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*Research article*

# On the number of spanning trees in the Cartesian product of star graphs and certain well-known graphs

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**Abstract:** The Cartesian product is a fundamental construction in graph theory that builds complex structures, such as grids and hypercubes, from simpler components. Many graph parameters, including the diameter and various topological indices used in mathematical chemistry, can be computed from the properties of the factors. In this paper, we studied the complexity, that is, the number of spanning trees, of the Cartesian product of a star graph with several important families of graphs. We first proved a general reduction: For any connected graph, the number of spanning trees of its product with a star was determined by two determinants of a shifted Laplacian matrix of that graph. Using this reduction together with linear algebra, block-matrix factorizations, and Chebyshev polynomials, we derived explicit closed-form formulas when the second factor is a path, cycle, complete graph, complete bipartite graph, fan, wheel, ladder, prism, Möbius ladder, gear graph, or hypercube.

**Keywords:** number of spanning trees; Cartesian product; star; path; cycle; complete; fan; wheel; ladder; prism; Möbius ladder; gear; hypercube; Chebyshev polynomials
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## 1. Introduction

A spanning tree of an undirected graph $G$ is a connected acyclic subgraph that contains all vertices of $G$. Spanning trees are of fundamental importance in computer networking, telecommunications, and transportation systems, since they form the minimal connected substructure that links every node without redundancy. By permitting exactly one active path between any two nodes, they eliminate loops and prevent broadcast storms, while backup spanning trees provide alternative routes when a link fails [1–3]. Beyond these applications, counting spanning trees, denoted by $\tau(G)$, is a central problem in graph theory and combinatorics. The number $\tau(G)$ measures the reliability and redundancy of a network, plays a role in the design of routing algorithms, and is tied to the Laplacian spectrum through Kirchhoff's Matrix-Tree Theorem [4,5].

The systematic enumeration of spanning trees began in 1847, when Kirchhoff [6] proved that $\tau(G)$ equals any cofactor of the Laplacian matrix $L(G) = D(G) - A(G)$. Here, $A(G)$ is the adjacency matrix of a graph $G$ on $n$ vertices, whose $(i,j)$ entry is $1$ if $v_i$ is adjacent to $v_j$ and $0$ otherwise, and $D(G)$ is the diagonal degree matrix whose $(i,i)$ entry is $\deg(v_i)$. Explicitly,

$$L(G)_{ij} = \begin{cases} \deg(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j. \\ 0 & \text{otherwise} \end{cases}$$

Classical counts include Cayley's formula $n^{n-2}$ for the complete graph on $n$ vertices [7], and the formula $p^{q-1}q^{p-1}$ for the complete bipartite graph $K_{p,q}$, due to Fiedler and Sedláček [8]. More generally, Lewis [9] showed that the complete $k$-partite graph with parts $V_1, \ldots, V_k$ of sizes $p_1, \ldots, p_k$, and $n = p_1 + \cdots + p_k$, has

$$n^{k-2} \prod_{i=1}^{k} (n - p_i)^{p_i - 1}$$

Spanning trees. Moreover, Daoud [10] derived exact formulas for the complexity of several products of complete and complete tripartite graphs.

Closed forms are also known for several other families. For the wheel $W_n$ on $n+1$ vertices $(n \geq 3)$, Myers [11] proved that

$$\tau(W_n) = \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2.$$

For the Möbius ladder $M_n$ on $2n$ vertices $(n \geq 2)$, Sedláček [12] proved that

$$\tau(M_n) = \frac{n}{2}\left[\left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n + 2\right].$$

For the prism $R_n$ on $2n$ vertices $(n \geq 3)$, Boesch and Bogdanowicz [13] proved that

$$\tau(R_n) = \frac{n}{2}\left[\left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n - 2\right].$$

For the ladder $L_n$ on $2n$ vertices $(n \geq 1)$, Sedláček [14] obtained

$$\tau(L_n) = \frac{\left[\left(2+\sqrt{3}\right)^n - \left(2-\sqrt{3}\right)^n\right]}{2\sqrt{3}}.$$

For the fan $F_n$ on $n+1$ vertices $(n \geq 2)$, Bogdanowicz [15] expressed the count through Fibonacci numbers:

$$\tau(F_n) = \frac{\left[\left(3+\sqrt{5}\right)^n - \left(3-\sqrt{5}\right)^n\right]}{2^n\sqrt{5}}.$$

For the gear graph $G_n$, obtained by inserting a vertex between every pair of adjacent outer vertices of the wheel $W_n$ $(n \geq 2)$, Daoud [16] proved that

$$\tau(G_n) = \left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n - 2.$$

Finally, for the $n$-dimensional hypercube $Q_n$, the Laplacian eigenvalues are $2k$ with multiplicity $\binom{n}{k}$, and the Matrix-Tree Theorem gives [17]

$$\tau(Q_n) = 2^{-n}\prod_{k=1}^{n}(2k)^{\binom{n}{k}} = 2^{2^n-n-1}\prod_{k=2}^{n}k^{\binom{n}{k}}.$$

Several techniques have been developed for computing $\tau(G)$. Spectral methods, based on the eigenvalues of the Laplacian matrix, are among the most elegant and general. Let $G$ be a connected graph on $n$ vertices with Laplacian eigenvalues $0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_n$. Kelmans and Chelnokov [18] showed that

$$\tau(G) = \frac{1}{n}\prod_{i=2}^{n}\mu_i.$$

Temperley [19] proved that $\tau(G) = \frac{\det(L+J)}{n^2}$, where $J$ is the $n \times n$ matrix all of whose entries are 1. Another classical technique is the deletion–contraction method, which computes $\tau(G)$ recursively through

$$\tau(G) = \tau(G-e) + \tau(G/e)$$

for any edge $e$, where $G-e$ and $G/e$ denote the graphs obtained by deleting and contracting $e$; contraction may create parallel edges, so multigraphs must be allowed [20,21]. A further approach uses electrically equivalent transformations, in which edge weights represent conductances and $\tau(G)$ is obtained by tracking the multiplicative factors produced by merging parallel edges, series reduction, and triangle–star $(\Delta - Y)$ conversions while the graph is progressively simplified [22,23]. Finally, modular decomposition partitions a graph recursively into modules; for suitable graph classes, the resulting decomposition tree reduces the computation to a product of values attached to the vertices, yielding an efficient, near-linear-time algorithm [24].

## 2. Essential proof tools

In this section, we list some properties of Chebyshev polynomials that are essential to our computations; see Boesch [13].

Chebyshev polynomials of the second kind, denoted by $U_n(x)$ for $n \in \mathbb{N}_0$, are orthogonal polynomials with respect to the weight function $w(x) = \sqrt{1-x^2}$ on the interval $[-1,1]$. Fundamentally, $U_n(x)$ can be defined via its exact product factorization over its $n$ distinct roots $x_k = \cos\left(\frac{k\pi}{n+1}\right)$ for $k = 1,2,\dots,n$:

$$U_n(x) = 2^n \prod_{k=1}^{n} \left(x - \cos\frac{k\pi}{n+1}\right).$$

By expressing the cosine factors via Euler's identity $2\cos\theta = e^{i\theta} + e^{-i\theta}$ and evaluating the corresponding product over the roots of unity in the complex plane, this finite product evaluates explicitly to the closed-form algebraic representation (Binet-like formula):

$$U_n(x) = \frac{\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}}{2\sqrt{x^2-1}}.$$

For $|x| \le 1$, setting $x = \cos\theta$ simplifies this explicit algebraic form directly to the classical trigonometric relation:

$$U_n(\cos\theta) = \frac{\sin\big((n+1)\theta\big)}{\sin\theta}.$$

Furthermore, the sequence satisfies the standard linear three-term recurrence relation:

$$U_0(x) = 1, U_1(x) = 2x, U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x) \quad \text{for } n \ge 1.$$

