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Research article

On eigenvalues and eigenvectors of the resultant graph of factor graphs

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Abstract: In this paper, we investigated the spectral properties of graphs obtained via the H-join operation on a family of factor graphs. We introduced the notion of strongly inherited eigenvalues and characterized the resultant graphs for which all eigenvalues, except possibly one, were strongly inherited from the factor graphs. We also introduced the concept of cross-eigenvectors and provided a characterization of the resultant graph that admitted such eigenvectors. Finally, we derived the characteristic polynomial for the universal adjacency matrix of the H_m-join of factor graphs.

Keywords: H-join; Graph eigenvalues; Graph eigenvectors
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1. Introduction

All the graphs considered in this paper are simple and undirected. For a graph G, we denote its vertex set and edge set by $V(G)$ and $E(G)$, respectively. For $u, v \in V(G)$, we denote $uv \in E(G)$ if u and v are adjacent in G. Let $n = |V(G)|$. The adjacency matrix of G, denoted by $A(G) = [a_{uv}]$ and indexed by the vertices of G, is the $n \times n$ symmetric matrix defined by $a_{uv} = 1$ if $uv \in E(G)$ and $a_{uv} = 0$ otherwise.

Let R be a graph. Two subgraphs G_1 and G_2 of R are said to be disjoint if $V(G_1) \cap V(G_2) = \emptyset$, that is, if they have no common vertices. Let $\mathcal{F} = \{G_1, G_2, \ldots, G_k\}$ be a family of graphs. We say that $\mathcal{F}$ is a family of factor graphs of R if each G_i is an induced subgraph of R (i.e., $G_i = R[V(G_i)]$), any two subgraphs G_i and G_j with $i \neq j$ are disjoint, and $V(R) = \bigcup_{1 \leq i \leq k} V(G_i)$. For each $1 \leq i \leq k$, let

$$V(G_i) = \{v_{i1}, v_{i2}, \ldots, v_{in_i}\}. \tag{1.1}$$

Then

$$V(R) = \{v_{11}, v_{12}, \ldots, v_{1n_1}, v_{21}, v_{22}, \ldots, v_{2n_2}, \ldots, v_{k1}, v_{k2}, \ldots, v_{kn_k}\}. \tag{1.2}$$

Under this ordering, the adjacency matrix of R can be written in block form as

$$A(R) = [B_{ij}], \tag{1.3}$$

where each block B_{ij} is an $n_i \times n_j$ matrix. For each $1 \le i \le k$, the diagonal block B_{ii} is the adjacency matrix $A(G_i)$, while for $i \ne j$, the off-diagonal block $B_{ij} = [b_{uv}]$ is the connecting matrix between G_i and G_j, defined by

$$b_{uv} = \begin{cases} 1, & \text{if } u \in V(G_i),\ v \in V(G_j), \text{ and } uv \in E(R), \\ 0, & \text{otherwise.} \end{cases}$$

Here, the adjacency matrix $A(G_i)$ is indexed according to the vertex ordering $v_{i1}, v_{i2}, \ldots, v_{in_i}$. Note that the transpose of B_{ij} is given by $B_{ij}^T = B_{ji}$.

Now, let $\mathcal{F} = \{G_1, G_2, \ldots, G_k\}$ be a family of graphs and let $\mathcal{B} = \{B_{ij} : 1 \le i < j \le k\}$ be a family of matrices, where each B_{ij} is an $n_i \times n_j$ $(0, 1)$-matrix whose rows and columns are indexed by the vertices of G_i and G_j, respectively. We define the graph R by specifying its vertex set as

$$V(R) = \bigcup_{1 \le i \le k} V(G_i), \tag{1.4}$$

and its edge set as

$$E(R) = \bigcup_{1 \le i \le k} E(G_i) \ \cup \ \bigcup_{1 \le i < j \le k} E'(B_{ij}), \tag{1.5}$$

where $B_{ij} = [b_{uv}]$ and

$$E'(B_{ij}) = \{uv : u \in V(G_i), v \in V(G_j), b_{uv} = 1\}. \tag{1.6}$$

Note that $\mathcal{F}$ is a family of factor graphs of the resultant graph R. The resultant graph R is called the joint of the family $\mathcal{F}$ with respect to the family of connecting matrices $\mathcal{B}$. Now, suppose that the vertices of each G_i are ordered as in Eq (1.1) and the vertices of R are ordered as in Eq (1.2). If we set $B_{ji} = B_{ij}^\top$ for $1 \le j < i \le k$ and $B_{ii} = A(G_i)$ for $1 \le i \le k$, then the adjacency matrix of R is given by Eq (1.3). It is readily seen that every graph can be realized as the joint of a suitable family of graphs with respect to a suitable family of connecting matrices.

Let H be a graph with vertex set $V(H) = \{1, 2, \ldots, k\}$. For $1 \le i < j \le k$, let $J_{n_i n_j}$ be the $n_i \times n_j$ matrix with all entries 1 and $J_{n_i} = J_{n_i n_i}$. Set $B_{ij} = \rho_{ij} J_{n_i n_j}$, where $\rho_{ij} = 1$ if i and j are adjacent in H and $\rho_{ij} = 0$ otherwise. A graph R is said to be a H-join of the family $\mathcal{F} = \{G_1, G_2, \ldots, G_k\}$ if R is the joint of the family $\mathcal{F}$ with respect to the family of connecting matrices $\mathcal{B} = \{B_{ij} : 1 \le i < j \le k\}$. The concept of the H-join operation was originally introduced by Schwenk [1]. Cardoso et al. [2] determined the spectrum of R by using a generalization of Fiedler's lemma [3]. When all the factor graphs G_i are isomorphic to the same graph G, the resultant graph R is called the lexicographic product of H and G [4]. Abreu et al. [5] determined the spectrum for the lexicographic product of H and G when G is a regular graph and H is the lexicographic product of a regular graph by itself. In [6], Wang and Wong obtained the characteristic polynomial for the adjacency matrix of the lexicographic product of arbitrary graphs. See also [7–9] for other related results in a more general setting.

In this paper, we characterize the connecting matrices $\mathcal{B}$ under certain conditions (see Theorems 2.6 and 3.6). We then consider the H_m-join of factor graphs, which is discussed in Section 4 and we derive the characteristic polynomial for the universal adjacency matrix of the H_m-join of factor graph (Theorem 4.1).

2. Strongly inherited eigenvalues

This section defines and explores the conditions under which eigenvalues of individual factor graphs are strongly inherited by the overall resultant graph structure.

Throughout this section, let R denote the joint of the family $\mathcal{F} = \{G_1, G_2, \ldots, G_k\}$ with respect to the family of connecting matrices $\mathcal{B} = \{B_{ij} : 1 \leq i < j \leq k\}$. We assume that Eqs (1.1)–(1.3) hold.

Let w be an eigenvalue of a matrix A and $\mathbf{e}$ be its corresponding eigenvector, i.e., $A\mathbf{e} = w\mathbf{e}$. The pair $(w, \mathbf{e})$ is called an eigenpair of A. The set of all eigenvalues of A is denoted by $\mathcal{S}(A)$. If A is an $n \times n$ matrix, then $\mathcal{S}(A)$ consists of all the roots of the characteristic polynomial of A, namely,

$$\phi_A(x) = \det(xI_n - A).$$

An $n \times n$ matrix A is said to have an eigenbasis $\mathcal{E}(A)$ if $\mathcal{E}(A)$ consists of n linearly independent eigenvectors of A. The eigenbasis $\mathcal{E}(A)$ is said to be orthogonal if $\mathbf{u}^\top \mathbf{v} = 0$ for every pair of distinct vectors $\mathbf{u}, \mathbf{v} \in \mathcal{E}(A)$. It is said to be orthonormal if it is orthogonal and $\mathbf{u}^\top \mathbf{u} = 1$ for every $\mathbf{u} \in \mathcal{E}(A)$. The spectral theorem states that every real symmetric matrix has an orthonormal eigenbasis consisting of real eigenvectors.

Let i be a fixed index and $\mathbf{e}$ be an $n_i \times 1$ matrix (i.e., an n_i-column vector). Define

$$\widehat{\mathbf{e}}^{(i)} = [\mathbf{0}_{n_1}^\top, \ldots, \mathbf{0}_{n_{i-1}}^\top, \mathbf{e}^\top, \mathbf{0}_{n_{i+1}}^\top, \ldots, \mathbf{0}_{n_k}^\top]^\top, \tag{2.1}$$

where $\mathbf{0}_n$ denotes the zero n-column vector. Let w be an eigenvalue of both $A(R)$ and $A(G_i)$ for some i, i.e., $w \in \mathcal{S}(A(R)) \cap \mathcal{S}(A(G_i))$. We say that w is strongly inherited from G_i if $(w, \widehat{\mathbf{e}}^{(i)})$ is an eigenpair of $A(R)$ for every eigenvector $\mathbf{e}$ of $A(G_i)$ corresponding to the eigenvalue w. Using this definition, we define the set

$$\mathrm{Str}(R, G_i) = \{w \in \mathcal{S}(A(R)) \cap \mathcal{S}(A(G_i)) \ : \ w \text{ is strongly inherited from } G_i\}. \tag{2.2}$$

Studying this new concept provides a framework for understanding how the spectral properties of individual factor graphs G_i behave when they are transferred to the resultant joint graph R. By analyzing these eigenvalues, it becomes possible to characterize the exact structural conditions required for this inheritance (see Theorem 2.6).

An isolated subgraph of a graph is a subgraph that is completely disconnected from the rest of the graph, that is, it has no edges connecting its vertices to vertices outside the subgraph.

Let $\mathbf{0}_{y_1 y_2}$ denote the zero $y_1 \times y_2$ matrix and $\mathbf{1}_n$ denote the n-column vector whose entries are all equal to 1. We denote the dimension of a vector space S by $\dim(S)$.

Lemma 2.1. *Let i_0 be a fixed index. If $\mathcal{S}(A(G_{i_0})) = \mathrm{Str}(R, G_{i_0})$, then $B_{ii_0} = \mathbf{0}_{n_i n_{i_0}}$ for all $1 \leq i < i_0$ and $B_{i_0 i} = \mathbf{0}_{n_{i_0} n_i}$ for all $i_0 < i \leq k$. Furthermore, G_{i_0} is an isolated subgraph of R.*

Proof. First, note that the conditions $B_{ii_0} = \mathbf{0}_{n_i n_{i_0}}$ for all $1 \leq i < i_0$ and $B_{i_0 i} = \mathbf{0}_{n_{i_0} n_i}$ for all $i_0 < i \leq k$ imply that G_{i_0} is an isolated subgraph of R.

Since $A(G_{i_0})$ is a real symmetric $n_{i_0} \times n_{i_0}$ matrix, it has an orthonormal eigenbasis consisting of real eigenvectors of $A(G_{i_0})$. Let

$$\mathcal{E}(A(G_{i_0})) = \{\mathbf{e}_{i_0 1}, \mathbf{e}_{i_0 2}, \ldots, \mathbf{e}_{i_0 n_{i_0}}\}$$

be such an eigenbasis. For each $1 \leq j \leq n_{i_0}$, let w_j be the eigenvalue associated with the eigenvector $\mathbf{e}_{i_0 j}$. Now, $w_j \in \mathrm{Str}(R, G_{i_0})$ implies that $(w_j, \widehat{\mathbf{e}}_{i_0 j}^{(i_0)})$ is an eigenpair of $A(R)$. Since

$$A(R)\mathbf{e}_{i_0 j}^{(i_0)} = w_j \mathbf{e}_{i_0 j}^{(i_0)},$$

it follows from Eq (2.1) that

$$B_{ti_0}\mathbf{e}_{i_0 j} = \mathbf{0}_{n_t}, \text{ for all } 1 \leq t \leq k, t \neq i_0. \tag{2.3}$$

Now, every row of B_{ti_0} can be expressed as a linear combination of the transposes of the eigenvectors in $\mathcal{E}(A(G_{i_0}))$, because $\mathcal{E}(A(G_{i_0}))$ is an an eigenbasis. A particular row of B_{ti_0} is of the form

$$c_1\mathbf{e}_{i_0 1}^\top + c_2\mathbf{e}_{i_0 2}^\top + \cdots + c_{n_{i_0}}\mathbf{e}_{i_0 n_{i_0}}^\top,$$

for some scalars $c_1, c_2, \ldots, c_{n_{i_0}}$. By considering this row in Eq (2.3), we have

$$(c_1\mathbf{e}_{i_0 1}^\top + c_2\mathbf{e}_{i_0 2}^\top + \cdots + c_{n_{i_0}}\mathbf{e}_{i_0 n_{i_0}}^\top)\mathbf{e}_{i_0 j} = 0.$$

It follows from

$$\mathbf{e}_{i_0 j'}^\top \mathbf{e}_{i_0 j} = 0 \quad \text{for all } j' \neq j, \text{ and } \quad \mathbf{e}_{i_0 j}^\top \mathbf{e}_{i_0 j} = 1$$

that $c_j = 0$. Since Eq (2.3) holds for every $1 \leq j \leq n_{i_0}$, we have $c_j = 0$ for all j. This means every row of B_{ti_0} is the zero row. Hence, $B_{ti_0} = \mathbf{0}_{n_t n_{i_0}}$.

