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*Research article*

# On second-order linear differential equations with quadratic coefficients

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**Abstract:** For a fixed integer $N \geq 1$, we study degree-$N$ polynomial solutions of $A(x)y'' + B(x)y' + C(x)y = 0$, where $\deg A \leq 2$, $\deg B = 2$, and $\deg C \leq 2$. Such solutions require $C(x) = c_0 - Nb_2x$, and they exist exactly when $c_0$ is a zero of the constraint polynomial $D_N(c_0) = \det(K_N + c_0 I)$, where $K_N$ is an unreduced upper Hessenberg matrix. Using scaling, translation, and homogeneity of the equation, we show that the discriminant locus of $D_N$ depends on three essential parameters, give an explicit normal form, and relate the family to the confluent Heun equations and to Turbiner's $\mathfrak{sl}_2$ description. We prove a reality-and-simplicity criterion for the zeros of $D_N$ valid for every $N$ when $A$ has a real zero, compute $D_N$ for $N \leq 4$, show that the Galois group of $D_4$ is $S_5$, and show that every multiple zero of $D_N$ is a defective eigenvalue with a single Jordan block, near which, under an analytic perturbation satisfying a transversality condition equivalent to the Lidskii condition, the admissible values split according to a Puiseux series of order equal to the block size.

**Keywords:** polynomial solutions; confluent Heun equations; constraint polynomials; quasi-exact solvability; discriminants; exceptional points
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**Abbreviations:** QES: quasi-exactly solvable.

## 1. Introduction

Polynomial solutions of second-order differential equations with polynomial coefficients occur in the theory of classical orthogonal polynomials, in Heun-type equations and in quasi-exactly solvable (QES) spectral problems. Bochner's theorem classifies the classical orthogonal polynomial systems satisfying such an equation [7, 29]; linear-algebraic criteria for polynomial solutions of general linear equations are given in [3, 4, 17]; and in mathematical physics, polynomial solutions are tied to Heun polynomials, series termination, and quasi-exact solvability [2, 8, 9, 14, 19, 25, 26, 30, 31, 34]. The Rabi-

model motivation for the constraint-polynomial construction studied here is developed in [11, 22, 23, 33, 35].

We consider

$$A(x)y''(x) + B(x)y'(x) + C(x)y(x) = 0, \tag{1.1}$$

$$A(x) = a_0 + a_1x + a_2x^2, \qquad B(x) = b_0 + b_1x + b_2x^2, \qquad C(x) = c_0 + c_1x + c_2x^2, \tag{1.2}$$

with $b_2 \neq 0$ throughout. If $y(x) = u_Nx^N + u_{N-1}x^{N-1} + \cdots$ with $u_N \neq 0$ solves (1.1), the coefficient of $x^{N+2}$ gives $c_2 = 0$, and then the coefficient of $x^{N+1}$ gives $c_1 = -Nb_2$. These *termination conditions*

$$c_2 = 0, \qquad c_1 = -Nb_2 \tag{1.3}$$

reduce $C$ to $C(x) = c_0 - Nb_2x$. With $\mathcal{P}_N = \operatorname{span}\{1, x, \dots, x^N\}$ and

$$L_N = A(x)\frac{d^2}{dx^2} + B(x)\frac{d}{dx} + (c_0 - Nb_2x),$$

one has $L_N(\mathcal{P}_N) \subseteq \mathcal{P}_N$ (Lemma 2.2). In the monomial basis, $L_N|_{\mathcal{P}_N}$ is represented by the $(N+1)\times(N+1)$ matrix

$$M_N(c_0) = K_N + c_0I_{N+1}, \tag{1.4}$$

where $K_N$ is the matrix of $A\,d^2/dx^2 + B\,d/dx - Nb_2x$ on $\mathcal{P}_N$. The *$N$-th constraint polynomial* is

$$D_N(c_0) := \det M_N(c_0) = \det(K_N + c_0I_{N+1}). \tag{1.5}$$

**Definition 1.1.** A number $c_0 \in \mathbb{C}$ is *admissible* for the $N$-th sector if $L_Ny = 0$ for some nonzero $y \in \mathcal{P}_N$, equivalently $D_N(c_0) = 0$, equivalently $-c_0$ is an eigenvalue of $K_N$.

**Definition 1.2.** Let $T(\omega)$ be a square matrix depending on a parameter $\omega$. An eigenvalue $\lambda$ of $T(\omega^*)$ is *defective* if its geometric multiplicity $\dim\ker(T(\omega^*) - \lambda I)$ is smaller than its algebraic multiplicity. Following the usage of non-Hermitian spectral theory [6, 36], we call $\omega^*$ an *exceptional point* of $T$ if $T(\omega^*)$ has a defective eigenvalue.

In Kato's perturbation theory [13], the term exceptional point has a different meaning: It refers to a parameter value at which the number of distinct eigenvalues of an analytic family drops, and such an eigenvalue need not be defective. For the constraint matrices, every repeated eigenvalue is defective (Lemma 10.1). Along analytic curves with generically simple spectrum, this identifies the crossings of the discriminant locus with Kato's exceptional values; degeneracies that persist along a curve require the distinction explained in Remark 10.3.

If $P(t) = \alpha\prod_{j=1}^{m}(t - r_j)$, its discriminant is $\operatorname{Disc}_t P = \alpha^{2m-2}\prod_{i<j}(r_i - r_j)^2$, and

$$\operatorname{Disc}_t P = (-1)^{m(m-1)/2}\alpha^{-1}\operatorname{Res}_t(P, \partial_t P). \tag{1.6}$$

The central object of the paper is

$$\Delta_N = \operatorname{Disc}_{c_0} D_N(c_0). \tag{1.7}$$

We write $\omega = (a_0, a_1, a_2, b_0, b_1, b_2)$ for a coefficient vector and set

$$\Omega_{\mathbb{C}} = \{\omega \in \mathbb{C}^6 : \ b_2 \neq 0\}, \qquad \Omega_{\mathbb{R}} = \Omega_{\mathbb{C}} \cap \mathbb{R}^6, \qquad \Sigma_N = \{\omega \in \Omega_{\mathbb{C}} : \ \Delta_N(\omega) = 0\}.$$

Then $\Sigma_N$ is exactly the set of coefficient vectors for which two admissible values coincide. Let

$$\mathcal{R}_N = \{\omega \in \Omega_{\mathbb{R}} : \ D_N(\cdot\,;\omega) \text{ has } N+1 \text{ distinct real zeros}\}.$$

$\mathcal{R}_N$ is open in $\Omega_{\mathbb{R}}$, and $\partial_{\Omega_{\mathbb{R}}}\mathcal{R}_N \subseteq \Sigma_N \cap \Omega_{\mathbb{R}}$, because along a path of real parameters, zeros can leave the real axis only after two real zeros collide. For $N = 1, 2$ equality holds: A multiple zero of a real polynomial of degree at most three is real, and every point of $\Sigma_N \cap \Omega_{\mathbb{R}}$ is a limit of points of $\mathcal{R}_N$ (Section 6 and Proposition 7.1). For $N \geq 3$ the inclusion can be strict. A point of $\Sigma_N \cap \Omega_{\mathbb{R}}$ at which a multiple real zero coexists with a nonreal pair is not a limit of points of $\mathcal{R}_N$, since a small real perturbation keeps that pair nonreal. For example, for $N = 3$ the equation with $A = x^2 + x + 52$ and $B = x^2 + 14$ has

$$D_3(c_0) = (c_0 + 6)^2\,(c_0^2 - 4c_0 + 174),$$

with the double real zero $-6$ and the nonreal pair $2 \pm i\sqrt{170}$.

**Relation to earlier work and contributions.** The termination conditions and the constraint polynomial $D_N$ were introduced in this generality by Moroz [22]; Moroz and Miroshnichenko [23] showed that the zeros of $D_N$ are real and simple whenever an associated moment functional is positive definite. The subfamily $A(x) = x(x - \alpha)$ with $c_1 = -Nb_2$, together with its $\mathfrak{sl}_2$ algebraization, was treated by Agboola and Zhang [2] via the Bethe ansatz (see also [34]), and the subfamily $A \equiv \text{const}$ appears in [1] as the auxiliary equation for quartic potentials. For $A \not\equiv 0$, up to affine changes of variable, the family (1.1) is the union of the four confluent Heun equations (Proposition 3.7), for which the condition $c_1 = -Nb_2$ is the standard polynomial condition [25, 26]. Against this background, the contributions of the paper are the following.

(C1) *Symmetry reduction.* $D_N$ depends on $(a_0, a_1, b_0, b_2)$ only through the scaling invariants $I_1 = a_0 b_2^2$, $I_2 = a_1 b_2$, $I_3 = b_0 b_2$ (Theorem 3.1); combined with translations and homogeneity, the discriminant locus depends on three essential parameters, and an explicit normal form is given (Theorem 3.3, Corollary 3.4). We also record the $\mathfrak{sl}_2$ form of $L_N$ for general $A$ (Proposition 3.6), extending [2, eq. (4.3)].

(C2) *A reality criterion for all N.* If $A$ has a real zero $\alpha$ and $b_2 B(\alpha) < 0$, $b_2((N-1)A'(\alpha) + B(\alpha)) < 0$, then all $N + 1$ admissible values are real and simple (Theorem 5.1, Corollary 5.2). This is an explicit, checkable instance of the positivity condition of [23]. As an application, the nontrivial roots of the Juddian constraint polynomial of the quantum Rabi model are real and simple in $\Delta^2$ for every $N$ (Example 11.1).

(C3) *Low-degree sectors.* Explicit formulas for $D_N$, $N \leq 4$, in invariant coordinates and in normal form; the exact real-root interval in the cubic sector (Proposition 7.1); the real-root test in the quartic sector, obtained from the classical criterion [18, 24]; and a proof that the Galois group of $D_4$ is $S_5$ (Theorem 9.1).

(C4) *Spectral interpretation.* Every multiple zero of $D_N$ is a defective eigenvalue with a single Jordan block. This is a direct consequence of the classical fact that unreduced Hessenberg matrices are nonderogatory. Applying the classical adjugate identity within this family, we identify the transversality hypothesis of the Newton–Puiseux splitting law with the Lidskii condition [20, 21] (Proposition 10.4, Theorem 10.5).

Classical tools (the cubic and quartic discriminant criteria, Sturm sequences, the discriminant–resultant identity, Wilkinson's condition number, and the Newton–Puiseux theorem) are quoted from the literature and used without claim of novelty.