Chebyshev polynomials of the first kind, denoted by $T_n(x)$ for $n \in \mathbb{N}_0$, are orthogonal polynomials with respect to the weight function $w(x) = (1-x^2)^{-1/2}$on $[-1,1]$. Fundamentally, $T_n(x)$ is defined via its product factorization over its $n$ distinct roots $x_j = \cos\left(\frac{(2j+1)\pi}{2n}\right)$ for $j = 0,1,\dots,n-1$:

$$T_n(x) = 2^{n-1} \prod_{j=0}^{n-1} \left(x - \cos\frac{(2j+1)\pi}{2n}\right).$$

By expressing the cosine terms via complex exponentials $2\cos\theta = e^{i\theta} + e^{-i\theta}$ and identifying $e^{i\frac{(2j+1)\pi}{2n}}$ as the $2n$-th roots of $-1$, this finite product evaluates explicitly to the closed-form algebraic representation (Binet-like formula):

$$T_n(x) = \frac{\left(x+\sqrt{x^2-1}\right)^{n} + \left(x-\sqrt{x^2-1}\right)^{n}}{2}.$$

For $x = \cos\theta$, this algebraic form simplifies directly to the classic trigonometric relation:

$$T_n(\cos\theta) = \cos(n\theta)$$

The sequence satisfies the standard three-term linear recurrence relation:

$$T_0(x) = 1, T_1(x) = x, T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad \text{for } n \geq 1.$$

The trigonometric product identity $\Omega_n(y)$, defined for a real parameter $y$ and positive integer $n \in \mathbb{N}$, naturally arises in the spectral decomposition of symmetric block-circulant matrices. Fundamentally, $\Omega_n(y)$ is defined via its finite product factorization over odd-indexed cosine phase factors:

$$\Omega_n(y) = \prod_{j=0}^{n-1}\left(y - 2\cos\frac{(2j+1)\pi}{n}\right).$$

By expressing the cosine terms via Euler's identity $2\cos\theta = e^{i\theta} + e^{-i\theta}$ and identifying $e^{i\frac{(2j+1)\pi}{n}}$ as the $n$-th roots of $-1$ in the complex plane, this finite product evaluates explicitly to the closed-form algebraic representation (Binet-like formula):

$$\Omega_n(y) = \frac{\left(y + \sqrt{y^2-4}\right)^n + \left(y - \sqrt{y^2-4}\right)^n + 2^{n+1}}{2^n}.$$

In terms of Chebyshev polynomials of the first kind $T_n(x)$, setting the change of variable $x = y/2$ connects the product directly to $T_n(x)$ via the exact functional identity:

$$\Omega_n(y) = 2\,T_n\left(\frac{y}{2}\right) + 2$$

For $|y| > 2$, setting $y = 2\cosh\theta$ yields the hyperbolic counterpart $\Omega_n(2\cosh\theta) = 2\cosh(n\theta) + 2 = 4\cosh^2\left(\frac{n\theta}{2}\right)$, illustrating the exponential growth behavior of the identity as $n \to \infty$.

## 3. Major results

The Cartesian product of two graphs $G_1$ and $G_2$, denoted $G_1 \square G_2$, is defined on the vertex set $V(G_1) \times V(G_2)$, where two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent if and only if either $u_1 = u_2$ with $v_1 \sim v_2$ in $G_2$, or $v_1 = v_2$ with $u_1 \sim u_2$ in $G_1$ [25]. Equivalently, $G_1 \square G_2$ can be visualized by replacing each vertex of $G_1$ with an isomorphic copy of $G_2$ and connecting corresponding vertices across adjacent copies, a construction that is commutative up to isomorphism, satisfying $G_1 \square G_2 \cong G_2 \square G_1$. Far beyond a purely combinatorial construction, the Cartesian product serves as the natural mathematical model for a broad spectrum of real-world structures, from parallel computing architectures, where hypercube and torus interconnection networks exploit the product structure to optimize data routing and fault tolerance to mathematical chemistry, where topological indices of product graphs encode structural properties of composite molecular compounds, to network design, where the layered regularity of $G_1 \square G_2$ enables the construction of highly connected,

low-diameter systems with provable reliability guarantees. In this work, we exploit the rich spectral structure of the Cartesian product, governed by the identity $L(G_1 \square G_2) = L(G_1) \otimes I_{n_2} + I_{n_1} \otimes L(G_2)$, together with tools from linear algebra, matrix theory, and the theory of Chebyshev polynomials, to derive exact closed-form expressions for the complexity $\tau(G_1 \square G_2)$ of the Cartesian product of the star graph $S_m = K_{1,m}$ with each of the following significant graph families: the path $P_n$, cycle $C_n$, complete $K_n$, complete bipartite $K_{n,n}$, fan $F_n = K_1 \vee P_n$, wheel $W_n = K_1 \vee C_n$, ladder $L_n = K_2 \square P_n$, prism $R_n = K_2 \square C_n$, Möbius ladder $M_n$, formed from an $n$-cycle by adding rungs connecting diametrically opposite vertices, and gear graph $G_n$, derived from the wheel graph $W_n$ by subdividing each edge of its outer cycle.

**Theorem1.** For any connected graph $H$ of order $n$ with Laplacian matrix $L$, and any integer $m \geq 1$, the number of spanning trees of the Cartesian product $S_m \square H$ is given by

$$\tau(S_m \square H) = \frac{\tau(H)}{m+1} \cdot (det(I + L))^{m-1} \cdot det((m+1)I + L)$$

*Proof*. The proof rests on the Matrix-Tree Theorem of Kirchhoff, together with two standard facts concerning the Laplacian spectrum of the star graph and the Laplacian spectrum of a Cartesian product of graphs. Recall first that for any connected graph $G$ on $N$ vertices, if the eigenvalues of its Laplacian matrix $L(G)$ are ordered as

$$0 = \nu_1 < \nu_2 \leq \nu_3 \leq \cdots \leq \nu_N$$

then the number of spanning trees of $G$ is given by

$$\tau(G) = \frac{1}{N} \prod_{i=2}^{N} \nu_i$$

that is, one over the order of the graph times the product of all the nonzero Laplacian eigenvalues. It is this identity that we shall apply to the graph $S_m \square H$, once its Laplacian spectrum has been determined in terms of the spectrum of $H$.

We therefore begin by recalling the Laplacian spectrum of the star $S_m$, which consists of $m+1$ vertices, one central vertex of degree $m$ and $m$ leaves of degree one. Its Laplacian eigenvalues are known to be 0, together with the eigenvalue 1 occurring with multiplicity $m-1$, and finally the eigenvalue $m+1$ occurring once:

$$0, 1, 1, \ldots, 1(m - 1 \text{ times}), m + 1.$$

This can be seen by noting that $L(S_m)$ has rank $m$, since $S_m$ is connected, and that its trace equals the sum of the vertex degrees, namely $2m$, a value which is indeed recovered as

$$0 + (m-1) \cdot 1 + (m+1) = 2m$$

thereby confirming both the values and their multiplicities.

Having established the spectrum of stars, we now turn to the second ingredient, namely the behaviors of the Laplacian spectrum under the Cartesian product. If $G$ and $H$ are graphs with Laplacian eigenvalues $\lambda_1, \ldots, \lambda_p$ and $\mu_1, \ldots, \mu_n$, respectively, then, because the eigenvectors of $L(G \square H)$ arise as tensor products of the eigenvectors of $L(G)$ and $L(H)$, the eigenvalues of $L(G \square H)$ are precisely all the pairwise sums

$$\lambda_i + \mu_j$$

taken over every choice of $i$ and $j$. Applying this fact with $G = S_m$, and writing $\mu_1 = 0 < \mu_2 \leq \mu_3 \leq \cdots \leq \mu_n$ for the Laplacian eigenvalues of $H$, we obtain the full spectrum of $S_m \square H$, a graph on $N = (m+1)n$ vertices, by adding each of the three types of eigenvalues of $K_{1,m}$ to every eigenvalue of $H$ in turn. Doing so, the eigenvalue $0$ of the star contributes to the values

$$\mu_1, \mu_2, \ldots, \mu_n$$

each of the $m-1$ copies of the eigenvalue $1$ contributes the values $1+\mu_1, 1+\mu_2, \ldots, 1+\mu_n$ and the eigenvalue $m+1$ contributes the values $(m+1)+\mu_1, (m+1)+\mu_2, \ldots, (m+1)+\mu_n$.