The lemma follows by noting that $B_{i_0 i} = B_{i i_0}^\top$ for every $i_0 < i \leq k$. □

Remark 2.2. Lemma 2.1 states that if every eigenvalue of a specific factor graph G_{i_0} is strongly inherited by the resultant graph R, i.e., $\mathcal{S}(A(G_{i_0})) = \mathrm{Str}(R, G_{i_0})$, then the necessary consequence for the graph's structure is that the connecting matrices to and from G_{i_0} must be zero. Building upon this, Theorem 2.3 provides the reverse consequence as well as the forward consequence, but applied to the entire collection of subgraphs rather than just one subgraph.

Theorem 2.3. *$\mathcal{S}(A(G_i)) = \mathrm{Str}(R, G_i)$ for every $1 \leq i \leq k$, if and only if, R is the disjoint union of the graphs $G_1, G_2, \ldots, G_k$.*

Proof. Suppose $\mathcal{S}(A(G_i)) = \mathrm{Str}(R, G_i)$ for every $1 \leq i \leq k$. By Lemma 2.1, each G_i is an isolated subgraph of R. The converse is obvious. □

Let A be an $n \times n$ symmetric matrix, and let w_1, w_2 be distinct eigenvalues of A with corresponding eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively. Then, $A\mathbf{e}_1 = w_1\mathbf{e}_1$ and $A\mathbf{e}_2 = w_2\mathbf{e}_2$. Taking the transpose of the second equation yields $\mathbf{e}_2^\top A = w_2\mathbf{e}_2^\top$. Therefore,

$$w_2\mathbf{e}_2^\top\mathbf{e}_1 = (\mathbf{e}_2^\top A)\mathbf{e}_1 = \mathbf{e}_2^\top(A\mathbf{e}_1) = w_1\mathbf{e}_2^\top\mathbf{e}_1.$$

Since $w_2 \neq w_1$, it follows that $\mathbf{e}_2$ and $\mathbf{e}_1$ are orthogonal, i.e.,

$$\mathbf{e}_2^\top\mathbf{e}_1 = 0. \tag{2.4}$$

For each $w \in \mathcal{S}(A)$, let $\xi_w(A)$ denote the real eigenspace corresponding to the eigenvalue w, i.e.,

$$\xi_w(A) = \{\mathbf{e} \in \mathbb{R}^n : A\mathbf{e} = w\mathbf{e}\}.$$

Lemma 2.4. *Let G be a connected l-regular graph with n vertices, where $l \geq 0$. Then, l is an eigenvalue of $A(G)$, with corresponding eigenvector $\mathbf{1}_n$, and the eigenspace $\xi_l(A(G))$ has dimension 1. Furthermore, if $\mathbf{e}$ is an eigenvector of $A(G)$ corresponding to an eigenvalue different from l, then*

$$\mathbf{1}_n^\top \mathbf{e} = 0.$$

Proof. If $l = 0$, then G being connected and 0-regular implies $n = 1$ and $A(G) = [0]$. The sole eigenvalue of $A(G)$ is $l = 0$ with eigenvector $\mathbf{1}_1 = [1]$, $\dim(\xi_0(A(G))) = 1$, and the condition $\mathbf{1}_n^\top \mathbf{e} = 0$ holds as there are no other eigenvalues. So, it is left to consider the case $l \geq 1$.

We may assume that $V(G) = \{1, 2, \ldots, n\}$. Clearly, $(l, \mathbf{1}_n)$ is an eigenpair of $A(G) = [a_{ij}]$. Let $\mathbf{x} = [x_1, x_2, \ldots, x_n]^\top$ be an eigenvector of $A(G)$ corresponding to the eigenvalue l. Choose an index i_0 such that

$$|x_{i_0}| = \max\{|x_i| : 1 \leq i \leq n\}.$$

Considering the i_0-th row of the equation $A(G)\mathbf{x} = l\mathbf{x}$, we obtain

$$a_{i_0 1}x_1 + a_{i_0 2}x_2 + \cdots + a_{i_0 n}x_n = lx_{i_0}. \tag{2.5}$$

Since $\sum_{1 \leq j \leq n} a_{i_0 j} = l$, it follows that

$$\begin{aligned} l|x_{i_0}| &\leq a_{i_0 1}|x_1| + a_{i_0 2}|x_2| + \cdots + a_{i_0 n}|x_n|, \\ &\leq (a_{i_0 1} + a_{i_0 2} + \cdots + a_{i_0 n})\,|x_{i_0}|, \\ &\leq l|x_{i_0}|. \end{aligned}$$

This implies that $|x_j| = |x_{i_0}|$ for every vertex j adjacent to the vertex i_0 in G. As G is connected, it follows that $|x_j| = |x_{i_0}|$ for every $1 \leq j \leq n$.

Let's consider Eq (2.5) again. Since $\sum_{1 \leq j \leq n} a_{i_0 j} = l$, we have $a_{i_0 j} = 1$ for $j \in \{j_1, j_2, \ldots, j_l\}$ and $a_{i_0 j} = 0$ for all other j, including $j = i_0$. Therefore,

$$x_{j_1} + x_{j_2} + \cdots + x_{j_l} = lx_{i_0}. \tag{2.6}$$

If x_{j_r} and $x_{j_{r'}}$ have opposite signs for some $1 \leq r < r' \leq l$, then $x_{j_r} = -x_{j_{r'}}$. Taking the absolute value of the sum on the the left yields

$$|x_{j_1} + x_{j_2} + \cdots + x_{j_l}| = \Big|\sum_{1 \leq s \leq l} x_{j_s}\Big| = \Big|\sum_{\substack{1 \leq s \leq l, \\ s \notin \{r, r'\}}} x_{j_s}\Big| \leq (l-2)|x_{i_0}|.$$

By Eq (2.5), we get $l|x_{i_0}| \leq (l-2)|x_{i_0}|$, a contradiction. Hence, x_{j_r} and $x_{j_{r'}}$ must have the same sign for all $1 \leq r < r' \leq l$. It follows from Eq (2.5) that x_{i_0} and x_{j_r} are of the same sign. So, $x_j = x_{i_0}$ for every vertex j adjacent to the vertex i_0 in G. As G is connected, it follows that $x_j = x_{i_0}$ for every $1 \leq j \leq n$.

Thus, $\mathbf{x} = x_{i_0}\mathbf{1}_n$ and the eigenspace $\xi_l(A(G))$ has dimension 1. This completes the proof of the lemma, with the final statement following from Eq (2.4). □

Lemma 2.5. *Let G be a graph with n vertices. If $A(G)$ has an eigenvector $\mathbf{e} = [x_1, x_2, \ldots, x_n]^\top$ with entries in $\{0, 1\}$ and $x_{i_0} = 0$ for some i_0, then G is not connected.*

Proof. We may assume that $V(G) = \{1, 2, \ldots, n\}$. Let w be the eigenvalue corresponding to the eigenvector $\mathbf{e}$. Then $A(G)\mathbf{e} = w\mathbf{e}$. Let $A(G) = [a_{ij}]$ and define $D = \{i \ : x_i = 0\}$. Note that $D \neq \emptyset$ since $i_0 \in D$. Since $\mathbf{e}$ is an eigenvector of $A(G)$, it is a nonzero vector. So, $D \subsetneq V(G)$. Consider the i-th row of the equation $A(G)\mathbf{e} = w\mathbf{e}$, where $i \in D$. Then

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = 0.$$

Since $a_{ij}, x_j \in \{0, 1\}$ for all j, every term in the above sum is nonnegative. Therefore, $a_{ij}x_j = 0$ for all j, which implies that $a_{ij} = 0$ for all $j \notin D$. Thus, i is not adjacent to any vertex $j \notin D$. Since i was chosen arbitrarily from D, no vertex in D is adjacent to any vertex in $V(G) \setminus D$. Hence, there are no edges between D and $V(G) \setminus D$, so G is not connected. □

Theorem 2.6. *Suppose that G_i is connected for each $1 \leq i \leq k$ with $k \geq 2$, and R is connected. Then, $\mathcal{S}(A(G_i)) \setminus \mathrm{Str}(R, G_i) = \{w_{i1}\}$ and $\dim(\xi_{w_{i1}}(A(G_i))) = 1$ for every $1 \leq i \leq k$, if and only if, every G_i is a w_{i1}-regular graph and $B_{ij} = \mathbf{0}_{n_i n_j}$ or $B_{ij} = \mathbf{1}_{n_i}\mathbf{1}_{n_j}^\top$ for all $1 \leq i, j \leq k$ with $i \neq j$.*

Proof. Suppose $\mathcal{S}(A(G_i)) \setminus \mathrm{Str}(R, G_i) = \{w_{i1}\}$ and $\dim(\xi_{w_{i1}}(A(G_i))) = 1$ for every $1 \leq i \leq k$. Let

$$\mathcal{E}(A(G_i)) = \{\mathbf{e}_{i1}, \mathbf{e}_{i2}, \ldots, \mathbf{e}_{in_i}\}$$

be an orthonormal real eigenbasis of $A(G_i)$. By relabeling if necessary, we may assume that $\mathbf{e}_{i1}$ is an eigenvector of $A(G_i)$ corresponding to the eigenvalue w_{i1}.

Let

$$\mathrm{Support}(\mathcal{B}) = \{(i, j) : B_{ij} \neq \mathbf{0}_{n_i n_j}, 1 \leq i, j \leq k, i \neq j\}.$$

Since R is connected, $\mathrm{Support}(\mathcal{B}) \neq \emptyset$. Let $(i_0, j_0) \in \mathrm{Support}(\mathcal{B})$. For $2 \leq j \leq n_{j_0}$, let $w_{j_0 j}$ be the eigenvalue of $A(G_{j_0})$ corresponding to the eigenvector $\mathbf{e}_{j_0 j}$. Since $\mathcal{S}(A(G_{j_0})) \setminus \mathrm{Str}(R, G_{j_0}) = \{w_{j_0 1}\}$ and $\dim(\xi_{w_{j_0 1}}(A(G_{j_0}))) = 1$, it follows that $w_{j_0 j}$ is an eigenvalue of $A(R)$ and is strongly inherited from G_{j_0} for every $2 \leq j \leq j_0$. This means that $(w_{j_0 j}, \widehat{\mathbf{e}}_{j_0 j}^{(j_0)})$ is an eigenpair of $A(R)$. Since

$$A(R)\widehat{\mathbf{e}}_{j_0 j}^{(j_0)} = w_{j_0 j}\widehat{\mathbf{e}}_{j_0 j}^{(j_0)},$$

it follows from Eqs (1.3) and (2.1) that

$$B_{i_0 j_0}\mathbf{e}_{j_0 j} = \mathbf{0}_{n_{i_0}}. \tag{2.7}$$

Note that Eq(2.7) holds for every $2 \leq j \leq n_{j_0}$.