## 2. Polynomial sectors: recurrence and reconstruction

The coefficients may be taken in $\mathbb{C}$; statements about real zeros assume real coefficients. Throughout, $N \geq 1$, $b_2 \neq 0$, and (1.3) holds, so $\deg_{c_0} D_N = N + 1$.

**Remark 2.1.** For $N = 0$ the termination conditions are $c_1 = c_2 = 0$ and $D_0(c_0) = c_0$; a constant solution exists precisely for $c_0 = 0$.

Let $y = \sum_{n=0}^{N} u_n x^n$. Comparing coefficients of $x^n$ in (1.1) and using (1.3) gives

$$a_0(n+2)(n+1)u_{n+2} + (a_1 n(n+1) + b_0(n+1))u_{n+1} + (a_2 n(n-1) + b_1 n + c_0)u_n + b_2(n-1-N)u_{n-1} = 0, \quad (2.1)$$

with $u_{-1} = 0$ and $u_j = 0$ for $j > N$.

**Lemma 2.2.** *Let $L = A\partial_x^2 + B\partial_x + C$. Then $L(\mathcal{P}_N) \subseteq \mathcal{P}_N$ if and only if* (1.3) *holds.*

*Proof.* If $L(\mathcal{P}_N) \subseteq \mathcal{P}_N$, the coefficients of $x^{N+2}$ and $x^{N+1}$ in $Lx^N$ vanish, giving $c_2 = 0$ and then $c_1 = -Nb_2$. Conversely, under (1.3), the only term of $Lx^j$ of degree exceeding $j$ is $b_2(j-N)x^{j+1}$, which has degree $\leq N$ for $j < N$ and vanishes for $j = N$. □

**Proposition 2.3.** *Let $c_0$ be admissible and $y = \sum_{n=0}^{N} u_n x^n$ a nonzero solution. Then $u_N \neq 0$. With $u_N = 1$, the remaining coefficients are determined by*

$$u_{n-1} = \frac{a_0(n+2)(n+1)u_{n+2} + (a_1 n(n+1) + b_0(n+1))u_{n+1} + (a_2 n(n-1) + b_1 n + c_0)u_n}{b_2(N+1-n)}, \quad (2.2)$$

*$n = N, N-1, \ldots, 1$, with $u_{N+1} = u_{N+2} = 0$, and the remaining equation*

$$2a_0 u_2 + b_0 u_1 + c_0 u_0 = 0 \quad (2.3)$$

*holds if and only if $D_N(c_0) = 0$. In particular the polynomial solution is unique up to a scalar factor.*

*Proof.* If $u_k = \cdots = u_N = 0$ for some $1 \leq k \leq N$, then (2.1) with $n = k$ reduces to $b_2(k-1-N)u_{k-1} = 0$, so $u_{k-1} = 0$. Descending induction from $k = N$ shows that $u_N = 0$ forces $y = 0$.

The system $M_N(c_0)u = 0$ consists of the $N + 1$ Eqs (2.1), $n = 0, \ldots, N$. For $n = N, \ldots, 1$, the coefficient of $u_{n-1}$ is $-b_2(N+1-n) \neq 0$, so after fixing $u_N = 1$ these $N$ equations determine $u_{N-1}, \ldots, u_0$ uniquely by (2.2); this step imposes no condition on $c_0$. The vector $u(c_0)$ so obtained satisfies $M_N(c_0)u(c_0) = 0$ if and only if the remaining equation $n = 0$, which is (2.3), holds. Since every nonzero solution of $M_N(c_0)u = 0$ has $u_N \neq 0$ and is therefore a multiple of $u(c_0)$, (2.3) holds exactly when $M_N(c_0)$ is singular, that is, when $D_N(c_0) = 0$. □

## 3. Symmetries of the constraint polynomial

### 3.1. Matrix structure and scaling

A matrix $H = (h_{jk})_{0\le j,k\le N}$ is *upper Hessenberg* if $h_{jk} = 0$ for $j > k+1$ and *unreduced* if in addition $h_{j,j-1} \neq 0$ for $1 \le j \le N$.

**Theorem 3.1.** *Fix $N \ge 1$ and impose* (1.3)*. The nonzero entries of $K_N$ lie in the bands*

$$(K_N)_{j,j} = a_2 j(j-1) + b_1 j, \qquad 0 \le j \le N, \tag{3.1}$$
$$(K_N)_{j,j+1} = a_1 j(j+1) + b_0(j+1), \qquad 0 \le j \le N-1, \tag{3.2}$$
$$(K_N)_{j,j+2} = a_0(j+1)(j+2), \qquad 0 \le j \le N-2, \tag{3.3}$$
$$(K_N)_{j,j-1} = b_2(j-1-N), \qquad 1 \le j \le N. \tag{3.4}$$

*Hence $K_N$ is unreduced upper Hessenberg. With $S = \mathrm{diag}(1, b_2^{-1}, \dots, b_2^{-N})$ and*

$$I_1 = a_0 b_2^2, \qquad I_2 = a_1 b_2, \qquad I_3 = b_0 b_2, \tag{3.5}$$

*the matrix $\widetilde{K}_N := S K_N S^{-1}$ has entries*

$$\begin{aligned} (\widetilde{K}_N)_{j,j} &= a_2 j(j-1) + b_1 j, \quad (\widetilde{K}_N)_{j,j+1} = I_2 j(j+1) + I_3(j+1), \\ (\widetilde{K}_N)_{j,j+2} &= I_1(j+1)(j+2), \quad (\widetilde{K}_N)_{j,j-1} = j-1-N. \end{aligned} \tag{3.6}$$

*Consequently $D_N(c_0) = \det(\widetilde{K}_N + c_0 I)$ is a polynomial in $c_0, a_2, b_1, I_1, I_2, I_3$.*

*Proof.* The bands follow from (2.1). Since $(S K_N S^{-1})_{jk} = b_2^{k-j}(K_N)_{jk}$, the diagonal is unchanged, the first and second superdiagonals are multiplied by $b_2$ and $b_2^2$, and the subdiagonal by $b_2^{-1}$, which gives (3.6). Similarity preserves determinants. □

**Proposition 3.2.** *Under $x = \rho t$, $\rho \neq 0$, the equation keeps the form* (1.1) *with coefficients $a_0/\rho^2$, $a_1/\rho$, $a_2$, $b_0/\rho$, $b_1$, $\rho b_2$, $c_0$, and $\rho c_1 = -N(\rho b_2)$. Hence $a_2, b_1, c_0, I_1, I_2, I_3$ are invariant, and $\rho = b_2^{-1}$ normalizes $b_2 = 1$, $(a_0, a_1, b_0) = (I_1, I_2, I_3)$.*

*Proof.* Since $\partial_x = \rho^{-1}\partial_t$, the new coefficients of $v''$ and $v'$ are $\rho^{-2}A(\rho t)$ and $\rho^{-1}B(\rho t)$, and that of $v$ is $C(\rho t) = c_0 + \rho c_1 t$. □

We call $(a_2, b_1, I_1, I_2, I_3)$ the *invariant coordinates*, regard $\Delta_N = \mathrm{Disc}_{c_0} D_N$ as a polynomial in them, and set

$$V_N = \{(a_2, b_1, I_1, I_2, I_3) \in \mathbb{C}^5 : \ \Delta_N = 0\}. \tag{3.7}$$

### 3.2. Translation and homogeneity

The scaling of Proposition 3.2 is one of three natural operations on (1.1). The other two are the translation $x \mapsto x + s$, which also preserves $\mathcal{P}_N$, and multiplication of the equation by a nonzero constant $\kappa$.

**Theorem 3.3.** *For $s \in \mathbb{C}$, let $\omega_s$ denote the coefficient vector of $A(x+s)$ and $B(x+s)$. Then*

$$D_N(c_0; \omega) = D_N(c_0 - Nb_2 s; \omega_s). \tag{3.8}$$

*In invariant coordinates, with*

$$\hat{c}_0 = c_0 + \tfrac{N}{2} b_1, \qquad \hat{I}_1 = I_1 - \tfrac{1}{2} b_1 I_2 + \tfrac{1}{4} a_2 b_1^2, \qquad \hat{I}_2 = I_2 - a_2 b_1, \qquad \hat{I}_3 = I_3 - \tfrac{1}{4} b_1^2, \tag{3.9}$$

*one has*

$$D_N(c_0; a_2, b_1, I_1, I_2, I_3) = D_N(\hat{c}_0; a_2, 0, \hat{I}_1, \hat{I}_2, \hat{I}_3). \tag{3.10}$$

*Moreover $D_N$ is weighted homogeneous: For $\kappa \neq 0$,*

$$D_N(\kappa c_0; \kappa a_2, \kappa b_1, \kappa^3 I_1, \kappa^2 I_2, \kappa^2 I_3) = \kappa^{N+1} D_N(c_0; a_2, b_1, I_1, I_2, I_3). \tag{3.11}$$

*Proof.* Let $(T_s y)(x) = y(x+s)$, an automorphism of $\mathcal{P}_N$. If $L_{\omega, c_0}$ denotes $L_N$ with coefficients $\omega$ and constant term $c_0$, then, since $C(x+s) = (c_0 - Nb_2 s) - Nb_2 x$,

$$T_s \, L_{\omega, c_0} = L_{\omega_s,\, c_0 - Nb_2 s} \, T_s \qquad \text{on } \mathcal{P}_N.$$

Hence the two restricted operators are similar and have the same determinant, which is (3.8). Choosing $s = -b_1/(2b_2)$ gives $b_1(\omega_s) = 0$, $c_0 - Nb_2 s = \hat{c}_0$, $a_0(\omega_s) = A(s)$, $a_1(\omega_s) = A'(s)$, $b_0(\omega_s) = B(s)$, and $b_2(\omega_s) = b_2$; computing $A(s)b_2^2$, $A'(s)b_2$, and $B(s)b_2$ gives (3.9) and (3.10). Multiplying (1.1) by $\kappa$ multiplies $M_N(c_0)$ by $\kappa$ and replaces $(a_i, b_i, c_0)$ by $(\kappa a_i, \kappa b_i, \kappa c_0)$; this multiplies $I_1, I_2, I_3$ by $\kappa^3, \kappa^2, \kappa^2$, and $\det(\kappa M_N) = \kappa^{N+1} \det M_N$. □