Since $H$ is connected, the eigenvalue $\mu_1 = 0$ is simple, so that among all $(m+1)n$ values listed above, exactly one of them, namely $0+\mu_1$, is equal to zero, in accordance with the fact that $S_m \square H$ is connected and hence possesses a single zero Laplacian eigenvalue. We may therefore form the product of all the remaining nonzero eigenvalues, and this product splits naturally into three factors corresponding to the three blocks described above. From the first block, omitting the single zero term $\mu_1$, we obtain

$$\prod_{j=2}^{n} \mu_j = n \cdot \tau(H),$$

which by the Matrix-Tree Theorem applied directly to $H$ equals $n \cdot \tau(H)$. From the second block, recalling that

$$det(I + L) = \prod_{j=1}^{n} (1 + \mu_j).$$

The $m-1$ repeated copies together contribute a factor of $(det(I + L))^{m-1}$. From the third block, in the same way, we obtain:

$$\prod_{j=1}^{n} ((m+1) + \mu_j) = det((m+1)I + L).$$

Multiplying these three contributions together, the product of all nonzero Laplacian eigenvalues of $S_m \square H$ is found to equal

$$n \cdot \tau(H) \cdot (\det(I + L))^{m-1} \cdot det((m+1)I + L).$$

It remains only to invoke the Matrix-Tree Theorem once more, this time for graph $S_m \square H$, according to which $\tau(S_m \square H)$ is obtained by dividing the above product by the total number of vertices $N = (m+1)n$. Carrying out this division, and cancelling the common factor of $n$ that appears in the numerator and the denominator, we arrive at

$$\tau(S_m \square H) = \frac{\tau(H)}{m+1} \cdot (det(I + L))^{m-1} \cdot det((m+1)I + L).$$

The proof is complete. ■

**Theorem 2.** For $m \geq 0$ and $n \geq 2$, the number of spanning trees of the Cartesian product $S_m \square P_n$

is given by

$$\tau(S_m \square P_n) = \frac{1}{2^{mn} \cdot 5^{(m-1)/2} \cdot \sqrt{(m+1)(m+5)}} \cdot [(3+\sqrt{5})^n - (3-\sqrt{5})^n]^{m-1}$$

$$\cdot \left[\left(m+3+\sqrt{(m+1)(m+5)}\right)^n - \left(m+3-\sqrt{(m+1)(m+5)}\right)^n\right].$$

*Proof.* We first dispose of the degenerate case $m = 0$: Here, $S_0$ consists of a single vertex, so that $S_0 \square P_n$ is isomorphic to $P_n$, and since $P_n$ is a tree, we have $\tau(S_0 \square P_n) = 1$, which agrees with the stated formula as $m = 0$. We may therefore assume throughout that $m \geq 1$.

Since $S_m$ is precisely the star $K_{1,m}$, and $P_n$ is a tree so that $\tau(P_n) = 1$, applying the general Theorem 1 established previously for $\tau(S_m \square H)$ immediately reduces it to

$$\tau(S_m \square P_n) = \frac{1}{m+1} \cdot (det(I + L(P_n)))^{m-1} \cdot det\big((m+1)I + L(P_n)\big),$$

where $L(P_n)$ denotes the Laplacian matrix of the path $P_n$. It remains only to evaluate the two determinants on the right-hand side explicitly.

The Laplacian Matrix of a path graph $P_n$ is defined as:

$$L(P_n) = \begin{pmatrix} 1 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 1 \end{pmatrix}_{n\times n}$$

Adding $x$ to every diagonal entry, we have

$$xI + L(P_n) = \begin{pmatrix} x+1 & -1 & 0 & \cdots & \cdots 0 & 0 \\ -1 & x+2 & -1 & \ddots & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & x+2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & x+1 \end{pmatrix}$$

Let $D_n(x) = \det\big(xI + L(P_n)\big)$.

We can expand along the last row to obtain a three-term recurrence relation.

Starting from the base cases $D_1 = x+1$, $D_2 = (x+1)(x+2) - 1 = x^2 + 3x + 1$, the recurrence for the interior terms is

$$D_k = (x+2)D_{k-1} - D_{k-2}\,,\ 2 \leq k \leq n-1,$$

and the final step, where the diagonal entry becomes $x+1$ instead of $x+2$, gives

$$D_n = (x+1)D_{n-1} - D_{n-2}.$$

To solve this recurrence explicitly, let $y = x+2$ and define the Chebyshev-like polynomials $u_k(y)$ by

$$u_0 = 1, u_1 = y, u_k = yu_{k-1} - u_{k-2}$$

Working through the recurrence with these building blocks yields the closed form

$$D_n(x) = u_n(x+2) - 2u_{n-1}(x+2) + u_{n-2}(x+2)$$

An equivalent and often cleaner way to express the same result comes from the fact that $L(P_n)$ is symmetric, so its eigenvalues are known explicitly:

$$\lambda_k = 2 - 2cos\frac{k\pi}{n}, k = 0,1, \dots, n-1.$$

Since $\det(xI + L(P_n)) = \prod_k (x + \lambda_k)$, this gives the compact product formula

$$\det\big(xI + L(P_n)\big) = \prod_{k=0}^{n-1}\left(x + 2 - 2cos\frac{k\pi}{n}\right) = xU_{n-1}\left(\frac{\mathrm{x}+2}{2}\right).$$

Applying this with $x = 1$ gives

$$det\big(I + L(P_n)\big) = U_{n-1}\left(\frac{3}{2}\right),$$

and applying it with $x = m + 1$ gives

$$\det\big((m+1)I + L(P_n)\big) = (m+1)U_{n-1}\left(\frac{m+3}{2}\right).$$

Substituting these two evaluations into the expression obtained above for $\tau(S_m \square P_n)$, the factor of $m + 1$ cancels against the denominator, leaving

$$\tau(S_m \square P_n) = (U_{n-1}(3/2))^{m-1} \cdot U_{n-1}((m+3)/2).$$

It remains to express each Chebyshev value in closed form. For any $x > 2$, the polynomial $U_{n-1}(\mathrm{x}/2)$ admits the Binet-type representation

$$U_{n-1}\left(\frac{x}{2}\right) = \frac{\left(x + \sqrt{x^2 - 4}\right)^n - \left(x - \sqrt{x^2 - 4}\right)^n}{2^n\sqrt{x^2 - 4}}.$$

Applying this at $x = 3$, where $x^2 - 4 = 5$, gives

$$U_{n-1}(3/2) = \frac{(3+\sqrt{5})^n - (3-\sqrt{5})^n}{2^n\sqrt{5}},$$

and applying it at $x = m + 3$, where $x^2 - 4 = (m+1)(m+5)$, gives

$$U_{n-1}((m+3)/2) = \frac{(m+3+\sqrt{(m+1)(m+5)})^n - (m+3-\sqrt{(m+1)(m+5)})^n}{2^n\sqrt{(m+1)(m+5)}}$$

Substituting these two closed forms into the expression for $\tau(S_m \square P_n)$ obtained above, we have

$$\tau(S_m \square P_n) = \frac{1}{2^{mn} \cdot 5^{(m-1)/2} \cdot \sqrt{(m+1)(m+5)}} \cdot [(3+\sqrt{5})^n - (3-\sqrt{5})^n]^{m-1} \cdot [(m+3+\sqrt{(m+1)(m+5)})^n - (m+3-\sqrt{(m+1)(m+5)})^n].$$

The proof is complete. ■

**Theorem 3.** For $m \geq 0$ and $n \geq 3$, the number of spanning trees of the Cartesian product $S_m \square C_n$ is given by

$$\tau(S_m \square C_n) = \frac{n}{2^{mn} \cdot (m+1)} \cdot [(3+\sqrt{5})^n + (3-\sqrt{5})^n - 2^{n+1}]^{m-1} \cdot [(m+3+\sqrt{(m+1)(m+5)})^n + (m+3-\sqrt{(m+1)(m+5)})^n - 2^{n+1}].$$

*Proof*. We first dispose of $m = 0$: Here, $S_0$ is a single vertex, so $S_0 \square C_n$ is isomorphic to $C_n$. A cycle on $n$ vertices has exactly $n$ spanning trees. In particular, $\tau(S_0 \square C_n) = n$, consistent with the stated formula as $m = 0$, so we may therefore assume $m \geq 1$.

