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Research article

On complex picture fuzzy subgroups: a Lattice-ordered and sharpened homomorphism theorems

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Abstract: Abbas and Malik recently introduced complex picture fuzzy subgroups (CPic-FSGs) and developed their basic theory. Their development rests on a standing *homogeneity* assumption: in every component, the amplitude and the phase are required to be comonotone, so that the two induce the same order and a comparison of complex grades becomes one-dimensional. This paper removes that assumption. We replace homogeneity by an explicit order. The set of admissible complex grades is shown to be a complete lattice under the componentwise order, and every inequality used in this theory is identified as an inequality in that lattice. On that basis, we prove that a complex picture fuzzy set is a CPic-FSG if and only if both its amplitude triple and its phase triple are picture fuzzy subgroups, without any comonotonicity hypothesis. Four worked examples are exhibited, each of which fails the homogeneity condition and therefore lies outside the scope of Abbas and Malik; in one of them, the amplitude triple is a picture fuzzy subgroup while the phase triple is not, which cannot occur under homogeneity. Then, we treat the two questions that the admissibility constraints are non-trivial, which are not addressed in the prior literature. First, we explicitly define the intersection and union that preserves admissibility. Second, we prove that the homomorphic image is a complex picture fuzzy set, that outright fails for the variant of the definition in which the neutral grade is collected by a supremum. Finally, we sharpen the homomorphism theory: Direct images are shown to transport the subgroup and normality conditions under epimorphisms, and inverse images under arbitrary homomorphisms, in each case weakening the isomorphism hypothesis previously used.

Keywords: complex picture fuzzy set; complex picture fuzzy subgroup; lattice order on complex grades; homogeneity; admissibility; complex picture fuzzy normal subgroup; homomorphism of complex picture fuzzy subgroups
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1. Introduction

Zadeh laid the foundational framework for the fuzzy set in 1965 [1], a notion that plays a crucial role in addressing uncertainty across many fields. In 1986, Atanassov [2] defined the intuitionistic fuzzy set (IFS), which attaches a membership and a nonmembership grade whose sum does not exceed one to each element. An IFS still records no degree of *neutrality*: An element is only described by how much it belongs and how much it does not. Cuong and Kreinovich [3] and Cuong [4] introduced the picture fuzzy set (Pic-FS), which adds a third, neutral grade subject to the constraint that the three grades sum to at most one. Therefore, a Pic-FS models situations of the "yes / abstain / no" type, where abstention is a positive datum rather than the absence of information. For general background on the fuzzy set theory, see [5].

In a parallel development, Ramot et al. [6] proposed the complex fuzzy set (CFS) in 2002. Each element is assigned a complex-valued grade $r\,e^{i\theta}$ consisting of an *amplitude* term $r \in [0, 1]$ and a *phase* term θ ranging over $[0, 2\pi)$. The phase is not decoration: It carries a second, independent layer of information — typically periodicity or recurrence — that an ordinary fuzzy grade cannot express. Then, Alkouri and Salleh introduced the complex intuitionistic fuzzy set (CIFS) [7, 8], and Akram et al. [9] worked with complex picture fuzzy sets (CPic-FS), in which each of the three picture grades is complex valued. Picture fuzzy environments have been studied from many angles; see [10] for aggregation operators and their use in multi-attribute decision making.

Fuzzy sets can be realized inside algebraic structures, and in particular inside groups. Rosenfeld [11] proposed fuzzy groups in 1971, and the corresponding subgroup notions were developed for each later extension: intuitionistic fuzzy subgroups (IFSGs) by Biswas [12], picture fuzzy subgroups (Pic-FSGs) by Dogra and Pal [13], complex fuzzy subgroups (CFSGs) by AbuHijleh et al. [14], and complex intuitionistic fuzzy subgroups (CIFSGs) by Gulzar et al. [15]. Further work in this direction includes Pythagorean fuzzy subgroups [16], harmonization of fuzzy subgroups [17], complex Pythagorean fuzzy subgroups [18], complex Fermatean fuzzy subgroups [19], and harmonized fuzzy subgroups via θ-open sets [20].

Several frameworks generalize or run parallel to the picture fuzzy set. Here, we only discuss the one that bears directly on the technical content of this paper, and we say why it does. The constraint that makes the present work non-trivial is admissibility: The requirement that the three amplitudes sum to at most one and, separately, that the three phases do. Almost every difficulty resolved below, which operator to use for the neutral grade of an intersection (Proposition 6), and whether a homomorphic image is a complex picture fuzzy set at all (Example 6, Proposition 10), is a consequence of that constraint rather than of the complex-valued form. Therefore, it is relevant that there exist complex-valued, three-grade frameworks in which the constraint is absent. The complex neutrosophic set of Ali and Smarandache [21] is such a framework: It carries complex-valued truth, indeterminacy, and falsity grades but imposes no joint summation bound on them. Consequently, the operations studied here are unconditionally well defined in that setting, and the phenomena recorded in Proposition 6 and Example 6 simply do not arise. The contrast precisely locates where the difficulty in the picture fuzzy case comes from.

While this manuscript was under review, Abbas and Malik [22] published a study of complex picture fuzzy subgroups (CPFSG) in which the notion was introduced with the same axioms used here, and a substantial theory was developed: level sets, support and core, conjugacy and conjugacy classes,

normalizers, a complex picture fuzzy order and index, an analogue of Lagrange's theorem, normal subgroups, and the behavior of all of these under homomorphisms, together with an application to the modeling of a parallel robotic gripper on $SE(3)$. We became aware of [22] after submitting the first revision of this paper. The two works were carried out independently; we claim no priority for the notion of a complex picture fuzzy subgroup, which belongs to [22], and the present paper has been re-scoped accordingly.

The development in [22] rests on a hypothesis that was imposed once and then carried throughout. Complex grades do not come with an order, and [22] obtained one by restricting the objects: A complex picture fuzzy set is called *homogeneous* when, in each of the three components, the amplitude and the phase are comonotone,

$$\ell(\varrho) \leq \ell(\varkappa) \iff \vartheta(\varrho) \leq \vartheta(\varkappa) \qquad \text{for all } \varrho, \varkappa, \tag{1.1}$$

for the pairs (ℓ_p, ℓ_α), (ℓ_r, ℓ_ω) and (ℓ_q, ℓ_γ); then, all developments in [22] are stated for homogeneous complex picture fuzzy sets. Under (1.1), the amplitude order and the phase order coincide, so comparison of complex grades collapses to a single linear comparison.

The assumption is a strong one. It forces the phase to be a monotone reparameterization of the amplitude in each component, so the phase carries no ordering information of its own; in particular, two elements with equal amplitudes must have equal phases. Every example appearing in the present paper violates (1.1) (Remark 7), and one of them — Example 2, where the amplitude triple is a picture fuzzy subgroup while the phase triple is not — cannot be formulated at all under homogeneity, since Eq (1.1) would force the two triples to order identically.

In this paper, we remove the homogeneity assumption and supply what is then needed in its place.

(C1) *An order instead of a restriction.* Section 2 shows that the admissible complex grades form a complete lattice under the componentwise order (Definition 1, Proposition 1), so that every inequality in this theory becomes a precise lattice statement with no hypothesis on the objects. The price is that the order is partial, which we state and handle explicitly.

(C2) *The decomposition theorem without hypotheses.* Theorem 1 separately characterizes CPic-FSGs by the amplitude and phase triples. The corresponding result of [22] assumes homogeneity; ours does not, and Example 2 shows the difference is not vacuous.

(C3) *Admissibility of the operations.* Intersection and union are explicitly defined and proven to preserve both constraints, and the minimum convention for the neutral grade is shown to be the only admissible one (Proposition 6, Remark 8). This question is not addressed in [22].

(C4) *Well-definedness of the homomorphic image.* Proposition 10 proves that the direct image is a complex picture fuzzy set. The corresponding definition in [22] asserts this without proof, and Example 6 shows that the proof is not vacuous: A natural variant of the definition returns an inadmissible triple.

(C5) *Sharper homomorphism hypotheses.* Theorems 5–8 obtain direct images under epimorphisms and inverse images under *arbitrary* homomorphisms, where [22] assumes an isomorphism in three of the four corresponding statements (Remark 12).

We do not duplicate the material of [22] listed above — support, core, conjugacy, normalizer, order, index, and Lagrange's theorem. Extending those notions to the non-homogeneous setting is a natural continuation and is recorded in Section 6 as further work.

1.1. Why the neutral and phase components are not redundant

Independently of [22], it should be shown that a CPic-FSG is not a relabeling of a CIFSG or of a Pic-FSG. We prove this. Theorem 1 makes the question precise, and Examples 2 and 3 answer it: There is a complex picture fuzzy set whose amplitude triple is a Pic-FSG but which is not a CPic-FSG, and one whose membership/nonmembership part is a CIFSG but which is not a CPic-FSG. Neither component can be dropped, and CPic-FSG results are therefore not specializations of CIFSG or Pic-FSG results.

Table 1 places CPic-FSGs among the known subgroup structures; where it is a comparison of fuzzy subgroup structures: "Cond." counts the real-valued closure and inverse inequalities the definition imposes (two per scalar function). The last two rows distinguish the homogeneous theory of [22] from the present one. Figure 1 displays the specialization relations, and Table 2 fixes the notation.

Table 1. Comparison of fuzzy subgroup structures.

Structure	Grade(s)	Phase	Neutral	Admissibility	Cond.	Source
FSG	ξ	no	no	$\xi \in [0,1]$	2	[11]
IFSG	$\ell_p,\ \ell_q$	no	no	$\ell_p + \ell_q \le 1$	4	[12]
Pic-FSG	$\ell_p,\ \ell_r,\ \ell_q$	no	yes	$\ell_p + \ell_r + \ell_q \le 1$	6	[13]
CFSG	$\ell_p e^{2\pi i \ell_\alpha}$	yes	no	$\ell_p,\ \ell_\alpha \in [0,1]$	4	[14]
CIFSG	$\ell_p e^{2\pi i \ell_\alpha},\ \ell_q e^{2\pi i \ell_\gamma}$	yes	no	$\ell_p + \ell_q \le 1,\ \ell_\alpha + \ell_\gamma \le 1$	8	[15]
Complex picture fuzzy subgroups						
CPFSG (homogeneous)	$\ell_p e^{2\pi i \ell_\alpha},\ \ell_r e^{2\pi i \ell_\omega},\ \ell_q e^{2\pi i \ell_\gamma}$	yes	yes	$\sum \ell \le 1,\ \sum \vartheta \le 1$, (1.1)	12	[22]
CPic-FSG (general)	$\ell_p e^{2\pi i \ell_\alpha},\ \ell_r e^{2\pi i \ell_\omega},\ \ell_q e^{2\pi i \ell_\gamma}$	yes	yes	$\sum \ell \le 1,\ \sum \vartheta \le 1$	12	this paper

In Figure 1, an arrow $X \to Y$ means X is recovered from Y by the degeneration of Corollary 2. Examples 2 and 3 show that the two black arrows into CPic-FSG are strict. The remaining arrow records that the homogeneous theory of [22] is the special case (1.1) of the theory developed here; Remark 7 shows the inclusion is proper.

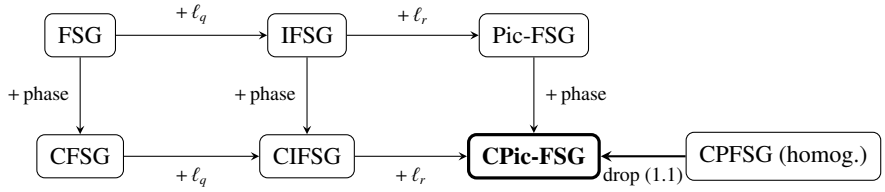


Figure 1. Specialization relations.

Table 2. Notation.

Symbol	Meaning
$(\mathsf{G}, \blacksquare)$	a group with identity ι
$\ell_p, \ell_r, \ell_q : \mathsf{G} \to [0,1]$	amplitudes of membership, neutral membership, nonmembership
$\ell_\alpha, \ell_\omega, \ell_\gamma : \mathsf{G} \to [0,1]$	corresponding phase parameters
$\ell_p e^{2\pi i \ell_\alpha}$	the grade with amplitude ℓ_p and phase $2\pi\ell_\alpha$
$\mathbb{L}$	the lattice $[0,1] \times [0,1]$ of complex grades (Definition 1)
$\preceq$, $\wedge$, $\vee$	order, meet and join of $\mathbb{L}$
$\mathsf{D}^{\mathrm{amp}}$, D^{ph}	amplitude triple and phase triple of D
$F^{-1}(\varkappa)$	the fibre $\{\varrho \in \mathbb{U} : F(\varrho) = \varkappa\}$

1.2. Organization

Section 2 fixes the order on complex grades and recalls the definitions that precede CPic-FSGs. Section 3 introduces the CPic-FSG in the general (non-homogeneous) setting, proves the decomposition and level-set theorems, settles independence and reduction, and defines the set operations and proves their admissibility. Section 4 treats cosets and normal subgroups. Section 5 develops the homomorphism theory. Section 6 concludes and lists further work.

2. Preliminaries

Throughout, $(\mathsf{G}, \blacksquare)$ is a group with identity ι, and $\mathbb{U}$ denotes a nonempty universe. First, we settle what the inequalities between complex grades mean, since every later definition depends on it.

2.1. Complex grades, phase uniqueness, and the order

A complex grade is written $\ell\, e^{2\pi i\vartheta}$ with $\ell \in [0,1]$ and $\vartheta \in [0,1]$; the real number $2\pi\vartheta$ is the phase. The following remark is needed because the exponential is periodic.

Remark 1. *(Phase uniqueness) The map $\vartheta \mapsto e^{2\pi i\vartheta}$ is not injective on $[0,1]$: it identifies $\vartheta = 0$ with $\vartheta = 1$. Consequently, the complex number $\ell\, e^{2\pi i\vartheta}$ does not by itself determine the pair (ℓ, ϑ), and it determines it not at all when $\ell = 0$. Therefore, we take the* pair $(\ell, \vartheta) \in [0,1] \times [0,1]$ *as the primitive object and regard $\ell\, e^{2\pi i\vartheta}$ as the notation for that pair; equivalently, one may normalize the phase parameter to $[0,1)$, on which the map is injective. All statements below are statements about pairs, and no complex arithmetic — no addition or multiplication of grades — is used anywhere in this paper.*

Definition 1. *(The lattice of complex grades) Let*

$$\mathbb{L} \;=\; \{\, \ell\, e^{2\pi i\vartheta} \;:\; \ell, \vartheta \in [0,1] \,\} \;\cong\; [0,1] \times [0,1],$$

with the identification of Remark 1. For $z_1 = \ell_1 e^{2\pi i\vartheta_1}$ and $z_2 = \ell_2 e^{2\pi i\vartheta_2}$ in $\mathbb{L}$, define

$$z_1 \preceq z_2 \quad\Longleftrightarrow\quad \ell_1 \le \ell_2 \;\text{ and }\; \vartheta_1 \le \vartheta_2,$$

and write $z_2 \succeq z_1$ for $z_1 \preceq z_2$. For a nonempty family $\{z_j\}_{j\in J} \subseteq \mathbb{L}$, $z_j = \ell_j e^{2\pi i\vartheta_j}$, put

$$\bigwedge_{j\in J} z_j \;=\; \Big(\inf_{j\in J} \ell_j\Big)\, e^{2\pi i\left(\inf_{j\in J}\vartheta_j\right)}, \qquad \bigvee_{j\in J} z_j \;=\; \Big(\sup_{j\in J} \ell_j\Big)\, e^{2\pi i\left(\sup_{j\in J}\vartheta_j\right)}.$$

For two elements, we write $z_1 \wedge z_2$ and $z_1 \vee z_2$.

Proposition 1. *$(\mathbb{L}, \preceq)$ is a complete lattice with the least element $0e^{2\pi i\,0}$ and the greatest element $1e^{2\pi i\,1}$, and the operations of Definition 1 are its infimum and supremum. Moreover, for $z_1, z_2, w \in \mathbb{L}$,*

$$z_1 \wedge z_2 \succeq w \iff (z_1 \succeq w \text{ and } z_2 \succeq w), \qquad z_1 \vee z_2 \preceq w \iff (z_1 \preceq w \text{ and } z_2 \preceq w).$$

Proof. Under the identification $\mathbb{L} \cong [0,1] \times [0,1]$, the relation $\preceq$ is the product order of two copies of the complete lattice $([0,1], \leq)$. A product of complete lattices is a complete lattice whose infima and suprema are computed componentwise, which is exactly Definition 1; the bounds are $(0,0)$ and $(1,1)$. The two displayed equivalences are the universal properties of the meet and the join. □

Notation. *$\preceq$ is a partial order: $0.5e^{2\pi i\,0.2}$ and $0.3e^{2\pi i\,0.9}$ are incomparable. Accordingly, "$z_1 \not\succeq z_2$" means that $z_1 \succeq z_2$ fails, not that $z_1 \preceq z_2$ holds. For readability, and to keep the formulas of CFSGs [14], CIFSGs [15], CPic-FSs [9], and CPic-FSGs [22] recognizable, we continue to write $\geq$, $\leq$, $\wedge$, $\vee$ for $\succeq$, $\preceq$ and the lattice meet and join on $\mathbb{L}$; by Definition 1, these symbols are now completely specified. Whenever a step uses the partiality of the order, we say so explicitly.*

Corollary 1. *For $z_j = \ell_j e^{2\pi i \vartheta_j} \in \mathbb{L}$ $(j = 1, 2, 3)$,*

$$z_1 \succeq z_2 \wedge z_3 \iff (\ell_1 \geq \ell_2 \wedge \ell_3 \text{ and } \vartheta_1 \geq \vartheta_2 \wedge \vartheta_3),$$

and dually with $\preceq$ and $\vee$. In particular, every "where …" clause in Definitions 7–9 is an equivalent *reformulation of the preceding complex inequality, not an additional hypothesis.*

Proof. Immediate from Definition 1, since $z_2 \wedge z_3 = (\ell_2 \wedge \ell_3) e^{2\pi i(\vartheta_2 \wedge \vartheta_3)}$. □

For comparison with [22], we record the alternative device for ordering complex grades.