**Corollary 3.4.** *$\Delta_N(a_2, b_1, I_1, I_2, I_3) = \Delta_N(a_2, 0, \hat{I}_1, \hat{I}_2, \hat{I}_3)$, and $\Delta_N$ is weighted homogeneous of weighted degree $N(N+1)$ for the weights $(1, 1, 3, 2, 2)$. Consequently, after an affine change of variable and multiplication by a constant, every equation of the family can be brought to the normal form*

$$b_2 = 1, \qquad b_1 = 0, \qquad a_2 \in \{0, 1\},$$

*with remaining parameters*

$$(a_0^{\text{new}}, a_1^{\text{new}}, b_0^{\text{new}}) = (\kappa^3 \hat{I}_1, \kappa^2 \hat{I}_2, \kappa^2 \hat{I}_3), \qquad c_0^{\text{new}} = \kappa \hat{c}_0,$$

*where $\kappa = a_2^{-1}$ if $a_2 \neq 0$ and $\kappa = 1$ otherwise, and $\hat{c}_0, \hat{I}_k$ are given by* (3.9)*. The discriminant locus is therefore determined by its intersections with the two three-dimensional slices $a_2 = 1$ and $a_2 = 0$; over $\mathbb{R}$ the same normalization holds with $\kappa, \rho, s$ real. On the slice $a_2 = 0$, the multiplication by $\kappa$ is not used by the normalization and remains free: It acts by $(\hat{I}_1, \hat{I}_2, \hat{I}_3, \hat{c}_0) \mapsto (\kappa^3 \hat{I}_1, \kappa^2 \hat{I}_2, \kappa^2 \hat{I}_3, \kappa \hat{c}_0)$, and $\Delta_N$ is weighted homogeneous for this action. The part of the discriminant locus in this slice is therefore a weighted cone, determined by two essential parameters; for instance, if $\hat{I}_3 \neq 0$, one may normalize $\hat{I}_3 = \pm 1$ with a real $\kappa$. Thus "three essential parameters" refers to the slice $a_2 = 1$.*

*Proof.* Translation shifts all zeros of $D_N$ by the same amount, so the discriminant is unchanged; weighted homogeneity of $\Delta_N$ follows from (3.11) since each difference of zeros has weight 1 and $\Delta_N$ is a product of $N(N+1)$ such differences. For the normal form, translate by $s = -b_1/(2b_2)$, multiply by $\kappa$, which replaces the quadratic coefficient of $B$ by $\kappa b_2$, and then scale by $\rho = (\kappa b_2)^{-1}$. By Proposition 3.2 the scaling leaves $\kappa^3 \hat{I}_1, \kappa^2 \hat{I}_2, \kappa^2 \hat{I}_3$, and $\kappa \hat{c}_0$ unchanged and makes them equal to $a_0, a_1, b_0, c_0$. □

**Remark 3.5.** The quantities $a_2$, $b_2$, $\operatorname{Disc} A = a_1^2 - 4a_0a_2$, $\operatorname{Disc} B = b_1^2 - 4b_0b_2$, $J = a_1b_1 - 2(a_0b_2 + a_2b_0)$, and $R = a_1b_2 - a_2b_1$ are invariant under translation, and satisfy the single quadratic relation

$$R^2 = b_2^2 \operatorname{Disc} A + a_2^2 \operatorname{Disc} B - 2a_2b_2J.$$

In terms of them, $\hat{I}_2 = R$, $\hat{I}_3 = -\frac{1}{4}\operatorname{Disc} B$, and $\hat{I}_1 = b_2^2 A(-b_1/2b_2)$ is $b_2^2$ times the value of $A$ at the vertex of $B$. By (3.10) the coefficients of $D_N$, written as a polynomial in the translation-invariant spectral variable $\hat{c}_0 = c_0 + \frac{N}{2}b_1$, are translation invariant.

### *3.3. The $\mathfrak{sl}_2$ structure*

Let $J^- = \partial_x$, $J^0 = x\partial_x - \frac{N}{2}$, $J^+ = x^2\partial_x - Nx$. These operators span a copy of $\mathfrak{sl}_2$ acting irreducibly on $\mathcal{P}_N$, and every endomorphism of $\mathcal{P}_N$ is represented by a polynomial in these generators [30, 31]. For differential operators before restriction to $\mathcal{P}_N$, this statement is understood modulo operators annihilating $\mathcal{P}_N$.

**Proposition 3.6.** *Under* (1.3)*,*

$$L_N = a_2J^0J^0 + a_1J^0J^- + a_0J^-J^- + b_2J^+ + (b_1 + (N-1)a_2)J^0 + (b_0 + \tfrac{N}{2}a_1)J^- + \hat{c}_0 + \tfrac{1}{4}N(N-2)a_2. \quad (3.12)$$

*Proof.* Direct expansion; see Appendix A. □

For $A = x(x - \alpha)$, this reduces to [2, Eq (4.3)]. In this language, $J^+$ is the only raising generator present, which is the Hessenberg structure; the scaling invariants are the weight-zero combinations for the $J^0$-grading; translations act by conjugation with $e^{sJ^-}$; and the constant term contains the translation-invariant spectral variable $\hat{c}_0$.

### *3.4. Singular points and the confluent Heun family*

**Proposition 3.7.** *Assume $A \not\equiv 0$ and* (1.3)*. Up to an affine change of the independent variable and multiplication by a constant,* (1.1) *is one of the following equations of [26, §31.12] (see also [25, Parts B–E]), multiplied by the displayed factor:*

1) *$A$ has two simple zeros: the confluent Heun equation $w'' + (\frac{\gamma}{z} + \frac{\delta}{z-1} + \epsilon)w' + \frac{\alpha z - q}{z(z-1)}w = 0$, times $z(z-1)$, with $b_2 = \epsilon$ and termination condition $\alpha = -N\epsilon$;*
2) *$A$ has a double zero: the doubly-confluent Heun equation $w'' + (\frac{\delta}{z^2} + \frac{\gamma}{z} + 1)w' + \frac{\alpha z - q}{z^2}w = 0$, times $z^2$, with $\alpha = -N$;*
3) *$\deg A = 1$: the biconfluent Heun equation $w'' - (\frac{\gamma}{z} + \delta + z)w' + \frac{\alpha z - q}{z}w = 0$, times $z$, with $\alpha = N$;*
4) *$A$ is a nonzero constant: the triconfluent Heun equation $w'' + (\gamma + z)zw' + (\alpha z - q)w = 0$, with $\alpha = -N$.*

*In each case, $c_0 = -q$ is the accessory parameter, and $D_N(c_0) = 0$ is the classical polynomial condition on it. The identification is over $\mathbb{C}$: For a real equation, the affine change of variable may have to be complex, as in case 1 when $A$ has a pair of complex-conjugate zeros, and in case 3 when $a_1b_2 > 0$, since the normalization there requires $\rho^2 = -a_1/b_2$. In case 2, a real change of variable suffices. In*

*case 4, the scaling can be chosen real, since it requires $\rho^3 = a_0/b_2$, but the translation that moves a zero of $B$ to the origin is real only if $B$ has a real zero; thus a real change of variable exists in case 4 if and only if* $\operatorname{Disc} B \geq 0$. *For example, $A = 1$, $B = x^2 + 1$ admits no real change of variable to the triconfluent form.*

*Proof.* Move the zeros of $A$ to 0 and 1 (case 1) or to 0 (cases 2 and 3) by an affine map, normalize the leading coefficients by scaling and by multiplication with a constant, and compare coefficients. In case 4, the only affine freedom left after normalizing $A$ and $b_2$ is translation, which moves a zero of $B$ to the origin; this is why $B = \gamma z + z^2$ has no constant term. The termination condition is $c_1 = -Nb_2$ in each case. □

**Remark 3.8.** Let $A \not\equiv 0$; the case $A \equiv 0$ is a first-order degeneration and is not treated by division by $A$. Dividing by $A$, the equation becomes $y'' + \frac{B}{A}y' + \frac{c_0 - Nb_2x}{A}y = 0$. Since $\deg B = 2$ and $\deg A \leq 2$, $B/A$ is not $O(x^{-1})$ at infinity, so $\infty$ is always irregular; in particular (1.1) is never the (general) Heun equation. A simple zero $\alpha$ of $A$ is a regular singular point (or an ordinary point). If $\alpha$ is a double zero of $A$, then $(x - \alpha)B/A$ is analytic at $\alpha$ if and only if $B(\alpha) = 0$; thus $\alpha$ is irregular when $B(\alpha) \neq 0$ (the generic doubly-confluent case) and regular singular when $B(\alpha) = 0$, because $(x - \alpha)^2(c_0 - Nb_2x)/A$ is always analytic there. Example 4.3 is of the latter type.

### *3.5. Sensitivity of simple admissible values*

We use $\|\cdot\|_2$ for the Euclidean norm and the induced matrix norm, and $v^H = \bar{v}^T$.

**Proposition 3.9.** *Let $\tau \mapsto (a_2, b_1, I_1, I_2, I_3)(\tau)$ be differentiable on an interval, write $\widetilde{K}_N(\tau)$ for the matrix* (3.6) *and $D_N(c_0, \tau) = \det(\widetilde{K}_N(\tau) + c_0 I)$, and let $c_0(\tau)$ be a differentiable branch of simple zeros. If $v(\tau)$, $\ell(\tau)$ are right and left eigenvectors, $\widetilde{K}_N v = -c_0 v$ and $\ell^T \widetilde{K}_N = -c_0 \ell^T$, then*

$$\frac{dc_0}{d\tau} = -\frac{\partial_\tau D_N(c_0(\tau), \tau)}{\partial_{c_0} D_N(c_0(\tau), \tau)} = -\frac{\ell^T \dot{\widetilde{K}}_N v}{\ell^T v}, \qquad \left|\frac{dc_0}{d\tau}\right| \leq \kappa \|\dot{\widetilde{K}}_N\|_2, \quad \kappa = \frac{\|\ell\|_2 \|v\|_2}{|\ell^T v|}. \tag{3.13}$$

*Moreover $\kappa = \sup_{\|X\|_2 = 1} |\ell^T X v|/|\ell^T v|$, the supremum being attained at $X_* = \bar{\ell} v^H / (\|\ell\|_2 \|v\|_2)$.*

*Proof.* The first identity is implicit differentiation of $D_N(c_0(\tau), \tau) = 0$. The second follows by differentiating $\widetilde{K}_N v = -c_0 v$, multiplying by $\ell^T$, and using $\ell^T \widetilde{K}_N = -c_0 \ell^T$; here $\ell^T v \neq 0$ because the eigenvalue is simple. The number $\kappa$ is Wilkinson's condition number of a simple eigenvalue [28, 32]; the bound and the extremal matrix are standard. □

**Remark 3.10.** Because the pairing $\ell^T X v$ is bilinear rather than sesquilinear, the extremal matrix contains the entrywise conjugate $\bar{\ell}$; with $\bar{\ell}$ one has $\ell^T \bar{\ell} = \|\ell\|_2^2$ for every complex $\ell$. When $\ell$ and $v$ are real, $X_* = \ell v^T / (\|\ell\|_2 \|v\|_2)$. The perturbations relevant here are structured ones, $\dot{\widetilde{K}}_N = \sum_\theta \partial_\theta \widetilde{K}_N \dot{\theta}$ over $\theta \in \{a_2, b_1, I_1, I_2, I_3\}$, with $\partial_\theta \widetilde{K}_N$ read off from (3.6); the structured sensitivity is $|\ell^T \dot{\widetilde{K}}_N v|/|\ell^T v|$ and can be much smaller than $\kappa \|\dot{\widetilde{K}}_N\|_2$. With the normalization $\|\ell\|_2 = \|v\|_2 = 1$, as a simple admissible value approaches a multiple value at a fixed finite parameter point, $|\ell^T v| \to 0$ (Proposition 10.4), and therefore $\kappa \to \infty$.