Applying the general theorem established previously for $\tau(S_m \square H)$ immediately reduces it to

$$\tau(S_m \square C_n) = \frac{n}{m+1} \cdot (det(I + L(C_n)))^{m-1} \cdot det((m+1)I + L(C_n))$$

where $L(C_n)$ denotes the Laplacian matrix of the cycle $C_n$. It remains only to evaluate the two determinants on the right-hand side explicitly.

The Laplacian Matrix of a cycle graph $C_n$ is defined as

$$L(C_n) = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}_{n \times n}$$

Adding $x$ to every diagonal entry gives

$$xI + L(C_n) = \begin{pmatrix} x+2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & -1 \\ -1 & 0 & \cdots & 0 & -1 & x+2 \end{pmatrix}$$

Since the eigenvalues of $xI + L$ are $x + \lambda$, where $\lambda$ ranges over the eigenvalues of $L$, we begin by finding the eigenvalues of $L(\mathrm{C_n})$. Because $L(\mathrm{C_n})$ is a circulant matrix, we try the vector $v_j = \omega^{jk}$, where $\omega = e^{2\pi i/n}$ and $\omega^n = 1$. Row $j$ of the equation $Lv = \lambda v$, then we get

$$-v_{j-1} + 2v_j - v_{j+1} = (2 - \omega^{-k} - \omega^k)v_j = \left(2 - 2cos\frac{2k\pi}{n}\right)v_j.$$

So, the eigenvalues are

$$\lambda_k = 2 - 2cos\frac{2k\pi}{n}.$$

Multiplying the eigenvalues of $xI + L$ then gives the determinant:

$$\det\big(xI + L(C_n)\big) = \prod_{k=0}^{n-1}(x + 2 - 2cos\frac{2k\pi}{n}) = 2T_n\left(\frac{x+2}{2}\right) - 2.$$

Applying this with $x = 1$ gives

$$det\big(I + L(C_n)\big) = 2\,T_n\left(\frac{3}{2}\right) - 2 = \frac{(3 + \sqrt{5})^n + (3 - \sqrt{5})^n - 2^{n+1}}{2^n}$$

and applying it with $x = m + 1$ gives

$$det\big((m + 1)I + L(C_n)\big) = 2\,T_n\left(\frac{\mathrm{m}+3}{2}\right) - 2$$

$$= \frac{\big(m + 3 + \sqrt{(m + 1)(m + 5)}\big)^n + \big(m + 3 - \sqrt{(m + 1)(m + 5)}\big)^n - 2^{n+1}}{2^n}.$$

Substituting these two evaluations into the reduced formula for $\tau(S_m \square C_n)$ obtained above, we get

$$\tau(S_m \square C_n) = \frac{n}{2^{mn}\cdot(m+1)} \cdot [(3 + \sqrt{5})^n + (3 - \sqrt{5})^n - 2^{n+1}]^{m-1} \cdot [(m + 3 +$$

$$\sqrt{(m + 1)(m + 5)})^n + (m + 3 - \sqrt{(m + 1)(m + 5)})^n - 2^{n+1}].$$

The proof is complete. ■

**Theorem 4.** For $m \geq 0$ and $n \geq 1$, the number of spanning trees of the Cartesian product $S_m \square K_n$ is given by

$$\tau(S_m \square K_n) = n^{n-2} \cdot (n + 1)^{(n-1)(m-1)} \cdot (m + n + 1)^{n-1}$$

*Proof*. As before, we first dispose of $m = 0$: Here, $S_0$ is a single vertex, so $S_0 \square K_n$ is isomorphic to $K_n$, and by Cayley's formula $\tau(K_n) = n^{n-2}$, which agrees with the stated formula as $m = 0$, since $(n + 1)^{-(n-1)}$ and $(n + 1)^{(n-1)}$ cancel. We may therefore assume m ≥ 1.

Applying the general theorem established previously for $\tau(S_m \square \mathrm{H})$ immediately reduces it to

$$\tau(S_m \square K_n) = \frac{n^{n-2}}{m + 1} \cdot (det(I + L(K_n)))^{m-1} \cdot det((m + 1)I + L(K_n))$$

where $L(K_n)$ denotes the Laplacian matrix of the complete graph $K_n$. It remains only to evaluate the two determinants on the right-hand side explicitly.

$$L(K_n) = \begin{pmatrix} n-1 & -1 & \cdots & \cdots & -1 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 \\ -1 & \cdots & \cdots & -1 & n-1 \end{pmatrix}_{n\times n} = nI - J,$$

where $J$ for the $n \times n$ all-ones matrix.

Adding $x$ to every diagonal entry gives

$$xI + L(K_n) = \begin{pmatrix} x+n-1 & -1 & \cdots & \cdots & -1 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 \\ -1 & \cdots & \cdots & -1 & x+n-1 \end{pmatrix} = (x+n)I - J$$

Since $xI + L = (x+n)I - J$, it is enough to know the eigenvalues of J. Because $J$ has rank 1, it has one nonzero eigenvalue. The all-ones vector satisfies $J\mathbf{1} = n\mathbf{1}$ so that the eigenvalue is $n$. The remaining $n-1$ eigenvalues are $0$, with eigenvectors such as $(1,-1,0,\ldots,0)^T$.

Consequently, the eigenvalues of $(x+n)I - J$ are

$$(x+n) - n = x \text{ (once)}, \ (x+n) - 0 = x + n \ ((n-1)\ times)$$

Multiplying them gives the determinant:

$$det(xI + L(K_n)) = x(x+n)^{n-1}.$$

Applying this with $x = 1$ gives

$$det(I + L(K_n)) = 1 \cdot (n+1)^{n-1} = (n+1)^{n-1},$$

and applying it with $x = m+1$ gives

$$det((m+1)I + L(K_n)) = (m+1)(m+n+1)^{n-1}.$$

Substituting these two evaluations into the reduced formula for $\tau(S_m \square K_n)$ obtained above, the factor of $m+1$ cancels against the denominator, and we obtain

$$\tau(S_m \square K_n) = n^{n-2} \cdot ((n+1)^{n-1})^{m-1} \cdot (m+n+1)^{n-1}$$

which, upon combining the two exponents of $(n+1)$, simplifies to exactly

$$\tau(S_m \square K_n) = n^{n-2} \cdot (n+1)^{(n-1)(m-1)} \cdot (m+n+1)^{n-1}.$$

The proof is complete. ■

**Theorem 5.** For $m \geq 0$ and $n \geq 2$, the number of spanning trees of the Cartesian product $S_m \square F_n$ is given by

$$\tau(S_m \square F_n) = \frac{(n+2)^{m-1}(m+n+2)}{2^{2n+m-1} \cdot \left(\sqrt{3}\right)^{m-1} \cdot \sqrt{5(m+2)(m+6)}} \cdot [(3+\sqrt{5})^n - (3-\sqrt{5})^n]$$
$$\cdot \left([(2+\sqrt{3})^n - (2-\sqrt{3})^n]\right)^{m-1} \cdot [(m+4+\sqrt{(m+2)(m+6)})^n$$
$$- (m+4-\sqrt{(m+2)(m+6)})^n]$$

*Proof.* We first dispose of the degenerate case $m = 0$: Here, $S_0$ consists of a single vertex, so that $S_0 \square F_n$ is isomorphic to $F_n$, we have $\tau(S_0 \square F_n) = U_{n-1}(3/2)$, which agrees with the stated formula as $m = 0$. We may therefore assume throughout that $m \geq 1$.