Now, each row of $B_{i_0 j_0}$ can be expressed as a linear combination of the transposes of eigenvectors in $\mathcal{E}(A(G_{j_0}))$. Specifically, each row of $B_{i_0 j_0}$ has the form

$$c_1\mathbf{e}_{j_0 1}^\top + c_2\mathbf{e}_{j_0 2}^\top + \cdots + c_{n_{j_0}}\mathbf{e}_{j_0 n_{j_0}}^\top \tag{2.8}$$

for some scalars $c_1, c_2, \ldots, c_{n_{j_0}}$. Since the eigenvectors in $\mathcal{E}(A(G_{j_0}))$ are orthonormal, we have

$$\mathbf{e}_{j_0 j'}^\top \mathbf{e}_{j_0 j} = 0 \quad \text{for all } j' \neq j, \text{ while } \mathbf{e}_{j_0 j}^\top \mathbf{e}_{j_0 j} = 1.$$

Combining this with Expression (2.8), a row of Eq (2.7) can be written as

$$(c_1\mathbf{e}_{j_0 1}^\top + c_2\mathbf{e}_{j_0 2}^\top + \cdots + c_{n_{j_0}}\mathbf{e}_{j_0 n_{j_0}}^\top)\mathbf{e}_{j_0 j} = 0,$$

which implies that $c_j = 0$. This is true for all $2 \leq j \leq n_{j_0}$. So, we conclude that all coefficients except the first one vanish for this particular row. Thus, each row of $B_{i_0 j_0}$ is a scalar multiple of $\mathbf{e}_{j_0 1}^\top$, i.e.,

$$B_{i_0 j_0} = \begin{bmatrix} d_1 \mathbf{e}_{j_0 1}^\top \\ d_2 \mathbf{e}_{j_0 1}^\top \\ \vdots \\ d_{n_{i_0}} \mathbf{e}_{j_0 1}^\top \end{bmatrix}, \tag{2.9}$$

for some scalars $d_1, d_2, \ldots, d_{n_{i_0}}$.

Let $\mathbf{e}_{j_0 1} = [y_1, y_2, \ldots, y_{n_{j_0}}]^\top$. Suppose that $y_{t_1}, y_{t_2} \neq 0$ for some $t_1 \neq t_2$. Choose a nonzero row of $B_{i_0 j_0}$. Since $B_{i_0 j_0}$ is a $n_{i_0} \times n_{j_0}$ $(0,1)$-matrix, each entry of this row is either 0 or 1. This row is of the form $c\mathbf{e}_{j_0 1}^\top$ for some nonzero scalar c. Therefore, the t_1-th and t_2-th entries of this row satisfy $cy_{t_1} = 1 = cy_{t_2}$, which implies that $y_{t_1} = y_{t_2}$. This means that all nonzero entries of $\mathbf{e}_{j_0 1}$ are equal. Hence, there exist a scalar c' and a n_{j_0}-column vector $\mathbf{x}$ with entries in $\{0, 1\}$ such that $\mathbf{e}_{j_0 1} = c'\mathbf{x}$. This also implies that $\mathbf{x}$ is an eigenvector of $A(G_{j_0})$. Since G_{j_0} is connected, it follows from Lemma 2.5 that $\mathbf{x} = \mathbf{1}_{n_{j_0}}$. Therefore, $A(G_{j_0})\mathbf{1}_{n_{j_0}} = w_{j_0 1}\mathbf{1}_{n_{j_0}}$, which implies that every vertex of G_{j_0} has degree $w_{j_0 1}$. Hence, G_{j_0} is a $w_{j_0 1}$-regular graph. Moreover, each nonzero row of $B_{i_0 j_0}$ is a scalar multiple of $\mathbf{1}_{n_{j_0}}^T$. It follows from Eq (2.9) that

$$B_{i_0 j_0} = \begin{bmatrix} d_1 \mathbf{1}_{n_{j_0}}^\top \\ d_2 \mathbf{1}_{n_{j_0}}^\top \\ \vdots \\ d_{n_{i_0}} \mathbf{1}_{n_{j_0}}^\top \end{bmatrix}.$$

Furthermore, $d_i \in \{0, 1\}$ for all i. Now,

$$B_{j_0 i_0} = B_{i_0 j_0}^\top = \begin{bmatrix} d_1 \mathbf{1}_{n_{j_0}} & d_2 \mathbf{1}_{n_{j_0}} & \cdots & d_{n_{i_0}} \mathbf{1}_{n_{j_0}} \end{bmatrix}.$$

By using a similar argument as above, each nonzero row of $B_{j_0 i_0}$ is a scalar multiple of $\mathbf{1}_{n_{i_0}}^\top$. Therefore, $d_i = 1$ for all i. Hence,

$$B_{i_0 j_0} = J_{n_{i_0}, n_{j_0}} = \mathbf{1}_{n_{i_0}} \mathbf{1}_{n_{j_0}}^\top.$$

For any j_0, the connectedness of R implies that there exists some $i_0 \neq j_0$ such that a vertex in G_{i_0} is adjacent to a vertex in G_{j_0}. So, $(i_0, j_0) \in \text{Support}(\mathcal{B})$. Applying the argument above to this (i_0, j_0), we conclude that G_{j_0} is a $w_{j_0 1}$-regular graph. Moreover, for any $1 \leq i, j \leq k$ with $i \neq j$, if $B_{ij} \neq \mathbf{0}_{n_i n_j}$, then $B_{ij} = \mathbf{1}_{n_i} \mathbf{1}_{n_j}^\top$.

Conversely, suppose that G_i is a w_{i1}-regular graph for every $1 \leq i \leq k$ and $B_{ij} = \mathbf{0}_{n_i n_j}$ or $B_{ij} = \mathbf{1}_{n_i} \mathbf{1}_{n_j}^\top$ for all $1 \leq i, j \leq k$ with $i \neq j$. Let w be an eigenvalue of $A(G_i)$ with $w \neq w_{i1}$. Consider the eigenspace $\xi_w(A(G_i))$. By Lemma 2.4, $\mathbf{1}_{n_i}^\top \mathbf{e} = 0$ for every eigenvector $\mathbf{e} \in \xi_w(A(G_i))$. Since $B_{ji} = \mathbf{0}_{n_j n_i}$ or $B_{ji} = \mathbf{1}_{n_j} \mathbf{1}_{n_i}^\top$ for every $j \neq i$, we have $B_{ji}\mathbf{e} = \mathbf{0}_{n_j}$. By Eqs (1.3) and (2.1), $(w, \widehat{\mathbf{e}^{(i)}})$ is an eigenpair of $A(R)$. Hence, $w \in \text{Str}(R, G_i)$ for every eigenvalue w of $A(G_i)$ other than w_{i1}.

By Lemma 2.4, $(w_{i1}, \mathbf{1}_{n_i})$ is an eigenpair of $A(G_i)$ and $\dim(\xi_{w_{i1}}(A(G_i))) = 1$. Suppose that $(w_{i1}, \widehat{\mathbf{1}_{n_i}^{(i)}})$ is an eigenpair of $A(R)$. Then, from $A(R)\widehat{\mathbf{1}_{n_i}^{(i)}} = w_{i1}\widehat{\mathbf{1}_{n_i}^{(i)}}$, we obtain

$$B_{i'i}\mathbf{1}_{n_i} = \mathbf{0}_{n_{i'}} \text{ for all } i' \neq i. \tag{2.10}$$

Since R is connected, there is an index $i_0 \neq i$ such that $B_{i_0 i} \neq \mathbf{0}_{n_{i_0} n_i}$. So, $B_{i_0 i} = \mathbf{1}_{n_{i_0}} \mathbf{1}_{n_i}^\top$. It then follows from Eq (2.10) that

$$n_i \mathbf{1}_{n_{i_0}} = B_{i_0 i} \mathbf{1}_{n_i} = \mathbf{0}_{n_{i'}},$$

a contradition. Hence, $(w_{i1}, \widehat{\mathbf{1}_{n_i}^{(i)}})$ is not an eigenpair of $A(R)$, and consequently,

$$\mathcal{S}(A(G_i)) \setminus \mathrm{Str}(R, G_i) = \{w_{i1}\}.$$

This completes the proof of the theorem. □

Example 2.7. Let G_1 be a triangle. Then, it is a 2-regular graph with adjacency matrix

$$A(G_1) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Its eigenvalues are $-1, -1, 2$. Let G_2 be a single edge. Then, it is a 1-regular graph with adjacency matrix

$$A(G_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Its eigenvalues are $-1, 1$. Let the connecting matrix be

$$B_{12} = \mathbf{1}_3 \mathbf{1}_2^\top = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then, the adjacency matrix of the resultant graph R is

$$A(R) = \begin{pmatrix} A(G_1) & B_{12} \\ B_{12}^\top & A(G_2) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix},$$

which has eigenvalues $-1, -1, -1, -1, 4$. By Theorem 2.6, it follows that $\mathcal{S}(A(G_1)) \setminus \mathrm{Str}(R, G_1) = \{2\}$, $\dim(\xi_2(A(G_1))) = 1$, $\mathcal{S}(A(G_2)) \setminus \mathrm{Str}(R, G_2) = \{1\}$, and $\dim(\xi_1(A(G_2))) = 1$.

3. Cross-eigenvectors

This section identifies and analyzes cross-eigenvectors that arise when identical factor graphs are structurally joined over a host graph H.

Let $\mathcal{F} = \{G_1, G_2, \ldots, G_k\}$ be a family of graphs, where the vertex set of each G_i is given as in Eq (1.1). Throughout this section, we assume that G_1 is isomorphic to each G_i under the mapping $v_{1j} \mapsto v_{ij}$ for all $1 \leq j \leq n_1$. Here, $n_1 = n_2 = \cdots = n_k$.

Let H be a graph with vertex set $V(H) = \{1, 2, \ldots, k\}$ and adjacency matrix

$$A(H) = [\rho_{ij}]. \tag{3.1}$$

Let B be an $n_1 \times n_1$ symmetric $(0,1)$-matrix, and define the family of connecting matrices $\mathcal{B} = \{B_{ij} : 1 \leq i < j \leq k\}$, where

$$B_{ij} = \rho_{ij} B. \tag{3.2}$$

The graph resulting from the joint of the family $\mathcal{F}$ with respect to the family of connecting matrices $\mathcal{B}$ is denoted by $(G_1; H; B)$. For convenience, we write $(G; H; B)$ in place of $(G_1; H; B)$ and let $n = n_1$. Note that

$$A((G;H;B)) = \begin{bmatrix} A(G) & \rho_{12}B & \rho_{13}B & \cdots & \rho_{1k}B \\ \rho_{21}B & A(G) & \rho_{23}B & \cdots & \rho_{2k}B \\ \rho_{31}B & \rho_{32}B & A(G) & \cdots & \rho_{3k}B \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{k1}B & \rho_{k2}B & \rho_{k3}B & \cdots & A(G) \end{bmatrix}, \tag{3.3}$$

and each block is an $n \times n$ matrix.

When $B = J_n$, the graph $(G; H; B)$ is precisely the H-join of the family $\mathcal{F}$, or equivalently, the lexicographic product of H and G. If H is the empty graph, then $(G; H; B)$ is the disjoint union of k copies of G. So, throughout the remainder of this section, we assume that H is not the empty graph.

Let $a_1, a_2, \ldots, a_{l_1}$ be all the distinct eigenvalues of $A(G)$, and let $b_1, b_2, \ldots, b_{l_2}$ be all the distinct eigenvalues of B. For each $1 \leq i \leq l_1$, there exists a subspace $U_i \subseteq \mathbb{R}^n$ such that

$$\xi_{a_i}(A(G)) = (\xi_{a_i}(A(G)) \cap \xi_{b_1}(B)) \oplus (\xi_{a_i}(A(G)) \cap \xi_{b_2}(B)) \oplus \cdots \oplus (\xi_{a_i}(A(G)) \cap \xi_{b_{l_2}}(B)) \oplus U_i. \tag{3.4}$$

Note that U_i may be the zero subspace. Let

$$U = U_1 \oplus U_2 \oplus \cdots \oplus U_{l_1}. \tag{3.5}$$

Given a vector $\mathbf{z} = [z_1, z_2, \ldots, z_k]^\top \in \mathbb{R}^k$, we denote its i-th entry by $\mathbf{z}[i]$, that is, $\mathbf{z}[i] = z_i$. A vector $\mathbf{v} \in \mathbb{R}^{nk}$ is called a cross-vector of $\mathbb{R}^k$ and $\mathbb{R}^n$ if there exist vectors $\mathbf{z} \in \mathbb{R}^k$ and $\mathbf{u} \in \mathbb{R}^n$ such that

$$\mathbf{v} = \begin{bmatrix} \mathbf{z}[1]\,\mathbf{u} \\ \mathbf{z}[2]\,\mathbf{u} \\ \vdots \\ \mathbf{z}[k]\,\mathbf{u} \end{bmatrix}. \tag{3.6}$$

We denote such a vector by $\mathrm{Cr}(\mathbf{z}, \mathbf{u})$.

