_2''), (u_1'', u_2'')) = y$$

is a path in $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ not containing v. If $(u_1, u_2) \cdot (u_1', u_2') = (1, 1)$ and $(e_1, e_2) \cdot (e_1'', e_2'') = (0, 0)$,

$$x = ((e_1', e_2'), (u_1', u_2')) \sim ((e_1', e_2'), (u_1, u_2)) \sim ((e_1, e_2), (u_1', u_2')) \sim ((e_1'', e_2''), (u_1'', u_2'')) = y$$

is a path in $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ not containing v. If $(u_1, u_2) \cdot (u_1', u_2') = (1, 1)$ and $(u_1, u_2) \cdot (u_1'', u_2'') = (1, 1)$,

$$x = ((e_1', e_2'), (u_1', u_2')) \sim ((e_1'', e_2''), (u_1, u_2)) \sim ((e_1'', e_2''), (u_1'', u_2'')) = y$$

is a path in $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ not containing v.

Next, we let x' be a vertex in G_1 and y' be a vertex in G_2. Since G_1 and G_2 are connected, there exist $x' - x$ and $y - y'$ paths. Joining these two paths with the $x - y$ path obtained above yields an $x' - y'$ path in which v is not a vertex. Hence, v is not a cut-vertex, which is a contradiction. Consequently, $\kappa(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = 2$. □

Observe that both vertex-connectivity and edge-connectivity equal the minimum degree of 2 according to Theorem 3.5, which implies $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ is connected relative to its minimum degree.

3.5. Further properties

Finally, we investigate miscellaneous properties of clean graphs such as planarity, clique number, chromatic number, as well as independence number.

For planarity, it is known from Kuratowski's theorem that a graph G is planar if and only if G contains no subdivision of either $K_{3,3}$ or K_5. For $Cl_2(\mathbb{F}_2 \times \mathbb{F}_2)$ and $Cl_2(\mathbb{F}_2 \times \mathbb{F}_3)$, they are planar by Figure 1. On the other hand, $Cl_2(\mathbb{F}_3 \times \mathbb{F}_3)$ is nonplanar as it has a subdivision of $K_{3,3}$ shown in Figure 6.

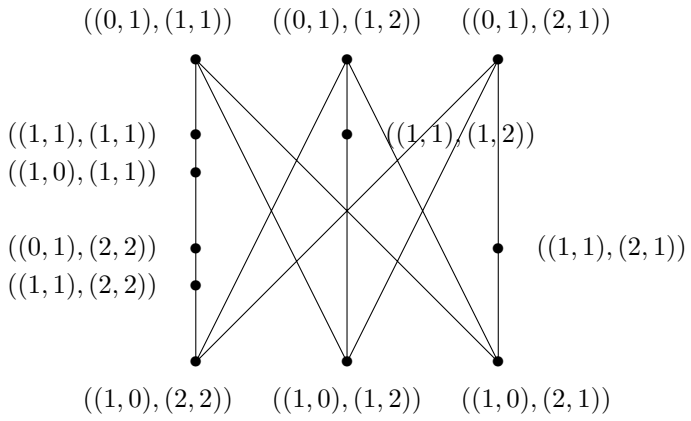


Figure 6. A subdivision of $K_{3,3}$ in $Cl_2(\mathbb{F}_3 \times \mathbb{F}_3)$.

For the case $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \neq \emptyset$, $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ is nonplanar as stated in the next proposition.

Proposition 3.18. *Suppose that $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \neq \emptyset$. Then $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ is nonplanar.*

Proof. Suppose that $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \neq \emptyset$. Then $p^n, q^m \in \{2, 3\}$. By Lemma 3.2, $U_1(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \neq \emptyset$. Thus, $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ contains at least one triangle and one 2-triangle. As a result, $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ contains a subgraph induced by a triangle and a 2-triangle, which also contains a subdivision of $K_{3,3}$ as shown in Figure 7. Hence, $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ is nonplanar. □

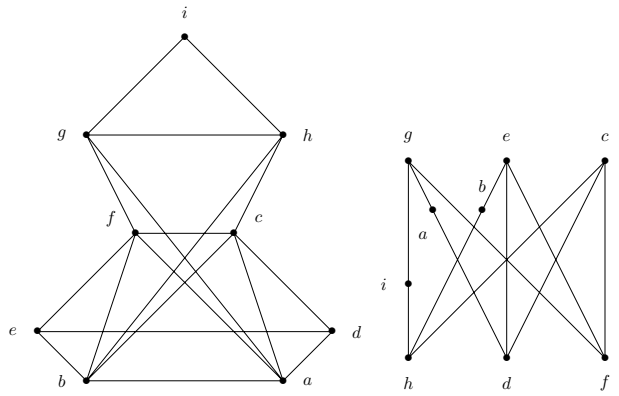


Figure 7. A subdivision of $K_{3,3}$ in $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$.