Applying the general theorem established previously for $\tau(S_m \square H)$ with $H = F_n$ reduces it to

$$\tau(S_m \square F_n) = \frac{U_{n-1}(3/2)}{m+1} \cdot (det(I + L(F_n)))^{m-1} \cdot det\big((m+1)I + L(F_n)\big).$$

The fan graph $F_n$ is obtained by joining one extra vertex (the hub) to every vertex of the path $P_n$, so it has $n+1$ vertices. The hub has degree $n$, the two endpoints of the path have degree 2, and every interior path vertex has degree 3. Ordering the hub first, we get the Laplacian matrix $L(F_n)$:

$$L(F_n) = \begin{pmatrix} n & -1 & -1 & \cdots & \cdots & -1 & -1 \\ -1 & 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & -1 & 3 & -1 & \ddots & \ddots & \vdots \\ \vdots & 0 & -1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ -1 & \vdots & \ddots & \ddots & -1 & 3 & -1 \\ -1 & 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}_{(n+1)\times(n+1)} = \begin{pmatrix} n & -\mathbf{1}^T \\ -\mathbf{1} & L(P_n) + I \end{pmatrix}.$$

Adding $x$ to every diagonal entry gives

$$xI + L(F_n) = \begin{pmatrix} x+n & -1 & \cdots & \cdots & \cdots & \cdots & -1 \\ -1 & x+2 & -1 & 0 & \cdots & \cdots & 0 \\ \vdots & -1 & x+3 & -1 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & x+3 & -1 \\ -1 & 0 & \cdots & \cdots & 0 & -1 & x+2 \end{pmatrix} = \begin{pmatrix} x+n & -\mathbf{1}^T \\ -\mathbf{1} & A \end{pmatrix},$$

where $A = (x+1)I + L(P_n)$ is the lower-right $n \times n$ block.

Because the matrix is written in block form, we can use the Schur complement with respect to the block $A$:

$$\det\big(xI + L(F_n)\big) = \det(A)(x + n - \mathbf{1}^T A^{-1}\mathbf{1})$$

Every row of $L(P_n)$ sums to zero, so $L(P_n)\mathbf{1} = 0$ and therefore $A\mathbf{1} = (x+1)\mathbf{1}$. This gives $A^{-1}\mathbf{1} = \frac{1}{x+1}\mathbf{1}$, and hence

$$\mathbf{1}^{\mathrm{T}}A^{-1}\mathbf{1} = \frac{n}{x+1},\ \ so\ \ \ det\big(xI + L(F_n)\big) = det(A)\left(x + n - \frac{n}{x+1}\right)$$

Next, the eigenvalues of $L(P_n)$ are $\lambda_k = 2 - 2cos\frac{k\pi}{n}, k = 0,1,\dots,n-1$, so the eigenvalues of $A$ are $x + 1 + \lambda_k$. In particular, $\lambda_0 = 0$ contributes the factor $x + 1$.

$$\det(A) = (x+1)\prod_{k=1}^{n-1}\left(x + 3 - 2cos\frac{k\pi}{n}\right).$$

Substituting both pieces into the Schur complement formula, factor $x + 1$ cancels against the denominator:

$$\det\left(xI + L(F_n)\right) = \prod_{k=1}^{n-1}\left(x + 3 - 2cos\frac{k\pi}{n}\right)[(x+n)(x+1) - n]$$

Since $(x+n)(x+1) - n = x(x+n+1)$, we get

$$\det\left(xI + L(F_n)\right) = x(x+n+1)\prod_{k=1}^{n-1}\left(x + 3 - 2cos\frac{k\pi}{n}\right)$$

and hence

$$det(xI\ +\ L(F_n))\ =\ x(x+\ n\ +\ 1)\ U_{n-1}((x\ +\ 3)/2)$$

Applying this with $x = 1$ gives

$$det(I\ +\ L(F_n))\ =\ (n\ +\ 2)\ U_{n-1}(2)$$

and applying it with $x = m + 1$ gives

$$det((m\ +\ 1)I\ +\ L(F_n))\ =\ (m\ +\ 1)(m\ +\ n\ +\ 2)\ U_{n-1}((m\ +\ 4)/2)$$

Substituting these two evaluations, together with $\tau(F_n) = U_{n-1}\left(\frac{3}{2}\right)$, into the reduced formula above, the factor of $m\ +\ 1$ cancels against the denominator, and we obtain

$$\tau(S_m \square F_n)\ =\ U_{n-1}(3/2)\ \cdot\ ((n\ +\ 2)\ U_{n-1}(2))^{m-1}\ \cdot\ (m\ +\ n\ +\ 2)\ U_{n-1}((m\ +\ 4)/2)$$

It remains to express each Chebyshev value in the Binet-type closed form established previously, $U_{n-1}\left(\frac{x}{2}\right) = \left[\left(x + \sqrt{x^2-4}\right)^n - \left(x - \sqrt{x^2-4}\right)^n\right]/2^n\sqrt{x^2-4}$. Applying it at $x = 3$ (where $x^2 - 4 = 5$ gives

$$U_{n-1}(3/2)\ =\ \frac{[(3\ +\ \sqrt{5})^n\ -\ (3\ -\ \sqrt{5})^n]}{2^n\ \sqrt{5}}$$

applying it at $x = 4$ (where $x^2 - 4 = 12 = (2\sqrt{3})^2$ gives

$$U_{n-1}(2) = \frac{[(2+\sqrt{3})^n - (2-\sqrt{3})^n]}{2\sqrt{3}}$$

and applying it at $x = m+4$ (where $x^2 - 4 = (m+2)(m+6)$) gives

$$U_{n-1}((m+4)/2) = \frac{[(m+4+\sqrt{(m+2)(m+6)})^n - (m+4-\sqrt{(m+2)(m+6)})^n]}{2^n\sqrt{(m+2)(m+6)}}$$

Substituting these three closed forms into the expression for $\tau(S_m \square F_n)$ obtained above, we get:

$$\tau(S_m \square F_n) = \frac{(n+2)^{m-1}(m+n+2)}{2^{2n+m-1}\cdot(\sqrt{3})^{m-1}\cdot\sqrt{5(m+2)(m+6)}}\cdot\left[(3+\sqrt{5})^n - (3-\sqrt{5})^n\right]$$
$$\cdot\left([(2+\sqrt{3})^n - (2-\sqrt{3})^n]\right)^{m-1}$$
$$\cdot\left[(m+4+\sqrt{(m+2)(m+6)})^n - (m+4-\sqrt{(m+2)(m+6)})^n\right].$$

The proof is complete. ■

**Theorem 6.** For $m \geq 0$ and $n \geq 3$, the number of spanning trees of the Cartesian product $S_m \square W_n$ is given by

$$\tau(S_m \square W_n) = \frac{(n+2)^{m-1}(m+n+2)}{2^{2n+m-1}(m+2)}\cdot[(3+\sqrt{5})^n + (3-\sqrt{5})^n - 2^{n+1}]$$
$$\cdot\left([(2+\sqrt{3})^n + (2-\sqrt{3})^n - 2]\right)^{m-1}\cdot[(m+4+\sqrt{(m+2)(m+6)})^n$$
$$+ (m+4-\sqrt{(m+2)(m+6)})^n - 2^{n+1}]$$

*Proof.* We first dispose of the degenerate case $m = 0$: Here, $S_0$ consists of a single vertex, so that $S_0 \square W_n$ is isomorphic to $W_n$, and we have $\tau(S_0 \square W_n) = T_n(3)$, which agrees with the stated formula as $m = 0$. We may therefore assume throughout that $m \geq 1$.