Previous studies on the standard H-join operation restricted the connecting matrix to $B_{ij} = J_n$. The concept of cross-eigenvectors is studied here because it enables the spectral analysis of resultant graphs using a much broader class of connecting matrices (see Theorem 3.6 and Corollary 3.8).

Lemma 3.1. *Let $\mathcal{M}_1 \subseteq \mathbb{R}^k$ and $\mathcal{M}_2 \subseteq \mathbb{R}^n$. If $\mathbf{a}$ is a linear combination of vectors in $\mathcal{M}_1$ and $\mathbf{b}$ is a linear combination of vectors in $\mathcal{M}_2$, then $\mathrm{Cr}(\mathbf{a}, \mathbf{b})$ is a linear combination of vectors in*

$$\{\mathrm{Cr}(\mathbf{z}, \mathbf{u}) : \mathbf{z} \in \mathcal{M}_1, \mathbf{u} \in \mathcal{M}_2\}.$$

Proof. Let

$$\mathbf{a} = \sum_{\mathbf{m}_1 \in \mathcal{M}_1} c_{\mathbf{m}_1} \mathbf{m}_1 \quad \text{and} \quad \mathbf{b} = \sum_{\mathbf{m}_2 \in \mathcal{M}_2} c_{\mathbf{m}_2} \mathbf{m}_2.$$

Then, for each $1 \leq i \leq k$,

$$\mathbf{a}[i] = \sum_{\mathbf{m}_1 \in \mathcal{M}_1} c_{\mathbf{m}_1} \mathbf{m}_1[i].$$

Therefore,

$$\begin{aligned}
\mathrm{Cr}(\mathbf{a}, \mathbf{b}) &= \begin{bmatrix} \mathbf{a}[1]\mathbf{b} \\ \mathbf{a}[2]\mathbf{b} \\ \vdots \\ \mathbf{a}[k]\mathbf{b} \end{bmatrix} = \sum_{\mathbf{m}_1 \in \mathcal{M}_1} c_{\mathbf{m}_1} \begin{bmatrix} \mathbf{m}_1[1]\mathbf{b} \\ \mathbf{m}_1[2]\mathbf{b} \\ \vdots \\ \mathbf{m}_1[k]\mathbf{b} \end{bmatrix} \\
&= \sum_{\mathbf{m}_1 \in \mathcal{M}_1} c_{\mathbf{m}_1} \left(\sum_{\mathbf{m}_2 \in \mathcal{M}_2} c_{\mathbf{m}_2} \begin{bmatrix} \mathbf{m}_1[1]\mathbf{m}_2 \\ \mathbf{m}_1[2]\mathbf{m}_2 \\ \vdots \\ \mathbf{m}_1[k]\mathbf{m}_2 \end{bmatrix} \right) \\
&= \sum_{\mathbf{m}_1 \in \mathcal{M}_1} c_{\mathbf{m}_1} \left(\sum_{\mathbf{m}_2 \in \mathcal{M}_2} c_{\mathbf{m}_2} \mathrm{Cr}(\mathbf{m}_1, \mathbf{m}_2) \right) \\
&= \sum_{\mathbf{m}_1 \in \mathcal{M}_1,\ \mathbf{m}_2 \in \mathcal{M}_2} c_{\mathbf{m}_1} c_{\mathbf{m}_2} \mathrm{Cr}(\mathbf{m}_1, \mathbf{m}_2).
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 3.2. *Let $\mathcal{M}_1 \subseteq \mathbb{R}^k$ and $\mathcal{M}_2 \subseteq \mathbb{R}^n$ be two linearly independent sets. Then*

$$\{\mathrm{Cr}(\mathbf{z}, \mathbf{u}) : \mathbf{z} \in \mathcal{M}_1, \mathbf{u} \in \mathcal{M}_2\}$$

is a linearly independent set.

Proof. Suppose that

$$\sum_{\mathbf{z} \in \mathcal{M}_1,\ \mathbf{u} \in \mathcal{M}_2} c_{\mathbf{z},\mathbf{u}} \mathrm{Cr}(\mathbf{z}, \mathbf{u}) = \mathbf{0}_{nk}. \tag{3.7}$$

Note that

$$\begin{aligned}
\sum_{\mathbf{z} \in \mathcal{M}_1,\ \mathbf{u} \in \mathcal{M}_2} c_{\mathbf{z},\mathbf{u}} \mathrm{Cr}(\mathbf{z}, \mathbf{u}) &= \sum_{\mathbf{u} \in \mathcal{M}_2} \left(\sum_{\mathbf{z} \in \mathcal{M}_1} c_{\mathbf{z},\mathbf{u}} \mathrm{Cr}(\mathbf{z}, \mathbf{u}) \right) \\
&= \sum_{\mathbf{u} \in \mathcal{M}_2} \left(\sum_{\mathbf{z} \in \mathcal{M}_1} c_{\mathbf{z},\mathbf{u}} \begin{bmatrix} \mathbf{z}[1]\mathbf{u} \\ \mathbf{z}[2]\mathbf{u} \\ \vdots \\ \mathbf{z}[k]\mathbf{u} \end{bmatrix} \right)
\end{aligned}$$

$$= \sum_{\mathbf{u}\in\mathcal{M}_2} \begin{pmatrix} \begin{bmatrix} \sum_{\mathbf{z}\in\mathcal{M}_1} c_{\mathbf{z},\mathbf{u}}\mathbf{z}[1]\mathbf{u} \\ \sum_{\mathbf{z}\in\mathcal{M}_1} c_{\mathbf{z},\mathbf{u}}\mathbf{z}[2]\mathbf{u} \\ \vdots \\ \sum_{\mathbf{z}\in\mathcal{M}_1} c_{\mathbf{z},\mathbf{u}}\mathbf{z}[k]\mathbf{u} \end{bmatrix} \end{pmatrix}.$$

Using the linear independence of $\mathcal{M}_2$ together with Eq (3.7), we obtain that for all $\mathbf{u} \in \mathcal{M}_2$ and $1 \le i \le k$,

$$\sum_{\mathbf{z}\in\mathcal{M}_1} c_{\mathbf{z},\mathbf{u}}\mathbf{z}[i] = 0.$$

This implies that, for every $\mathbf{u} \in \mathcal{M}_2$,

$$\sum_{\mathbf{z}\in\mathcal{M}_1} c_{\mathbf{z},\mathbf{u}}\mathbf{z} = \mathbf{0}_k.$$

By the linear independence of $\mathcal{M}_1$, we obtain $c_{\mathbf{z},\mathbf{u}} = 0$ for all $\mathbf{z} \in \mathcal{M}_1, \mathbf{u} \in \mathcal{M}_2$. Hence, the lemma follows. □

A vector $\mathbf{v}$ of $A((G; H; B))$ is called a cross-eigenvector of H and G if there exist an eigenvector $\mathbf{z}$ of $A(H)$ and an eigenvector $\mathbf{u}$ of $A(G)$ such that

$$\mathbf{v} = \mathrm{Cr}(\mathbf{z}, \mathbf{u}). \tag{3.8}$$

By Eq (3.4), we may choose an orthonormal basis for the eigenspace $\xi_{a_i}(A(G))$ consisting of orthonormal bases for the subspaces

$$\xi_{a_i}(A(G)) \cap \xi_{b_j}(B) \quad (1 \le j \le l_2)$$

and the subspace U_i. Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be the union of all the chosen orthonormal bases for the eigenspaces $\xi_{a_i}(A(G))$ $(1 \le i \le l_1)$. Then, $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthonormal eigenbasis of $A(G)$.

Let $\{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k\}$ be an orthonormal eigenbasis of $A(H)$. Define

$$\Xi_0 = \{\mathrm{Cr}(\mathbf{z}_i, \mathbf{u}_j) : 1 \le i \le k, 1 \le j \le n\}. \tag{3.9}$$

Note that

$$\mathrm{Cr}(\mathbf{z}_{i_1}, \mathbf{u}_{j_1})^\top \mathrm{Cr}(\mathbf{z}_{i_2}, \mathbf{u}_{j_2}) = (\mathbf{z}_{i_1}^\top \mathbf{z}_{i_2})(\mathbf{u}_{j_1}^\top \mathbf{u}_{j_2}).$$

Since $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ and $\{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k\}$ are orthonormal, $\mathrm{Cr}(\mathbf{z}_{i_1}, \mathbf{u}_{j_1})^\top \mathrm{Cr}(\mathbf{z}_{i_2}, \mathbf{u}_{j_2}) = 0$ for all $(i_1, j_1) \ne (i_2, j_2)$, and $\mathrm{Cr}(\mathbf{z}_{i_1}, \mathbf{u}_{j_1})^\top \mathrm{Cr}(\mathbf{z}_{i_2}, \mathbf{u}_{j_2}) = 1$ when $(i_1, j_1) = (i_2, j_2)$. Hence,

$$\Xi_0 \text{ forms an orthonormal basis for } \mathbb{R}^{nk}. \tag{3.10}$$

A matrix $A = [a_{ij}]$ is called a diagonal matrix if $a_{ij} = 0$ whenever $i \ne j$. We denote the diagonal matrix by $\mathrm{diag}(a_{11}, a_{22}, \ldots, a_{kk})$.

Lemma 3.3. *Let $(w, \mathbf{u})$ be an eigenpair of $A(G)$, and let $(x, \mathbf{u})$ be an eigenpair of B. If $(y, \mathbf{z})$ is an eigenpair of $A(H)$, then $(w + xy, \mathrm{Cr}(\mathbf{z}, \mathbf{u}))$ is an eigenpair of $A((G; H; B))$.*

Proof. Let $\mathbf{z} = [z_1, z_2, \ldots, z_k]^\top$. From $A(H) = [\rho_{ij}]$ and $A(H)\mathbf{z} = y\mathbf{z}$, we obtain

$$\begin{aligned} z_1\rho_{11} + z_2\rho_{12} + \cdots + z_k\rho_{1k} &= yz_1, \\ z_1\rho_{21} + z_2\rho_{22} + \cdots + z_k\rho_{2k} &= yz_2, \\ &\vdots \\ z_1\rho_{k1} + z_2\rho_{k2} + \cdots + z_k\rho_{kk} &= yz_k. \end{aligned} \tag{3.11}$$

By Eq (3.3) and the fact that $\rho_{ii} = 0$ for all i, we obtain

$$\begin{aligned} & A((G; H; B))\,\mathrm{Cr}(\mathbf{z}, \mathbf{u}) \\ &= \begin{bmatrix} z_1A(G)\mathbf{u} + z_2\rho_{12}B\mathbf{u} + z_3\rho_{13}B\mathbf{u} + \cdots + z_k\rho_{1k}B\mathbf{u} \\ z_1\rho_{21}B\mathbf{u} + z_2A(G)\mathbf{u} + z_3\rho_{23}B\mathbf{u} + \cdots + z_k\rho_{2k}B\mathbf{u} \\ \vdots \\ z_1\rho_{k1}B\mathbf{u} + z_2\rho_{k2}B\mathbf{u} + \cdots + z_{k-1}\rho_{k,k-1}B\mathbf{u} + z_kA(G)\mathbf{u} \end{bmatrix} \\ &= \begin{bmatrix} z_1w\mathbf{u} + z_2\rho_{12}x\mathbf{u} + z_3\rho_{13}x\mathbf{u} + \cdots + z_k\rho_{1k}x\mathbf{u} \\ z_1\rho_{21}x\mathbf{u} + z_2w\mathbf{u} + z_3\rho_{23}x\mathbf{u} + \cdots + z_k\rho_{2k}x\mathbf{u} \\ \vdots \\ z_1\rho_{k1}x\mathbf{u} + z_2\rho_{k2}x\mathbf{u} + \cdots + z_{k-1}\rho_{k,k-1}x\mathbf{u} + z_kw\mathbf{u} \end{bmatrix} \\ &= \begin{bmatrix} z_1w\mathbf{u} + z_1xy\mathbf{u} \\ z_2w\mathbf{u} + z_2xy\mathbf{u} \\ \vdots \\ z_kw\mathbf{u} + z_kxy\mathbf{u} \end{bmatrix} \\ &= (w + xy)\,\mathrm{Cr}(\mathbf{z}, \mathbf{u}). \end{aligned}$$