Definition 2. *(Homogeneity, [22]) A pair (ℓ, ϑ) of functions $\mathsf{G} \to [0,1]$ is* comonotone *if*

$$\ell(\varrho) \leq \ell(\varkappa) \iff \vartheta(\varrho) \leq \vartheta(\varkappa) \qquad \text{for all } \varrho, \varkappa \in \mathsf{G}.$$

A complex picture fuzzy set D (Definition 9 below) is homogeneous *if each of the three pairs (ℓ_p, ℓ_α), (ℓ_r, ℓ_ω), (ℓ_q, ℓ_γ) is comonotone.*

Remark 2. *Comonotonicity of (ℓ, ϑ) forces $\ell(\varrho) = \ell(\varkappa) \Rightarrow \vartheta(\varrho) = \vartheta(\varkappa)$, since $\ell(\varrho) \leq \ell(\varkappa)$ and $\ell(\varkappa) \leq \ell(\varrho)$ both hold. Thus, under homogeneity, the phase is constant on the level sets of the amplitude, and on the image of D the restriction of $\preceq$ is a total order. This is what makes the comparison one-dimensional in [22]; additionally, it is why homogeneity is a genuine restriction, as Remark 7 records.*

Definition 3. *([11]) Let $(\mathsf{G}, \blacksquare)$ be a group, and let $\xi\colon \mathsf{G} \to [0,1]$ define a fuzzy set D of G. Then, D is a fuzzy subgroup (FSG) of G if for all $\varrho, \varkappa \in \mathsf{G}$, the following hold:*

(i) $\xi(\varrho \blacksquare \varkappa) \geq \xi(\varrho) \wedge \xi(\varkappa)$;

(ii) $\xi(\varrho^{-1}) \geq \xi(\varrho)$.

Biswas [12] introduced the intuitionistic fuzzy subgroup (IFSG) in 1989. Twenty-four years later, Cuong and Kreinovich [3] defined the Pic-FS, and Dogra and Pal [13] presented the Pic-FSG in 2023.

Definition 4. *([12]) Let* $(\mathsf{G}, \blacksquare)$ *be a group, and let* $\ell_p, \ell_q \colon \mathsf{G} \to [0,1]$ *with* $\ell_p + \ell_q \leq 1$ *pointwise define an IFS* D *of* G*. Then,* D *is an IFSG of* G *if for all* $\varrho, \varkappa \in \mathsf{G}$*, the following hold:*

(i) $\ell_p(\varrho \blacksquare \varkappa) \geq \ell_p(\varrho) \wedge \ell_p(\varkappa)$;

(ii) $\ell_p(\varrho^{-1}) \geq \ell_p(\varrho)$;

(iii) $\ell_q(\varrho \blacksquare \varkappa) \leq \ell_q(\varrho) \vee \ell_q(\varkappa)$;

(iv) $\ell_q(\varrho^{-1}) \leq \ell_q(\varrho)$.

Definition 5. *([3]) A Pic-FS* D *on* $\mathbb{U}$ *is determined by three functions* $\ell_p, \ell_r, \ell_q \colon \mathbb{U} \to [0,1]$*, called the degrees of membership, neutral membership, and nonmembership, respectively, subject to* $\ell_p(\varrho) + \ell_r(\varrho) + \ell_q(\varrho) \leq 1$ *for every* $\varrho \in \mathbb{U}$*.*

Definition 6. *([13]) Let* $(\mathsf{G}, \blacksquare)$ *be a group, and let* $\mathsf{D} = (\ell_p, \ell_r, \ell_q)$ *be a Pic-FS of* G*. Then,* D *is a Pic-FSG of* G *if for all* $\varrho, \varkappa \in \mathsf{G}$*, the following hold:*

(i) $\ell_p(\varrho \blacksquare \varkappa) \geq \ell_p(\varrho) \wedge \ell_p(\varkappa)$, $\ell_r(\varrho \blacksquare \varkappa) \geq \ell_r(\varrho) \wedge \ell_r(\varkappa)$ *and* $\ell_q(\varrho \blacksquare \varkappa) \leq \ell_q(\varrho) \vee \ell_q(\varkappa)$;

(ii) $\ell_p(\varrho^{-1}) \geq \ell_p(\varrho)$, $\ell_r(\varrho^{-1}) \geq \ell_r(\varrho)$ *and* $\ell_q(\varrho^{-1}) \leq \ell_q(\varrho)$.

Definition 7. *([14]) Let* $(\mathsf{G}, \blacksquare)$ *be a group, and let* D *be the CFS of* G *given by* $\xi = \ell_p e^{2\pi i \ell_\alpha}$ *with* $\ell_p, \ell_\alpha \colon \mathsf{G} \to [0,1]$*, so* $\xi(\varrho) \in \mathbb{L}$ *for every* ϱ*. Then,* D *is a CFSG of* G *if for all* $\varrho, \varkappa \in \mathsf{G}$*, the following hold:*

(i) $\ell_p(\varrho \blacksquare \varkappa) e^{2\pi i \ell_\alpha(\varrho \blacksquare \varkappa)} \geq \ell_p(\varrho) e^{2\pi i \ell_\alpha(\varrho)} \wedge \ell_p(\varkappa) e^{2\pi i \ell_\alpha(\varkappa)}$*; equivalently (Corollary 1)* $\ell_p(\varrho \blacksquare \varkappa) \geq \ell_p(\varrho) \wedge \ell_p(\varkappa)$ *and* $\ell_\alpha(\varrho \blacksquare \varkappa) \geq \ell_\alpha(\varrho) \wedge \ell_\alpha(\varkappa)$;

(ii) $\ell_p(\varrho^{-1}) e^{2\pi i \ell_\alpha(\varrho^{-1})} \geq \ell_p(\varrho) e^{2\pi i \ell_\alpha(\varrho)}$*; equivalently* $\ell_p(\varrho^{-1}) \geq \ell_p(\varrho)$ *and* $\ell_\alpha(\varrho^{-1}) \geq \ell_\alpha(\varrho)$*.*

Definition 8. *([15]) Let* $(\mathsf{G}, \blacksquare)$ *be a group, and let* D *be the CIFS of* G *given by* $\xi = (\ell_p e^{2\pi i \ell_\alpha}, \ell_q e^{2\pi i \ell_\gamma})$ *with* $\ell_p, \ell_q, \ell_\alpha, \ell_\gamma \colon \mathsf{G} \to [0,1]$*,* $\ell_p + \ell_q \leq 1$ *and* $\ell_\alpha + \ell_\gamma \leq 1$ *pointwise. Then,* D *is a CIFSG of* G *if conditions* (i) *and* (ii) *of Definition 7 hold for* $\ell_p e^{2\pi i \ell_\alpha}$*, and for all* $\varrho, \varkappa \in \mathsf{G}$*, the following hold:*

(iii) $\ell_q(\varrho \blacksquare \varkappa) e^{2\pi i \ell_\gamma(\varrho \blacksquare \varkappa)} \leq \ell_q(\varrho) e^{2\pi i \ell_\gamma(\varrho)} \vee \ell_q(\varkappa) e^{2\pi i \ell_\gamma(\varkappa)}$;

(iv) $\ell_q(\varrho^{-1}) e^{2\pi i \ell_\gamma(\varrho^{-1})} \leq \ell_q(\varrho) e^{2\pi i \ell_\gamma(\varrho)}$.

3. Complex picture fuzzy subgroups

The following definition is constructed with a new lattice of a complex grade.

Definition 9. *Let* $(\mathsf{G}, \blacksquare)$ *be a group. A* complex picture fuzzy set *(CPic-FS)* D *of* G *consists of six functions*

$$\ell_p,\ \ell_r,\ \ell_q,\ \ell_\alpha,\ \ell_\omega,\ \ell_\gamma \colon \mathsf{G} \longrightarrow [0,1],$$

written together as

$$\xi(\varrho) = (\ell_p(\varrho) e^{2\pi i \ell_\alpha(\varrho)},\ \ell_r(\varrho) e^{2\pi i \ell_\omega(\varrho)},\ \ell_q(\varrho) e^{2\pi i \ell_\gamma(\varrho)}) \in \mathbb{L}^3,$$

subject to the two admissibility constraints

$$\ell_p(\varrho) + \ell_r(\varrho) + \ell_q(\varrho) \leq 1 \quad \textit{and} \quad \ell_\alpha(\varrho) + \ell_\omega(\varrho) + \ell_\gamma(\varrho) \leq 1 \qquad \textit{for all } \varrho \in \mathsf{G}. \tag{3.1}$$

We call $\mathsf{D}^{\mathrm{amp}} = (\ell_p, \ell_r, \ell_q)$ *the* amplitude triple *and* $\mathsf{D}^{\mathrm{ph}} = (\ell_\alpha, \ell_\omega, \ell_\gamma)$ *the* phase triple*; by* (3.1)*, each is a Pic-FS of* G.

Then, D *is a complex picture fuzzy subgroup (CPic-FSG) of* G *if for all* $\varrho, \varkappa \in \mathsf{G}$*, with* $\geq, \leq, \wedge, \vee$ *as in Definition 1, the following hold:*

(ia) $\ell_p(\varrho\varkappa)e^{2\pi i\ell_\alpha(\varrho\varkappa)} \geq \ell_p(\varrho)e^{2\pi i\ell_\alpha(\varrho)} \wedge \ell_p(\varkappa)e^{2\pi i\ell_\alpha(\varkappa)}$*; equivalently* $\ell_p(\varrho\varkappa) \geq \ell_p(\varrho) \wedge \ell_p(\varkappa)$ *and* $\ell_\alpha(\varrho\varkappa) \geq \ell_\alpha(\varrho) \wedge \ell_\alpha(\varkappa)$.

(iia) $\ell_p(\varrho^{-1})e^{2\pi i\ell_\alpha(\varrho^{-1})} \geq \ell_p(\varrho)e^{2\pi i\ell_\alpha(\varrho)}$*; equivalently* $\ell_p(\varrho^{-1}) \geq \ell_p(\varrho)$ *and* $\ell_\alpha(\varrho^{-1}) \geq \ell_\alpha(\varrho)$.

(ib) $\ell_r(\varrho\varkappa)e^{2\pi i\ell_\omega(\varrho\varkappa)} \geq \ell_r(\varrho)e^{2\pi i\ell_\omega(\varrho)} \wedge \ell_r(\varkappa)e^{2\pi i\ell_\omega(\varkappa)}$*; equivalently* $\ell_r(\varrho\varkappa) \geq \ell_r(\varrho) \wedge \ell_r(\varkappa)$ *and* $\ell_\omega(\varrho\varkappa) \geq \ell_\omega(\varrho) \wedge \ell_\omega(\varkappa)$.

(iib) $\ell_r(\varrho^{-1})e^{2\pi i\ell_\omega(\varrho^{-1})} \geq \ell_r(\varrho)e^{2\pi i\ell_\omega(\varrho)}$*; equivalently* $\ell_r(\varrho^{-1}) \geq \ell_r(\varrho)$ *and* $\ell_\omega(\varrho^{-1}) \geq \ell_\omega(\varrho)$.

(ic) $\ell_q(\varrho\varkappa)e^{2\pi i\ell_\gamma(\varrho\varkappa)} \leq \ell_q(\varrho)e^{2\pi i\ell_\gamma(\varrho)} \vee \ell_q(\varkappa)e^{2\pi i\ell_\gamma(\varkappa)}$*; equivalently* $\ell_q(\varrho\varkappa) \leq \ell_q(\varrho) \vee \ell_q(\varkappa)$ *and* $\ell_\gamma(\varrho\varkappa) \leq \ell_\gamma(\varrho) \vee \ell_\gamma(\varkappa)$.

(iic) $\ell_q(\varrho^{-1})e^{2\pi i\ell_\gamma(\varrho^{-1})} \leq \ell_q(\varrho)e^{2\pi i\ell_\gamma(\varrho)}$*; equivalently* $\ell_q(\varrho^{-1}) \leq \ell_q(\varrho)$ *and* $\ell_\gamma(\varrho^{-1}) \leq \ell_\gamma(\varrho)$.

Remark 3. *(Attribution) Conditions* (ia)–(iic) *are the axioms of [22, Definition 3.7], where the notion was introduced. The difference from [22] is not in the axioms but in the class of objects to which they are applied: Authors in [22] applied them to* homogeneous *complex picture fuzzy sets (Definition 2), whereas Definition 9 imposes no comonotonicity, the order having been supplied instead by Definition 1. All results below are stated in this general setting.*

Remark 4. *(Why the neutral grade obeys the membership inequalities) Conditions* (ib)*,* (iib) *have the same direction as* (ia)*,* (iia) *rather than that of* (ic)*,* (iic)*. This is not a free choice. In a Pic-FS, the neutral grade records* positive *evidence of abstention, so it behaves like the membership grade and unlike the nonmembership grade; accordingly Dogra and Pal [13] imposed* $\geq$ *on* ℓ_r *in Definition 6, and Definition 9 is the complex-valued extension of exactly those conditions. Two consequences confirm the choice. By Theorem 1, the resulting notion restricts to the Pic-FSG of [13] in the phase-free case, which would fail under the reversed direction; then, Proposition 3 gives* $\ell_r(\iota) \geq \ell_r(\varrho)$ *(i.e., abstention is maximal at the identity).*

3.1. Decomposition, reduction, and independence of the conditions

Theorem 1. *(Decomposition) Let* D *be a CPic-FS of a group* $(\mathsf{G}, \blacksquare)$*. Then,* D *is a CPic-FSG of* G *if and only if both* $\mathsf{D}^{\mathrm{amp}}$ *and* D^{ph} *are Pic-FSGs of* G *in the sense of Definition 6.*

Proof. By Corollary 1, condition (ia) is equivalent to the conjunction of $\ell_p(\varrho \blacksquare \varkappa) \geq \ell_p(\varrho) \wedge \ell_p(\varkappa)$ and $\ell_\alpha(\varrho \blacksquare \varkappa) \geq \ell_\alpha(\varrho) \wedge \ell_\alpha(\varkappa)$, the first being condition (i) of Definition 6 for $\mathsf{D}^{\mathrm{amp}}$ in its membership component and the second the same condition for D^{ph}. The same argument applied to (ib), (ic) and, using the dual half of Corollary 1, to (iia)–(iic), shows that the twelve real inequalities of Definition 9 are precisely the six inequalities of Definition 6 for $\mathsf{D}^{\mathrm{amp}}$ together with the six for D^{ph}. Finally, $\mathsf{D}^{\mathrm{amp}}$ and D^{ph} are Pic-FSs by (3.1). □

Remark 5. *Theorem 1 corresponds to [22, Theorem 3.8], which states the same characterization for* homogeneous *complex picture fuzzy sets. There, the proof uses homogeneity to pass between a complex*

inequality and the pair of real inequalities. The proof above does not: By Corollary 1, that passage is an identity in the lattice $\mathbb{L}$ *and holds for every CPic-FS. Therefore, Theorem 1 is the non-homogeneous generalization of [22, Theorem 3.8], and by Remark 7, the generalization is proper.*

Remark 6. *Theorem 1 must not be read as saying that the CPic-FSG theory reduces to the Pic-FSG theory. The two triples are* coupled*: The admissibility constraints* (3.1) *tie the three amplitudes to one another and the three phases to one another, and every set operation and homomorphic image must simultaneously respect both constraints. It is exactly this coupling that fails in Proposition 6 for one of the two intersection conventions in the literature, and in Example 6 for the direct image. Neither phenomenon is visible in the Pic-FSG setting.*

Corollary 2. *(Reduction to known structures) Let* D *be a CPic-FSG of* $(\mathsf{G}, \blacksquare)$.

(1) *If* $\ell_\alpha \equiv \ell_\omega \equiv \ell_\gamma \equiv 0$*, then all grades are real and* $\mathsf{D}^{\mathrm{amp}}$ *is exactly a Pic-FSG* (Definition 6).

(2) *If* $\ell_r \equiv 0$ *and* $\ell_\omega \equiv 0$*, then* $(\ell_p e^{2\pi i \ell_\alpha}, \ell_q e^{2\pi i \ell_\gamma})$ *is exactly a CIFSG* (Definition 8).