Applying the general theorem established previously for $\tau(S_m \square H)$ with $H = W_n$ reduces it to

$$\tau(S_m \square W_n) = \frac{T_n(3)}{m+1}\cdot(det(I + L(W_n)))^{m-1}\cdot det\big((m+1)I + L(W_n)\big).$$

The wheel $W_n$ is obtained by joining one extra vertex (the hub) to every vertex of the cycle $C_n$, so it has $n+1$ vertices. The hub has degree n, and every rim vertex has degree 3 (two rim

neighbours plus the hub). Ordering the hub first, we get the Laplacian matrix $L(W_n)$:

$$L(W_n) = \begin{pmatrix} n & -1 & -1 & \cdots & \cdots & -1 & -1 \\ -1 & 3 & -1 & 0 & \cdots & 0 & -1 \\ -1 & -1 & 3 & -1 & \ddots & \ddots & 0 \\ \vdots & 0 & -1 & 3 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ -1 & 0 & \ddots & \ddots & -1 & 3 & -1 \\ -1 & -1 & 0 & \cdots & 0 & -1 & 3 \end{pmatrix}_{(n+1)\times(n+1)} = \begin{pmatrix} n & -\mathbf{1}^T \\ -\mathbf{1} & L(C_n) + I \end{pmatrix}.$$

Adding $x$ to every diagonal entry gives

$$xI + L(W_n) = \begin{pmatrix} x+n & -1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & x+3 & -1 & 0 & \cdots & 0 & -1 \\ -1 & -1 & x+3 & -1 & \cdots & 0 & 0 \\ -1 & 0 & -1 & x+3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & -1 & x+3 & -1 \\ -1 & -1 & 0 & \cdots & 0 & -1 & x+3 \end{pmatrix} = \begin{pmatrix} x+n & -\mathbf{1}^T \\ -\mathbf{1} & A \end{pmatrix},$$

where $A = (x+1)I + L(C_n)$ is the lower-right $n \times n$ block.

Since the matrix is written in block form, we can use the Schur complement with respect to block $A$:

$$\det\big(xI + L(W_n)\big) = \det(A)(x + n - \mathbf{1}^T A^{-1}\mathbf{1})$$

Every row of $L(C_n)$ sums to zero, so $A\mathbf{1} = (x+1)\mathbf{1}$. This gives $A^{-1}\mathbf{1} = \frac{1}{x+1}\mathbf{1}$, and hence

$$\mathbf{1}^{\mathrm{T}}\boldsymbol{A^{-1}}\mathbf{1} = \frac{\boldsymbol{n}}{\boldsymbol{x+1}}, \ \ \boldsymbol{so} \ \ \boldsymbol{det\big(xI + L(W_n)\big) = det(A)\left(x + n - \frac{n}{x+1}\right)}$$

Next, the eigenvalues of $L(C_n)$ are $\lambda_k = 2 - 2cos\frac{2k\pi}{n}, k = 0,1,\dots,n-1$, so the eigenvalues of $A$ are $x + 1 + \lambda_k$. In particular, $\lambda_0 = 0$ contributes the factor $x + 1$.

$$\det(A) = (x+1)\prod_{k=1}^{n-1}(x + 3 - 2cos\frac{2k\pi}{n}).$$

Substituting both pieces, factor $x + 1$ cancels against the denominator:

$$\det\big(xI + L(W_n)\big) = \prod_{k=1}^{n-1}\left(x + 3 - 2cos\frac{2k\pi}{n}\right)[(x+n)(x+1) - n].$$

Since $(x+n)(x+1) - n = x(x+n+1)$, we have

$$\det(xI + L(W_n)) = x(x+n+1)\prod_{k=1}^{n-1}\left(x + 3 - 2cos\frac{2k\pi}{n}\right)$$

and hence

$$det(xI + L(W_n)) = x(x + n + 1)\frac{det((x + 1)I + L(C_n))}{x + 1}$$

By the cycle identity, $det\big((x+1)I + L(C_n)\big) = T_n(x+3)$, so that

$$det(xI + L(W_n)) = \frac{x(x + n + 1)\,T_n(x + 3)}{x + 1}$$

Applying this with $x = 1$ gives

$$det(I + L(W_n)) = \frac{(n + 2)\,T_n(4)}{2}$$

and applying it with $x = m + 1$ gives

$$det((m + 1)I + L(W_n)) = \frac{(m + 1)(m + n + 2)\,T_n(m + 4)}{m + 2}$$

Substituting these two evaluations, together with $\tau(W_n) = T_n(3)$, into the reduced formula above, the factor of $m + 1$ cancels against the denominator, and we obtain

$$\tau(S_m \square W_n) = T_n(3) \cdot \left(\frac{(n + 2)\,T_n(4)}{2}\right)^{m-1} \cdot \frac{(m + n + 2)\,T_n(m + 4)}{m + 2}$$

It remains to express $T_n(4)$ and $T_n(m+4)$ in closed form, using the same evaluation $T_n(y) = \left[\left(y + \sqrt{y^2-4}\right)^n + \left(y - \sqrt{y^2-4}\right)^n - 2^{n+1}\right]/2^n$. Applying it at $y = 4$ (where $y^2 - 4 = 12$) gives

$$T_n(4) = \frac{(4 + \sqrt{12})^n + (4 - \sqrt{12})^n - 2^{n+1}}{2^n} = [(2 + \sqrt{3})^n + (2 - \sqrt{3})^n - 2]$$

and applying it at $y = m + 4$ (where $y^2 - 4 = (m+2)(m+6)$) gives

$$T_n(m + 4) = \frac{[(m + 4 + \sqrt{(m + 2)(m + 6)})^n + (m + 4 - \sqrt{(m + 2)(m + 6)})^n - 2^{n+1}]}{2^n}$$

Substituting these closed forms, together with the earlier evaluation $T_n(3) = \left[\left(3 + \sqrt{5}\right)^n + \left(3 - \sqrt{5}\right)^n - 2^{n+1}\right]/2^n$, into the expression for $\tau(S_m \square W_n)$ obtained above, we get:

$$\tau(S_m \square W_n) = \frac{(n+2)^{m-1}(m+n+2)}{2^{2n+m-1}(m+2)} \cdot \left[\left(3+\sqrt{5}\right)^n + \left(3-\sqrt{5}\right)^n - 2^{n+1}\right]$$
$$\cdot \left(\left[(2+\sqrt{3})^n + (2-\sqrt{3})^n - 2\right]\right)^{m-1}$$
$$\cdot \left[\left(m+4+\sqrt{(m+2)(m+6)}\right)^n + \left(m+4-\sqrt{(m+2)(m+6)}\right)^n - 2^{n+1}\right].$$

The proof is complete. ■

**Theorem 7.** For $m \geq 0$ and $n \geq 1$, the number of spanning trees of the Cartesian product $S_m \square L_n$ is given by

$$\tau(S_m \square L_n) = \frac{m+3}{2^{2nm+1}\left(\sqrt{3}\right)^{m-1}\sqrt{3(m+1)(m+3)(m+5)(m+7)}} \cdot \left[\left(2+\sqrt{3}\right)^n\right.$$
$$\left. - \left(2-\sqrt{3}\right)^n\right] \cdot \left(\left[\left(3+\sqrt{5}\right)^n - \left(3-\sqrt{5}\right)^n\right]\right)^{m-1}$$
$$\cdot \left(\left[\left(5+\sqrt{21}\right)^n - \left(5-\sqrt{21}\right)^n\right]\right)^{m-1} \cdot \left[\left(m+3+\sqrt{(m+1)(m+5)}\right)^n\right.$$
$$\left. - \left(m+3-\sqrt{(m+1)(m+5)}\right)^n\right] \cdot \left[\left(m+5+\sqrt{(m+3)(m+7)}\right)^n\right.$$
$$\left. - \left(m+5-\sqrt{(m+3)(m+7)}\right)^n\right]$$

*Proof.* We first dispose of the degenerate case $m = 0$: Here, $S_0$ consists of a single vertex, so that $S_0 \square L_n$ is isomorphic to $L_n$, and we have $\tau(S_0 \square L_n) = U_{n-1}(2)$, which agrees with the stated formula as $m = 0$. We may therefore assume throughout that $m \geq 1$.