Hence, the lemma follows. □

Lemma 3.4. *Let* $(w, \mathbf{u})$ *be an eigenpair of* $A(G)$*, where* $\mathbf{u}$ *is not an eigenvector of* B*. Then,* $(w', \mathrm{Cr}(\mathbf{z}, \mathbf{u}))$ *is an eigenpair of* $A((G; H; B))$*, if and only if,* $(0, \mathbf{z})$ *is an eigenpair of* $A(H)$ *and* $w' = w$*.*

Proof. Let $\mathbf{z} = [z_1, z_2, \ldots, z_k]^\top$. By Eq (3.3) and the fact that $\rho_{ii} = 0$ for all i, we obtain

$$\begin{aligned} & A((G; H; B))\,\mathrm{Cr}(\mathbf{z}, \mathbf{u}) \\ &= \begin{bmatrix} z_1A(G)\mathbf{u} + z_2\rho_{12}B\mathbf{u} + z_3\rho_{13}B\mathbf{u} + \cdots + z_k\rho_{1k}B\mathbf{u} \\ z_1\rho_{21}B\mathbf{u} + z_2A(G)\mathbf{u} + z_3\rho_{23}B\mathbf{u} + \cdots + z_k\rho_{2k}B\mathbf{u} \\ \vdots \\ z_1\rho_{k1}B\mathbf{u} + z_2\rho_{k2}B\mathbf{u} + \cdots + z_{k-1}\rho_{k,k-1}B\mathbf{u} + z_kA(G)\mathbf{u} \end{bmatrix} \\ &= \begin{bmatrix} z_1w\mathbf{u} + (z_1\rho_{11} + z_2\rho_{12} + \cdots + z_k\rho_{1k})B\mathbf{u} \\ z_2w\mathbf{u} + (z_1\rho_{21} + z_2\rho_{22} + \cdots + z_k\rho_{2k})B\mathbf{u} \\ \vdots \\ z_kw\mathbf{u} + (z_1\rho_{k1} + z_2\rho_{k2} + \cdots + z_k\rho_{kk})B\mathbf{u} \end{bmatrix}, \end{aligned} \tag{3.12}$$

where the last equation follows by substituting $A(G)\mathbf{u} = w\mathbf{u}$ and adding the zero term $z_i\rho_{ii}B\mathbf{u} = \mathbf{0}_{|V(G)|}$.

Suppose $(w', \mathrm{Cr}(\mathbf{z}, \mathbf{u}))$ is an eigenpair of $A((G; H; B))$. Then,

$$A((G; H; B))\,\mathrm{Cr}(\mathbf{z}, \mathbf{u}) = w'\,\mathrm{Cr}(\mathbf{z}, \mathbf{u}) = \begin{bmatrix} z_1 w'\mathbf{u} \\ z_2 w'\mathbf{u} \\ \vdots \\ z_k w'\mathbf{u} \end{bmatrix}.$$

Combining this with Eq (3.12), we obtain, for each $1 \leq i \leq k$,

$$(z_1\rho_{i1} + z_2\rho_{i2} + \cdots + z_k\rho_{ik})\,B\mathbf{u} = z_i(w' - w)\mathbf{u}.$$

Since $\mathbf{u}$ is not an eigenvector of B, it follows that

$$\begin{aligned} z_1\rho_{11} + z_2\rho_{12} + \cdots + z_k\rho_{1k} &= 0, \\ z_1\rho_{21} + z_2\rho_{22} + \cdots + z_k\rho_{2k} &= 0, \\ &\vdots \\ z_1\rho_{k1} + z_2\rho_{k2} + \cdots + z_k\rho_{kk} &= 0, \end{aligned} \tag{3.13}$$

and $w' = w$. Note that Eq (3.13) is equivalent to $A(H)\mathbf{z} = \mathbf{0}_k = 0\mathbf{z}$. Hence, $(0, \mathbf{z})$ is an eigenpair of $A(H)$.

Conversely, suppose $(0, \mathbf{z})$ is an eigenpair of $A(H)$ and $w' = w$. Then, Eq (3.13) holds. It follows from Eq (3.12) that

$$A((G; H; B))\mathrm{Cr}(\mathbf{z}, \mathbf{u}) = w'\mathrm{Cr}(\mathbf{z}, \mathbf{u}) = \begin{bmatrix} z_1 w'\mathbf{u} \\ z_2 w'\mathbf{u} \\ \vdots \\ z_k w'\mathbf{u} \end{bmatrix}.$$

Hence, $(w, \mathrm{Cr}(\mathbf{z}, \mathbf{u}))$ is an eigenpair of $A((G; H; B))$.

This completes the proof of the lemma. □

Lemma 3.5. *Let $W_1, W_2, W_3 \subseteq \mathbb{R}^t$ be finite sets of vectors, and let Z be the vector space spanned by $W_1 \cup W_2$. Suppose that $W_1 \cup W_2$ and $W_2 \cup W_3$ are both linearly independent sets. If $W_3 \subseteq Z$, then there exists a subset $W_4 \subseteq W_1$ such that $|W_4| = |W_3|$, $(W_1 \setminus W_4) \cup W_2 \cup W_3$ is a linearly independent set, and Z is spanned by $(W_1 \setminus W_4) \cup W_2 \cup W_3$.*

Proof. Note that if W_3 is the empty set, then the lemma holds by taking W_4 to be the empty set. So, we may assume that $|W_3| \geq 1$. We prove the lemma by induction on $|W_3|$.

Let $\mathbf{u} \in W_3$. Since $\mathbf{u} \in Z$, it follows that

$$\mathbf{u} = \sum_{\mathbf{v} \in W_1} c_{\mathbf{v}}\mathbf{v} + \sum_{\mathbf{v} \in W_2} c_{\mathbf{v}}\mathbf{v}.$$

Next, the linear independence of $W_2 \cup W_3$ implies that there exists $\mathbf{v}_0 \in W_1$ such that $c_{\mathbf{v}_0} \neq 0$. It is straightforward to verify that $(W_1 \setminus \{\mathbf{v}_0\}) \cup W_2 \cup \{\mathbf{u}\}$ is a linearly independent set and spans Z.

If $|W_3| = 1$, then the lemma follows by taking $W_4 = \{\mathbf{v}_0\}$. So, we may assume that $|W_3| \geq 2$. Now, let $W_3' = W_3 \setminus \{\mathbf{u}\}$, $W_1' = W_1 \setminus \{\mathbf{v}_0\}$, and $W_2' = W_2 \cup \{\mathbf{u}\}$. Then, both $W_1' \cup W_2'$ and $W_2' \cup W_3'$ are linearly independent sets, and $W_3' \subseteq Z$. By the induction hypothesis, there exists a subset $W_4' \subseteq W_1'$ such that $|W_4'| = |W_3'|$, $(W_1' \setminus W_4') \cup W_2' \cup W_3'$ is a linearly independent set, and Z is spanned by $(W_1' \setminus W_4') \cup W_2' \cup W_3'$. The lemma follows by setting $W_4 = W_4' \cup \{\mathbf{v}_0\}$ and observing that $W_1' \setminus W_4' = W_1 \setminus W_4$ and $W_2' \cup W_3' = W_2 \cup W_3$. □

Theorem 3.6. *The following statements are equivalent.*
(a) $\dim(U) = 0$ *in Eq* (3.5).
(b) Ξ_0 *in Eq* (3.9) *is an orthonormal eigenbasis for* $A((G; H; B))$.
(c) $A((G; H; B))$ *has nk linearly independent eigenvectors, each of which is a cross-eigenvector of H and G.*
(d) $A(G)B = BA(G)$.

Proof.
(a) ⇒ (b) Since $\dim(U) = 0$, every vector in $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is also an eigenvector of B. By Lemma 3.3, every vector in Ξ_0 is an eigenvector of $A((G; H; B))$. Hence, (b) follows from Observation (3.10).

(b) ⇒ (c) This follows immediately.

(c) ⇒ (a) Let Ξ denote the set of nk linearly independent eigenvectors, each of which is a cross-eigenvector of H and G. Let

$$\mathcal{T}_1(\Xi) = \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u} \text{ is an eigenvector of } B \text{ such that } \mathrm{Cr}(\mathbf{z}, \mathbf{u}) \in \Xi \text{ for some eigenvector } \mathbf{z} \text{ of } A(H)\}.$$

Choose a maximal linearly independent subset of $\mathcal{T}_1(\Xi)$, and denote it by $\mathcal{U}_1$.

Next, choose a basis for the eigenspace $\xi_0(A(H))$ and denote it by $\mathcal{V}_0$. If 0 is not an eigenvalue of $A(H)$, then $\xi_0(A(H))$ is the zero space. Extend this basis to an eigenbasis of $A(H)$ and denote the resulting eigenbasis by $\mathcal{V}$. Clearly, $\mathcal{V}_0 \subseteq \mathcal{V}$, and $\mathcal{V}$ consists of k linearly independent eigenvectors. Moreover, $\mathcal{V}$ spans $\mathbb{R}^k$.

By Lemma 3.3, $\mathrm{Cr}(\mathbf{z}, \mathbf{u})$ is an eigenvector of $A((G; H; B))$ for all $\mathbf{z} \in \mathcal{V}$ and $\mathbf{u} \in \mathcal{U}_1$. By Lemma 3.2,

$$\mathcal{Q}_1 = \{\mathrm{Cr}(\mathbf{z}, \mathbf{u}) : \mathbf{z} \in \mathcal{V}, \mathbf{u} \in \mathcal{U}_1\}$$

is a linearly independent set. Since Ξ spans $\mathbb{R}^{nk}$, it follows from Lemma 3.5 that there exists a subset $W_4 \subseteq \Xi$ such that $|W_4| = |\mathcal{Q}_1|$, $(\Xi \setminus W_4) \cup \mathcal{Q}_1$ is a linearly independent set, and $\mathbb{R}^{nk}$ is spanned by $(\Xi \setminus W_4) \cup \mathcal{Q}_1$. Observe that $(\Xi \setminus W_4) \cup \mathcal{Q}_1$ consists of nk linearly independent eigenvectors of $A((G; H; B))$, each of which is a cross-eigenvector of H and G.

Suppose $\Xi \setminus W_4 \neq \emptyset$. Consider

$$\mathcal{T}_2(\Xi \setminus W_4) = \{\mathbf{u} \in \mathbb{R}^n : \mathrm{Cr}(\mathbf{z}, \mathbf{u}) \in \Xi \setminus W_4 \text{ for some eigenvector } \mathbf{z} \text{ of } A(H)\}.$$

Let $\mathbf{u}' \in \mathcal{T}_2(\Xi \setminus W_4)$. Then, there exists an eigenvector $\mathbf{z}'$ of $A(H)$ such that $\mathrm{Cr}(\mathbf{z}', \mathbf{u}') \in \Xi \setminus W_4$. If $\mathbf{u}'$ is an eigenvector of B, then $\mathbf{u}' \in \mathcal{T}_1(\Xi)$. By the maximality of $\mathcal{U}_1$, $\mathbf{u}'$ is a linear combination of vectors in $\mathcal{U}_1$. Moreover, since $\mathcal{V}$ spans $\mathbb{R}^k$, $\mathbf{z}'$ is a linear combination of vectors in $\mathcal{V}$. By Lemma 3.1, $\mathrm{Cr}(\mathbf{z}', \mathbf{u}')$ is a linear combination of vectors in $\mathcal{Q}_1$, which is a contradiction, as $(\Xi \setminus W_4) \cup \mathcal{Q}_1$ is a linearly independent set. Thus, no vector in $\mathcal{T}_2(\Xi \setminus W_4)$ is an eigenvector of B.

Choose a maximal subset $\mathcal{U}_2$ of $\mathcal{T}_2(\Xi \setminus W_4)$ such that $\mathcal{U}_1 \cup \mathcal{U}_2$ is a linearly independent set. Since $\mathcal{U}_1 \cup \mathcal{U}_2 \subseteq \mathbb{R}^n$, we have $|\mathcal{U}_1| + |\mathcal{U}_2| = |\mathcal{U}_1 \cup \mathcal{U}_2| \leq n$. Define

$$\mathcal{Q}_2 = \{\mathrm{Cr}(\mathbf{z}, \mathbf{u}) : \mathbf{z} \in \mathcal{V}_0, \mathbf{u} \in \mathcal{U}_2\}.$$

By Lemma 3.2,

$$\mathcal{Q}_3 = \{\mathrm{Cr}(\mathbf{z}, \mathbf{u}) : \mathbf{z} \in \mathcal{V}, \mathbf{u} \in \mathcal{U}_1 \cup \mathcal{U}_2\}$$

is a linearly independent set. Since $\mathcal{Q}_1 \cup \mathcal{Q}_2 \subseteq \mathcal{Q}_3$, it follows that $\mathcal{Q}_1 \cup \mathcal{Q}_2$ is a linearly independent set. By Lemma 3.5, there exists a subset $W_4' \subseteq \Xi \setminus W_4$ such that $|W_4'| = |\mathcal{Q}_2|$, $(\Xi \setminus W_4) \setminus W_4' \cup \mathcal{Q}_1 \cup \mathcal{Q}_2$ is a linearly independent set, and $\mathbb{R}^{nk}$ is spanned by $(\Xi \setminus W_4) \setminus W_4' \cup \mathcal{Q}_1 \cup \mathcal{Q}_2$. Note that $(\Xi \setminus W_4) \setminus W_4' \cup \mathcal{Q}_1 \cup \mathcal{Q}_2$ consists of nk linearly independent eigenvectors of $A((G; H; B))$, each of which is a cross-eigenvector of H and G.