## 4. The triangular slice $I_1 = I_2 = I_3 = 0$

**Proposition 4.1.** *If $I_1 = I_2 = I_3 = 0$, then*

$$D_N(c_0) = \prod_{j=0}^{N} (c_0 + j(b_1 + (j-1)a_2)). \tag{4.1}$$

*For real $a_2, b_1$, all admissible values are real, and the values indexed by $0 \le j < k \le N$ coincide exactly when $b_1 = -(j+k-1)a_2$.*

*Proof.* As $b_2 \neq 0$, $a_0 = a_1 = b_0 = 0$, so $K_N$ is lower bidiagonal with diagonal entries $d_j = j(b_1 + (j-1)a_2)$, and $d_j - d_k = (j-k)(b_1 + (j+k-1)a_2)$. □

**Remark 4.2.** On this slice $A = a_2x^2$ and $B = x(b_1 + b_2x)$, and when $a_2 \neq 0$ the point $x = 0$ is regular singular with indicial polynomial $a_2r(r-1) + b_1r + c_0$. Writing $p(r) = a_2r(r-1) + b_1r + c_0$, one has $L_Nx^j = p(j)x^j + b_2(j-N)x^{j+1}$, so the factors in (4.1) are exactly $p(j)$. For an admissible value of $c_0$, put

$$j_* = \max\{\, j \in \{0, \dots, N\} : \; p(j) = 0 \,\}.$$

Then the unique polynomial solution has the form $y = x^{j_*}Q(x)$ with $Q(0) \neq 0$ and $\deg Q = N - j_*$. Indeed, the coefficients of $y = \sum_{n=0}^{N} u_nx^n$ satisfy $p(n)u_n + b_2(n-1-N)u_{n-1} = 0$. At $n = j_*$ this gives $u_{j_*-1} = 0$, and downward induction gives $u_n = 0$ for all $n < j_*$. Since $u_N \neq 0$ (Proposition 2.3) and $p(n) \neq 0$ for $n > j_*$, the same relation read downward from $n = N$ shows that $u_N, u_{N-1}, \dots, u_{j_*}$ are all nonzero. When only one factor vanishes, $j_*$ is its index. When two factors indexed by $j < k$ vanish at the same admissible value, the polynomial solution starts at $x^k$: The two indicial roots are distinct integers in resonance, not coincident exponents, and only the larger one is the lowest exponent of the polynomial solution. If $a_2 = 0$, the factorization still holds, but the regular singular (Frobenius) interpretation is not asserted.

**Example 4.3.** Take $N = 3$, $a_2 = b_2 = 1$, and $a_0 = a_1 = b_0 = b_1 = 0$:

$$x^2y'' + x^2y' + (c_0 - 3x)y = 0, \qquad D_3(c_0) = c_0^2(c_0+2)(c_0+6). \tag{4.2}$$

Thus $c_0 = 0$ is a double zero. Recurrence (2.2) gives the unique monic polynomial solution $y = x^3 + 6x^2 + 6x$. By Lemma 10.1 the space of polynomial solutions in $\mathcal{P}_3$ is one-dimensional, although the algebraic multiplicity is two. Here $A$ has a double zero at 0 with $B(0) = 0$, so 0 is a regular singular point (Remark 3.8). At $c_0 = 0$ the indicial roots are 0 and 1, and the polynomial solution starts with $6x$, in accordance with Remark 4.2 ($j_* = 1$).

## 5. A reality criterion valid for every $N$

The explicit real-root loci of Sections 7–8 are limited to small $N$. When $A$ has a real zero, translation places it at the origin, $I_1 = 0$, and $\widetilde{K}_N$ becomes tridiagonal. This leads to a simple criterion for all $N$.

**Theorem 5.1.** *Let $a_2, b_1, I_2, I_3$ be real and $I_1 = 0$. If*

$$I_3 < 0 \qquad \text{and} \qquad (N-1)I_2 + I_3 < 0, \tag{5.1}$$

*then $D_N$ has $N+1$ distinct real zeros. In particular $\Delta_N \neq 0$, and there are no exceptional points in this region.*

*Proof.* For $I_1 = 0$, the matrix $\widetilde{K}_N$ in (3.6) is tridiagonal, and the products of its symmetric off-diagonal pairs are

$$(\widetilde{K}_N)_{j,j+1}(\widetilde{K}_N)_{j+1,j} = (j+1)(j-N)(I_2 j + I_3), \qquad 0 \le j \le N-1.$$

Since $j - N < 0$ and $I_2 j + I_3$ is affine in $j$, all these products are positive exactly when $I_2 j + I_3 < 0$ at $j = 0$ and $j = N - 1$, which is (5.1). A real tridiagonal matrix whose off-diagonal products are all positive is similar, through a real diagonal matrix, to a real symmetric tridiagonal matrix with nonzero off-diagonal entries (an unreduced Jacobi matrix), whose eigenvalues are real and simple. □

**Corollary 5.2.** *Let the coefficients be real and let $\alpha \in \mathbb{R}$ be a zero of $A$. If*

$$b_2 B(\alpha) < 0 \qquad \text{and} \qquad b_2\big((N-1)A'(\alpha) + B(\alpha)\big) < 0, \tag{5.2}$$

*then all $N + 1$ admissible values of $c_0$ are real and simple.*

*Proof.* Translate by $s = \alpha$. By (3.8) the zeros of $D_N$ are shifted by the real number $Nb_2\alpha$, and the translated equation has $a_0 = A(\alpha) = 0$, $a_1 = A'(\alpha)$, $b_0 = B(\alpha)$, so $I_1 = 0$, $I_2 = A'(\alpha)b_2$, $I_3 = B(\alpha)b_2$. Apply Theorem 5.1. □

**Remark 5.3.** Condition (5.1) is sufficient, not necessary. It is an explicit instance of the positive-definiteness hypothesis of [23]: A real tridiagonal matrix with positive off-diagonal products is diagonally similar to a real symmetric Jacobi matrix with nonzero off-diagonal entries, which is the recurrence matrix of a positive-definite moment functional (Favard's theorem). The criterion applies to the confluent, doubly-confluent, and biconfluent cases of Proposition 3.7 whenever the relevant zero of $A$ is real; it does not apply to the triconfluent case, where $A$ has no zeros. The root-centred coordinates used in Corollary 5.2 differ from the centred normal form of Corollary 3.4. Translating by $s = -b_1/(2b_2)$ removes the linear term of $B$, whereas translating by a zero $\alpha$ of $A$ removes the constant term of $A$. Starting from $b_1 = 0$, the translation $x = z + \alpha$ produces the new linear coefficient $2b_2\alpha$, so in general the two normalizations cannot be imposed simultaneously. Appendix A reports a numerical test on $10^4$ random parameter sets with $1 \le N \le 15$.

## 6. The quadratic sector $N = 1$

For $N = 1$, $c_1 = -b_2$ and

$$M_1(c_0) = \begin{pmatrix} c_0 & b_0 \\ -b_2 & b_1 + c_0 \end{pmatrix}, \qquad D_1(c_0) = c_0^2 + b_1 c_0 + I_3 = \hat{c}_0^2 + \hat{I}_3, \qquad \Delta_1 = b_1^2 - 4I_3 = \operatorname{Disc} B. \tag{6.1}$$

For real coefficients, the two admissible values are real exactly when $\operatorname{Disc} B \ge 0$ and coincide when $\operatorname{Disc} B = 0$. The coefficients of $A$ do not enter, since $y'' = 0$ on $\mathcal{P}_1$.

**Example 6.1.** For $y'' + (x^2 - 1)y' + (c_0 - x)y = 0$, one has $I_3 = -1$ and $D_1 = (c_0 - 1)(c_0 + 1)$, with solutions $y = x + 1$ for $c_0 = 1$ and $y = x - 1$ for $c_0 = -1$.

## 7. The cubic sector $N = 2$

For $N = 2$,

$$M_2(c_0) = \begin{pmatrix} c_0 & b_0 & 2a_0 \\ -2b_2 & b_1 + c_0 & 2(a_1 + b_0) \\ 0 & -b_2 & 2a_2 + 2b_1 + c_0 \end{pmatrix}, \qquad D_2(c_0) = c_0^3 + s_1 c_0^2 + s_2 c_0 + s_3, \tag{7.1}$$

$$s_1 = 2a_2 + 3b_1, \qquad s_2 = 2a_2 b_1 + 2b_1^2 + 2I_2 + 4I_3, \qquad s_3 = 4I_1 + 4(a_2 + b_1)I_3. \tag{7.2}$$

In the normal frame $b_1 = 0$ of Corollary 3.4, these become $s_1 = 2a_2$, $s_2 = 2\hat{I}_2 + 4\hat{I}_3$, $s_3 = 4\hat{I}_1 + 4a_2\hat{I}_3$ in the variable $\hat{c}_0 = c_0 + b_1$.