(3) *The component* $\ell_p e^{2\pi i \ell_\alpha}$ *alone is always a CFSG* (Definition 7).

(4) *If* $\ell_r \equiv \ell_q \equiv 0$ *and* $\ell_\alpha \equiv \ell_\omega \equiv \ell_\gamma \equiv 0$*, then* ℓ_p *is exactly a FSG* (Definition 3).

(5) *If* D *is homogeneous in the sense of Definition 2, then* D *is a CPFSG in the sense of [22, Definition 3.7].*

Conversely, each of (1)–(5) *produces a CPic-FSG from the corresponding structure.*

Proof. In each of (1)–(4), the stated functions satisfy (3.1), and the relevant subgroup inequalities are those of Definition 9 with the vanishing components removed; for a vanishing component, the inequality reads $0 \geq 0 \wedge 0$ or $0 \leq 0 \vee 0$ and holds trivially. For (2), $\ell_r \equiv 0$ forces $\ell_p + \ell_q \leq 1$, which is the CIFS constraint, and similarly for the phases. Part (5) is immediate, the axioms being the same by Remark 3. The converse direction is the same computation read backwards. □

Remark 7. *(The examples of this paper are not homogeneous) Every worked example, in the Table 3, fails the comonotonicity condition of Definition 2, so none of them lies in the class treated by [22].*

Table 3. Overview of Examples 1 to 4.

Example	Component	A pair violating comonotonicity
1	(ℓ_p, ℓ_α)	$\ell_p(c) = 0.35 > 0.30 = \ell_p(a)$ while $\ell_\alpha(c) = \ell_\alpha(a) = 0.40$
2	(ℓ_p, ℓ_α)	$\ell_p(0) = \ell_p(2) = 0.50$ while $\ell_\alpha(0) = 0.40 \neq 0.30 = \ell_\alpha(2)$
3	(ℓ_r, ℓ_ω)	$\ell_r(0) = 0.25 < 0.30 = \ell_r(1)$ while $\ell_\omega(0) = 0.30 > 0.25 = \ell_\omega(1)$
4	(ℓ_p, ℓ_α)	equal amplitudes 0.55 on $77\mathbb{Z}$ and on $7\mathbb{Z} \setminus 11\mathbb{Z}$ with phases $0.45 \neq 0.40$

By Remark 2, a homogeneous CPic-FS has equal phases wherever it has equal amplitudes, which is what each of the first, second, and fourth rows violates; the third row violates comonotonicity outright. In particular, Example 2, in which $\mathsf{D}^{\mathrm{amp}}$ *is a Pic-FSG and* D^{ph} *is not, cannot be realized by a homogeneous CPic-FS, since under* (1.1)*, the two triples induce the same order and are Pic-FSGs together. Therefore, the class treated here is strictly larger than that of [22].*

Proposition 2. *(Independence of the inverse conditions) Let* D *be a CPic-FS of* $(\mathsf{G}, \blacksquare)$ *satisfying* (ia)–(ic).

(1) *If every element of* G *has a finite order, then* (iia)–(iic) *automatically hold, so* D *is a CPic-FSG.*

(2) *In general* (iia)–(iic) *are not implied by* (ia)–(ic)*, respectively.*

Proof. (1) Let $\varrho \in \mathsf{G}$ have an order of $n \geq 1$, so $\varrho^{-1} = \varrho^{n-1}$. For $n = 1$, we have $\varrho = \iota = \varrho^{-1}$ and there is nothing to prove. For $n \geq 2$, conditions (ia)–(ic) alone give, by Proposition 5 — whose proof uses only the closure conditions — $\ell_p(\varrho^{n-1}) \geq \ell_p(\varrho)$, $\ell_\alpha(\varrho^{n-1}) \geq \ell_\alpha(\varrho)$, the analogous inequalities for ℓ_r, ℓ_ω, and $\ell_q(\varrho^{n-1}) \leq \ell_q(\varrho)$, $\ell_\gamma(\varrho^{n-1}) \leq \ell_\gamma(\varrho)$. Substituting $\varrho^{-1} = \varrho^{n-1}$ yields (iia)–(iic).

(2) Take $\mathsf{G} = (\mathbb{Z}, +)$, which is torsion free, and define the following:

$$\xi(\varrho) = \begin{cases} (0.50e^{2\pi i\, 0.40},\ 0.20e^{2\pi i\, 0.20},\ 0.20e^{2\pi i\, 0.20}), & \varrho \geq 0, \\ (0.20e^{2\pi i\, 0.10},\ 0.20e^{2\pi i\, 0.20},\ 0.50e^{2\pi i\, 0.50}), & \varrho < 0. \end{cases}$$

Both rows satisfy (3.1), the amplitude sums being 0.90 and the phase sums being 0.80. Each of $\ell_p, \ell_\alpha, \ell_r$, and ℓ_ω takes its smaller value exactly on $\varrho < 0$ and each of ℓ_q, ℓ_γ its larger value there; since $x, y \geq 0$ implies $x + y \geq 0$ and $x, y < 0$ implies $x + y < 0$, while in the mixed case, the right-hand side of (ia), (ib) is already the smaller value and that of (ic) already the larger, (ia)–(ic) hold for all $x, y \in \mathbb{Z}$. However, $\ell_p(-1) = 0.20 < 0.50 = \ell_p(1)$, so (iia) fails, and $\ell_q(-1) = 0.50 > 0.20 = \ell_q(1)$, so (iic) fails. □

3.2. Basic properties

The three results of this subsection correspond, in the homogeneous case, to [22, Propositions 3.9 and 3.10, Theorem 3.11]. The proofs below only use Definition 1 and so do not require homogeneity.

Proposition 3. *Let* D *be a CPic-FSG of a group* $(\mathsf{G}, \blacksquare)$ *with identity* ι*. Then, for every* $\varrho \in \mathsf{G}$*, the following hold:*

(1.1) $\ell_p(\iota)e^{2\pi i\ell_\alpha(\iota)} \geq \ell_p(\varrho)e^{2\pi i\ell_\alpha(\varrho)}$*, i.e.,* $\ell_p(\iota) \geq \ell_p(\varrho)$ *and* $\ell_\alpha(\iota) \geq \ell_\alpha(\varrho)$*;*

(1.2) $\ell_r(\iota)e^{2\pi i\ell_\omega(\iota)} \geq \ell_r(\varrho)e^{2\pi i\ell_\omega(\varrho)}$*, i.e.,* $\ell_r(\iota) \geq \ell_r(\varrho)$ *and* $\ell_\omega(\iota) \geq \ell_\omega(\varrho)$*;*

(1.3) $\ell_q(\iota)e^{2\pi i\ell_\gamma(\iota)} \leq \ell_q(\varrho)e^{2\pi i\ell_\gamma(\varrho)}$*, i.e.,* $\ell_q(\iota) \leq \ell_q(\varrho)$ *and* $\ell_\gamma(\iota) \leq \ell_\gamma(\varrho)$*;*

(2.1) $\ell_p(\varrho^{-1})e^{2\pi i\ell_\alpha(\varrho^{-1})} = \ell_p(\varrho)e^{2\pi i\ell_\alpha(\varrho)}$*;*

(2.2) $\ell_r(\varrho^{-1})e^{2\pi i\ell_\omega(\varrho^{-1})} = \ell_r(\varrho)e^{2\pi i\ell_\omega(\varrho)}$*;*

(2.3) $\ell_q(\varrho^{-1})e^{2\pi i\ell_\gamma(\varrho^{-1})} = \ell_q(\varrho)e^{2\pi i\ell_\gamma(\varrho)}$.

Proof. **(1.1)** Using $\iota = \varrho \blacksquare \varrho^{-1}$, condition (ia), Definition 1, and condition (iia), in turn,

$$\begin{aligned} \ell_p(\iota)e^{2\pi i\ell_\alpha(\iota)} &= \ell_p(\varrho \blacksquare \varrho^{-1})e^{2\pi i\ell_\alpha(\varrho\blacksquare\varrho^{-1})} \\ &\geq \ell_p(\varrho)e^{2\pi i\ell_\alpha(\varrho)} \wedge \ell_p(\varrho^{-1})e^{2\pi i\ell_\alpha(\varrho^{-1})} && \text{by (ia)} \\ &= (\ell_p(\varrho) \wedge \ell_p(\varrho^{-1}))e^{2\pi i\left(\ell_\alpha(\varrho)\wedge\ell_\alpha(\varrho^{-1})\right)} && \text{by Definition 1} \\ &= \ell_p(\varrho)e^{2\pi i\ell_\alpha(\varrho)}, \end{aligned}$$

the last step because (iia) gives $\ell_p(\varrho^{-1}) \geq \ell_p(\varrho)$ and $\ell_\alpha(\varrho^{-1}) \geq \ell_\alpha(\varrho)$, so both minima are attained at ϱ.

(1.2) Identical, with (ℓ_r, ℓ_ω) in place of (ℓ_p, ℓ_α) and (ib), (iib) in place of (ia), (iia):

$$\ell_r(\iota)e^{2\pi i\ell_\omega(\iota)} \;\geq\; (\ell_r(\varrho) \wedge \ell_r(\varrho^{-1}))e^{2\pi i\left(\ell_\omega(\varrho)\wedge\ell_\omega(\varrho^{-1})\right)} \;=\; \ell_r(\varrho)e^{2\pi i\ell_\omega(\varrho)}.$$

(1.3) Here, the inequalities are reversed, so the maxima rather than the minima occur. By (ic) and then (iic),

$$\begin{aligned}\ell_q(\iota)e^{2\pi i\ell_\gamma(\iota)} &= \ell_q(\varrho \blacksquare \varrho^{-1})e^{2\pi i\ell_\gamma(\varrho\blacksquare\varrho^{-1})}\\ &\leq \ell_q(\varrho)e^{2\pi i\ell_\gamma(\varrho)} \vee \ell_q(\varrho^{-1})e^{2\pi i\ell_\gamma(\varrho^{-1})}\\ &= (\ell_q(\varrho) \vee \ell_q(\varrho^{-1}))e^{2\pi i\left(\ell_\gamma(\varrho)\vee\ell_\gamma(\varrho^{-1})\right)}\\ &= \ell_q(\varrho)e^{2\pi i\ell_\gamma(\varrho)},\end{aligned}$$

since (iic) gives $\ell_q(\varrho^{-1}) \leq \ell_q(\varrho)$ and $\ell_\gamma(\varrho^{-1}) \leq \ell_\gamma(\varrho)$, so both maxima are attained at ϱ.

(2.1) Applying (iia) to ϱ^{-1} and using $(\varrho^{-1})^{-1} = \varrho$,

$$\ell_p(\varrho)e^{2\pi i\ell_\alpha(\varrho)} = \ell_p((\varrho^{-1})^{-1})e^{2\pi i\ell_\alpha((\varrho^{-1})^{-1})} \;\geq\; \ell_p(\varrho^{-1})e^{2\pi i\ell_\alpha(\varrho^{-1})}.$$

Together with (iia) applied to ϱ, which gives the reverse inequality, and the antisymmetry of $\preceq$ (Proposition 1), equality follows. Explicitly, $\ell_p(\varrho^{-1}) = \ell_p(\varrho)$ and $\ell_\alpha(\varrho^{-1}) = \ell_\alpha(\varrho)$.

(2.2) The same argument with (iib) gives $\ell_r(\varrho^{-1}) = \ell_r(\varrho)$ and $\ell_\omega(\varrho^{-1}) = \ell_\omega(\varrho)$.

(2.3) The same argument with (iic), both inequalities now being $\leq$, gives $\ell_q(\varrho^{-1}) = \ell_q(\varrho)$ and $\ell_\gamma(\varrho^{-1}) = \ell_\gamma(\varrho)$. □

Proposition 4. *Let* $(\mathsf{G}, \blacksquare)$ *be a group and* D *be a CPic-FS of* G*. Then,* D *is a CPic-FSG of* G *if and only if for all* $\varrho, \varkappa \in \mathsf{G}$*, the following hold:*

(I) $\ell_p(\varrho \blacksquare \varkappa^{-1})e^{2\pi i\ell_\alpha(\varrho\blacksquare\varkappa^{-1})} \;\geq\; \ell_p(\varrho)e^{2\pi i\ell_\alpha(\varrho)} \wedge \ell_p(\varkappa)e^{2\pi i\ell_\alpha(\varkappa)}$;

(II) $\ell_r(\varrho \blacksquare \varkappa^{-1})e^{2\pi i\ell_\omega(\varrho\blacksquare\varkappa^{-1})} \;\geq\; \ell_r(\varrho)e^{2\pi i\ell_\omega(\varrho)} \wedge \ell_r(\varkappa)e^{2\pi i\ell_\omega(\varkappa)}$;

(III) $\ell_q(\varrho \blacksquare \varkappa^{-1})e^{2\pi i\ell_\gamma(\varrho\blacksquare\varkappa^{-1})} \;\leq\; \ell_q(\varrho)e^{2\pi i\ell_\gamma(\varrho)} \vee \ell_q(\varkappa)e^{2\pi i\ell_\gamma(\varkappa)}$.

By Corollary 1, each of (I)–(III) *is equivalent to the corresponding pair of amplitude and phase inequalities.*

Proof. **Necessity** ($\Rightarrow$). Let D be a CPic-FSG. By Proposition 3 (2.1)–(2.3),

$$\ell_p(\varkappa^{-1})e^{2\pi i\ell_\alpha(\varkappa^{-1})} = \ell_p(\varkappa)e^{2\pi i\ell_\alpha(\varkappa)}, \quad \ell_r(\varkappa^{-1})e^{2\pi i\ell_\omega(\varkappa^{-1})} = \ell_r(\varkappa)e^{2\pi i\ell_\omega(\varkappa)}, \tag{3.2}$$

$$\ell_q(\varkappa^{-1})e^{2\pi i\ell_\gamma(\varkappa^{-1})} = \ell_q(\varkappa)e^{2\pi i\ell_\gamma(\varkappa)}. \tag{3.3}$$

Applying (ia) to the pair $(\varrho, \varkappa^{-1})$ and substituting the first of these equalities gives (I); applying (ib) and (ic) to the same pair and substituting the second and third equalities gives (II) and (III).

Sufficiency ($\Leftarrow$). Assume (I)–(III).

Membership component. Taking $\varkappa = \varrho$ in (I),

$$\ell_p(\iota)e^{2\pi i\ell_\alpha(\iota)} = \ell_p(\varrho \blacksquare \varrho^{-1})e^{2\pi i\ell_\alpha(\varrho\blacksquare\varrho^{-1})} \;\geq\; \ell_p(\varrho)e^{2\pi i\ell_\alpha(\varrho)} \wedge \ell_p(\varrho)e^{2\pi i\ell_\alpha(\varrho)} = \ell_p(\varrho)e^{2\pi i\ell_\alpha(\varrho)}. \tag{3.4}$$

Taking $\varrho = \iota$, $\varkappa = \varrho$ in (I) and using (3.4),

$$\ell_p(\varrho^{-1})e^{2\pi i\ell_\alpha(\varrho^{-1})} = \ell_p(\iota \blacksquare \varrho^{-1})e^{2\pi i\ell_\alpha(\iota\blacksquare\varrho^{-1})} \;\geq\; \ell_p(\iota)e^{2\pi i\ell_\alpha(\iota)} \wedge \ell_p(\varrho)e^{2\pi i\ell_\alpha(\varrho)} = \ell_p(\varrho)e^{2\pi i\ell_\alpha(\varrho)}, \tag{3.5}$$

which is (iia). Finally, replacing $\varkappa$ by $\varkappa^{-1}$ in (I) and using (3.5) for $\varkappa$,

$$\ell_p(\varrho \blacksquare \varkappa)e^{2\pi i\ell_\alpha(\varrho\blacksquare\varkappa)} \;\geq\; \ell_p(\varrho)e^{2\pi i\ell_\alpha(\varrho)} \wedge \ell_p(\varkappa^{-1})e^{2\pi i\ell_\alpha(\varkappa^{-1})} \;\geq\; \ell_p(\varrho)e^{2\pi i\ell_\alpha(\varrho)} \wedge \ell_p(\varkappa)e^{2\pi i\ell_\alpha(\varkappa)},$$

which is (ia). The second inequality uses that $\wedge$ is monotone in each argument (Proposition 1).

Neutral component. Replace (ℓ_p, ℓ_α) by (ℓ_r, ℓ_ω) and (I) by (II). The three displayed steps go through verbatim and yield (iib) and (ib).

Nonmembership component. Replace (ℓ_p, ℓ_α) by (ℓ_q, ℓ_γ), (I) by (III), and reverse every inequality, with $\vee$ in place of $\wedge$: taking $\varkappa = \varrho$ in (III) gives $\ell_q(\iota)e^{2\pi i\ell_\gamma(\iota)} \leq \ell_q(\varrho)e^{2\pi i\ell_\gamma(\varrho)}$; then, taking $\varrho = \iota$ gives $\ell_q(\varrho^{-1})e^{2\pi i\ell_\gamma(\varrho^{-1})} \leq \ell_q(\varrho)e^{2\pi i\ell_\gamma(\varrho)}$, which is (iic); and replacing $\varkappa$ by $\varkappa^{-1}$ gives (ic), using that $\vee$ is monotone in each argument.