Suppose $(\Xi \setminus W_4) \setminus W_4' \neq \emptyset$. Let $\mathrm{Cr}(\mathbf{z}, \mathbf{u}) \in (\Xi \setminus W_4) \setminus W_4'$ for some eigenvectors $\mathbf{z}$ of $A(H)$ and $\mathbf{u}$ of $A(G)$. Then, $\mathrm{Cr}(\mathbf{z}, \mathbf{u})$ is an eigenvector of $A((G; H; B))$. By the choice of $\mathcal{U}_2$, $\mathbf{u}$ is a linear combination of vectors in $\mathcal{U}_1 \cup \mathcal{U}_2$. Since $\mathbf{u} \in \mathcal{T}_2(\Xi \setminus W_4)$, $\mathbf{u}$ is not an eigenvector of B. By Lemma 3.4, $(0, \mathbf{z})$ is an eigenpair of $A(H)$. So, $\mathbf{z}$ is a linear combination of vectors in $\mathcal{V}_0$. By Lemma 3.1, $\mathrm{Cr}(\mathbf{z}, \mathbf{u})$ is a linear combination of vectors in $\mathcal{Q}_1 \cup \mathcal{Q}_2$. This is a contradiction, as $(\Xi \setminus W_4) \setminus W_4' \cup \mathcal{Q}_1 \cup \mathcal{Q}_2$ is a linearly independent set. Hence, $(\Xi \setminus W_4) \setminus W_4' = \emptyset$ and $\mathcal{Q}_1 \cup \mathcal{Q}_2$ consists of nk linearly independent eigenvectors of $A((G; H; B))$.

Now,

$$nk = |\mathcal{Q}_1 \cup \mathcal{Q}_2| = |\mathcal{Q}_1| + |\mathcal{Q}_2| = |\mathcal{U}_1||\mathcal{V}| + |\mathcal{U}_2||\mathcal{V}_0| \leq (|\mathcal{U}_1| + |\mathcal{U}_2|)|\mathcal{V}| \leq nk.$$

Therefore, equality must hold throughout the above inequalities. Since H is not the empty graph, we have $\mathcal{V}_0 \subsetneq \mathcal{V}$. If $|\mathcal{U}_2| > 0$, then

$$|\mathcal{U}_1||\mathcal{V}| + |\mathcal{U}_2||\mathcal{V}_0| < (|\mathcal{U}_1| + |\mathcal{U}_2|)|\mathcal{V}|,$$

a contradition. Thus, $\mathcal{Q}_2 = \emptyset = \mathcal{U}_2$. Consequently, $\mathcal{Q}_1$ consists of nk linearly independent eigenvectors of $A((G; H; B))$. Moreover, $|\mathcal{U}_1| = n$, and thus U_1 forms the common eigenbasis for $A(G)$ and B, resulting in $\dim(U) = 0$.

(a) $\Rightarrow$ (d) Suppose $\dim(U) = 0$. Then, every vector in $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is also an eigenvector of B. Let $P = [\mathbf{u}_1 \mathbf{u}_2 \cdots \mathbf{u}_n]$ be the orthogonal matrix whose columns are these eigenvectors. Then, $P^\top A(G) P = D_1$ and $P^\top B P = D_2$ for some diagonal matrices D_1 and D_2. Since $D_1 D_2 = D_2 D_1$, we have

$$A(G)B = (PD_1P^\top)(PD_2P^\top) = PD_1D_2P^\top = PD_2D_1P^\top = (PD_2P^\top)(PD_1P^\top) = BA(G).$$

(d) $\Rightarrow$ (a) Suppose $A(G)B = BA(G)$. Since $A(G)$ and B are real symmetric matrices that commute, they are simultaneously orthogonally diagonalizable. Thus, there exists an orthonormal basis of $\mathbb{R}^n$ consisting of common eigenvectors of $A(G)$ and B. This implies that each eigenspace $\xi_{a_i}(A(G))$ is spanned entirely by eigenvectors of B, so $U_i = \{0\}$ for all $1 \leq i \leq l_1$. Hence, $\dim(U) = 0$.

This completes the proof of the theorem. □

Corollary 3.7. *Suppose that $A((G; H; B))$ has nk linearly independent eigenvectors, each of which is a cross-eigenvector of H and G. Then,*

$$\{(w + xy, \mathrm{Cr}(\mathbf{z}, \mathbf{u})) : \mathrm{Cr}(\mathbf{z}, \mathbf{u}) \in \Xi_0\}$$

is a complete system of eigenvectors and associated eigenvalues of $A((G;H;B))$, where $(w,\mathbf{u})$, $(x,\mathbf{u})$, and $(y,\mathbf{z})$ are eigenpairs of $A(G)$, B, and $A(H)$, respectively.

Proof. By Theorem 3.6, Ξ_0 defined in Eq (3.9) is an orthonormal eigenbasis for $A((G;H;B))$. The corollary then follows immediately from Lemma 3.3. □

Recall that when $B = J_n$, the graph $(G;H;B)$ is the H-join of the family $\mathcal{F}$ consisting of k copies of G. This case was studied in [8]. We now turn our attention to the cases $B = I_n$, $B = J_n - I_n$, and $B = A(G^c)$, where I_n denotes the $n \times n$ identity matrix and G^c denotes the complement of G.

Corollary 3.8. *The following statements hold.*
(a) *If $B = I_n$, then*

$$\{(w + y, \mathrm{Cr}(\mathbf{z},\mathbf{u})) : \mathrm{Cr}(\mathbf{z},\mathbf{u}) \in \Xi_0\}$$

is a complete system of eigenvectors and associated eigenvalues of $A((G;H;B))$, where $(w,\mathbf{u})$ and $(y,\mathbf{z})$ are eigenpairs of $A(G)$ and $A(H)$, respectively.
(b) *If $B = J_n - I_n$ and G is a connected l-regular graph, then*

$$\{(l + (n-1)y, \mathrm{Cr}(\mathbf{z}, (1/\sqrt{n})\mathbf{1}_n)) : \mathrm{Cr}(\mathbf{z}, (1/\sqrt{n})\mathbf{1}_n) \in \Xi_0\}$$

$$\cup \{(w - y, \mathrm{Cr}(\mathbf{z},\mathbf{u})) : \mathrm{Cr}(\mathbf{z},\mathbf{u}) \in \Xi_0, \mathbf{u} \neq (1/\sqrt{n})\mathbf{1}_n\}$$

is a complete system of eigenvectors and associated eigenvalues of $A((G;H;B))$, where $(w,\mathbf{u})$ and $(y,\mathbf{z})$ are eigenpairs of $A(G)$ and $A(H)$, respectively, with $w \neq l$.
(c) *If $B = A(G^c)$ and G is a connected l-regular graph, then*

$$\{(l + (n-1-l)y, \mathrm{Cr}(\mathbf{z}, (1/\sqrt{n})\mathbf{1}_n)) : \mathrm{Cr}(\mathbf{z}, (1/\sqrt{n})\mathbf{1}_n) \in \Xi_0\}$$

$$\cup \{(w - (1+w)y, \mathrm{Cr}(\mathbf{z},\mathbf{u})) : \mathrm{Cr}(\mathbf{z},\mathbf{u}) \in \Xi_0, \mathbf{u} \neq (1/\sqrt{n})\mathbf{1}_n\}$$

is a complete system of eigenvectors and associated eigenvalues of $A((G;H;B))$, where $(w,\mathbf{u})$ and $(y,\mathbf{z})$ are eigenpairs of $A(G)$ and $A(H)$, respectively, with $w \neq l$.

Proof.
(a) Clearly, $B\mathbf{u} = \mathbf{u}$ for every eigenvector $\mathbf{u}$ of $A(G)$. This means that $\dim(U) = 0$ in Eq (3.5). Hence, (a) follows from Theorem 3.6 and Corollary 3.7.

(b) Note that $B\mathbf{1}_n = (n-1)\mathbf{1}_n$. Let $\mathbf{u}$ be an eigenvector of G corresponding to an eigenvalue different from l. By Lemma 2.4, $J_n\mathbf{u} = (\mathbf{1}_n\mathbf{1}_n^\top)\mathbf{u} = \mathbf{1}_n(\mathbf{1}_n^\top\mathbf{u}) = 0$. Thus, $B\mathbf{u} = -\mathbf{u}$ for every eigenvector u of $A(G)$ corresponding to an eigenvalue different from l. So, $\dim(U) = 0$ in Eq (3.5). Hence, (b) follows from Theorem 3.6 and Corollary 3.7.

(c) Note that $B = A(G^c) = J_n - I_n - A(G)$. Therefore, $B\mathbf{1}_n = (n-1-l)\mathbf{1}_n$ and $B\mathbf{u} = -(1+w)\mathbf{u}$ for every eigenvector $\mathbf{u}$ of $A(G)$ corresponding to an eigenvalue $w \neq l$. Hence, (c) follows from Theorem 3.6 and Corollary 3.7. □

Remark 3.9. We note here that when $B = I_n$, the graph structure corresponds to placing a perfect matching between adjacent copies of the factor graphs G_i and G_j. Specifically, a vertex in G_i is connected only to its exact mirror vertex in G_j. When $B = J_n - I_n$, the graph structure corresponds to a complete bipartite connection between G_i and G_j, except that the mirror edges are removed.

Specifically, a vertex in G_i connects to every vertex in G_j except its own counterpart. When $B = A(G^c)$, the graph structure is such that connections between G_i and G_j perfectly mimic the missing edges of the base graph G. Two vertices across different layers are connected, if and only if, they are not connected within their own layer.

Example 3.10. Let H be the complete graph with 2 vertices and G be the 4-cycle. Then, their adjacency matrices are

$$A(H) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A(G) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

The eigenpairs of $A(H)$ are

$$\left(-1, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) \qquad \text{and} \qquad \left(1, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

The eigenpairs of $A(G)$ are

$$\left(-2, \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}\right), \left(2, \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right), \left(0, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}\right), \left(0, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}\right).$$

By part (a) of Corollary 3.8, the eigenpairs of $A((G; H; I_4))$ are

$$\left(-3, \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}\right), \left(1, \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}\right), \left(-1, \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right), \left(-1, \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}\right),$$

$$\left(-1, \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}\right), \left(3, \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right), \left(1, \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}\right), \left(1, \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}\right).$$

Now, we determine the eigenpairs of $A((G; H; J_4 - I_4))$ by using part (b) of Corollary 3.8. Note that G is a 2-regular graph and $l + (n-1)y = 2 + 3y$. So, by using the eigenpair

$$\left(-1, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) \quad \text{of } H,$$

we obtain the eigenvalue $2 + 3(-1) = -1$. Thus, one of the eigenpairs of $A((G; H; J_4 - I_4))$ is

$$\left(-1, \frac{1}{2\sqrt{2}}\begin{pmatrix}1\\1\\1\\1\\-1\\-1\\-1\\-1\end{pmatrix}\right).$$

Similarly, by using the eigenpair

$$\left(1, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\right) \quad \text{of } H,$$

we obtain one of the eigenpairs of $A((G; H; J_4 - I_4))$, that is,

$$\left(5, \frac{1}{2\sqrt{2}}\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}\right).$$