For a real cubic $f(t) = t^3 + s_1 t^2 + s_2 t + s_3$, the classical criterion [10] states that

$$\operatorname{Disc} f = s_1^2 s_2^2 - 4s_2^3 - 4s_1^3 s_3 - 27s_3^2 + 18s_1 s_2 s_3 \tag{7.3}$$

is positive if $f$ has three distinct real zeros, negative if $f$ has one real zero and a nonreal pair, and zero if $f$ has a multiple zero. Hence $\Delta_2$ is (7.3) with (7.2) substituted, and $V_2 = \{\Delta_2 = 0\}$.

### 7.1. Real-zero classification as $I_1$ varies

Fix real $a_2, b_1, I_2, I_3$ and set $\eta = s_1^2 - 3s_2$. Only $s_3$ depends on $I_1$, and $\Delta_2 = -27s_3^2 + (18s_1 s_2 - 4s_1^3)s_3 + s_1^2 s_2^2 - 4s_2^3$ has discriminant $16\eta^3$ as a quadratic in $s_3$. For $\eta \geq 0$, its roots are $s_3^{\pm} = (-2s_1^3 + 9s_1 s_2 \pm 2\eta^{3/2})/27$, so $\Delta_2 = 27(s_3^+ - s_3)(s_3 - s_3^-)$. Solving $s_3 = s_3^{\pm}$ for $I_1$ gives

$$I_1^{\pm} = \frac{-2s_1^3 + 9s_1 s_2 \pm 2\eta^{3/2}}{108} - (a_2 + b_1)I_3, \qquad I_1^- \leq I_1^+. \tag{7.4}$$

If $\eta > 0$, then $D_2'(c_0) = 3c_0^2 + 2s_1 c_0 + s_2$ has the two zeros $r_\pm = (-s_1 \pm \sqrt{\eta})/3$, and $s_3 = s_3^{\pm}$ is equivalent to $D_2(r_\pm) = 0$. Indeed $D_2(r_\pm) = s_3 + \frac{2s_1^3 - 9s_1 s_2 \mp 2\eta^{3/2}}{27}$, which one checks by reducing $r^3 + s_1 r^2 + s_2 r$ modulo $3r^2 + 2s_1 r + s_2$ and substituting $r = r_\pm$. Thus $I_1 = I_1^{\pm}$ is exactly the parameter value at which $r_\pm$ is a double zero.

**Proposition 7.1.** *Fix real $a_2, b_1, I_2, I_3$ and let $I_1 \in \mathbb{R}$.*

1) *If $\eta > 0$, $D_2$ has three distinct real zeros exactly when $I_1^- < I_1 < I_1^+$; at $I_1 = I_1^{\pm}$ it has a double real zero $r_\pm$ and a simple real zero; outside $[I_1^-, I_1^+]$ it has one real zero and a nonreal pair.*

2) *If $\eta = 0$, $D_2$ has a triple zero for $I_1 = s_1^3/108 - (a_2 + b_1)I_3$, and one real zero and a nonreal pair otherwise.*

3) *If $\eta < 0$, $D_2$ has one real zero and a nonreal pair for every $I_1$.*

*Proof.* Since $s_3 - s_3^{\pm} = 4(I_1 - I_1^{\pm})$, $\Delta_2 = 432(I_1^+ - I_1)(I_1 - I_1^-)$, and the classical criterion applies. At an endpoint in case 1, the zero is not triple, since $\operatorname{Disc} D_2' = 4\eta > 0$. In case 2, $s_2 = s_1^2/3$ and $\Delta_2 = -27(s_3 - s_1^3/27)^2$, with $D_2 = (c_0 + s_1/3)^3$ at equality. In case 3, $D_2' > 0$ on $\mathbb{R}$. □

**Proposition 7.2.** *$D_2$ has a triple zero if and only if $s_1^2 = 3s_2$ and $I_1 = s_1^3/108 - (a_2 + b_1)I_3$. On this codimension-two set, the hypersurfaces $I_1 = I_1^+$ and $I_1 = I_1^-$ meet.*

*Proof.* A monic cubic has a triple zero exactly when it equals $(c_0 + s_1/3)^3$. □

For $I_2 = I_3 = 0$, one finds $\eta = 4a_2^2 + 6a_2b_1 + 3b_1^2$ and

$$I_1^{\pm} = \frac{-a_2(2a_2 + 3b_1)(4a_2 + 3b_1) \pm (4a_2^2 + 6a_2b_1 + 3b_1^2)^{3/2}}{54}. \tag{7.5}$$

**Example 7.3.** Let $a_2 = b_1 = 1$, $I_2 = I_3 = 0$. Then $s_1 = 5$, $s_2 = 4$, $\eta = 13$, and $I_1^{\pm} = (-35 \pm 13\sqrt{13})/54$, i.e., $I_1^- \approx -1.5162$, $I_1^+ \approx 0.2199$. At $I_1^+$, $D_2$ has the double zero $c_0^* = (-5 + \sqrt{13})/3 \approx -0.4648$ and the simple zero $\approx -4.0704$. With $b_2 = 1$, $a_0 = I_1^+$, $a_1 = b_0 = 0$, the equation $(I_1^+ + x^2)y'' + (x + x^2)y' + (c_0 - 2x)y = 0$ has at $c_0 = c_0^*$ the unique monic polynomial solution $y_* = x^2 + \frac{7+\sqrt{13}}{3}x + \frac{5\sqrt{13}-1}{18}$, so $c_0^*$ is a defective eigenvalue of $-K_2$.

**Proposition 7.4.** *Fix $a_2, b_1, I_2, I_3$, and suppose $c_0^*$ is a zero of multiplicity exactly two of $D_2(\cdot, I_1^*)$, so $\nu := 3c_0^* + s_1 \neq 0$. Then the two zeros near $c_0^*$ satisfy*

$$c_{0,\pm}(I_1) = c_0^* \pm \Lambda\sqrt{I_1 - I_1^*} + O(I_1 - I_1^*), \qquad \Lambda^2 = -\frac{8}{6c_0^* + 2s_1}. \tag{7.6}$$

*Proof.* With $\delta = c_0 - c_0^*$ and $\varepsilon = I_1 - I_1^*$, and since $\partial_{I_1} D_2 = 4$, one has exactly $D_2(c_0^* + \delta, I_1^* + \varepsilon) = \nu\delta^2 + \delta^3 + 4\varepsilon$. Hence $\delta^2(\nu + \delta) = -4\varepsilon$, so $\delta = O(|\varepsilon|^{1/2})$ and $\delta^2 = -4\varepsilon/\nu + O(|\varepsilon|^{3/2})$. This is the case $m = 2$ of Theorem 10.5. □

For real data, the two zeros near $c_0^*$ are real when $\Lambda^2(I_1 - I_1^*) > 0$ and form a nonreal pair when $\Lambda^2(I_1 - I_1^*) < 0$.

## 8. The quartic sector $N = 3$

For $N = 3$,

$$M_3(c_0) = \begin{pmatrix} c_0 & b_0 & 2a_0 & 0 \\ -3b_2 & b_1 + c_0 & 2(a_1 + b_0) & 6a_0 \\ 0 & -2b_2 & 2a_2 + 2b_1 + c_0 & 6a_1 + 3b_0 \\ 0 & 0 & -b_2 & 6a_2 + 3b_1 + c_0 \end{pmatrix}, \tag{8.1}$$

and $D_3(c_0) = c_0^4 + s_1c_0^3 + s_2c_0^2 + s_3c_0 + s_4$ with

$$s_1 = 8a_2 + 6b_1, \tag{8.2}$$
$$s_2 = 10I_2 + 10I_3 + 12a_2^2 + 26a_2b_1 + 11b_1^2, \tag{8.3}$$
$$s_3 = 24I_1 + 24I_2a_2 + 18I_2b_1 + 48I_3a_2 + 30I_3b_1 + 12a_2^2b_1 + 18a_2b_1^2 + 6b_1^3, \tag{8.4}$$
$$s_4 = 72I_1a_2 + 36I_1b_1 + 18I_2I_3 + 9I_3^2 + 36I_3a_2^2 + 54I_3a_2b_1 + 18I_3b_1^2. \tag{8.5}$$

The substitution $c_0 = z - s_1/4$ gives $D_3 = z^4 + gz^2 + hz + k$ with

$$g = 10I_2 + 10I_3 - 12a_2^2 - 10a_2b_1 - \tfrac{5}{2}b_1^2, \tag{8.6}$$
$$h = 24I_1 + 16(-I_2a_2 - \tfrac{3}{4}I_2b_1 + \tfrac{1}{2}I_3a_2 + a_2^3 + a_2^2b_1 + \tfrac{1}{4}a_2b_1^2), \tag{8.7}$$
$$k = \tfrac{1}{16}(384I_1a_2 + 288I_2I_3 - 128I_2a_2^2 - 192I_2a_2b_1 - 72I_2b_1^2 + 144I_3^2 - 320I_3a_2^2$$

$$- 288I_3a_2b_1 - 72I_3b_1^2 + 128a_2^3b_1 + 176a_2^2b_1^2 + 72a_2b_1^3 + 9b_1^4), \tag{8.8}$$

and

$$\Delta_3 = 256k^3 - 128g^2k^2 + 144gh^2k - 27h^4 + 16g^4k - 4g^3h^2. \tag{8.9}$$

In the normal frame $b_1 = 0$, with the spectral variable $\hat{c}_0 = c_0 + \frac{3}{2}b_1$ in place of $c_0$, the coefficients reduce to $s_1 = 8a_2$, $s_2 = 10\hat{I}_2 + 10\hat{I}_3 + 12a_2^2$, $s_3 = 24\hat{I}_1 + 24\hat{I}_2a_2 + 48\hat{I}_3a_2$, $s_4 = 72\hat{I}_1a_2 + 18\hat{I}_2\hat{I}_3 + 9\hat{I}_3^2 + 36\hat{I}_3a_2^2$, and $g, h, k$ reduce accordingly.

**Theorem 8.1** (classical; [18, 24])**.** *Let $g, h, k \in \mathbb{R}$ and $\Delta_3 \neq 0$. Then $z^4 + gz^2 + hz + k$ has*

*(i) four distinct real zeros if and only if $\Delta_3 > 0$, $g < 0$ and $g^2 - 4k > 0$;*

*(ii) two distinct real zeros and a nonreal pair if and only if $\Delta_3 < 0$;*

*(iii) no real zeros if and only if $\Delta_3 > 0$ and ($g \geq 0$ or $g^2 - 4k \leq 0$).*

*Proof.* We sketch the argument for completeness. For simple zeros, the sign of the discriminant is $(-1)^p$, where $p$ is the number of nonreal conjugate pairs; so $\Delta_3 < 0$ gives (ii), and $\Delta_3 > 0$ leaves exactly two possibilities, four real zeros, or none. It therefore suffices to show that the inequalities in (i) hold for four distinct real zeros and fail when there are no real zeros; the "if" directions of (i) and (iii) then follow from this dichotomy.