Hence, all six conditions of Definition 9 hold and D is a CPic-FSG. □

Proposition 5. *Let* D *be a CPic-FS of* $(\mathsf{G}, \blacksquare)$ *satisfying* (ia)–(ic) *of Definition 9, and write* $\varrho^n = \varrho\blacksquare\cdots\blacksquare\varrho$ *(*n *factors). Then, for every* $\varrho \in \mathsf{G}$ *and every integer* $n \geq 1$*, the following hold:*

(1) $\ell_p(\varrho^n)e^{2\pi i\ell_\alpha(\varrho^n)} \;\geq\; \ell_p(\varrho)e^{2\pi i\ell_\alpha(\varrho)}$*;*

(2) $\ell_r(\varrho^n)e^{2\pi i\ell_\omega(\varrho^n)} \;\geq\; \ell_r(\varrho)e^{2\pi i\ell_\omega(\varrho)}$*;*

(3) $\ell_q(\varrho^n)e^{2\pi i\ell_\gamma(\varrho^n)} \;\leq\; \ell_q(\varrho)e^{2\pi i\ell_\gamma(\varrho)}$.

Proof. We prove (1) by induction on n; (2) and (3) are identical, using (ib), (ic), and, in (3), reversing the inequalities and replacing $\wedge$ by $\vee$.

Base case $n = 1$. Here, $\varrho^1 = \varrho$ and the assertion is the reflexivity of $\preceq$.

Induction hypothesis. Suppose that for some $n \geq 1$,

$$\ell_p(\varrho^n)e^{2\pi i\ell_\alpha(\varrho^n)} \geq \ell_p(\varrho)e^{2\pi i\ell_\alpha(\varrho)}.$$

Induction step. Since $\varrho^{n+1} = \varrho^n \blacksquare \varrho$, condition (ia) gives

$$\ell_p(\varrho^{n+1})e^{2\pi i\ell_\alpha(\varrho^{n+1})} \;\geq\; \ell_p(\varrho^n)e^{2\pi i\ell_\alpha(\varrho^n)} \wedge \ell_p(\varrho)e^{2\pi i\ell_\alpha(\varrho)} \;=\; \ell_p(\varrho)e^{2\pi i\ell_\alpha(\varrho)},$$

the last equality by the induction hypothesis and Proposition 1. By induction, the assertion holds for all $n \geq 1$. □

The level-set characterization below corresponds to [22, Proposition 4.2 and Theorem 4.6], where level sets are developed considerably further for homogeneous CPic-FSs.

Theorem 2. *(Level-set characterization) Let* D *be a CPic-FS of a group* $(\mathsf{G}, \blacksquare)$*. For* $s, t, u \in \mathbb{L}$*, put*

$$\mathsf{D}_{(s,t,u)} = \{\varrho \in \mathsf{G} \;:\; \ell_p(\varrho)e^{2\pi i\ell_\alpha(\varrho)} \geq s,\; \ell_r(\varrho)e^{2\pi i\ell_\omega(\varrho)} \geq t,\; \ell_q(\varrho)e^{2\pi i\ell_\gamma(\varrho)} \preceq u\}.$$

Then, D *is a CPic-FSG of* G *if and only if* $\mathsf{D}_{(s,t,u)}$ *is a subgroup of* G *for every choice of* $s, t, u \in \mathbb{L}$ *for which it is nonempty.*

Proof. ($\Rightarrow$) Let D be a CPic-FSG, and let $\varrho, \varkappa \in \mathsf{D}_{(s,t,u)} \neq \emptyset$. By Proposition 4 (I) and monotonicity of $\wedge$,

$$\ell_p(\varrho \blacksquare \varkappa^{-1}) e^{2\pi i \ell_\alpha(\varrho \blacksquare \varkappa^{-1})} \succeq \ell_p(\varrho) e^{2\pi i \ell_\alpha(\varrho)} \wedge \ell_p(\varkappa) e^{2\pi i \ell_\alpha(\varkappa)} \succeq s \wedge s = s,$$

and likewise $\succeq t$ for the neutral component by (II) and $\preceq u$ for the nonmembership component by (III). Hence, $\varrho \blacksquare \varkappa^{-1} \in \mathsf{D}_{(s,t,u)}$, and a nonempty subset closed under $(\varrho, \varkappa) \mapsto \varrho \blacksquare \varkappa^{-1}$ is a subgroup.

($\Leftarrow$) Let $\varrho, \varkappa \in \mathsf{G}$ and put

$$\begin{aligned} s &= \ell_p(\varrho) e^{2\pi i \ell_\alpha(\varrho)} \wedge \ell_p(\varkappa) e^{2\pi i \ell_\alpha(\varkappa)}, \\ t &= \ell_r(\varrho) e^{2\pi i \ell_\omega(\varrho)} \wedge \ell_r(\varkappa) e^{2\pi i \ell_\omega(\varkappa)}, \\ u &= \ell_q(\varrho) e^{2\pi i \ell_\gamma(\varrho)} \vee \ell_q(\varkappa) e^{2\pi i \ell_\gamma(\varkappa)}. \end{aligned}$$

Then, $\varrho, \varkappa \in \mathsf{D}_{(s,t,u)}$, so $\mathsf{D}_{(s,t,u)}$ is nonempty and hence a subgroup; therefore, $\varrho \blacksquare \varkappa^{-1} \in \mathsf{D}_{(s,t,u)}$, which is exactly (I)–(III) of Proposition 4 for this pair. As $\varrho, \varkappa$ were arbitrary, Proposition 4 shows that D is a CPic-FSG. □

3.3. Examples

Example 1. *Let* $\mathbb{U} = \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{\iota, a, b, c\}$ *with* $\iota = (0,0)$, $a = (0,1)$, $b = (1,0)$, $c = (1,1)$, *and componentwise addition* $\oplus$, *and let* D *be the CPic-FS given by the following:*

$$\begin{aligned} \xi(\iota) &= (0.40e^{2\pi i\, 0.45},\ 0.30e^{2\pi i\, 0.30},\ 0.30e^{2\pi i\, 0.25}), \\ \xi(a) = \xi(b) &= (0.30e^{2\pi i\, 0.40},\ 0.25e^{2\pi i\, 0.20},\ 0.45e^{2\pi i\, 0.35}), \\ \xi(c) &= (0.35e^{2\pi i\, 0.40},\ 0.25e^{2\pi i\, 0.25},\ 0.40e^{2\pi i\, 0.35}). \end{aligned}$$

Admissibility Eq (3.1) *holds: The amplitude sums are* 1.00, 1.00, 1.00 *and the phase sums are* 1.00, 0.95, 1.00.

By Theorem 1 it suffices to separately verify the six scalar functions, and since every element satisfies $\varrho = \varrho^{-1}$, *the inverse conditions* (iia)–(iic) *hold with equality. For the closure conditions, the group has* ι *as identity and* $a \oplus b = c$, $a \oplus c = b$, $b \oplus c = a$, $a \oplus a = b \oplus b = c \oplus c = \iota$. *Any product involving* ι *is immediate from Proposition 3, so only the six products of two non-identity elements need checking. Table 4 records all of them.*

Table 4. Complete verification of Example 1. For $\ell \in \{\ell_p, \ell_\alpha, \ell_r, \ell_\omega\}$, the requirement is $\ell(x \oplus y) \geq \ell(x) \wedge \ell(y)$; for $\ell \in \{\ell_q, \ell_\gamma\}$, it is $\ell(x \oplus y) \leq \ell(x) \vee \ell(y)$.

x, y	$x \oplus y$	ℓ_p	ℓ_α	ℓ_r	ℓ_ω	ℓ_q	ℓ_γ
a, b	c	$0.35 \geq 0.30$	$0.40 \geq 0.40$	$0.25 \geq 0.25$	$0.25 \geq 0.20$	$0.40 \leq 0.45$	$0.35 \leq 0.35$
b, a	c	$0.35 \geq 0.30$	$0.40 \geq 0.40$	$0.25 \geq 0.25$	$0.25 \geq 0.20$	$0.40 \leq 0.45$	$0.35 \leq 0.35$
a, c	b	$0.30 \geq 0.30$	$0.40 \geq 0.40$	$0.25 \geq 0.25$	$0.20 \geq 0.20$	$0.45 \leq 0.45$	$0.35 \leq 0.35$
c, a	b	$0.30 \geq 0.30$	$0.40 \geq 0.40$	$0.25 \geq 0.25$	$0.20 \geq 0.20$	$0.45 \leq 0.45$	$0.35 \leq 0.35$
b, c	a	$0.30 \geq 0.30$	$0.40 \geq 0.40$	$0.25 \geq 0.25$	$0.20 \geq 0.20$	$0.45 \leq 0.45$	$0.35 \leq 0.35$
c, b	a	$0.30 \geq 0.30$	$0.40 \geq 0.40$	$0.25 \geq 0.25$	$0.20 \geq 0.20$	$0.45 \leq 0.45$	$0.35 \leq 0.35$
a, a	ι	$0.40 \geq 0.30$	$0.45 \geq 0.40$	$0.30 \geq 0.25$	$0.30 \geq 0.20$	$0.30 \leq 0.45$	$0.25 \leq 0.35$
b, b	ι	$0.40 \geq 0.30$	$0.45 \geq 0.40$	$0.30 \geq 0.25$	$0.30 \geq 0.20$	$0.30 \leq 0.45$	$0.25 \leq 0.35$
c, c	ι	$0.40 \geq 0.35$	$0.45 \geq 0.40$	$0.30 \geq 0.25$	$0.30 \geq 0.25$	$0.30 \leq 0.40$	$0.25 \leq 0.35$

Every entry of Table 4 holds, so D *is a CPic-FSG of* $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

The next two examples are the ones promised in the Introduction. They show that the phase triple and the neutral component each carry algebraic information that the remaining data do not determine, so that the CPic-FSG theory is not a relabeling of Pic-FSG or the CIFSG theory.

Example 2. *(The phase affects the algebraic conclusion) Let* $\mathsf{G} = (\mathbb{Z}_4, \oplus_4)$*, and let* D *be the CPic-FS as follows:*

$$\begin{aligned}\xi(0) &= (0.50e^{2\pi i\,0.40},\ 0.30e^{2\pi i\,0.30},\ 0.20e^{2\pi i\,0.25}),\\ \xi(2) &= (0.50e^{2\pi i\,0.30},\ 0.30e^{2\pi i\,0.25},\ 0.20e^{2\pi i\,0.30}),\\ \xi(1) = \xi(3) &= (0.40e^{2\pi i\,0.45},\ 0.25e^{2\pi i\,0.20},\ 0.30e^{2\pi i\,0.30}).\end{aligned}$$

Admissibility Eq (3.1) *holds throughout: The amplitude sums are* $1.00, 1.00, 0.95$ *and the phase sums are* $0.95, 0.85, 0.95$*, respectively.*

The amplitude triple $\mathsf{D}^{\mathrm{amp}}$ *is the two-level Pic-FS attached to the subgroup* $\{0, 2\} \leq \mathbb{Z}_4$*, with* $(\ell_p, \ell_r, \ell_q) = (0.50, 0.30, 0.20)$ *on* $\{0, 2\}$ *and* $(0.40, 0.25, 0.30)$ *off it. A direct check of the sixteen products shows that* $\mathsf{D}^{\mathrm{amp}}$ *is a Pic-FSG of* $\mathbb{Z}_4$.

Nevertheless, D *is* not *a CPic-FSG. Taking* $\varrho = \varkappa = 1$*, so that* $\varrho \oplus_4 \varkappa = 2$*, condition* (ia) *would require*

$$0.50\, e^{2\pi i\,0.30} \ \geq\ 0.40\, e^{2\pi i\,0.45} \wedge 0.40\, e^{2\pi i\,0.45} = 0.40\, e^{2\pi i\,0.45},$$

which fails, because $\ell_\alpha(2) = 0.30 < 0.45 = \ell_\alpha(1) \wedge \ell_\alpha(1)$*. Equivalently, by Theorem 1, the phase triple is not a Pic-FSG, even though the amplitude triple is; additionally, note that* $\ell_\alpha(\iota) = 0.40 < 0.45 = \ell_\alpha(1)$*, which contradicts Proposition 3 (1.1). Hence, no information contained in* $\mathsf{D}^{\mathrm{amp}}$ *can decide whether* D *is a CPic-FSG: The phase is an independent carrier of algebraic structure, and Example 1 is not an instance of a Pic-FSG in disguise.*

Example 3. *(The neutral grade affects the algebraic conclusion) Let* $\mathsf{G} = (\mathbb{Z}_4, \oplus_4)$*, and let* D *be the CPic-FS as follows:*

$$\begin{aligned}\xi(0) &= (0.50e^{2\pi i\,0.40},\ 0.25e^{2\pi i\,0.30},\ 0.20e^{2\pi i\,0.20}),\\ \xi(2) &= (0.50e^{2\pi i\,0.40},\ 0.20e^{2\pi i\,0.25},\ 0.20e^{2\pi i\,0.20}),\\ \xi(1) = \xi(3) &= (0.40e^{2\pi i\,0.30},\ 0.30e^{2\pi i\,0.25},\ 0.30e^{2\pi i\,0.30}).\end{aligned}$$

Admissibility holds: The amplitude sums are $0.95, 0.90, 1.00$ *and the phase sums are* $0.90, 0.85, 0.85$*, respectively.*

Deleting the neutral component leaves $(\ell_p e^{2\pi i \ell_\alpha},\ \ell_q e^{2\pi i \ell_\gamma})$ *with* $\ell_p + \ell_q \leq 1$ *and* $\ell_\alpha + \ell_\gamma \leq 1$*, which is the two-level CIFS attached to* $\{0, 2\} \leq \mathbb{Z}_4$ *and is a CIFSG in the sense of Definition 8, as a check of all sixteen products confirms.*

Yet D *is not a CPic-FSG: Taking* $\varrho = \varkappa = 1$*, condition* (ib) *would require* $\ell_r(2) \geq \ell_r(1) \wedge \ell_r(1)$ *(i.e.,* $0.20 \geq 0.30$*), which is false. Hence, the neutral grade, too, carries algebraic information that the membership/nonmembership pair does not determine, and a CPic-FSG result cannot be obtained from a CIFSG result by adjoining an arbitrary neutral grade.*

3.4. Set operations and their admissibility

Before intersecting two CPic-FSGs, we must say what the intersection of two CPic-FSs is, and check that the result is again a CPic-FS.

Definition 10. *Let* M *and* N *be CPic-FSs of* G*, with components* $\ell_{p,\mathsf{M}}, \ell_{\alpha,\mathsf{M}}, \dots$ *and* $\ell_{p,\mathsf{N}}, \ell_{\alpha,\mathsf{N}}, \dots$*, respectively. Their* intersection $\mathsf{M} \cap \mathsf{N}$ *and* union $\mathsf{M} \cup \mathsf{N}$ *are the CPic-FSs with components, for every* $\varrho \in \mathsf{G}$*,*

$$\begin{aligned} \ell_{p,\mathsf{M}\cap\mathsf{N}} &= \ell_{p,\mathsf{M}} \wedge \ell_{p,\mathsf{N}}, & \ell_{p,\mathsf{M}\cup\mathsf{N}} &= \ell_{p,\mathsf{M}} \vee \ell_{p,\mathsf{N}}, \\ \ell_{r,\mathsf{M}\cap\mathsf{N}} &= \ell_{r,\mathsf{M}} \wedge \ell_{r,\mathsf{N}}, & \ell_{r,\mathsf{M}\cup\mathsf{N}} &= \ell_{r,\mathsf{M}} \wedge \ell_{r,\mathsf{N}}, \\ \ell_{q,\mathsf{M}\cap\mathsf{N}} &= \ell_{q,\mathsf{M}} \vee \ell_{q,\mathsf{N}}, & \ell_{q,\mathsf{M}\cup\mathsf{N}} &= \ell_{q,\mathsf{M}} \wedge \ell_{q,\mathsf{N}}, \end{aligned}$$

and the same formulas for the phase components ℓ_α, ℓ_ω*, and* ℓ_γ*, respectively. These are the conventions of Cuong and Kreinovich [3, Definitions 3 and 6] applied separately to the amplitude triple and to the phase triple.*

Proposition 6. *(Admissibility of the set operations) Let* M, N *be CPic-FSs of* G*. Then,* $\mathsf{M} \cap \mathsf{N}$ *and* $\mathsf{M} \cup \mathsf{N}$ *of Definition 10 are CPic-FSs (i.e., they satisfy* (3.1)*). However, if the neutral component of the intersection is formed with* $\vee$ *instead of* $\wedge$*, then the result need not satisfy* (3.1)*.*

Proof. Intersection. Fix ϱ and suppress it. First, suppose that $\ell_{q,\mathsf{M}} \vee \ell_{q,\mathsf{N}} = \ell_{q,\mathsf{M}}$. Then,

$$\ell_{p,\mathsf{M}} \wedge \ell_{p,\mathsf{N}} + \ell_{r,\mathsf{M}} \wedge \ell_{r,\mathsf{N}} + \ell_{q,\mathsf{M}} \vee \ell_{q,\mathsf{N}} \;\le\; \ell_{p,\mathsf{M}} + \ell_{r,\mathsf{M}} + \ell_{q,\mathsf{M}} \;\le\; 1,$$

since each of the first two terms is at most the corresponding M-component. Instead, if the maximum is $\ell_{q,\mathsf{N}}$, then the same computation with N in place of M applies. The phase constraint is identical.