Now, the remaining eigenpairs are of the form $(w - y, \mathrm{Cr}(\mathbf{z}, \mathbf{u}))$. Hence, the remaining eigenpairs of $A((G; H; J_4 - I_4))$ are

$$\left(-1, \frac{1}{2\sqrt{2}}\begin{pmatrix}1\\-1\\1\\-1\\-1\\1\\-1\\1\end{pmatrix}\right), \left(1, \frac{1}{2}\begin{pmatrix}1\\0\\-1\\0\\-1\\0\\1\\0\end{pmatrix}\right), \left(1, \frac{1}{2}\begin{pmatrix}0\\1\\0\\-1\\0\\-1\\0\\1\end{pmatrix}\right), \left(-3, \frac{1}{2\sqrt{2}}\begin{pmatrix}1\\-1\\1\\-1\\1\\-1\\1\\-1\end{pmatrix}\right), \left(-1, \frac{1}{2}\begin{pmatrix}1\\0\\-1\\0\\1\\0\\-1\\0\end{pmatrix}\right), \left(-1, \frac{1}{2}\begin{pmatrix}0\\1\\0\\-1\\0\\1\\0\\-1\end{pmatrix}\right).$$

Now, we determine the eigenpairs of $A((G; H; A(G^c)))$ by using part (c) of Corollary 3.8. The first two eigenpairs of $A((G; H; A(G^c)))$ are

$$\left(1, \frac{1}{2\sqrt{2}}\begin{pmatrix}1\\1\\1\\1\\-1\\-1\\-1\\-1\end{pmatrix}\right) \quad \text{and} \quad \left(3, \frac{1}{2\sqrt{2}}\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}\right).$$

The remaining eigenpairs of $A((G;H;A(G^c)))$ are

$$\left(-3, \frac{1}{2\sqrt{2}}\begin{pmatrix}1\\-1\\1\\-1\\-1\\1\\-1\\1\end{pmatrix}\right), \left(1, \frac{1}{2}\begin{pmatrix}1\\0\\-1\\0\\-1\\0\\1\\0\end{pmatrix}\right), \left(1, \frac{1}{2}\begin{pmatrix}0\\1\\0\\-1\\0\\-1\\0\\1\end{pmatrix}\right), \left(-1, \frac{1}{2\sqrt{2}}\begin{pmatrix}1\\-1\\1\\-1\\1\\-1\\1\\-1\end{pmatrix}\right), \left(-1, \frac{1}{2}\begin{pmatrix}1\\0\\-1\\0\\1\\0\\-1\\0\end{pmatrix}\right), \left(-1, \frac{1}{2}\begin{pmatrix}0\\1\\0\\-1\\0\\1\\0\\-1\end{pmatrix}\right).$$

4. Universal adjacency matrix

This section extends the spectral analysis to derive the characteristic polynomial of the universal adjacency matrix for the generalized H_m-join of factor graphs.

For a graph G that is the H-join of regular graphs, Chen and Chen [9] derived the characteristic polynomial of the matrix $A(G) - tD(G)$, where $D(G)$ is the degree matrix of G. Subsequently, Wang and Wong [6] established the characteristic polynomial of the adjacency matrix for the lexicographic product of arbitrary graphs. The universal adjacency matrix of a graph G with n vertices is defined as

$$U(G) = \alpha A(G) + \beta I_n + \gamma J_n + \delta D(G), \tag{4.1}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $\alpha \neq 0$. Exploring the spectral properties of this matrix, Haemers and Oboudi [10] determined the spectra of the universal adjacency matrix for the disjoint union of regular graphs. Furthermore, Saravanan et al. [8] derived the characteristic polynomial of the universal adjacency matrix for the H-join of arbitrary graphs. Most recently, Arunkumar and Ganeshbabu [11] introduced the concept of the H_m-join, a generalization that naturally reduces to the standard H-join when $m = 1$.

Let H be a graph with vertex set $V(H) = \{1, 2, \ldots, k\}$ and $\mathcal{F} = \{G_1, \ldots, G_k\}$ be a family of graphs, where the vertex set of each G_i is given as in Eq (1.1). For each $1 \leq i \leq k$, let $\chi_i : V(G_i) \to \{1, 2, \ldots, m\}$ be a labeling function. The H_m-join of the family $\mathcal{F}$ with respect to the labeling functions $\mathcal{X} = \{\chi_1, \chi_2, \ldots, \chi_k\}$ is the graph with vertex set $V(G) = \bigcup_{i=1}^{k} V(G_i)$ and edge set

$$E(G) = \bigcup_{i=1}^{k} E(G_i) \cup \bigcup_{j_1 j_2 \in E(H)} \{uv : u \in V(G_{j_1}), v \in V(G_{j_2}) \text{ with } \chi_{j_1}(u) = \chi_{j_2}(v)\}. \tag{4.2}$$

Clearly, when $m = 1$, the H_m-join reduces to the standard H-join. Moreover, every join of the family $\mathcal{F}$ with respect to a family of connecting matrices $\mathcal{B}$ (see Eqs (1.4) and (1.5)) can be realized as an H_m-join of the family $\mathcal{F}$ for a suitable positive integer m and an appropriate collection of labeling functions, provided that $B_{ij} = \rho_{ij} X_i X_j^{\top}$. We denote the H_m-join of $\mathcal{F}$ with respect to the labeling functions $\mathcal{X}$ by

$$(\mathcal{F}; H; \mathcal{X}; m). \tag{4.3}$$

For each labeling function χ_i, define the corresponding $n_i \times m$ labeling matrix $X_i = [y_{jt}]$ by

$$y_{jt} = \begin{cases} 1, & \text{if } \chi_i(v_{ij}) = t, \\ 0, & \text{otherwise,} \end{cases} \tag{4.4}$$

where $v_{ij} \in V(G_i)$ and $t \in \{1, 2, \ldots, m\}$.

Let $A(H)$ be as given in Eq (3.1). Then, the adjacency matrix $A((\mathcal{F}; H; \mathcal{X}; m))$ has the following block representation

$$A((\mathcal{F}; H; \mathcal{X}; m)) = \begin{pmatrix} A(G_1) & \rho_{12}X_1X_2^\top & \rho_{13}X_1X_3^\top & \cdots & \rho_{1k}X_1X_k^\top \\ \rho_{21}X_2X_1^\top & A(G_2) & \rho_{23}X_2X_3^\top & \cdots & \rho_{2k}X_2X_k^\top \\ \rho_{31}X_3X_1^\top & \rho_{32}X_3X_2^\top & A(G_3) & \cdots & \rho_{3k}X_3X_k^\top \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{k1}X_kX_1^\top & \rho_{k2}X_kX_2^\top & \rho_{k3}X_kX_3^\top & \cdots & A(G_k) \end{pmatrix}, \tag{4.5}$$

where each (i, j)-th block is an $n_i \times n_j$ matrix.

Arunkumar and Ganeshbabu [11] derived the characteristic polynomial of $A((\mathcal{F}; H; \mathcal{X}; m))$. In this section, we extend their approach by deriving the characteristic polynomial of the universal adjacency matrix for the graph $(\mathcal{F}; H; \mathcal{X}; m)$.

Let A, B be matrices of sizes $m\times n$ and $n\times m$, respectively. By the multiplicativity of the determinant,

$$\begin{aligned} \det\begin{pmatrix} I_m & A \\ B & I_n \end{pmatrix} &= \det\begin{pmatrix} I_m & A \\ B & I_n \end{pmatrix}\det\begin{pmatrix} I_m & -A \\ \mathbf{0}_{nm} & I_n \end{pmatrix} \\ &= \det\left(\begin{pmatrix} I_m & A \\ B & I_n \end{pmatrix}\begin{pmatrix} I_m & -A \\ \mathbf{0}_{nm} & I_n \end{pmatrix}\right) \\ &= \det\begin{pmatrix} I_m & \mathbf{0}_{mn} \\ B & I_n - BA \end{pmatrix} \\ &= \det(I_n - BA). \end{aligned}$$

Similarly,

$$\begin{aligned} \det\begin{pmatrix} I_m & A \\ B & I_n \end{pmatrix} &= \det\begin{pmatrix} I_m & A \\ B & I_n \end{pmatrix}\det\begin{pmatrix} I_m & \mathbf{0}_{mn} \\ -B & I_n \end{pmatrix} \\ &= \det(I_m - AB). \end{aligned}$$

Hence,

$$\det(I_n - BA) = \det(I_m - AB). \tag{4.6}$$

Given an $n \times m$ labeling matrix X, we define the corresponding $n \times (m + 1)$ extended labeling matrix by

$$\widehat{X} = [X\ \mathbf{1}_n]. \tag{4.7}$$

Let M be an $n \times n$ matrix, and let P, Q be $n \times m$ matrices. The $m \times m$ rational matrix function in the indeterminate x,

$$\Gamma_M(P, Q)(x) = Q^\top(xI_n - M)^{-1}P,$$

is called the (P, Q)-main function of M. This rational matrix function was introduced in [8, 11]. When $P = Q$, we simply write $\Gamma_M(P)(x)$ and refer to it as the P-main function of M. Note that

$$\Gamma_M(\widehat{X})(x) = \widehat{X}^\top(xI_n - M)^{-1}\widehat{X}$$

is an $(m+1)\times(m+1)$ rational matrix function. Moreover, the rational matrix function has the block decomposition

$$\Gamma_M(\widehat{X})(x) = \begin{pmatrix} X^\top(xI_n - M)^{-1}X & X^\top(xI_n - M)^{-1}\mathbf{1}_n \\ \mathbf{1}_n^\top(xI_n - M)^{-1}X & \mathbf{1}_n^\top(xI_n - M)^{-1}\mathbf{1}_n \end{pmatrix}. \tag{4.8}$$

For each factor G_i in $(\mathcal{F}; H; \mathcal{X}; m)$, the vertex degree of a vertex $v_{is} \in V(G_i)$ in G is given by

$$\deg_G(v_{is}) = \deg_{G_i}(v_{is}) + d_{is}, \tag{4.9}$$

where

$$d_{is} = \sum_{j\in N_H(i)} |\{v_{jr} \in V(G_j) : \chi_j(v_{jr}) = \chi_i(v_{is})\}| = \sum_{j\in N_H(i)} |\chi_j^{-1}(\chi_i(v_{is}))|.$$

Here, the set $N_H(i)$ is the neighborhood of vertex i in H, which consists of all the vertices j that are adjacent to i in H. Let

$$D_i = \mathrm{diag}(d_{i1}, d_{i2}, \ldots, d_{in_i}). \tag{4.10}$$

Then, D_i is an $n_i \times n_i$ diagonal matrix.

Consider the matrix

$$M_i = U(G_i) + \delta D_i = \alpha A(G_i) + \beta I_{n_i} + \gamma J_{n_i} + \delta(D(G_i) + D_i). \tag{4.11}$$

Then, the characteristic polynomial of M_i is given by

$$\phi_i(x) = \det(xI_{n_i} - M_i). \tag{4.12}$$

Theorem 4.1. *Let $G = (\mathcal{F}; H; \mathcal{X}; m)$. For all $1 \le i, j \le k$ with $i \ne j$, define the $(m+1)\times(m+1)$ block matrix*

$$K_{ij} = \begin{pmatrix} \alpha\rho_{ij}I_m & \mathbf{0}_m \\ \mathbf{0}_m^\top & \gamma \end{pmatrix}.$$

Then, the characteristic polynomial of the universal adjacency matrix $U(G)$ is

$$\det(xI_n - U(G)) = \left(\prod_{i=1}^{k} \phi_i(x)\right) \det(\widetilde{A}(x)), \tag{4.13}$$

where $\widetilde{A}(x)$ is the $k(m+1)\times k(m+1)$ block matrix whose (i, j)-th block is

$$\widetilde{A}(x)_{ij} = \begin{cases} I_{m+1}, & \text{if } i = j, \\ -\Gamma_i(x)K_{ij}, & \text{if } i \ne j, \end{cases} \tag{4.14}$$

and $\Gamma_i(x) = \widehat{X}_i^\top(xI_{n_i} - M_i)^{-1}\widehat{X}_i$ is the $\widehat{X}_i$-main function of M_i.