If the zeros $z_1, \ldots, z_4$ are real and distinct, then $\sum z_j = 0$ gives $g = -\frac{1}{2}\sum z_j^2 < 0$. If $k \leq 0$, then $g^2 - 4k > 0$. If $k > 0$, the arithmetic–geometric mean inequality gives $(\sum z_j^2)^2 \geq 16k$, with equality only if all $|z_j|$ coincide, which is impossible for four distinct reals with zero sum; hence $g^2 > 4k$.

If there are no real zeros, write $z^4 + gz^2 + hz + k = (z^2 + uz + v)(z^2 - uz + w)$ with $v > u^2/4$, $w > u^2/4$, so $g = v + w - u^2$ and $k = vw$. If $u = 0$, then $g = v + w > 0$ and (i) fails. If $u \neq 0$ and $g < 0$, put $\sigma = v + w$, $d = v - w$. Then $\sigma - u^2/2 > |d|$ and $u^2/2 < \sigma < u^2$, and

$$g^2 - 4k = d^2 - 2\sigma u^2 + u^4 < (\sigma - \tfrac{u^2}{2})^2 - 2\sigma u^2 + u^4 = u^4\Big[(\tfrac{\sigma}{u^2})^2 - 3\tfrac{\sigma}{u^2} + \tfrac{5}{4}\Big] < 0,$$

because the bracket is negative for $\frac{1}{2} < \sigma/u^2 < \frac{5}{2}$. Thus (i) fails in this case as well. □

When $\Delta_3 = 0$, the multiplicity pattern follows from $\gcd(D_3, D_3')$. For larger $N$, the number of distinct real zeros of $D_N(\cdot; \omega)$ for fixed real parameters is given by a Sturm sequence [5].

## 9. The quintic sector $N = 4$

For $N = 4$, $D_4 = \det M_4(c_0)$ with

$$M_4(c_0) = \begin{pmatrix} c_0 & b_0 & 2a_0 & 0 & 0 \\ -4b_2 & b_1 + c_0 & 2(a_1 + b_0) & 6a_0 & 0 \\ 0 & -3b_2 & 2a_2 + 2b_1 + c_0 & 6a_1 + 3b_0 & 12a_0 \\ 0 & 0 & -2b_2 & 6a_2 + 3b_1 + c_0 & 12a_1 + 4b_0 \\ 0 & 0 & 0 & -b_2 & 12a_2 + 4b_1 + c_0 \end{pmatrix}, \tag{9.1}$$

and $D_4(c_0) = c_0^5 + s_1c_0^4 + s_2c_0^3 + s_3c_0^2 + s_4c_0 + s_5$, where

$$s_1 = 10(2a_2 + b_1), \tag{9.2}$$

$$s_2 = 30I_2 + 20I_3 + 108a_2^2 + 130a_2b_1 + 35b_1^2, \tag{9.3}$$

$$s_3 = 2(42I_1 + 138I_2a_2 + 69I_2b_1 + 134I_3a_2 + 60I_3b_1 + 72a_2^3 + 186a_2^2b_1 + 127a_2b_1^2 + 25b_1^3), \tag{9.4}$$

$$\begin{aligned} s_4 = 8(&108I_1a_2 + 42I_1b_1 + 9I_2^2 + 24I_2I_3 + 54I_2a_2^2 + 66I_2a_2b_1 + 18I_2b_1^2 + 8I_3^2 \\ &+ 108I_3a_2^2 + 110I_3a_2b_1 + 26I_3b_1^2 + 18a_2^3b_1 + 33a_2^2b_1^2 + 18a_2b_1^3 + 3b_1^4), \end{aligned} \tag{9.5}$$

$$\begin{aligned} s_5 = 32(&9I_1I_2 + 6I_1I_3 + 54I_1a_2^2 + 45I_1a_2b_1 + 9I_1b_1^2 + 21I_2I_3a_2 + 9I_2I_3b_1 \\ &+ 10I_3^2a_2 + 4I_3^2b_1 + 18I_3a_2^3 + 33I_3a_2^2b_1 + 18I_3a_2b_1^2 + 3I_3b_1^3). \end{aligned} \tag{9.6}$$

In the normal frame $b_1 = 0$, with the spectral variable $\hat{c}_0 = c_0 + 2b_1$ in place of $c_0$ and $\hat{I}_k$ in place of $I_k$, every term containing $b_1$ drops out. By (1.6), $\Delta_4 = \mathrm{Res}_{c_0}(D_4, \partial_{c_0}D_4)$.

**Theorem 9.1.** *The Galois group of $D_4$ over $K = \mathbb{Q}(a_2, b_1, I_1, I_2, I_3)$ is the symmetric group $S_5$. In particular there is no formula in radicals over K for the admissible values of $c_0$ in the quintic sector.*

*Proof.* Take $A = 1 + 2x + 3x^2$, $B = 1 - x + x^2$, i.e., $(a_2, b_1, I_1, I_2, I_3) = (3, -1, 1, 2, 1)$. Then

$$D_4 = c_0^5 + 50c_0^4 + 697c_0^3 + 3400c_0^2 + 11312c_0 + 24256.$$

This polynomial is irreducible modulo 29, hence irreducible over $\mathbb{Q}$. Its discriminant is

$$-2^{19} \cdot 3^8 \cdot 4310995176457 < 0,$$

and for a quintic with simple zeros, this means exactly one nonreal conjugate pair, so there are three real zeros. An irreducible polynomial of prime degree $p$ with exactly two nonreal zeros has Galois group $S_p$, since the group contains a $p$-cycle and complex conjugation acts as a transposition. The specialization is separable, so its Galois group over $\mathbb{Q}$ is isomorphic to a subgroup of the Galois group of $D_4$ over $K$, acting compatibly on the roots [16, Ch. VII, §2]. Therefore the generic group contains $S_5$, and it is contained in $S_5$ because $\deg D_4 = 5$. □

**Remark 9.2.** Theorem 9.1 concerns the generic member. Special parameter values may make $D_4$ reducible (for example on the slice of Proposition 4.1), and after fixing four parameters the equation $\Delta_4 = 0$ may be solvable in radicals in the fifth.

## 10. Multiple zeros and exceptional points

**Lemma 10.1.** *Assume $b_2 \neq 0$. Every eigenvalue of $K_N$ has geometric multiplicity one.*

*Proof.* Deleting the first row and last column of $K_N - \lambda I$ leaves an upper triangular matrix with diagonal $-b_2N, \ldots, -b_2$, whose determinant is $(-1)^N N!\, b_2^N \neq 0$. Hence $\mathrm{rank}(K_N - \lambda I) \geq N$ for every $\lambda$. This is the classical fact that unreduced Hessenberg matrices are nonderogatory. □

**Theorem 10.2.** *Assume $b_2 \neq 0$. If $c_0^*$ is a zero of $D_N$ of multiplicity $m \geq 2$, then $-c_0^*$ is a defective eigenvalue of $K_N$ with a single Jordan block of size m. Thus every point of $V_N$ is an exceptional point of $K_N$ in the sense of Definition 1.2. The multiplicity is at least m exactly when $D_N^{(i)}(c_0^*) = 0$ for $0 \leq i \leq m - 1$; for $N = 2$, multiplicity three is the triple-zero locus of Proposition 7.2.*

*Proof.* The algebraic multiplicity of $-c_0^*$ is $m$, and by Lemma 10.1 its geometric multiplicity is one, so there is one Jordan block of size $m$. □

**Remark 10.3.** For an analytic one-parameter family $\widetilde{K}_N(\gamma(\tau))$, Kato's exceptional values [13, Ch. II, §1] are those at which the number of distinct eigenvalues drops below its generic value along that family. By Lemma 10.1 every repeated eigenvalue of $\widetilde{K}_N$ is defective, so every Kato exceptional value of such a family is an exceptional point in the sense of Definition 1.2. The converse holds for analytic curves with generically simple spectrum, but not for curves along which a repeated eigenvalue persists. For example, $N = 3$, $A = (1+\tau)x^2$, $B = x^2$ gives $D_3 = c_0^2(c_0 + 2(1+\tau))(c_0 + 6(1+\tau))$; the eigenvalue 0 is defective for every $\tau$ near 0, but the number of distinct eigenvalues stays equal to three, so $\tau = 0$ is not a Kato exceptional value of this curve. Under the transversality hypothesis of Theorem 10.5, the multiple eigenvalue splits into simple branches, and the parameter is exceptional in both senses.

The next proposition makes the role of the Jordan structure explicit.

**Proposition 10.4.** *Let $\xi^* \in V_N$ be a point in invariant coordinates, let $c_0^*$ be a zero of $D_N(\cdot;\xi^*)$ of multiplicity $m \geq 1$, and let $v, \ell$ be right and left eigenvectors of $\widetilde{K}_N^* = \widetilde{K}_N(\xi^*)$ for $-c_0^*$. Then $\operatorname{adj}(\widetilde{K}_N^* + c_0^* I) = \theta\, v\ell^T$ with $\theta \neq 0$. Consequently, for any differentiable family $\widetilde{K}_N(\tau)$ with $\widetilde{K}_N(0) = \widetilde{K}_N^*$ and $F(c_0, \tau) = \det(\widetilde{K}_N(\tau) + c_0 I)$,*

$$\partial_\tau F(c_0^*, 0) = \theta\, \ell^T \dot{\widetilde{K}}_N(0) v. \tag{10.1}$$

*If $m \geq 2$, then $\ell^T v = 0$.*

*Proof.* Put $M = \widetilde{K}_N^* + c_0^* I$. By Lemma 10.1, $\operatorname{rank} M = N$, so $\operatorname{adj} M$ has rank one and $M \operatorname{adj} M = \operatorname{adj} M\, M = \det M \cdot I = 0$. Hence the columns of $\operatorname{adj} M$ lie in $\ker M = \mathbb{C}v$ and its rows in the left kernel $\mathbb{C}\ell^T$, which gives $\operatorname{adj} M = \theta v \ell^T$ with $\theta \neq 0$. Jacobi's formula $\frac{d}{d\tau} \det X = \operatorname{tr}(\operatorname{adj} X\, \dot{X})$ then gives (10.1). Finally, $\partial_{c_0} D_N(c_0^*) = \operatorname{tr} \operatorname{adj} M = \theta\, \ell^T v$, which vanishes when $m \geq 2$. □

In words, the next theorem says the following. At an exceptional point of order $m$, a single admissible value $c_0^*$ of multiplicity $m$ is present. Under an analytic perturbation of the coefficients of size $\tau$ satisfying the transversality condition (10.2), it splits into $m$ simple admissible values. These leave $c_0^*$ at the rate $|\tau|^{1/m}$, much faster than the rate $|\tau|$ of a simple value, and in $m$ directions that are equally spaced in angle. For real data and $m = 2$, this is the familiar square-root law, under which two real admissible values collide and become a complex-conjugate pair.