Union. Suppose $\ell_{p,\mathsf{M}} \vee \ell_{p,\mathsf{N}} = \ell_{p,\mathsf{M}}$. Then,

$$\ell_{p,\mathsf{M}} + \ell_{r,\mathsf{M}} \wedge \ell_{r,\mathsf{N}} + \ell_{q,\mathsf{M}} \wedge \ell_{q,\mathsf{N}} \le \ell_{p,\mathsf{M}} + \ell_{r,\mathsf{M}} + \ell_{q,\mathsf{M}} \le 1,$$

and symmetrically otherwise; again, the phases are identical.

Failure of the $\vee$ *convention for the neutral grade.* Take ϱ with

$$(\ell_{p,\mathsf{M}}, \ell_{r,\mathsf{M}}, \ell_{q,\mathsf{M}})(\varrho) = (0.5,\, 0.4,\, 0.1), \qquad (\ell_{p,\mathsf{N}}, \ell_{r,\mathsf{N}}, \ell_{q,\mathsf{N}})(\varrho) = (0.1,\, 0.1,\, 0.8).$$

Both triples sum to 1, so both are admissible. Forming the intersection with $\wedge$ on ℓ_p, $\vee$ on ℓ_r, and $\vee$ on ℓ_q gives (0.1, 0.4, 0.8), whose sum is $1.3 > 1$. Hence, the result is not a CPic-FS. □

Remark 8. *(Which convention, and why) Two conventions for the neutral grade of an intersection appear in the literature we build on: [3, Definition 3] uses the minimum, while the complex picture fuzzy setting of [9, Definition 6] uses the maximum. Proposition 6 shows that this is not a matter of notation. With the minimum, Admissibility* (3.1) *is preserved for all inputs; with the maximum, it fails already in the two-point example above, so the "intersection" of two CPic-FSs need not be a CPic-FS and the class is not closed under the operation. Therefore, we adopt the minimum throughout, as in [3]. The same computation shows why the union in Definition 10 takes the minimum for* both ℓ_r *and* ℓ_q*, and it is the reason for the repair of the direct image in Section 5.*

Proposition 7 corresponds to [22, Theorem 3.12], proven there for mutually homogeneous CPic-FSGs and for arbitrary families. The admissibility of the intersection, established in Proposition 6 above, is not addressed in [22].

Proposition 7. *The intersection of two CPic-FSGs of a group* $(\mathsf{G}, \blacksquare)$ *is a CPic-FSG of* G.

Proof. Let M, N be CPic-FSGs of G. By Proposition 6, $\mathsf{M} \cap \mathsf{N}$ is a CPic-FS, so by Proposition 4, it suffices to verify (I)–(III). Let $\varrho, \varkappa \in \mathsf{G}$.

(I) Membership. Using Definition 10, then Proposition 4 (I) for M and for N, then the commutativity and associativity of $\wedge$, the following holds:

$$\begin{aligned}
\ell_{p,\mathsf{M}\cap\mathsf{N}}(\varrho \blacksquare \varkappa^{-1}) e^{2\pi i \ell_{\alpha,\mathsf{M}\cap\mathsf{N}}(\varrho\blacksquare\varkappa^{-1})} &= [\ell_{p,\mathsf{M}}(\varrho \blacksquare \varkappa^{-1}) \wedge \ell_{p,\mathsf{N}}(\varrho \blacksquare \varkappa^{-1})] e^{2\pi i [\ell_{\alpha,\mathsf{M}}(\varrho\blacksquare\varkappa^{-1}) \wedge \ell_{\alpha,\mathsf{N}}(\varrho\blacksquare\varkappa^{-1})]} \\
&\geq [(\ell_{p,\mathsf{M}}(\varrho) \wedge \ell_{p,\mathsf{M}}(\varkappa)) \wedge (\ell_{p,\mathsf{N}}(\varrho) \wedge \ell_{p,\mathsf{N}}(\varkappa))] e^{2\pi i [(\ell_{\alpha,\mathsf{M}}(\varrho)\wedge\ell_{\alpha,\mathsf{M}}(\varkappa)) \wedge (\ell_{\alpha,\mathsf{N}}(\varrho)\wedge\ell_{\alpha,\mathsf{N}}(\varkappa))]} \\
&= [(\ell_{p,\mathsf{M}}(\varrho) \wedge \ell_{p,\mathsf{N}}(\varrho)) \wedge (\ell_{p,\mathsf{M}}(\varkappa) \wedge \ell_{p,\mathsf{N}}(\varkappa))] e^{2\pi i [(\ell_{\alpha,\mathsf{M}}(\varrho)\wedge\ell_{\alpha,\mathsf{N}}(\varrho)) \wedge (\ell_{\alpha,\mathsf{M}}(\varkappa)\wedge\ell_{\alpha,\mathsf{N}}(\varkappa))]} \\
&= [\ell_{p,\mathsf{M}\cap\mathsf{N}}(\varrho) \wedge \ell_{p,\mathsf{M}\cap\mathsf{N}}(\varkappa)] e^{2\pi i [\ell_{\alpha,\mathsf{M}\cap\mathsf{N}}(\varrho)\wedge\ell_{\alpha,\mathsf{M}\cap\mathsf{N}}(\varkappa)]} \\
&= \left[\ell_{p,\mathsf{M}\cap\mathsf{N}}(\varrho) e^{2\pi i \ell_{\alpha,\mathsf{M}\cap\mathsf{N}}(\varrho)}\right] \wedge \left[\ell_{p,\mathsf{M}\cap\mathsf{N}}(\varkappa) e^{2\pi i \ell_{\alpha,\mathsf{M}\cap\mathsf{N}}(\varkappa)}\right],
\end{aligned}$$

the last step by Definition 1.

(II) Neutral membership. Replace (ℓ_p, ℓ_α) throughout by (ℓ_r, ℓ_ω); since Definition 10 also uses $\wedge$ for the neutral component of an intersection, every step above applies unchanged and yields (II).

(III) Nonmembership. Replace (ℓ_p, ℓ_α) by (ℓ_q, ℓ_γ), replace every $\wedge$ by $\vee$, and reverse the single inequality, as follows:

$$\begin{aligned}
\ell_{q,\mathsf{M}\cap\mathsf{N}}(\varrho \blacksquare \varkappa^{-1}) e^{2\pi i \ell_{\gamma,\mathsf{M}\cap\mathsf{N}}(\varrho\blacksquare\varkappa^{-1})} &= [\ell_{q,\mathsf{M}}(\varrho \blacksquare \varkappa^{-1}) \vee \ell_{q,\mathsf{N}}(\varrho \blacksquare \varkappa^{-1})] e^{2\pi i [\ell_{\gamma,\mathsf{M}}(\varrho\blacksquare\varkappa^{-1}) \vee \ell_{\gamma,\mathsf{N}}(\varrho\blacksquare\varkappa^{-1})]} \\
&\leq [(\ell_{q,\mathsf{M}}(\varrho) \vee \ell_{q,\mathsf{M}}(\varkappa)) \vee (\ell_{q,\mathsf{N}}(\varrho) \vee \ell_{q,\mathsf{N}}(\varkappa))] e^{2\pi i [(\ell_{\gamma,\mathsf{M}}(\varrho)\vee\ell_{\gamma,\mathsf{M}}(\varkappa)) \vee (\ell_{\gamma,\mathsf{N}}(\varrho)\vee\ell_{\gamma,\mathsf{N}}(\varkappa))]} \\
&= \left[\ell_{q,\mathsf{M}\cap\mathsf{N}}(\varrho) e^{2\pi i \ell_{\gamma,\mathsf{M}\cap\mathsf{N}}(\varrho)}\right] \vee \left[\ell_{q,\mathsf{M}\cap\mathsf{N}}(\varkappa) e^{2\pi i \ell_{\gamma,\mathsf{M}\cap\mathsf{N}}(\varkappa)}\right].
\end{aligned}$$

Hence, (I)–(III) all hold and $\mathsf{M} \cap \mathsf{N}$ is a CPic-FSG. □

Contrastingly, the union need not be a CPic-FSG.

The [22, Proposition 3.13] gives a sufficient condition under which a union of CPic-FSGs is again a CPic-FSG. The example below is complementary: It exhibits a union that is not a CPic-FSG, and the failure occurs simultaneously in amplitude and phase.

Example 4. *Let* $\mathsf{G} = (\mathbb{Z}, +)$*, and let* $\mathsf{D}_1, \mathsf{D}_2$ *be the CPic-FSs*

$$\mathsf{D}_1 = \begin{cases} (0.55e^{2\pi i\,0.40},\ 0.25e^{2\pi i\,0.30},\ 0.15e^{2\pi i\,0.25}), & \varrho \in 7\mathbb{Z}, \\ (0.45e^{2\pi i\,0.30},\ 0.20e^{2\pi i\,0.25},\ 0.30e^{2\pi i\,0.40}), & \textit{otherwise}, \end{cases}$$

and

$$\mathsf{D}_2 = \begin{cases} (0.50e^{2\pi i\,0.45},\ 0.30e^{2\pi i\,0.30},\ 0.15e^{2\pi i\,0.20}), & \varrho \in 11\mathbb{Z}, \\ (0.45e^{2\pi i\,0.35},\ 0.25e^{2\pi i\,0.25},\ 0.25e^{2\pi i\,0.30}), & \textit{otherwise}. \end{cases}$$

All four rows satisfy Eq (3.1)*: The amplitude sums are* 0.95, 0.95, 0.95, 0.95 *and the phase sums are* 0.95, 0.95, 0.95, 0.90*, respectively.*

Each D_j *is a CPic-FSG. Indeed, each is the two-level CPic-FS attached to a subgroup* $H \leq \mathbb{Z}$ *taking its larger values of* $\ell_p, \ell_\alpha, \ell_r, \ell_\omega$ *and its smaller values of* ℓ_q, ℓ_γ *on* H*. For such a set, if* $x, y \in H$*, then* $x + y \in H$ *and the required inequalities hold with equality. If exactly one of* x, y *lies in* H*, then the right-hand side of* (ia), (ib) *is already the smaller value and that of* (ic) *already the larger value. If neither lies in* H*, then the left-hand side is either the* H*-value, which is more favorable, or the off-*H *value, which again gives equality. Since* $-x \in H$ *exactly when* $x \in H$*, the inverse conditions hold with equality. Note that the identity* 0 *lies in both* $7\mathbb{Z}$ *and* $11\mathbb{Z}$*, so Proposition 3 is respected:* $\ell_p(0) = 0.55$ *is the maximum value of* ℓ_p *in* D_1 *and* $\ell_q(0) = 0.15$ *its minimum, and similarly for* D_2 *and for all phase components.*

Now, form $\mathsf{D} = \mathsf{D}_1 \cup \mathsf{D}_2$ *as in Definition 10. Writing the four regions* $77\mathbb{Z}$*,* $7\mathbb{Z} \setminus 11\mathbb{Z}$*,* $11\mathbb{Z} \setminus 7\mathbb{Z}$ *and the complement of* $7\mathbb{Z} \cup 11\mathbb{Z}$*,*

$$\mathsf{D} = \begin{cases} (0.55e^{2\pi i\,0.45},\ 0.25e^{2\pi i\,0.30},\ 0.15e^{2\pi i\,0.20}), & \varrho \in 77\mathbb{Z}, \\ (0.55e^{2\pi i\,0.40},\ 0.25e^{2\pi i\,0.25},\ 0.15e^{2\pi i\,0.25}), & \varrho \in 7\mathbb{Z} \setminus 11\mathbb{Z}, \\ (0.50e^{2\pi i\,0.45},\ 0.20e^{2\pi i\,0.25},\ 0.15e^{2\pi i\,0.20}), & \varrho \in 11\mathbb{Z} \setminus 7\mathbb{Z}, \\ (0.45e^{2\pi i\,0.35},\ 0.20e^{2\pi i\,0.25},\ 0.25e^{2\pi i\,0.30}), & \textit{otherwise}. \end{cases}$$

By Proposition 6, D *is a CPic-FS; the amplitude sums are* 0.95, 0.95, 0.85, 0.90 *and the phase sums are* 0.95, 0.90, 0.90, 0.90*, respectively.*

However, D *is not a CPic-FSG. Take* $\varrho_1 = 7$ *and* $\varrho_2 = 11$*, so* $\varrho_1 + \varrho_2 = 18 \notin 7\mathbb{Z} \cup 11\mathbb{Z}$*. Then, condition* (ia) *fails in both components:*

$$\ell_p(18)\, e^{2\pi i \ell_\alpha(18)} = 0.45\, e^{2\pi i\,0.35} \not\geq 0.55\, e^{2\pi i\,0.40} \wedge 0.50\, e^{2\pi i\,0.45} = 0.50\, e^{2\pi i\,0.40},$$

since $0.45 < 0.50$ *and* $0.35 < 0.40$*. Condition* (ic) *fails as well:*

$$\ell_q(18)\, e^{2\pi i \ell_\gamma(18)} = 0.25\, e^{2\pi i\,0.30} \not\leq 0.15\, e^{2\pi i\,0.25} \vee 0.15\, e^{2\pi i\,0.20} = 0.15\, e^{2\pi i\,0.25}.$$

Hence, the union of two CPic-FSGs need not be a CPic-FSG, in contrast with Proposition 7. The failure is not an artefact of the amplitudes alone: It simultaneously occurs in the amplitude and the phase of the membership grade and in both components of the nonmembership grade, which is a phenomenon with no counterpart in the Pic-FSG setting.

Theorem 3. *Let* D *be a CPic-FSG of a group* $(\mathsf{G}, \blacksquare)$ *with identity* ι*. Then,*

$$\mathbb{B} = \{\varrho \in \mathsf{G} :\ \xi(\varrho) = \xi(\iota)\} = \left\{ \varrho \in \mathsf{G} : \begin{array}{r} \ell_p(\varrho)e^{2\pi i \ell_\alpha(\varrho)} = \ell_p(\iota)e^{2\pi i \ell_\alpha(\iota)}, \\ \ell_r(\varrho)e^{2\pi i \ell_\omega(\varrho)} = \ell_r(\iota)e^{2\pi i \ell_\omega(\iota)}, \\ \ell_q(\varrho)e^{2\pi i \ell_\gamma(\varrho)} = \ell_q(\iota)e^{2\pi i \ell_\gamma(\iota)} \end{array} \right\}$$

is a subgroup of G*.*

Proof. Clearly $\iota \in \mathbb{B}$, so $\mathbb{B} \neq \emptyset$. Let $\varrho, \varkappa \in \mathbb{B}$. We show $\varrho \blacksquare \varkappa^{-1} \in \mathbb{B}$ componentwise.

Membership. By Proposition 4 (I) and the defining property of $\mathbb{B}$,

$$\ell_p(\varrho \blacksquare \varkappa^{-1})e^{2\pi i \ell_\alpha(\varrho \blacksquare \varkappa^{-1})} \geq \ell_p(\varrho)e^{2\pi i \ell_\alpha(\varrho)} \wedge \ell_p(\varkappa)e^{2\pi i \ell_\alpha(\varkappa)} = \ell_p(\iota)e^{2\pi i \ell_\alpha(\iota)}.$$

On the other hand, $\varrho \blacksquare \varkappa^{-1} \in \mathsf{G}$, so Proposition 3 (1.1) gives the reverse inequality as follows:

$$\ell_p(\iota)e^{2\pi i\ell_\alpha(\iota)} \succeq \ell_p(\varrho \blacksquare \varkappa^{-1})e^{2\pi i\ell_\alpha(\varrho\blacksquare\varkappa^{-1})}.$$

By antisymmetry of $\preceq$, the two grades are equal.

Neutral membership. Identical, using Proposition 4 (II) and Proposition 3 (1.2), the following holds:

$$\ell_r(\iota)e^{2\pi i\ell_\omega(\iota)} \;\succeq\; \ell_r(\varrho \blacksquare \varkappa^{-1})e^{2\pi i\ell_\omega(\varrho\blacksquare\varkappa^{-1})} \;\succeq\; \ell_r(\varrho)e^{2\pi i\ell_\omega(\varrho)} \wedge \ell_r(\varkappa)e^{2\pi i\ell_\omega(\varkappa)} = \ell_r(\iota)e^{2\pi i\ell_\omega(\iota)}.$$

Nonmembership. Here, both inequalities are reversed. By Proposition 4 (III),

$$\ell_q(\varrho \blacksquare \varkappa^{-1})e^{2\pi i\ell_\gamma(\varrho\blacksquare\varkappa^{-1})} \;\preceq\; \ell_q(\varrho)e^{2\pi i\ell_\gamma(\varrho)} \vee \ell_q(\varkappa)e^{2\pi i\ell_\gamma(\varkappa)} \;=\; \ell_q(\iota)e^{2\pi i\ell_\gamma(\iota)},$$

while Proposition 3 (1.3) gives

$$\ell_q(\iota)e^{2\pi i\ell_\gamma(\iota)} \preceq \ell_q(\varrho \blacksquare \varkappa^{-1})e^{2\pi i\ell_\gamma(\varrho\blacksquare\varkappa^{-1})}.$$

Again, antisymmetry yields equality.