Proof. Let $n = n_1 + n_2 + \cdots + n_k$. By Eq (4.1), $U(G) = \alpha A(G) + \beta I_n + \gamma J_n + \delta D(G)$. Furthermore, it follows from Eq (4.11) that the diagonal blocks of $U(G)$ are given by

$$U(G)_{ii} = \alpha A(G_i) + \beta I_{n_i} + \gamma J_{n_i} + \delta(D(G_i) + D_i) = M_i. \tag{4.15}$$

For off-diagonal blocks, with $i \neq j$, it follows from Eq (4.5) that

$$U(G)_{ij} = \alpha\rho_{ij}X_iX_j^\top + \gamma\mathbf{1}_{n_i}\mathbf{1}_{n_j}^\top. \tag{4.16}$$

By Eqs (4.7) and (4.16), it follows that

$$\begin{aligned}\widehat{X}_iK_{ij}\widehat{X}_j^\top &= \begin{bmatrix}X_i & \mathbf{1}_{n_i}\end{bmatrix}\begin{pmatrix}\alpha\rho_{ij}I_m & \mathbf{0}_m\\ \mathbf{0}_m^\top & \gamma\end{pmatrix}\begin{bmatrix}X_j^\top\\ \mathbf{1}_{n_j}^\top\end{bmatrix}\\ &= \alpha\rho_{ij}X_iX_j^\top + \gamma\mathbf{1}_{n_i}\mathbf{1}_{n_j}^\top\\ &= U(G)_{ij}.\end{aligned}$$

Combining this with Eq (4.15), we obtain the following block representation of

$$xI_n - U(G) = \begin{pmatrix} xI_{n_1} - M_1 & -\widehat{X}_1K_{12}\widehat{X}_2^\top & \cdots & -\widehat{X}_1K_{1k}\widehat{X}_k^\top\\ -\widehat{X}_2K_{21}\widehat{X}_1^\top & xI_{n_2} - M_2 & \cdots & -\widehat{X}_2K_{2k}\widehat{X}_k^\top\\ \vdots & \vdots & \ddots & \vdots\\ -\widehat{X}_kK_{k1}\widehat{X}_1^\top & -\widehat{X}_kK_{k2}\widehat{X}_2^\top & \cdots & xI_{n_k} - M_k\end{pmatrix}. \tag{4.17}$$

Define the following block diagonal matrices:

$$\Phi(x) = \operatorname{diag}(xI_{n_1} - M_1, xI_{n_2} - M_2, \ldots, xI_{n_k} - M_k); \tag{4.18}$$

$$Y = \operatorname{diag}(\widehat{X}_1, \widehat{X}_2, \ldots, \widehat{X}_k). \tag{4.19}$$

Define $K_{ii} = \mathbf{0}_{m+1,m+1}$ for $1 \leq i \leq k$ and let $K = [K_{ij}]$. Note that K is a $k(m+1) \times k(m+1)$ matrix.

Now, consider the $n \times n$ matrix $YKY^\top$. Note that the (i, j)-th block of $YKY^\top$ is given by

$$(YKY^\top)_{i,j} = \begin{cases}\mathbf{0}_{n_in_i}, & \text{if } i = j,\\ \widehat{X}_iK_{ij}\widehat{X}_j^\top, & \text{if } i \neq j.\end{cases} \tag{4.20}$$

By Eqs (4.17), (4.18), and (4.20), we obtain

$$xI_n - U(G) = \Phi(x) - YKY^\top = \Phi(x)\left(I_n - \Phi(x)^{-1}YKY^\top\right). \tag{4.21}$$

Note that

$$\det(\Phi(x)) = \prod_{i=1}^{k}\det(xI_{n_i} - M_i) = \prod_{i=1}^{k}\phi_i(x).$$

In addition, $\Phi(x)^{-1}Y$ is an $n \times k(m+1)$ matrix and $KY^\top$ is a $k(m+1) \times n$ matrix. By Eq (4.6), we have

$$\det\left(I_n - \Phi(x)^{-1}YKY^\top\right) = \det\left(I_{k(m+1)} - KY^\top\Phi(x)^{-1}Y\right). \tag{4.22}$$

Furthermore,

$$Y^\top\Phi(x)^{-1}Y = \operatorname{diag}(\Gamma_1(x), \Gamma_2(x), \ldots, \Gamma_k(x)),$$

which is a $k(m+1) \times k(m+1)$ matrix. Again, by Eq (4.6),

$$\det(I_{k(m+1)} - KY^\top\Phi(x)^{-1}Y) = \det(I_{k(m+1)} - Y^\top\Phi(x)^{-1}YK) = \det(\widetilde{A}(x)),$$

where the last equality follows from Eq (4.14). Hence, Eq (4.13) follows from Eqs (4.21) and (4.22).

This completes the proof of the theorem. □

Remark 4.2. The determinant formula proved above is valid for any x in the complement to the finite union of spectra of the M_i. The left-hand side of this equation is a polynomial; therefore, the righthand side rational function coincides with this polynomial on an infinite set of points and can be uniquely extended to a polynomial.

While Theorem 4.1 determines the characteristic polynomial of $U(G)$, the underlying matrix operations can be efficiently implemented by using the fast matrix multiplication algorithm in [12]. One may refer to [13] on handling large-scale matrix equations without incurring prohibitive computational costs.

Example 4.3. Let H be an edge. Then, its adjacency matrix is

$$A(H) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This means that $\rho_{12} = \rho_{21} = 1$. Let G_1 be a triangle and G_2 be a star on 3 vertices. Then, their adjacency matrices are given by

$$A(G_1) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad A(G_2) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and their degree matrices are given by

$$D(G_1) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad D(G_2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with respect to the vertex orderings in $V(G_1) = \{v_1, v_2, v_3\}$ and $V(G_2) = \{u_1, u_2, u_3\}$.

Let the labeling functions $\chi_i : V(G_i) \to \{1, 2\}$ for $i \in \{1, 2\}$ be defined as $\chi_1(v_1) = 1, \chi_1(v_2) = 1$, and $\chi_1(v_3) = 2$ for G_1, and $\chi_2(u_1) = 1, \chi_2(u_2) = 2$, and $\chi_2(u_3) = 2$ for G_2. Then, the labeling matrices are

$$X_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

By Eq (4.10), we get $D_1 = \text{diag}(1, 1, 2)$ and $D_2 = \text{diag}(2, 1, 1)$. Therefore, it follows that

$$M_1 = A(G_1) - I_3 + 2J_3 - 2(D(G_1) + D_1) = \begin{pmatrix} -5 & 3 & 3 \\ 3 & -5 & 3 \\ 3 & 3 & -7 \end{pmatrix}$$

and

$$M_2 = A(G_2) - I_3 + 2J_3 - 2(D(G_2) + D_2) = \begin{pmatrix} -7 & 3 & 3 \\ 3 & -3 & 2 \\ 3 & 2 & -3 \end{pmatrix}.$$

Thus,

$$\phi_1(x) = \det \begin{pmatrix} x+5 & -3 & -3 \\ -3 & x+5 & -3 \\ -3 & -3 & x+7 \end{pmatrix} = (x+8)(x^2 + 9x - 4),$$

$$\phi_2(x) = \det\begin{pmatrix} x+7 & -3 & -3 \\ -3 & x+3 & -2 \\ -3 & -2 & x+3 \end{pmatrix} = (x+5)(x^2+8x-11).$$

Since $\rho_{12} = \rho_{21} = 1$, it follows that $K_{12} = K_{21}$. Therefore,

$$K_{12} = \begin{pmatrix} \alpha I_2 & \mathbf{0}_2 \\ \mathbf{0}_2^T & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

By Eq (4.7),

$$\widehat{X_1} = [X_1\ \mathbf{1}_3] = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \widehat{X_2} = [X_2\ \mathbf{1}_3] = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \Gamma_1(x) &= \widehat{X_1}^\top (xI_3 - M_1)^{-1}\widehat{X_1} = \frac{1}{\phi_1(x)}\widehat{X_1}^\top(\mathrm{adj}(xI_3 - M_1))\widehat{X_1} \\ &= \frac{1}{\phi_1(x)}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x^2+12x+26 & 3x+30 & 3x+24 \\ 3x+30 & x^2+12x+26 & 3x+24 \\ 3x+24 & 3x+24 & x^2+10x+16 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\ &= \frac{1}{(x+8)(x^2+9x-4)}\begin{pmatrix} 2x^2+30x+112 & 6x+48 & 2x^2+36x+160 \\ 6x+48 & x^2+10x+16 & x^2+16x+64 \\ 2x^2+36x+160 & x^2+16x+64 & 3x^2+52x+224 \end{pmatrix}. \end{aligned}$$

Similarly,

$$\Gamma_2(x) = \frac{1}{(x+5)(x^2+8x-11)}\begin{pmatrix} x^2+6x+5 & 6x+30 & x^2+12x+35 \\ 6x+30 & 2x^2+24x+70 & 2x^2+30x+100 \\ x^2+12x+35 & 2x^2+30x+100 & 3x^2+42x+135 \end{pmatrix}.$$

Thus,

$$\Gamma_1(x)K_{12} = \frac{1}{(x+8)(x^2+9x-4)}\begin{pmatrix} 2x^2+30x+112 & 6x+48 & 4x^2+72x+320 \\ 6x+48 & x^2+10x+16 & 2x^2+32x+128 \\ 2x^2+36x+160 & x^2+16x+64 & 6x^2+104x+448 \end{pmatrix}$$

and

$$\Gamma_2(x)K_{21} = \frac{1}{(x+5)(x^2+8x-11)}\begin{pmatrix} x^2+6x+5 & 6x+30 & 2x^2+24x+70 \\ 6x+30 & 2x^2+24x+70 & 4x^2+60x+200 \\ x^2+12x+35 & 2x^2+30x+100 & 6x^2+84x+270 \end{pmatrix}.$$

Now,

$$\widetilde{A}(x) = \begin{pmatrix} I_3 & -\Gamma_1(x)K_{12} \\ -\Gamma_2(x)K_{21} & I_3 \end{pmatrix}.$$

Therefore,

$$\begin{aligned}\det(\widetilde{A}(x)) &= \det\left(I_3 - \Gamma_1(x)K_{12}\Gamma_2(x)K_{21}\right)\\ &= \frac{1}{(\phi_1(x)\phi_2(x))^3}\det\left(\phi_1(x)\phi_2(x)I_3 - \phi_1(x)\phi_2(x)\Gamma_1(x)K_{12}\Gamma_2(x)K_{21}\right)\\ &= \frac{(x+5)^3(x+8)^3(x^2+8x-11)^2(x^2+9x-4)^2(x+9)(x^3+8x^2-71x-466)}{(x+5)(x+8)(x^2+8x-11)(x^2+9x-4)^3}\\ &= \frac{(x+9)(x^3+8x^2-71x-466)}{(x^2+8x-11)(x^2+9x-4)}.\end{aligned}$$

By Theorem 4.1,

$$\begin{aligned}\det(xI_n - U(G)) &= \phi_1(x)\phi_2(x)\det(\widetilde{A}(x))\\ &= (x+5)(x+8)(x+9)(x^3+8x^2-71x-466).\end{aligned}$$

5. Conclusions

The study of graph spectra under various join operations has been extensively developed in previous works. Schwenk [1] originally introduced the concept of the H-join operation. Following this, Cardoso et al. [2] determined the spectrum of the resultant graph using a generalization of Fiedler's lemma. Later, Abreu et al. [5] determined the spectrum of the lexicographic product when both the factor graph and the H graph are regular. Wang and Wong [6] subsequently obtained the characteristic polynomial of the adjacency matrix for the lexicographic product of arbitrary graphs. As for the universal adjacency matrices, Chen and Chen [9] derived the characteristic polynomial for the H-join of regular graphs, while Haemers and Oboudi [10] determined the spectra of the universal adjacency matrix for the disjoint union of regular graphs. Saravanan et al. [8] advanced this by deriving the characteristic polynomial of the universal adjacency matrix for the standard H-join of arbitrary graphs. Recently, Arunkumar and Ganeshbabu [11] introduced the generalized H_m-join concept and derived the characteristic polynomial of its adjacency matrix.

Building upon these foundational works, this paper presents several original contributions to the spectral theory of factor graphs. First, we introduced the notion of strongly inherited eigenvalues. We successfully characterized the resultant graphs for which all eigenvalues, except possibly one, are strongly inherited from the factor graphs (Theorems 2.3 and 2.6). Second, we introduced the concept of cross-eigenvectors. We provided equivalent conditions and a full characterization of the resultant graph that admits these cross-eigenvectors (Theorem 3.6). Finally, we extended the existing literature on universal adjacency matrices and H_m-joins by deriving the characteristic polynomial for the universal adjacency matrix of the H_m-join of factor graphs (Theorem 4.1).

Author contributions

Both authors contributed equally to the preparation of this manuscript. Both authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest.

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