**Theorem 10.5.** *Let $U \subset \mathbb{C}$ be a neighbourhood of 0 and $\gamma : U \to \mathbb{C}^5$ holomorphic with $\gamma(0) = \xi^* \in V_N$. Let $c_0^*$ be a zero of $D_N(\cdot;\xi^*)$ of multiplicity exactly $m \geq 2$, put $F(c_0, \tau) = D_N(c_0; \gamma(\tau))$ and $a_m = \partial_{c_0}^m F(c_0^*, 0)/m! \neq 0$, and assume the transversality condition*

$$\partial_\tau F(c_0^*, 0) \neq 0, \qquad \textit{equivalently} \qquad \ell^T \dot{\widetilde{K}}_N(0) v \neq 0. \tag{10.2}$$

*Let $\mu = -\partial_\tau F(c_0^*, 0)/a_m$, fix $\lambda$ with $\lambda^m = \mu$, and let $\zeta = e^{2\pi i/m}$. Then there are $r, \rho > 0$ and a convergent power series*

$$\phi(t) = \sum_{j \geq 1} \beta_j t^j, \qquad \beta_1 = \lambda,$$

*such that for every $0 < |\tau| < r$, the polynomial $F(\cdot, \tau)$ has exactly $m$ zeros in $|c_0 - c_0^*| < \rho$. These zeros are simple, and for any $t$ with $t^m = \tau$, their unordered set is*

$$\{\, c_0^* + \phi(\zeta^k t) : \ k = 0, \dots, m-1 \,\}. \tag{10.3}$$

*On a slit punctured disc with a fixed holomorphic branch $t = \tau^{1/m}$, they can be labeled as*

$$c_{0,k}(\tau) = c_0^* + \phi(\zeta^k \tau^{1/m}) = c_0^* + \lambda \zeta^k \tau^{1/m} + O(|\tau|^{2/m}), \qquad k = 0, \dots, m-1;$$

*changing the branch permutes the labels. The same conclusion holds for real-analytic curves $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^5$, which extend holomorphically to a complex neighbourhood of 0.*

*Proof.* $F$ is holomorphic near $(c_0^*, 0)$, and $F(c_0^* + \delta, 0)$ has a zero of order exactly $m$ at $\delta = 0$. By the Weierstrass preparation theorem, $F(c_0^* + \delta, \tau) = u(\delta, \tau) W(\delta, \tau)$ near $(0, 0)$ with $u(0, 0) = a_m \neq 0$ and $W = \delta^m + w_{m-1}(\tau)\delta^{m-1} + \cdots + w_0(\tau)$, $w_i(0) = 0$. Setting $\delta = 0$ gives $w_0(\tau) = F(c_0^*, \tau)/u(0, \tau)$, so $w_0'(0) = \partial_\tau F(c_0^*, 0)/a_m = -\mu \neq 0$. The Newton polygon of $W$ in the $(\delta, \tau)$-exponent plane therefore consists of the single segment from $(0, 1)$ to $(m, 0)$, since every other monomial $\delta^j \tau^e$ with $1 \leq j \leq m-1$ has $e \geq 1$ and lies above it. This segment contains no interior lattice points and has slope $-1/m$, so by the Newton–Puiseux theorem, the $m$ zeros of $W$ near 0 form one cycle: There is a convergent series $\phi(t)$ in the covering parameter $t$, $t^m = \tau$, such that the zeros are $\phi(\zeta^k t)$, $k = 0, \dots, m-1$, and the leading coefficient $\beta_1$ is a root of the edge equation $\lambda^m - \mu = 0$. Replacing $t$ by $\zeta t$ permutes these zeros, so the unordered set does not depend on the choice of $t$, while a labeling requires a fixed branch of $\tau^{1/m}$ on a slit disc. The leading terms $\lambda \zeta^k t$ are distinct, so the $m$ zeros are simple for small $\tau \neq 0$. The equivalence in (10.2) is (10.1). □

**Remark 10.6.** By Theorem 10.2, the $m$ zeros in (10.3) are exactly the eigenvalues of $-\widetilde{K}_N(\gamma(\tau))$ that emerge from the size-$m$ Jordan block, and (10.2) is the Lidskii condition for this block [20, 21]; the expansion (10.3) agrees with the Lidskii–Vishik–Lyusternik theory in the generic case. In the cubic sector $N = 2$, if the selected zero is double and $\gamma(\tau) = (a_2, b_1, I_1^* + \tau, I_2, I_3)$, then $\partial_\tau F = 4$ and $a_m = 3c_0^* + s_1$ with $m = 2$, so (10.3) reduces to (7.6). (For $N \geq 3$, $\partial_{I_1} D_N$ depends on the parameters and on $c_0$.) If (10.2) fails, the full Newton polygon of $W$ and its edge equations determine the possible leading exponents and the subsequent expansions: There may be several cycles, a single cycle with a different exponent, or multiplicities that persist along the curve. For example, in the cubic sector with $a_2 = 1$, $b_1 = 0$, $I_2 = \frac{2}{3}$, $I_3 = 0$, and $I_1 = \frac{2}{27} + \frac{\tau^2}{4}$, one finds $D_2(c_0; \tau) = (c_0 + \frac{2}{3})^3 + \tau^2$. Here $\partial_\tau F(c_0^*, 0) = 0$, and the three zeros form a single cycle with leading exponent $2/3$. Thus the transverse exponent $1/m$ is not available without (10.2).

**Remark 10.7.** The standing assumption $b_2 \neq 0$ is essential for the spectral statements. If $b_2 = 0$, the termination conditions become $c_1 = c_2 = 0$, the subdiagonal (3.4) vanishes, the operator preserves every $\mathcal{P}_n$, and its restriction is upper triangular, so $D_N(c_0) = \prod_{j=0}^{N} (c_0 + d_j)$ with $d_j = j(b_1 + (j-1)a_2)$ as in (4.1). For $c_0 = -d_n$, the restriction to $\mathcal{P}_n$ is singular, so a nonzero polynomial solution of degree *at most* $n$ exists. A solution of exact degree $n$ is guaranteed if $d_n \neq d_j$ for all $0 \leq j < n$: After the leading coefficient is chosen, the remaining coefficients are determined successively. If diagonal entries coincide, additional compatibility conditions arise, and a solution of exact degree $n$ need not exist; for example, $y'' + y' = 0$ has $d_j = 0$ for all $j$ but no nonconstant polynomial solution. The formula (4.1) is also the limit of $D_N$ as $b_2 \to 0$ with the other coefficients fixed, since then $I_1, I_2, I_3 \to 0$. The Jordan structure, however, may change in this limit: For $b_2 = 0$, a repeated eigenvalue can be semisimple or defective. For example, with $A = x^2$, $B = 0$, $N = 1$, the matrix $K_1$ is the zero matrix, and $c_0 = 0$ is a semisimple double eigenvalue, whereas for every $b_2 \neq 0$ a double zero of $D_N$ is defective. Consequently, the conclusion that every multiple admissible value has a one-dimensional eigenspace requires $b_2 \neq 0$.

## 11. Relation to previous work

The existence problem for polynomial solutions of linear equations with polynomial coefficients is treated by linear-algebraic methods in [3, 4, 17], and determinant and recurrence criteria for second-order equations are given in [8, 9, 27]. For $A \not\equiv 0$, up to affine changes of variable, the family (1.1) is the union of the confluent Heun equations (Proposition 3.7), whose polynomial solutions are discussed in [25, Parts B–E] and [26]. From the QES point of view, $L_N$ is an element of the universal enveloping algebra of Turbiner's $\mathfrak{sl}_2$ [30, 31], and for $A = x(x-\alpha)$ Agboola and Zhang [2] gave the algebraization (3.12) and Bethe-ansatz formulas for the admissible values; related Bethe-ansatz treatments are in [19, 34]. The triconfluent subfamily $A \equiv$ const appears in [1], where constraint polynomials for quartic potentials are tabulated. Rabi-type models lead to the confluent case of Proposition 3.7 [22,35]; Example 11.1 works this out for the quantum Rabi model.

**Example 11.1** (Quantum Rabi model)**.** Consider $H = a^\dagger a + g(a + a^\dagger)\sigma_x + \Delta\sigma_z$ (units with $\omega = 1$, $g > 0$). In the Bargmann representation $a^\dagger \to z$, $a \to d/dz$, and in the basis in which $\sigma_x$ is diagonal, $H\psi = E\psi$ becomes

$$(z+g)f_1' + gzf_1 + \Delta f_2 = Ef_1, \qquad (z-g)f_2' - gzf_2 + \Delta f_1 = Ef_2.$$

Assume $\Delta \neq 0$; for $\Delta = 0$, the system decouples and is treated separately. Put $E = X - g^2$ and use the gauge $f_1 = e^{-gz}u$. The first equation gives $\Delta f_2 = e^{-gz}(Xu - (z+g)u')$, and substituting into the second gives an equation of the form (1.1) for $u$, with

$$A = z^2 - g^2, \qquad B = -2gz^2 + (1-2X)z + 2g^3 - g, \qquad C = X^2 - 2g^2X - \Delta^2 + 2gXz.$$

Thus $A$ has the two simple zeros $\pm g$ (the confluent case of Proposition 3.7), $b_2 = -2g$, and $c_1 = 2gX$. The termination condition $c_1 = -Nb_2$ is $X = N$, i.e., $E = N - g^2$, the energies of the Juddian isolated exact solutions [12, 15]. The remaining data are

$$a_2 = 1, \quad b_1 = 1 - 2N, \quad I_1 = -4g^4, \quad I_2 = 0, \quad I_3 = 2g^2(1-2g^2), \quad c_0 = N^2 - 2Ng^2 - \Delta^2,$$

and $D_N(c_0) = 0$ is the Juddian constraint polynomial in $(g, \Delta)$. For $\Delta \neq 0$, this scalar elimination is equivalent to the coupled system. It always contains the trivial factor $\Delta^2$. For $N = 1$, the remaining factor is $\Delta^2 + 4g^2 - 1$, and for $N = 2$ it is $\Delta^4 + (12g^2 - 5)\Delta^2 + 32g^4 - 32g^2 + 4$.