Hence, $\varrho\blacksquare\varkappa^{-1} \in \mathbb{B}$, and a nonempty subset of a group closed under $(\varrho, \varkappa) \mapsto \varrho\blacksquare\varkappa^{-1}$ is a subgroup. □

4. Complex picture fuzzy cosets and normal subgroups

The coset and normality notions of this section correspond to [22, Definitions 5.9 and 6.1, Theorems 6.3 and 6.6]. As before, the proofs below require no homogeneity hypothesis.

Definition 11. *Let* D *be a CPic-FSG of a group* $(\mathsf{G}, \blacksquare)$*, and let* $\varrho \in \mathsf{G}$*. The* complex picture fuzzy left coset $\varrho\cdot\mathsf{D}$ *is the CPic-FS of* G *whose six component functions are the translates*

$$\begin{aligned}
(\varrho\cdot\ell_p)(\varkappa) &= \ell_p(\varrho^{-1} \blacksquare \varkappa), \qquad & (\varrho\cdot\ell_\alpha)(\varkappa) &= \ell_\alpha(\varrho^{-1} \blacksquare \varkappa),\\
(\varrho\cdot\ell_r)(\varkappa) &= \ell_r(\varrho^{-1} \blacksquare \varkappa), \qquad & (\varrho\cdot\ell_\omega)(\varkappa) &= \ell_\omega(\varrho^{-1} \blacksquare \varkappa), \qquad & \varkappa \in \mathsf{G},\\
(\varrho\cdot\ell_q)(\varkappa) &= \ell_q(\varrho^{-1} \blacksquare \varkappa), \qquad & (\varrho\cdot\ell_\gamma)(\varkappa) &= \ell_\gamma(\varrho^{-1} \blacksquare \varkappa),
\end{aligned}$$

so that, in the grouped notation of Definition 9,

$$(\varrho\cdot\xi)(\varkappa) = \Big(\ell_p(\varrho^{-1} \blacksquare \varkappa)e^{2\pi i\ell_\alpha(\varrho^{-1}\blacksquare\varkappa)},\ \ell_r(\varrho^{-1} \blacksquare \varkappa)e^{2\pi i\ell_\omega(\varrho^{-1}\blacksquare\varkappa)},\ \ell_q(\varrho^{-1} \blacksquare \varkappa)e^{2\pi i\ell_\gamma(\varrho^{-1}\blacksquare\varkappa)}\Big).$$

The complex picture fuzzy right coset $\mathsf{D}\cdot\varrho$ *is defined by* $(\ell\cdot\varrho)(\varkappa) = \ell(\varkappa \blacksquare \varrho^{-1})$ *for each of the six components, i.e.,*

$$(\xi\cdot\varrho)(\varkappa) = \Big(\ell_p(\varkappa \blacksquare \varrho^{-1})e^{2\pi i\ell_\alpha(\varkappa\blacksquare\varrho^{-1})},\ \ell_r(\varkappa \blacksquare \varrho^{-1})e^{2\pi i\ell_\omega(\varkappa\blacksquare\varrho^{-1})},\ \ell_q(\varkappa \blacksquare \varrho^{-1})e^{2\pi i\ell_\gamma(\varkappa\blacksquare\varrho^{-1})}\Big).$$

Both are CPic-FSs of G*, since each component is a composition of a component of* D *with a bijection of* G *and therefore has the same range; in particular,* (3.1) *is inherited pointwise.*

Definition 12. *Let* D *be a CPic-FSG of a group* $(\mathsf{G}, \blacksquare)$*. Then,* D *is a* complex picture fuzzy normal subgroup *(CPic-FNSG) of* G *if* $\varrho\cdot\mathsf{D} = \mathsf{D}\cdot\varrho$ *for every* $\varrho \in \mathsf{G}$*, that is, if* $(\varrho\cdot\xi)(\varkappa) = (\xi\cdot\varrho)(\varkappa)$ *for all* $\varrho, \varkappa \in \mathsf{G}$*, equality being equality of all six component functions.*

Example 5. *Let* $(\mathbb{Z}_3, \oplus_3)$ *be the group of integers modulo* 3*, and let* D *be the CPic-FS as follows:*

$$\mathsf{D} = \begin{cases} (0.50e^{2\pi i\,0.45},\ 0.35e^{2\pi i\,0.30},\ 0.10e^{2\pi i\,0.15}), & \varrho = 0, \\ (0.40e^{2\pi i\,0.35},\ 0.30e^{2\pi i\,0.25},\ 0.20e^{2\pi i\,0.20}), & \varrho \in \{1, 2\}. \end{cases}$$

Admissibility holds, the amplitude sums being 0.95 *and* 0.90 *and the phase sums being* 0.90 *and* 0.80*, respectively. As in Example 4,* D *is the two-level CPic-FS attached to the trivial subgroup* $\{0\} \leq \mathbb{Z}_3$ *and is therefore a CPic-FSG. Since* $\mathbb{Z}_3$ *is abelian,*

$$\varrho^{-1} \oplus_3 \varkappa = \varkappa \oplus_3 \varrho^{-1}$$

for all $\varrho, \varkappa$*, so every component of* $\varrho \cdot \xi$ *agrees with the corresponding component of* $\xi \cdot \varrho$ *and* D *is a CPic-FNSG. For instance with* $\varrho = 1$*, whose inverse is* 2*,*

$$(1 \cdot \xi)(\varkappa) = \xi(2 \oplus_3 \varkappa) = \xi(\varkappa \oplus_3 2) = (\xi \cdot 1)(\varkappa) \qquad \textit{for every } \varkappa \in \mathbb{Z}_3.$$

Proposition 8. *Let* D *be a CPic-FSG of a group* $(\mathsf{G}, \blacksquare)$*. Then,* D *is a CPic-FNSG of* G *if and only if for all* $\varrho, \varkappa \in \mathsf{G}$*, the following hold:*

(1) $\ell_p(\varrho \blacksquare \varkappa) e^{2\pi i \ell_\alpha(\varrho \blacksquare \varkappa)} = \ell_p(\varkappa \blacksquare \varrho) e^{2\pi i \ell_\alpha(\varkappa \blacksquare \varrho)}$;

(2) $\ell_r(\varrho \blacksquare \varkappa) e^{2\pi i \ell_\omega(\varrho \blacksquare \varkappa)} = \ell_r(\varkappa \blacksquare \varrho) e^{2\pi i \ell_\omega(\varkappa \blacksquare \varrho)}$;

(3) $\ell_q(\varrho \blacksquare \varkappa) e^{2\pi i \ell_\gamma(\varrho \blacksquare \varkappa)} = \ell_q(\varkappa \blacksquare \varrho) e^{2\pi i \ell_\gamma(\varkappa \blacksquare \varrho)}$.

Proof. ($\Rightarrow$) Let D be a CPic-FNSG. By Definitions 11 and 12, for all $\varrho, \varkappa \in \mathsf{G}$,

$$\ell_p(\varrho^{-1} \blacksquare \varkappa) e^{2\pi i \ell_\alpha(\varrho^{-1} \blacksquare \varkappa)} = (\varrho \cdot \xi)_p(\varkappa) = (\xi \cdot \varrho)_p(\varkappa) = \ell_p(\varkappa \blacksquare \varrho^{-1}) e^{2\pi i \ell_\alpha(\varkappa \blacksquare \varrho^{-1})}.$$

Replacing ϱ by ϱ^{-1} and using $(\varrho^{-1})^{-1} = \varrho$ gives (1). The same substitution in the neutral and nonmembership components gives (2) and (3).

($\Leftarrow$) Assume (1)–(3) and let $\varrho, \varkappa \in \mathsf{G}$. Put $\nu = \varrho^{-1}$. Applying (1) to the pair $(\nu, \varkappa)$,

$$\ell_p(\varrho^{-1} \blacksquare \varkappa) e^{2\pi i \ell_\alpha(\varrho^{-1} \blacksquare \varkappa)} = \ell_p(\nu \blacksquare \varkappa) e^{2\pi i \ell_\alpha(\nu \blacksquare \varkappa)} = \ell_p(\varkappa \blacksquare \nu) e^{2\pi i \ell_\alpha(\varkappa \blacksquare \nu)} = \ell_p(\varkappa \blacksquare \varrho^{-1}) e^{2\pi i \ell_\alpha(\varkappa \blacksquare \varrho^{-1})},$$

so the membership components of $\varrho \cdot \xi$ and $\xi \cdot \varrho$ agree at $\varkappa$. Conditions (2) and (3) give the same conclusion for the neutral and nonmembership components. As $\varrho, \varkappa$ were arbitrary, $\varrho \cdot \mathsf{D} = \mathsf{D} \cdot \varrho$ for every ϱ, so D is a CPic-FNSG. □

Proposition 9. *Let* D *be a CPic-FSG of a group* $(\mathsf{G}, \blacksquare)$*. Then,* D *is a CPic-FNSG of* G *if and only if for all* $\varrho, \varkappa \in \mathsf{G}$*, the following hold:*

(1) $\ell_p(\varrho) e^{2\pi i \ell_\alpha(\varrho)} = \ell_p(\varkappa \blacksquare \varrho \blacksquare \varkappa^{-1}) e^{2\pi i \ell_\alpha(\varkappa \blacksquare \varrho \blacksquare \varkappa^{-1})}$;

(2) $\ell_r(\varrho) e^{2\pi i \ell_\omega(\varrho)} = \ell_r(\varkappa \blacksquare \varrho \blacksquare \varkappa^{-1}) e^{2\pi i \ell_\omega(\varkappa \blacksquare \varrho \blacksquare \varkappa^{-1})}$;

(3) $\ell_q(\varrho) e^{2\pi i \ell_\gamma(\varrho)} = \ell_q(\varkappa \blacksquare \varrho \blacksquare \varkappa^{-1}) e^{2\pi i \ell_\gamma(\varkappa \blacksquare \varrho \blacksquare \varkappa^{-1})}$.

Proof. ($\Rightarrow$) Let D be a CPic-FNSG, and let $\varrho, \varkappa \in \mathsf{G}$. Using $\iota = \varkappa \blacksquare \varkappa^{-1}$, associativity, and Proposition 8 (1) applied to the pair $(\varrho \blacksquare \varkappa,\ \varkappa^{-1})$,

$$\begin{aligned}\ell_p(\varrho)e^{2\pi i\ell_\alpha(\varrho)} &= \ell_p((\varrho \blacksquare \varkappa) \blacksquare \varkappa^{-1})e^{2\pi i\ell_\alpha((\varrho\blacksquare\varkappa)\blacksquare\varkappa^{-1})}\\ &= \ell_p(\varkappa^{-1} \blacksquare (\varrho \blacksquare \varkappa))e^{2\pi i\ell_\alpha(\varkappa^{-1}\blacksquare(\varrho\blacksquare\varkappa))}\\ &= \ell_p(\varkappa^{-1} \blacksquare \varrho \blacksquare \varkappa)e^{2\pi i\ell_\alpha(\varkappa^{-1}\blacksquare\varrho\blacksquare\varkappa)}.\end{aligned}$$

Since $\varkappa$ ranges over all of G and $\varkappa = (\varkappa^{-1})^{-1}$, this is (1). Conditions (2) and (3) follow in the same way from Proposition 8 (2) and (3).

($\Leftarrow$) Assume (1)–(3). For $\varrho, \varkappa \in \mathsf{G}$, using $\iota = \varkappa \blacksquare \varkappa^{-1}$ and then (1) with $\varrho \blacksquare \varkappa$ in place of ϱ,

$$\begin{aligned}\ell_p(\varkappa \blacksquare \varrho)e^{2\pi i\ell_\alpha(\varkappa\blacksquare\varrho)} &= \ell_p(\varkappa \blacksquare \varrho \blacksquare (\varkappa \blacksquare \varkappa^{-1}))e^{2\pi i\ell_\alpha(\varkappa\blacksquare\varrho\blacksquare(\varkappa\blacksquare\varkappa^{-1}))}\\ &= \ell_p(\varkappa \blacksquare (\varrho \blacksquare \varkappa) \blacksquare \varkappa^{-1})e^{2\pi i\ell_\alpha(\varkappa\blacksquare(\varrho\blacksquare\varkappa)\blacksquare\varkappa^{-1})}\\ &= \ell_p(\varrho \blacksquare \varkappa)e^{2\pi i\ell_\alpha(\varrho\blacksquare\varkappa)}.\end{aligned}$$

The same computation with (2) and (3) gives

$$\ell_r(\varkappa \blacksquare \varrho)e^{2\pi i\ell_\omega(\varkappa\blacksquare\varrho)} = \ell_r(\varrho \blacksquare \varkappa)e^{2\pi i\ell_\omega(\varrho\blacksquare\varkappa)}$$

and

$$\ell_q(\varkappa \blacksquare \varrho)e^{2\pi i\ell_\gamma(\varkappa\blacksquare\varrho)} = \ell_q(\varrho \blacksquare \varkappa)e^{2\pi i\ell_\gamma(\varrho\blacksquare\varkappa)}.$$

By Proposition 8, D is a CPic-FNSG. □

Theorem 4. *Let* D *be a CPic-FNSG of a group* $(\mathsf{G}, \blacksquare)$ *with identity* ι*. Then,*

$$\mathbb{A} = \{\varrho \in \mathsf{G} :\ \xi(\varrho) = \xi(\iota)\}$$

is a normal subgroup of G*.*

Proof. In particular, a CPic-FNSG is a CPic-FSG, so $\mathbb{A} = \mathbb{B}$ is a subgroup of G by Theorem 3; in particular, $\iota \in \mathbb{A}$ and $\mathbb{A} \neq \emptyset$.

It remains to verify the normality criterion, namely that $\varkappa \blacksquare \varrho \blacksquare \varkappa^{-1} \in \mathbb{A}$ for every $\varkappa \in \mathsf{G}$ and every $\varrho \in \mathbb{A}$. Fix such $\varkappa$ and ϱ. Since D is a CPic-FNSG, Proposition 9 applies and gives, for the three components simultaneously,

$$\begin{aligned}\ell_p(\varkappa \blacksquare \varrho \blacksquare \varkappa^{-1})e^{2\pi i\ell_\alpha(\varkappa\blacksquare\varrho\blacksquare\varkappa^{-1})} &= \ell_p(\varrho)e^{2\pi i\ell_\alpha(\varrho)} = \ell_p(\iota)e^{2\pi i\ell_\alpha(\iota)},\\ \ell_r(\varkappa \blacksquare \varrho \blacksquare \varkappa^{-1})e^{2\pi i\ell_\omega(\varkappa\blacksquare\varrho\blacksquare\varkappa^{-1})} &= \ell_r(\varrho)e^{2\pi i\ell_\omega(\varrho)} = \ell_r(\iota)e^{2\pi i\ell_\omega(\iota)},\\ \ell_q(\varkappa \blacksquare \varrho \blacksquare \varkappa^{-1})e^{2\pi i\ell_\gamma(\varkappa\blacksquare\varrho\blacksquare\varkappa^{-1})} &= \ell_q(\varrho)e^{2\pi i\ell_\gamma(\varrho)} = \ell_q(\iota)e^{2\pi i\ell_\gamma(\iota)},\end{aligned}$$

the second equality in each line holding because $\varrho \in \mathbb{A}$. Thus, $\xi(\varkappa \blacksquare \varrho \blacksquare \varkappa^{-1}) = \xi(\iota)$ (i.e., $\varkappa \blacksquare \varrho \blacksquare \varkappa^{-1} \in \mathbb{A}$).

Hence, $\mathbb{A}$ is a subgroup of G closed under conjugation by arbitrary elements of G, so $\mathbb{A} \trianglelefteq \mathsf{G}$. □

5. Complex picture fuzzy subgroups and homomorphisms

Throughout this section $(\mathbb{U}, \blacksquare_1)$ and $(\mathbb{V}, \blacksquare_2)$ are groups with identities $\iota_{\mathbb{U}}$ and $\iota_{\mathbb{V}}$, and $F\colon \mathbb{U} \to \mathbb{V}$ is a group homomorphism. For $\varkappa \in \mathbb{V}$, we write $F^{-1}(\varkappa) = \{\varrho \in \mathbb{U}\colon F(\varrho) = \varkappa\}$ for the fibre over $\varkappa$. We repeatedly use the following elementary fact.