Applying the general theorem established previously for $\tau(S_m \square H)$ with $H = L_n$ reduces it to

$$\tau(S_m \square L_n) = \frac{U_{n-1}(2)}{m+1} \cdot (det(I + L(L_n)))^{m-1} \cdot det\big((m+1)I + L(L_n)\big).$$

The ladder $L_n = P_n \square K_2$ consists of two copies of the path $P_n$, with vertices $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$, joined by $n$ rungs $u_i v_i$. It has $2n$ vertices. The four corner vertices have degree $2$, and every other vertex has degree $3$ (two path neighbours plus its rung). Ordering the $u$-vertices first and then the $v$-vertices, the rungs contribute $-I$ in the off-diagonal blocks, so the Laplacian matrix $L(L_n)$:

$$L(L_n) = \begin{pmatrix} L(P_n)+I & -I \\ -I & L(P_n)+I \end{pmatrix}_{2n \times 2n}.$$

Adding $x$ to every diagonal entry gives

$$xI + L(L_n) = \begin{pmatrix} A & -I \\ -I & A \end{pmatrix}_{2n\times 2n},$$

where $I$ is the $n \times n$ identity matrix and $A$ is the tridiagonal $n \times n$ matrix

$$A = (x+1)I + L(P_n) = \begin{pmatrix} x+2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & x+3 & -1 & \ddots & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & x+3 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & x+2 \end{pmatrix}.$$

Since the four blocks of this matrix commute with one another (they are polynomials in $L(P_n)$), the determinant of the block matrix reduces to a determinant of size $n$:

$$\det\begin{pmatrix} A & -I \\ -I & A \end{pmatrix} = \det(A^2 - I) = \det(A - I)\ \det(A + I).$$

Here, $A - I = xI + L(P_n)$ and $A + I = (x+2)I + L(P_n)$, so the problem reduces to the path graph. Recalling that the eigenvalues of $L(P_n)$ are $\lambda_k = 2 - 2\cos\frac{k\pi}{n}$ for $k = 0,1,\ldots,n-1$, we get

$$\det\big(xI + L(P_n)\big) = \prod_{k=0}^{n-1}(x + \lambda_k), \qquad \det((x+2)I + L(P_n)) = \prod_{k=0}^{n-1}(x + 2 + \lambda_k).$$

Multiplying the two products gives

$$det(xI\ +\ L(L_n))\ =\ det(xI\ +\ L(P_n))\ \cdot\ det((x\ +\ 2)I\ +\ L(P_n)).$$

Thus,

$$\det\big(xI + L(L_n)\big) = \prod_{k=0}^{n-1}\left(x + 2 - 2\cos\frac{k\pi}{n}\right)\left(x + 4 - 2\cos\frac{k\pi}{n}\right).$$

Applying the path-determinant identity to each factor gives

$$det(xI\ +\ L(L_n))\ =\ x(x\ +\ 2)\ U_{n\ -\ 1}((x\ +\ 2)/2)\ U_{n\ -\ 1}((x\ +\ 4)/2)$$

Applying this with $x = 1$ gives

$$det\big(I\ +\ L(L_n)\big)\ =\ 3\ U_{n\ -\ 1}\left(\frac{3}{2}\right)\ U_{n\ -\ 1}\left(\frac{5}{2}\right),$$

and applying it with $x = m + 1$ gives

$$det((m\ +\ 1)I\ +\ L(L_n))\ =\ (m\ +\ 1)(m\ +\ 3)\ U_{n\ -\ 1}((m\ +\ 3)/2)\ U_{n\ -\ 1}((m\ +\ 5)/2)$$

Substituting these two evaluations, together with $\tau(L_n) = U_{n-1}(2)$, into the reduced formula above, the factor of $m + 1$ cancels against the denominator, and we obtain

$$\tau(S_m \square L_n) = U_{n-1}(2)\ \cdot\ (3\ U_{n-1}(3/2)\ U_{n-1}(5/2))^{m-1}\cdot\ (m + 3)\ U_{n-1}((m\ +\ 3)/2)\ U_{n-1}((m\ +\ 5)/2).$$

It remains to express each Chebyshev factor in Binet-type closed form, using the identity $U_{n-1}\left(\frac{x}{2}\right) = \left[\left(x+\sqrt{x^2-4}\right)^n - \left(x-\sqrt{x^2-4}\right)^n\right] / \left(2^n\sqrt{x^2-4}\right)$ established previously. We record the five evaluations needed, in each case with $x^2-4$ computed alongside:

$$U_{n-1}(2) = \frac{\left(4+\sqrt{12}\right)^n - \left(4-\sqrt{12}\right)^n}{2^n\sqrt{12}}$$

which, since $4 \pm \sqrt{12} = 2(2 \pm \sqrt{3})$, simplifies to

$$U_{n-1}(2) = \frac{\left[\left(2+\sqrt{3}\right)^n - \left(2-\sqrt{3}\right)^n\right]}{2\sqrt{3}}$$

$$U_{n-1}(3/2) = \frac{\left(3+\sqrt{5}\right)^n - \left(3-\sqrt{5}\right)^n}{2^n\sqrt{5}}$$

$$U_{n-1}(5/2) = \frac{\left(5+\sqrt{21}\right)^n - \left(5-\sqrt{21}\right)^n}{2^n\sqrt{21}}$$

$$U_{n-1}((m+3)/2) = \frac{\left(m+3+\sqrt{(m+1)(m+5)}\right)^n - \left(m+3-\sqrt{(m+1)(m+5)}\right)^n}{2^n\sqrt{(m+1)(m+5)}}$$

$$U_{n-1}((m+5)/2) = \frac{\left(m+5+\sqrt{(m+3)(m+7)}\right)^n - \left(m+5-\sqrt{(m+3)(m+7)}\right)^n}{2^n\sqrt{(m+3)(m+7)}}$$

Substituting these five closed forms into the expression for $\tau(S_m \square L_n)$ obtained above, we get:

$$\begin{aligned}\tau(S_m \square L_n) = {} & \frac{m+3}{2^{2nm+1}\left(\sqrt{3}\right)^{m-1}\sqrt{3(m+1)(m+3)(m+5)(m+7)}} \cdot \left[\left(2+\sqrt{3}\right)^n\right. \\ & \left. - \left(2-\sqrt{3}\right)^n\right] \cdot \left(\left[\left(3+\sqrt{5}\right)^n - \left(3-\sqrt{5}\right)^n\right]\right)^{m-1} \\ & \cdot \left(\left[\left(5+\sqrt{21}\right)^n - \left(5-\sqrt{21}\right)^n\right]\right)^{m-1} \cdot \left[\left(m+3+\sqrt{(m+1)(m+5)}\right)^n\right. \\ & \left. - \left(m+3-\sqrt{(m+1)(m+5)}\right)^n\right] \cdot \left[\left(m+5+\sqrt{(m+3)(m+7)}\right)^n\right. \\ & \left. - \left(m+5-\sqrt{(m+3)(m+7)}\right)^n\right]\end{aligned}$$