As $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ is an induced subgraph of $Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$, every subdivision of $K_{3,3}$ contained in $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ must be contained in $Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ as well. Consequently, the next corollary follows.

Corollary 3.19. *Suppose that $U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \neq \emptyset$. Then $Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ is nonplanar.*

Next, it is also clear that $Cl(\mathbb{F}_2 \times \mathbb{F}_2)$ is planar while $Cl(\mathbb{F}_2 \times \mathbb{F}_3)$ is nonplanar as it contains a subdivision of K_5 as shown in Figure 8. In addition, $Cl(\mathbb{F}_3 \times \mathbb{F}_3)$ is nonplanar because it contains a nonplanar subgraph $Cl_2(\mathbb{F}_3 \times \mathbb{F}_3)$.

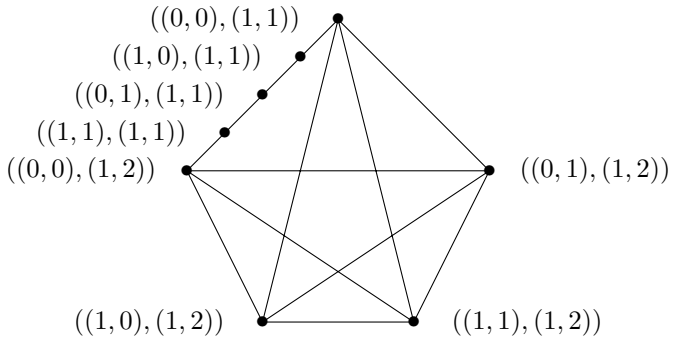


Figure 8. A subdivision of K_5 in $Cl(\mathbb{F}_2 \times \mathbb{F}_3)$.

The clique number and chromatic number of $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ can be obtained from [4, Corollary 3.2].

Corollary 3.20.

$$\omega(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = \chi(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = \begin{cases} 3 & \text{if } U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) = \emptyset, \\ 4 & \text{if } U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}) \neq \emptyset. \end{cases}$$

Finally, we consider the independence number of $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$.

Proposition 3.21. $\alpha(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) = (p^n - 1)(q^m - 1)$.

Proof. Since $\mathbb{F}_{p^n}$ and $\mathbb{F}_{q^m}$ are local rings, by [4, Theorem 3.4], we have

$$\begin{aligned}\alpha(Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})) &= |U_1(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})| + 2^{(2-1)} \cdot \frac{|U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})|}{2} \\ &= |U_1(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})| + |U_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})| \\ &= |U(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})|.\end{aligned}$$

Using Lemma 3.2, we obtain our result. □

4. Concluding remarks

This paper focuses on the clean graph of the product of two finite fields $\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}$. Two explicit structures were introduced: a vertex partition based on idempotent and unit types and a decomposition into triangles and 2-triangles grouped by inverse pairs of units. Utilizing these structures, we determined the order, size, maximum degree, minimum degree, exact diameter, vertex-connectivity, and edge-connectivity of both $Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ and $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$. We also completely characterized when $Cl(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ and $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ are Eulerian or planar. Additionally, the clique number, chromatic number, and independence number of $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$ were determined.

5. Future work

Several natural directions remain open for future investigation. It would be interesting to extend the vertex partition and the triangle and 2-triangle decomposition to the product of k finite fields for $k \geq 3$. In this setting, the number of nontrivial idempotents grows as $2^k - 1$. Moreover, the results on vertex-connectivity, edge-connectivity, and diameter may be further investigated for more general finite rings such as finite chain rings or products of finite local rings that are not necessarily fields. Furthermore, the graph-theoretic properties such as connectivity, clique structure, planarity, and independence number are fundamental parameters in applied graph settings.

Author contributions

Khajee Jantarakhajorn: Conceptualization, Investigation, Methodology, Writing-review, Validation; Borworn Khuhirun: Conceptualization, Investigation, Writing-original draft, Validation; Arnita Boonsawang: Validation, Writing-original draft. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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