Lemma 1. *(Fibre product) Let $F\colon \mathbb{U} \to \mathbb{V}$ be a group homomorphism, and let $\varkappa_1, \varkappa_2 \in F(\mathbb{U})$. Then,*

$$F^{-1}(\varkappa_1 \blacksquare_2 \varkappa_2) = F^{-1}(\varkappa_1) \blacksquare_1 F^{-1}(\varkappa_2) := \{\varrho_1 \blacksquare_1 \varrho_2 : \varrho_1 \in F^{-1}(\varkappa_1),\ \varrho_2 \in F^{-1}(\varkappa_2)\},$$

and $F^{-1}(\varkappa^{-1}) = \big(F^{-1}(\varkappa)\big)^{-1}$ for every $\varkappa \in F(\mathbb{U})$.

Proof. If $\varrho_j \in F^{-1}(\varkappa_j)$, then

$$F(\varrho_1 \blacksquare_1 \varrho_2) = F(\varrho_1) \blacksquare_2 F(\varrho_2) = \varkappa_1 \blacksquare_2 \varkappa_2$$

giving the inclusion $\supseteq$. Conversely, let $\varrho \in F^{-1}(\varkappa_1 \blacksquare_2 \varkappa_2)$ and choose any $\varrho_1 \in F^{-1}(\varkappa_1)$, which is nonempty since $\varkappa_1 \in F(\mathbb{U})$. Put $\varrho_2 = \varrho_1^{-1} \blacksquare_1 \varrho$. Then,

$$F(\varrho_2) = F(\varrho_1)^{-1} \blacksquare_2 F(\varrho) = \varkappa_1^{-1} \blacksquare_2 \varkappa_1 \blacksquare_2 \varkappa_2 = \varkappa_2$$

and $\varrho_1 \blacksquare_1 \varrho_2 = \varrho$, giving $\subseteq$. The second assertion follows from $F(\varrho^{-1}) = F(\varrho)^{-1}$ and the fact that $\varrho \mapsto \varrho^{-1}$ is an involution of $\mathbb{U}$. □

5.1. Why the direct image must be redefined

The direct image of a CPic-FS is formed fibrewise. The membership grade is naturally a supremum and the nonmembership grade an infimum. The question is what to do with the neutral grade, and the following example shows that the obvious answer is wrong.

Example 6. *(The supremum convention for the neutral grade is inadmissible) Let $\mathbb{U} = \mathbb{Z}_2 = \{0, 1\}$, $\mathbb{V} = \{\iota_{\mathbb{V}}\}$ be the trivial group, and F be the unique homomorphism, so that $F^{-1}(\iota_{\mathbb{V}}) = \{0, 1\}$ is a single fiber with two points. Let* D *have amplitude triples*

$$(\ell_p, \ell_r, \ell_q)(0) = (0.8,\ 0.1,\ 0.1), \qquad (\ell_p, \ell_r, \ell_q)(1) = (0.1,\ 0.8,\ 0.1),$$

both of which sum to 1 *and are therefore admissible. If the image is formed by taking the supremum of ℓ_p, the supremum of ℓ_r, and the infimum of ℓ_q over the fiber, then the resulting amplitude triple is*

$$(0.8,\ 0.8,\ 0.1), \qquad \textit{whose sum is } 1.7 > 1,$$

so the image violates Eq (3.1) *and is not a CPic-FS. Replacing the supremum of ℓ_r by its infimum gives* (0.8, 0.1, 0.1)*, with sum* 1.

Example 6 is the exact analogue, for images, of the failure recorded in Proposition 6 for intersections, and it is resolved the same way: The neutral grade follows the nonmembership grade, not the membership grade, when a family is collapsed to a single value. Therefore, we adopt the following definition.

The [22, Definition 7.2] defines the direct image with the same convention adopted below, supremum for the membership grade, infimum for the neutral and the nonmembership grades, and with the same degenerate branch off the image. It states that the result is a complex picture fuzzy set but does not verify the admissibility constraints (3.1). Example 6 shows that this verification is not vacuous, and Proposition 10 supplies it.

Definition 13. *(Direct and inverse image) Let $F\colon \mathbb{U} \to \mathbb{V}$ be a group homomorphism, let D be a CPic-FS of $\mathbb{U}$, and let R be a CPic-FS of $\mathbb{V}$.*

The direct image $F(\mathsf{D})$ *is the CPic-FS of $\mathbb{V}$ whose six component functions are, for $\varkappa \in \mathbb{V}$ with $F^{-1}(\varkappa) \neq \emptyset$,*

$$\ell_{p,F(\mathsf{D})}(\varkappa) = \sup_{\varrho\in F^{-1}(\varkappa)} \ell_{p,\mathsf{D}}(\varrho), \qquad \ell_{\alpha,F(\mathsf{D})}(\varkappa) = \sup_{\varrho\in F^{-1}(\varkappa)} \ell_{\alpha,\mathsf{D}}(\varrho),$$

$$\ell_{r,F(\mathsf{D})}(\varkappa) = \inf_{\varrho\in F^{-1}(\varkappa)} \ell_{r,\mathsf{D}}(\varrho), \qquad \ell_{\omega,F(\mathsf{D})}(\varkappa) = \inf_{\varrho\in F^{-1}(\varkappa)} \ell_{\omega,\mathsf{D}}(\varrho),$$

$$\ell_{q,F(\mathsf{D})}(\varkappa) = \inf_{\varrho\in F^{-1}(\varkappa)} \ell_{q,\mathsf{D}}(\varrho), \qquad \ell_{\gamma,F(\mathsf{D})}(\varkappa) = \inf_{\varrho\in F^{-1}(\varkappa)} \ell_{\gamma,\mathsf{D}}(\varrho),$$

and, for $\varkappa \notin F(\mathbb{U})$,

$$(\ell_p, \ell_r, \ell_q)_{F(\mathsf{D})}(\varkappa) = (0, 0, 1), \qquad (\ell_\alpha, \ell_\omega, \ell_\gamma)_{F(\mathsf{D})}(\varkappa) = (0, 0, 1).$$

In the grouped notation, and with $\bigvee$, $\bigwedge$, the lattice operations of Definition 1,

$$\xi_{F(\mathsf{D})}(\varkappa) = \Big(\bigvee_{\varrho\in F^{-1}(\varkappa)} \ell_{p,\mathsf{D}}(\varrho) e^{2\pi i \ell_{\alpha,\mathsf{D}}(\varrho)}, \bigwedge_{\varrho\in F^{-1}(\varkappa)} \ell_{r,\mathsf{D}}(\varrho) e^{2\pi i \ell_{\omega,\mathsf{D}}(\varrho)}, \bigwedge_{\varrho\in F^{-1}(\varkappa)} \ell_{q,\mathsf{D}}(\varrho) e^{2\pi i \ell_{\gamma,\mathsf{D}}(\varrho)} \Big).$$

The inverse image $F^{-1}(\mathsf{R})$ *is the CPic-FS of $\mathbb{U}$ obtained by composition: for every $\varrho \in \mathbb{U}$ and each of the six components ℓ,*

$$\ell_{F^{-1}(\mathsf{R})}(\varrho) = \ell_{\mathsf{R}}(F(\varrho)), \qquad \text{so} \qquad \xi_{F^{-1}(\mathsf{R})}(\varrho) = \xi_{\mathsf{R}}(F(\varrho)).$$

Remark 9. *(On the role of the suprema and infima) By Proposition 1, $\mathbb{L}$ is a* complete *lattice, so every nonempty family of grades has a join and a meet, computed componentwise. The expressions $\bigvee$ and $\bigwedge$ in Definition 13 are these lattice operations, and no appeal to an order on $\mathbb{C}$ is made.*

Two cautions are in order. First, the componentwise formula is a statement about joins and meets*, not about maxima and minima: The join of a family need not be one of its members. For instance, the fiber values $\{0.8e^{2\pi i\,0.2},\ 0.3e^{2\pi i\,0.9}\}$ join $0.8e^{2\pi i\,0.9}$, which is attained at neither point, because the amplitude and the phase extrema occur at different elements of the fiber. Second, no argument in this paper requires attainment: Every step only uses the universal properties of the join and meet recorded in Proposition 1, together with the identities*

$$\sup_{x\in X,\, y\in Y} \min\{f(x), g(y)\} = \min\Big\{\sup_X f,\ \sup_Y g\Big\}, \tag{5.1}$$

$$\inf_{x\in X,\, y\in Y} \min\{f(x), g(y)\} = \min\Big\{\inf_X f,\ \inf_Y g\Big\}, \tag{5.2}$$

$$\inf_{x\in X,\, y\in Y} \max\{f(x), g(y)\} = \max\Big\{\inf_X f,\ \inf_Y g\Big\}, \tag{5.3}$$

valid for arbitrary nonempty X, Y and bounded real f, g. Each is proven by noting that the right-hand side bounds the left in the stated direction pointwise, and is approached along sequences realizing the two one-variable extrema to within ε.

Proposition 10. *(Well-definedness of the images) Let $F\colon \mathbb{U} \to \mathbb{V}$ be a group homomorphism, D is a CPic-FS of $\mathbb{U}$, and R is a CPic-FS of $\mathbb{V}$. Then, $F(\mathsf{D})$ is a CPic-FS of $\mathbb{V}$ and $F^{-1}(\mathsf{R})$ is a CPic-FS of $\mathbb{U}$, that is, both satisfy the admissibility constraints* (3.1)*.*

Proof. Inverse image. For $\varrho \in \mathbb{U}$, the triples of $F^{-1}(\mathsf{R})$ at ϱ are literally the triples of R at $F(\varrho)$, so (3.1) is inherited pointwise.

Direct image. If $\varkappa \notin F(\mathbb{U})$, than both prescribed triples are $(0, 0, 1)$, with sum 1. Thus, let $\varkappa \in F(\mathbb{U})$ and write

$$P = \sup_{F^{-1}(\varkappa)} \ell_{p,\mathsf{D}}, \quad R = \inf_{F^{-1}(\varkappa)} \ell_{r,\mathsf{D}}, \quad Q = \inf_{F^{-1}(\varkappa)} \ell_{q,\mathsf{D}}.$$

Let $\varepsilon > 0$ and choose $\varrho_0 \in F^{-1}(\varkappa)$ with $\ell_{p,\mathsf{D}}(\varrho_0) > P - \varepsilon$. Since R and Q are infima over a set containing ϱ_0,

$$P - \varepsilon + R + Q \;<\; \ell_{p,\mathsf{D}}(\varrho_0) + \ell_{r,\mathsf{D}}(\varrho_0) + \ell_{q,\mathsf{D}}(\varrho_0) \;\le\; 1,$$

the last step by (3.1) for D. Letting $\varepsilon \downarrow 0$ gives $P + R + Q \le 1$. The identical argument applied to $\ell_\alpha, \ell_\omega, \ell_\gamma$ gives the phase constraint. Finally, all six values lie in $[0, 1]$, being suprema or infima of functions with values in $[0, 1]$. □

Remark 10. *Proposition 10 is exactly the step that fails for the supremum convention: In Example 6, the quantity R above would be a supremum, and the inequality $R \le \ell_{r,\mathsf{D}}(\varrho_0)$ — which is what makes the displayed chain work — is no longer available.*

Lemma 2. *With the notation of Definition 13, for every $\varkappa \in \mathbb{V}$ and every $\varrho \in \mathbb{U}$, the following hold:*

(1) $\displaystyle\bigvee_{\varrho' \in F^{-1}(\varkappa)} \ell_{p,\mathsf{D}}(\varrho') e^{2\pi i \ell_{\alpha,\mathsf{D}}(\varrho')} = \ell_{p,F(\mathsf{D})}(\varkappa)\, e^{2\pi i \ell_{\alpha,F(\mathsf{D})}(\varkappa)}$;

(2) $\displaystyle\bigwedge_{\varrho' \in F^{-1}(\varkappa)} \ell_{r,\mathsf{D}}(\varrho') e^{2\pi i \ell_{\omega,\mathsf{D}}(\varrho')} = \ell_{r,F(\mathsf{D})}(\varkappa)\, e^{2\pi i \ell_{\omega,F(\mathsf{D})}(\varkappa)}$;

(3) $\displaystyle\bigwedge_{\varrho' \in F^{-1}(\varkappa)} \ell_{q,\mathsf{D}}(\varrho') e^{2\pi i \ell_{\gamma,\mathsf{D}}(\varrho')} = \ell_{q,F(\mathsf{D})}(\varkappa)\, e^{2\pi i \ell_{\gamma,F(\mathsf{D})}(\varkappa)}$;

(4) $\ell_{p,F^{-1}(\mathsf{R})}(\varrho) e^{2\pi i \ell_{\alpha,F^{-1}(\mathsf{R})}(\varrho)} = \ell_{p,\mathsf{R}}(F(\varrho)) e^{2\pi i \ell_{\alpha,\mathsf{R}}(F(\varrho))}$, *and likewise for the pairs (ℓ_r, ℓ_ω) and (ℓ_q, ℓ_γ).*

Proof. Assertions (1)–(3) are immediate from Proposition 1: In the complete lattice $\mathbb{L} \cong [0, 1] \times [0, 1]$, joins and meets are computed componentwise, and the components are precisely the real suprema and infima appearing in Definition 13. Assertion (4) is the definition of $F^{-1}(\mathsf{R})$ read componentwise. As stressed in Remark 9, (1)–(3) assert equality of *lattice* extrema, and do not claim that the extremum is attained at a point of the fiber. □

5.2. Preservation theorems

Theorem 5. *Let $F\colon \mathbb{U} \to \mathbb{V}$ be an epimorphism of groups, and let D be a CPic-FSG of $\mathbb{U}$. Then, $F(\mathsf{D})$ is a CPic-FSG of $\mathbb{V}$.*

Proof. By Proposition 10, $F(\mathsf{D})$ is a CPic-FS of $\mathbb{V}$; surjectivity of F guarantees that every fiber is nonempty, so the second branch of Definition 13 never occurs. Let $\varkappa_1, \varkappa_2, \varkappa \in \mathbb{V}$ and abbreviate $X_j = F^{-1}(\varkappa_j)$.

Condition (ia). By Lemma 1, $F^{-1}(\varkappa_1 \blacksquare_2 \varkappa_2) = X_1 \blacksquare_1 X_2$, so

$$\begin{aligned}
\ell_{p,F(\mathsf{D})}(\varkappa_1 \blacksquare_2 \varkappa_2) &= \sup_{\varrho_1 \in X_1,\ \varrho_2 \in X_2} \ell_{p,\mathsf{D}}(\varrho_1 \blacksquare_1 \varrho_2) && \text{(Lemma 1)} \\
&\geq \sup_{\varrho_1 \in X_1,\ \varrho_2 \in X_2} \min\{\ell_{p,\mathsf{D}}(\varrho_1),\ \ell_{p,\mathsf{D}}(\varrho_2)\} && (\mathsf{D} \text{ satisfies (ia)}) \\
&= \min\Big\{\sup_{X_1} \ell_{p,\mathsf{D}},\ \sup_{X_2} \ell_{p,\mathsf{D}}\Big\} && \text{(by (5.1))} \\
&= \min\{\ell_{p,F(\mathsf{D})}(\varkappa_1),\ \ell_{p,F(\mathsf{D})}(\varkappa_2)\}.
\end{aligned}$$

The same chain with ℓ_α in place of ℓ_p gives

$$\ell_{\alpha,F(\mathsf{D})}(\varkappa_1 \blacksquare_2 \varkappa_2) \geq \ell_{\alpha,F(\mathsf{D})}(\varkappa_1) \wedge \ell_{\alpha,F(\mathsf{D})}(\varkappa_2).$$

By Corollary 1, the two together are condition (ia) for $F(\mathsf{D})$.

Condition (ib). This is identical, except that the outer extremum is an infimum and the second identity of Eq (5.1) is used as follows:

$$\begin{aligned}
\ell_{r,F(\mathsf{D})}(\varkappa_1 \blacksquare_2 \varkappa_2) &= \inf_{\varrho_1 \in X_1,\ \varrho_2 \in X_2} \ell_{r,\mathsf{D}}(\varrho_1 \blacksquare_1 \varrho_2) \\
&\geq \inf_{\varrho_1, \varrho_2} \min\{\ell_{r,\mathsf{D}}(\varrho_1),\ \ell_{r,\mathsf{D}}(\varrho_2)\} \\
&= \min\{\ell_{r,F(\mathsf{D})}(\varkappa_1),\ \ell_{r,F(\mathsf{D})}(\varkappa_2)\},
\end{aligned}$$

and likewise for ℓ_ω.

Condition (ic). Here, D gives an upper bound and (5.3) is used as follows:

$$\begin{aligned}
\ell_{q,F(\mathsf{D})}(\varkappa_1 \blacksquare_2 \varkappa_2) &= \inf_{\varrho_1 \in X_1,\ \varrho_2 \in X_2} \ell_{q,\mathsf{D}}(\varrho_1 \blacksquare_1 \varrho_2) \\
&\leq \inf_{\varrho_1, \varrho_2} \max\{\ell_{q,\mathsf{D}}(\varrho_1),\ \ell_{q,\mathsf{D}}(\varrho_2)\} \\
&= \max\{\ell_{q,F(\mathsf{D})}(\varkappa_1),\ \ell_{q,F(\mathsf{D})}(\varkappa_2)\},
\end{aligned}$$

and likewise for ℓ_γ.