Corollary 5.2 does not apply directly, because one of its strict inequalities becomes an equality, but an adaptation of the proof of Theorem 5.1 after a block reduction shows that the nontrivial roots are real. At the zero $\alpha = -g$ of $A$, one has $A'(\alpha)b_2 = 4g^2$ and $B(\alpha)b_2 = -4(N-1)g^2$, so in root-centered coordinates $I_2 j + I_3 = 4g^2(j - N + 1)$. This is negative for $0 \le j \le N-2$ and zero for $j = N-1$, so the second inequality of (5.2) holds with equality. The vanishing entry $(\widetilde{K}_N)_{N-1,N}$ splits off the trivial root $\Delta^2 = 0$, and the remaining $N \times N$ block has positive off-diagonal products, so it is diagonally similar to a real symmetric Jacobi matrix with nonzero off-diagonal entries. Hence, for every $N \ge 1$ and $g > 0$, the degree-$N$ factor of the Juddian constraint polynomial has $N$ real simple roots in $d = \Delta^2$, consistent with Kuś's analysis [15]. These roots need not be positive, and only positive roots give nonzero real $\Delta$; for instance, for $N = 2$ and $g = 0.9$, the roots are $d \approx -4.908$ and $d \approx 0.188$. A root of the degree-$N$ factor may also coincide with the separate factor $d = 0$, as for $N = 1$, $g = \frac{1}{2}$. For $1 \le N \le 3$ and

$g = 0.23$, every positive root $\Delta$ of $D_N$ was checked against direct diagonalization of $H$ in a truncated Fock basis, which contains the eigenvalue $N - g^2$ with absolute error below $10^{-12}$, using 160 oscillator basis states and both spin components (Appendix A).

Moroz [22] introduced the termination conditions and the constraint polynomial in the present generality. When $a_0 \neq 0$, the recurrence (2.1) has four terms. When $A$ has a zero $\alpha$, translating $\alpha$ to the origin, as in Section 5, gives a frame with $a_0 = 0$, in which (2.1) is a three-term recurrence. For a real frame, $\alpha$ must be real, and this translation is unavailable when $A$ is a nonzero constant. Moroz and Miroshnichenko [23] related three-term recurrences of this kind to orthogonal polynomials and proved reality and simplicity of the zeros under positive definiteness of a moment functional; Theorem 5.1 gives an explicit parameter region in which this holds for every $N$. Table 1 summarizes which ingredients are classical and which are specific to this paper.

**Table 1.** Classical ingredients and the specific results of this paper.

| Ingredient | Earlier or classical source | This paper |
|---|---|---|
| Termination conditions, $D_N$ | [22]; confluent Heun literature [25, 26] | recalled |
| $\mathfrak{sl}_2$ form of $L_N$ | [30]; [2] for $A = x(x - \alpha)$ | general $A$, Prop. 3.6 |
| Scaling, translation, homogeneity | elementary | explicit reduction, Thm. 3.3 |
| Reality of zeros | [23] (positivity) | explicit criterion for all $N$, Thm. 5.1 |
| Real-root tests $N = 2, 3$ | cubic/quartic criteria [10, 18, 24] | applied in invariant coordinates |
| Galois group of $D_4$ | — | $S_5$, Thm. 9.1 |
| Nonderogatory Hessenberg matrices | classical | applied, Thm. 10.2 |
| Puiseux splitting, Lidskii condition | [13, 20, 21] | identification (10.2) |

## 12. Conclusions

After the termination conditions, polynomial solutions of degree $N$ of (1.1) correspond to zeros of the constraint polynomial $D_N$, the characteristic polynomial of an unreduced Hessenberg matrix. Scaling, translation, and homogeneity reduce the dependence of the discriminant locus to three essential parameters and give a normal form $b_2 = 1$, $b_1 = 0$, $a_2 \in \{0, 1\}$, which also identifies the family, for $A \not\equiv 0$, with the confluent Heun equations. When $A$ has a real zero, the translated matrix is tridiagonal, and the explicit inequalities (5.2) guarantee $N + 1$ real simple admissible values for every $N$; for the quantum Rabi model, the same argument shows that the nontrivial Juddian constraint roots are real in $\Delta^2$. For $N \leq 4$, the constraint polynomials are explicit; the cubic sector has an exact real-root interval, the quartic sector has a complete real-root test, and the quintic sector has Galois group $S_5$. Finally, because $K_N$ is nonderogatory, every multiple zero of $D_N$ is a defective eigenvalue with one Jordan block, and under an analytic perturbation satisfying the Lidskii condition, the admissible values split along a single Puiseux cycle of order equal to the block size.

## Availability of data and material

No external datasets were used. The symbolic and numerical verification code (`appendix_check.py`) is provided as supplementary material.

## Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The author declares that he has no competing interests.

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## A. Symbolic and numerical verification

The supplementary script `appendix_check.py` (SymPy 1.14, NumPy) verifies: the displayed coefficient lists of $D_2$, $D_3$, $D_4$ against the determinants; the depressed-quartic coefficients $g, h, k$ and the discriminant (8.9); the identities (3.8), (3.10), and (3.11) for $N \le 4$; the identity (3.12) for symbolic $N$; the data used in Theorem 9.1; the example in Section 1 showing that the inclusion $\partial_{\Omega_{\mathbb{R}}}\mathcal{R}_N \subseteq \Sigma_N \cap \Omega_{\mathbb{R}}$ can be strict for $N = 3$; the examples of Remarks 4.2, 10.3, 10.6, and 10.7; and the Rabi reduction of Example 11.1, including the comparison with direct diagonalization. It also runs a numerical test of Theorem 5.1. These checks support the displayed computations; they do not replace the proofs. The listing below is abbreviated; the full coefficient comparisons are in the supplementary file. The matrix $M_N(c_0)$ is built column by column: Its $(i, j)$ entry is the coefficient of $x^i$ in $L_N x^j$. Each test printed `True`; the Galois-group data were as stated in Theorem 9.1, and the random test of Theorem 5.1 reported no failures in $10^4$ cases with $1 \le N \le 15$.

```
import sympy as sp, numpy as np, random
x, c0, s, kap = sp.symbols("x␣c0␣s␣kappa")
a0, a1, a2, b0, b1, b2 = sp.symbols("a0␣a1␣a2␣b0␣b1␣b2")
I1, I2, I3 = sp.symbols("I1␣I2␣I3")

def constraint_matrix(N, A, B, c):
b2_ = sp.Poly(B, x).coeff_monomial(x**2)
M = sp.zeros(N+1, N+1)
for j in range(N+1):
Ly = sp.expand(A*sp.diff(x**j, x, 2) + B*sp.diff(x**j, x) + (c - N*b2_*x)*x**j)
for i in range(N+1):
M[i, j] = Ly.coeff(x, i)
return M

A = a0 + a1*x + a2*x**2; B = b0 + b1*x + b2*x**2
def D(N): # D_N in invariant coordinates; b2 cancels
d = sp.expand(constraint_matrix(N, A, B, c0).det())
```

```
    return sp.expand(sp.cancel(d.subs({a0: I1/b2**2, a1: I2/b2, b0: I3/b2})))
DN = {N: D(N) for N in range(1, 5)}

# (1) compare DN[2], DN[3], DN[4] with the displayed coefficient lists s_1,...,s_{N+1}
# (2) compare g, h, k and Delta_3 with sp.discriminant(DN[3], c0)
# (full lists are in the supplementary file appendix_check.py)
# (3) translation, normal form and homogeneity
J1 = I1 - b1*I2/2 + a2*b1**2/4; J2 = I2 - a2*b1; J3 = I3 - b1**2/4
for N in range(1, 5):
As = sp.expand(A.subs(x, x+s)); Bs = sp.expand(B.subs(x, x+s))
trans = sp.expand(constraint_matrix(N, A, B, c0).det()
- constraint_matrix(N, As, Bs, c0 - N*b2*s).det()) == 0
nf = DN[N].subs(b1, 0).subs({c0: c0 + N*b1/2, I1: J1, I2: J2, I3: J3},
simultaneous=True)
hom = DN[N].subs({c0: kap*c0, a2: kap*a2, b1: kap*b1, I1: kap**3*I1,
       I2: kap**2*I2, I3: kap**2*I3}, simultaneous=True)
print(N, trans, sp.expand(DN[N]-nf) == 0, sp.expand(hom - kap**(N+1)*DN[N]) == 0)

# (4) sl(2) identity of Proposition 3.6
Nn = sp.Symbol("N"); f = sp.Function("f")(x)
Jm = lambda u: sp.diff(u, x); J0 = lambda u: x*sp.diff(u, x) - Nn*u/2
Jp = lambda u: x**2*sp.diff(u, x) - Nn*x*u
L = A*sp.diff(f, x, 2) + B*sp.diff(f, x) + (c0 - Nn*b2*x)*f
R = (a2*J0(J0(f)) + a1*J0(Jm(f)) + a0*Jm(Jm(f)) + b2*Jp(f) + (b1+(Nn-1)*a2)*J0(f)
+ (b0 + Nn*a1/2)*Jm(f) + (c0 + Nn*b1/2 + a2*Nn*(Nn-2)/4)*f)
print(sp.expand(L - R) == 0)

# (5) data for the Galois-group theorem
P = sp.Poly(sp.expand(constraint_matrix(4, 1+2*x+3*x**2, 1-x+x**2, c0).det()), c0)
print(P.as_expr(), sp.factor_list(P.as_expr(), c0, modulus=29),
sp.discriminant(P), len(sp.real_roots(P)))
```

The random test of Theorem 5.1 builds $\widetilde{K}_N$ from (3.6) with $I_1 = 0$ and random $a_2, b_1, I_2, I_3$ satisfying (5.1), and checks with `numpy.linalg.eigvals` that the spectrum is real and simple.




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