Conditions (iia)–(iic). By Lemma 1, $F^{-1}(\varkappa^{-1}) = (F^{-1}(\varkappa))^{-1}$, so the substitution $\varrho = \sigma^{-1}$ is a bijection of $F^{-1}(\varkappa^{-1})$ onto $F^{-1}(\varkappa)$. Hence, using Proposition 3 (2.1) for D in the middle step,

$$\ell_{p,F(\mathsf{D})}(\varkappa^{-1}) = \sup_{\varrho \in F^{-1}(\varkappa^{-1})} \ell_{p,\mathsf{D}}(\varrho) = \sup_{\sigma \in F^{-1}(\varkappa)} \ell_{p,\mathsf{D}}(\sigma^{-1}) = \sup_{\sigma \in F^{-1}(\varkappa)} \ell_{p,\mathsf{D}}(\sigma) = \ell_{p,F(\mathsf{D})}(\varkappa),$$

and identically for ℓ_α, and with infima and Proposition 3 (2.2), (2.3) for the pairs (ℓ_r, ℓ_ω) and (ℓ_q, ℓ_γ). In particular, each of (iia)–(iic) holds with equality.

Therefore, all six conditions of Definition 9 are satisfied, and $F(\mathsf{D})$ is a CPic-FSG of $\mathbb{V}$. □

Theorem 6. *Let* $F\colon \mathbb{U} \to \mathbb{V}$ *be* any *group homomorphism, and let* R *be a CPic-FSG of* $\mathbb{V}$. *Then,* $F^{-1}(\mathsf{R})$ *is a CPic-FSG of* $\mathbb{U}$.

Proof. By Proposition 10, $F^{-1}(\mathsf{R})$ is a CPic-FS of $\mathbb{U}$. Write $\mathsf{S} = F^{-1}(\mathsf{R})$ for brevity, and let $\varrho_1, \varrho_2, \varrho \in \mathbb{U}$. Recall from Definition 13 that $\ell_{\bullet,\mathsf{S}}(\varrho) = \ell_{\bullet,\mathsf{R}}(F(\varrho))$ for each of the six components.

Condition (ia). Using Definition 13, F is a homomorphism, and condition (ia) for R,

$$\begin{aligned}\ell_{p,\mathsf{S}}(\varrho_1 \blacksquare_1 \varrho_2)e^{2\pi i\ell_{\alpha,\mathsf{S}}(\varrho_1\blacksquare_1\varrho_2)} &= \ell_{p,\mathsf{R}}(F(\varrho_1 \blacksquare_1 \varrho_2))e^{2\pi i\ell_{\alpha,\mathsf{R}}(F(\varrho_1\blacksquare_1\varrho_2))}\\ &= \ell_{p,\mathsf{R}}(F(\varrho_1) \blacksquare_2 F(\varrho_2))e^{2\pi i\ell_{\alpha,\mathsf{R}}(F(\varrho_1)\blacksquare_2 F(\varrho_2))}\\ &\geq \left[\ell_{p,\mathsf{R}}(F(\varrho_1))e^{2\pi i\ell_{\alpha,\mathsf{R}}(F(\varrho_1))}\right] \wedge \left[\ell_{p,\mathsf{R}}(F(\varrho_2))e^{2\pi i\ell_{\alpha,\mathsf{R}}(F(\varrho_2))}\right]\\ &= \left[\ell_{p,\mathsf{S}}(\varrho_1)e^{2\pi i\ell_{\alpha,\mathsf{S}}(\varrho_1)}\right] \wedge \left[\ell_{p,\mathsf{S}}(\varrho_2)e^{2\pi i\ell_{\alpha,\mathsf{S}}(\varrho_2)}\right],\end{aligned}$$

the last step by Lemma 2 (4).

Condition (iia). Using $F(\varrho^{-1}) = F(\varrho)^{-1}$ and then Proposition 3 (2.1) applied to the CPic-FSG R,

$$\begin{aligned}\ell_{p,\mathsf{S}}(\varrho^{-1})e^{2\pi i\ell_{\alpha,\mathsf{S}}(\varrho^{-1})} &= \ell_{p,\mathsf{R}}(F(\varrho)^{-1})e^{2\pi i\ell_{\alpha,\mathsf{R}}\left(F(\varrho)^{-1}\right)}\\ &= \ell_{p,\mathsf{R}}(F(\varrho))e^{2\pi i\ell_{\alpha,\mathsf{R}}(F(\varrho))}\\ &= \ell_{p,\mathsf{S}}(\varrho)e^{2\pi i\ell_{\alpha,\mathsf{S}}(\varrho)}.\end{aligned}$$

Conditions (ib), (iib). Replace (ℓ_p, ℓ_α) by (ℓ_r, ℓ_ω) and use (ib) for R and Proposition 3 (2.2); every step above is unchanged.

Conditions (ic), (iic). Replace (ℓ_p, ℓ_α) by (ℓ_q, ℓ_γ), reverse the inequality, and replace $\wedge$ by $\vee$, using (ic) for R and Proposition 3 (2.3).

No step used injectivity or surjectivity of F, so $F^{-1}(\mathsf{R})$ is a CPic-FSG of $\mathbb{U}$ for every homomorphism F. □

Theorem 7. *Let $F\colon \mathbb{U} \to \mathbb{V}$ be an epimorphism of groups, and let D be a CPic-FNSG of $\mathbb{U}$. Then, $F(\mathsf{D})$ is a CPic-FNSG of $\mathbb{V}$.*

Proof. By Theorem 5, $F(\mathsf{D})$ is a CPic-FSG of $\mathbb{V}$, so by Proposition 8, it suffices to verify the three commutation identities. Let $\varkappa_1, \varkappa_2 \in \mathbb{V}$ and put $X_j = F^{-1}(\varkappa_j)$, which are nonempty since F is onto.

Membership. By Lemma 1, $F^{-1}(\varkappa_1 \blacksquare_2 \varkappa_2) = X_1 \blacksquare_1 X_2$ and $F^{-1}(\varkappa_2 \blacksquare_2 \varkappa_1) = X_2 \blacksquare_1 X_1$. Since D is a CPic-FNSG, Proposition 8 (1) gives $\ell_{p,\mathsf{D}}(\varrho_1 \blacksquare_1 \varrho_2) = \ell_{p,\mathsf{D}}(\varrho_2 \blacksquare_1 \varrho_1)$ and $\ell_{\alpha,\mathsf{D}}(\varrho_1 \blacksquare_1 \varrho_2) = \ell_{\alpha,\mathsf{D}}(\varrho_2 \blacksquare_1 \varrho_1)$ for all $\varrho_1, \varrho_2 \in \mathbb{U}$. Hence,

$$\begin{aligned}\ell_{p,F(\mathsf{D})}(\varkappa_1 \blacksquare_2 \varkappa_2) &= \sup_{\substack{\varrho_1\in X_1\\ \varrho_2\in X_2}} \ell_{p,\mathsf{D}}(\varrho_1 \blacksquare_1 \varrho_2) \;=\; \sup_{\substack{\varrho_1\in X_1\\ \varrho_2\in X_2}} \ell_{p,\mathsf{D}}(\varrho_2 \blacksquare_1 \varrho_1)\\ &= \sup_{\varrho\in X_2\blacksquare_1 X_1} \ell_{p,\mathsf{D}}(\varrho) \;=\; \ell_{p,F(\mathsf{D})}(\varkappa_2 \blacksquare_2 \varkappa_1),\end{aligned}$$

and identically for ℓ_α. By Definition 1, the two together give Proposition 8 (1) for $F(\mathsf{D})$.

Neutral membership. The same display with sup replaced by inf and (ℓ_p, ℓ_α) by (ℓ_r, ℓ_ω), using Proposition 8 (2) for D, gives $\ell_{r,F(\mathsf{D})}(\varkappa_1 \blacksquare_2 \varkappa_2) = \ell_{r,F(\mathsf{D})}(\varkappa_2 \blacksquare_2 \varkappa_1)$ and the same for ℓ_ω.

Nonmembership. Likewise with inf and (ℓ_q, ℓ_γ), using Proposition 8 (3) for D.

All three identities hold, so $F(\mathsf{D})$ is a CPic-FNSG of $\mathbb{V}$. □

Theorem 8. *Let $F\colon \mathbb{U} \to \mathbb{V}$ be* any *group homomorphism, and let R be a CPic-FNSG of $\mathbb{V}$. Then, $F^{-1}(\mathsf{R})$ is a CPic-FNSG of $\mathbb{U}$.*

Proof. By Theorem 6, $F^{-1}(\mathsf{R})$ is a CPic-FSG of $\mathbb{U}$, so by Proposition 8, it is enough to verify the three commutation identities. Write $\mathsf{S} = F^{-1}(\mathsf{R})$, and let $\varrho_1, \varrho_2 \in \mathbb{U}$.

Membership. Using that F is a homomorphism, Proposition 8 (1) for R applied to the pair $(F(\varrho_1), F(\varrho_2))$, and then again that F is a homomorphism,

$$\begin{aligned}\ell_{p,\mathsf{S}}(\varrho_1 \blacksquare_1 \varrho_2)e^{2\pi i\ell_{\alpha,\mathsf{S}}(\varrho_1\blacksquare_1\varrho_2)} &= \ell_{p,\mathsf{R}}(F(\varrho_1) \blacksquare_2 F(\varrho_2))e^{2\pi i\ell_{\alpha,\mathsf{R}}(F(\varrho_1)\blacksquare_2 F(\varrho_2))}\\ &= \ell_{p,\mathsf{R}}(F(\varrho_2) \blacksquare_2 F(\varrho_1))e^{2\pi i\ell_{\alpha,\mathsf{R}}(F(\varrho_2)\blacksquare_2 F(\varrho_1))}\\ &= \ell_{p,\mathsf{S}}(\varrho_2 \blacksquare_1 \varrho_1)e^{2\pi i\ell_{\alpha,\mathsf{S}}(\varrho_2\blacksquare_1\varrho_1)}.\end{aligned}$$

Neutral membership and nonmembership. The same three-line computation with (ℓ_r, ℓ_ω) and Proposition 8 (2), and with (ℓ_q, ℓ_γ) and Proposition 8 (3), gives the remaining two identities.

Again, no step used injectivity or surjectivity, so the conclusion holds for an arbitrary homomorphism F. □

Remark 11. *(Sharpness of the hypotheses) Theorems 6 and 8 only require that F be a homomorphism. Surjectivity is used in Theorems 5 and 7 for two distinct purposes: It makes every fiber nonempty, so that the degenerate branch of Definition 13 is never invoked and $F(\mathsf{D})$ is determined by suprema and infima over nonempty sets; moreover, it is what allows Lemma 1 to be applied to every pair $\varkappa_1, \varkappa_2 \in \mathbb{V}$. If F is not surjective, then the same proofs show that the restriction of $F(\mathsf{D})$ to the subgroup $F(\mathbb{U}) \leq \mathbb{V}$ is a CPic-FSG of $F(\mathbb{U})$, and a CPic-FNSG when D is a CPic-FNSG. Injectivity is nowhere required.*

Remark 12. *(Comparison with [22]) The four preservation theorems above correspond to [22, Theorems 7.4–7.6]. Besides holding without the homogeneity hypothesis, three of them weaken the map hypothesis, see Table 5.*

Table 5. The four preservation theorems above correspond to theorems in [22].

Statement	Here	Hypothesis here	Hypothesis in [22]
image of a subgroup	Thm. 5	epimorphism	isomorphism (Thm. 7.4)
preimage of a subgroup	Thm. 6	homomorphism	homomorphism (Thm. 7.5)
image of a normal sg.	Thm. 7	epimorphism	isomorphism (Thm. 7.6(1))
preimage of a normal sg.	Thm. 8	homomorphism	isomorphism (Thm. 7.6(2))

The weakening is not an artefact of a different proof strategy: As the proofs record, the only properties of F used in Theorems 6 and 8 are $F(\varrho_1 \blacksquare_1 \varrho_2) = F(\varrho_1) \blacksquare_2 F(\varrho_2)$ and $F(\varrho^{-1}) = F(\varrho)^{-1}$, and the only role of surjectivity in Theorems 5 and 7 is the one isolated in Remark 11. In particular, Theorem 8 applies to every group homomorphism, including non-injective and non-surjective ones, where [22, Theorem 7.6 (2)] requires a bijection.

6. Conclusions and further work

6.1. Summary

Complex picture fuzzy subgroups were introduced by Abbas and Malik [22], whose theory was developed for *homogeneous* complex picture fuzzy sets — those in which for each component, the

amplitude and the phase are comonotone (Definition 2). This paper removes that hypothesis and supplies what is needed in its place. Our findings are the following.

(1) *An order in place of a restriction.* The admissible complex grades form a complete lattice under the componentwise order (Definition 1, Proposition 1), and every inequality in this theory is a $\preceq$-inequality in that lattice, equivalent to a pair of real inequalities between amplitudes and between phases (Corollary 1). Because $e^{2\pi i\vartheta}$ is periodic, the primitive datum is the pair (ℓ, ϑ) rather than the complex number (Remark 1).

(2) *Decomposition without hypotheses.* A CPic-FS is a CPic-FSG if and only if its amplitude triple and its phase triple are both Pic-FSGs (Theorem 1), with no comonotonicity assumption; this is the non-homogeneous generalization of [22, Theorem 3.8] (Remark 5). The generalization is proper: Every example in this paper violates homogeneity (Remark 7), and Example 2 cannot be realized by a homogeneous CPic-FS at all.

(3) *Admissibility of the operations.* Intersection and union are explicitly defined and preserve both constraints of (3.1); the minimum convention for the neutral grade of an intersection is the only one of the two conventions in the literature that does so (Proposition 6, Remark 8). The class of CPic-FSGs is closed under intersection (Proposition 7) but not under union (Example 4).

(4) *Well-definedness of the homomorphic image.* The direct image formed with suprema for both the membership and the neutral grade need not be admissible (Example 6); the image of Definition 13 always is (Proposition 10). This verification is absent from the prior literature.

(5) *Sharper homomorphism hypotheses.* Direct images preserve the subgroup and normality conditions under epimorphisms, and inverse images under *arbitrary* homomorphisms; injectivity is never needed and the role of surjectivity is isolated in Remark 11. Three of the four corresponding statements in [22] assume an isomorphism (Remark 12).

(6) *Independence and non-redundancy.* The closure conditions imply the inverse conditions on torsion groups but not in general (Proposition 2), and neither the neutral nor the phase component can be dropped (Examples 2 and 3), so CPic-FSG results are not specializations of CIFSG or Pic-FSG results.

6.2. Limitations

Three limitations should be stated plainly.

First, this paper is group-theoretic: Factor structures, rings and fields are not treated, and the quotient of a group by the normal subgroup of Theorem 4 has not been equipped with an induced CPic-FS structure.

Second, this paper contains *no application*. In particular, we make no claim about cryptographic use. A previous version of this manuscript suggested that the algebraic structure of CPic-FSGs could be used to design cryptographic systems for complex picture fuzzy data; since no such construction is given here and no security analysis is offered, that suggestion is withdrawn. Whether such a scheme can be constructed, and whether it would offer any advantage over existing schemes, are open questions that require a concrete construction together with a security argument, neither of which is in scope.

Third, we do not reproduce the parts of [22] that this paper does not improve on: complex picture fuzzy support and core, conjugacy and conjugacy classes, normalizers, the complex picture fuzzy order

and index, the analogue of Lagrange's theorem, and the application to robotic grasping. Those results stand as published, for the homogeneous case.

6.3. Further work

(a) *Extending the published theory to the non-homogeneous setting.* The most direct continuation is to ask which of the notions of [22] listed above survive without comonotonicity. Some should transfer by the method used here — replacing homogeneity by the lattice of Definition 1 — but others are likely to require genuine modification. The complex picture fuzzy order and index are defined in [22] through the core, which is a level set; by Theorem 2, the level sets remain subgroups in the general case, so the order and index remain defined, and whether the analogue of Lagrange's theorem survives is a concrete open question.

(b) *Within group theory.* Factor structures and an analogue of the isomorphism theorems; the lattice of CPic-FSGs of a fixed group, which by Proposition 7 and Example 4 is closed under meets but not under the pointwise join, so that its join operation remains to be identified.

(c) *Other algebraic structures.* Complex picture fuzzy subrings, ideals, modules, and subfields, following the pattern established here: fix the order, prove a decomposition theorem, then check admissibility of every operation.

(d) *Frameworks without the summation constraint.* The analysis of Section 3 turns on the admissibility constraints (3.1). In complex-valued frameworks that impose no such bound, such as the complex neutrosophic sets of [21], the corresponding operations are unconditionally well defined, and it would be of interest to determine exactly how much of the subgroup theory depends on the constraint rather than on the complex-valued form.

Author contributions

Lulwah AL-Essa: validated the research outputs and authored the original draft; Eman A. AbuHijleh: analyzed the formal methodology and concepts, formulating the overarching research goals; Lulwah AL-Essa and Eman A. AbuHijleh: compiled and finalized the research after the revision stages. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors used an AI-assisted tool for language editing and for checking the numerical examples and the consistency of the reference list only.

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Conflict of interest

The authors declare that they have no competing interests.

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