The proof is complete. ■

**Theorem 8.** For $m \geq 0$ and $n \geq 3$, the number of spanning trees of the Cartesian product $S_m \square R_n$ is given by

$$\begin{aligned}\tau(S_m \square R_n) = {} & \frac{n}{2^{2nm+1}(m+1)} \cdot [(2+\sqrt{3})^n + (2-\sqrt{3})^n - 2] \\ & \cdot \left([(3+\sqrt{5})^n + (3-\sqrt{5})^n - 2^{n+1}]\right)^{m-1} \\ & \cdot \left([(5+\sqrt{21})^n + (5-\sqrt{21})^n - 2^{n+1}]\right)^{m-1} \\ & \cdot [(m+3+\sqrt{(m+1)(m+5)})^n + (m+3-\sqrt{(m+1)(m+5)})^n \\ & - 2^{n+1}] \cdot [(m+5+\sqrt{(m+3)(m+7)})^n \\ & + (m+5-\sqrt{(m+3)(m+7)})^n - 2^{n+1}].\end{aligned}$$

*Proof.* We first dispose of the degenerate case $m = 0$: Here, $S_0$ consists of a single vertex, so that $S_0 \square R_n$ is isomorphic to $R_n$, and we have $\tau(S_0 \square R_n) = \frac{n}{2} T_n(4)$ which agrees with the stated formula as $m = 0$. We may therefore assume throughout that $m \geq 1$.

Applying the general theorem established previously for $\tau(S_m \square H)$ with $H = R_n$ reduces it to

$$\tau(S_m \square R_n) = \frac{\frac{n}{2} T_n(4)}{m+1} \cdot (det(I + L(R_n)))^{m-1} \cdot det\big((m+1)I + L(R_n)\big).$$

The prism $R_n = C_n \square K_2$ consists of two copies of the cycle $C_n$, with vertices $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$, joined by $n$ rungs $u_i v_i$. It has $2n$ vertices, and every vertex has degree 3 (two cycle neighbours plus its rung). Ordering the u-vertices first and then the $v$-vertices, the rungs contribute $-I$ in the off-diagonal blocks, so the Laplacian matrix $L(R_n)$:

$$L(R_n) = \begin{pmatrix} L(C_n) + I & -I \\ -I & L(C_n) + I \end{pmatrix}_{2n \times 2n}$$

Adding $x$ to every diagonal entry gives

$$xI + L(R_n) = \begin{pmatrix} A & -I \\ -I & A \end{pmatrix}, \qquad A = (x+1)I + L(C_n) = \begin{pmatrix} x+3 & -1 & \cdots & -1 \\ -1 & x+3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & x+3 \end{pmatrix}_{n \times n}.$$

The four blocks are polynomials in $L(C_n)$, so they commute with one another. The determinant of the block matrix therefore reduces to a determinant of size $n$:

$$\det\begin{pmatrix} A & -I \\ -I & A \end{pmatrix} = \det(A^2 - I) = \det(A - I)\ \det(A + I).$$

Here, $A - I = xI + L(C_n)$ and $A + I = (x + 2)I + L(C_n)$, so the problem is reduced to the cycle graph. The eigenvalues of $L(C_n)$ are $\lambda_k = 2 - 2\cos\frac{2\pi k}{n}$ for $k = 0,1, \dots, n - 1$, and therefore

$$\det\big(xI + L(R_n)\big) = \prod_{k=0}^{n-1}\left(x + 2 - 2\cos\frac{2\pi k}{n}\right)\left(x + 4 - 2\cos\frac{2\pi k}{n}\right)$$

$$det(xI\ +\ L(R_n))\ =\ det(xI\ +\ L(C_n))\ \cdot\ det((x\ +\ 2)I\ +\ L(C_n))\ =\ T_n(x\ +\ 2)\,T_n(x\ +\ 4)$$

Applying this with $x = 1$ gives

$$det(I\ +\ L(R_n))\ =\ T_n(3)\,T_n(5)$$

and applying it with $x = m + 1$ gives

$$det((m\ +\ 1)I\ +\ L(R_n)) = T_n(m\ +\ 3)\,T_n(m\ +\ 5)$$

Substituting these two evaluations, together with $\tau(R_n) = (\frac{n}{2})T_n(4)$, into the reduced formula above, we obtain

$$\tau(S_m \square R_n)\ =\ \frac{n}{2(\mathrm{m}+1)}\ T_n(4)\cdot\ \big(T_n(3)T_n(5)\big)^{m\,-\,1}\ \cdot\ T_n(m\ +\ 3)T_n(m\ +\ 5).$$

Finally, substituting the Binet-type closed form $T_n(y) = [\big(y + \sqrt{y^2 - 4}\big)^n + \big(y - \sqrt{y^2 - 4}\big)^n - 2^{n+1}]/2^n$ established in the cycle proof, at $y = 3, 5, m + 3$, and $m + 5$, together with $T_n(4) = \big(2 + \sqrt{3}\big)^n + \big(2 - \sqrt{3}\big)^n - 2$ at $y = 4$, we have

$$\begin{aligned}\tau(S_m \square R_n)\ =\ &\frac{n}{2^{2nm\,+\,1}\,(m\ +\ 1)}\ \cdot\ [\big(2\ +\ \sqrt{3}\big)^n\ +\ \big(2\ -\ \sqrt{3}\big)^n\ -\ 2] \\ &\cdot\ \Big([\big(3\ +\ \sqrt{5}\big)^n\ +\ \big(3\ -\ \sqrt{5}\big)^n\ -\ 2^{n\,+\,1}]\Big)^{m\,-\,1} \\ &\cdot\ \Big([\big(5\ +\ \sqrt{21}\big)^n\ +\ \big(5\ -\ \sqrt{21}\big)^n\ -\ 2^{n\,+\,1}]\Big)^{m\,-\,1} \\ &\cdot\ [\big(m\ +\ 3\ +\ \sqrt{(m\ +\ 1)(m\ +\ 5)}\big)^n\ +\ \big(m\ +\ 3\ -\ \sqrt{(m\ +\ 1)(m\ +\ 5)}\big)^n \\ &-\ 2^{n\,+\,1}]\ \cdot\ [\big(m\ +\ 5\ +\ \sqrt{(m\ +\ 3)(m\ +\ 7)}\big)^n \\ &+\ \big(m\ +\ 5\ -\ \sqrt{(m\ +\ 3)(m\ +\ 7)}\big)^n\ -\ 2^{n\,+\,1}].\end{aligned}$$

The proof is complete. ■

**Theorem 9.** For $m \geq 0$ and $n \geq 1$, the number of spanning trees of the Cartesian product $S_m \square M_n$ is given by

$$\begin{aligned}\tau(S_m \square M_n) = {} & \frac{n}{2^{2nm+1}(m+1)} \cdot \left[\left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n + 2\right] \\ & \cdot \left(\left[\left(3+\sqrt{5}\right)^n + \left(3-\sqrt{5}\right)^n - 2^{n+1}\right]\right)^{m-1} \\ & \cdot \left(\left[\left(5+\sqrt{21}\right)^n + \left(5-\sqrt{21}\right)^n + 2^{n+1}\right]\right)^{m-1} \\ & \cdot \left[\left(m+3+\sqrt{(m+1)(m+5)}\right)^n + \left(m+3-\sqrt{(m+1)(m+5)}\right)^n\right. \\ & \left. - 2^{n+1}\right] \cdot \left[\left(m+5+\sqrt{(m+3)(m+7)}\right)^n\right. \\ & \left. + \left(m+5-\sqrt{(m+3)(m+7)}\right)^n + 2^{n+1}\right].\end{aligned}$$

*Proof.* We first dispose of the degenerate case $m = 0$: Here, $S_0$ consists of a single vertex, so that $S_0 \square M_n$ is isomorphic to $M_n$, and we have $\tau(S_0 \square M_n) = \frac{n}{2}\,\Omega_n(4)$, which agrees with the stated formula as $m = 0$. We may therefore assume throughout that $m \geq 1$.

Applying the general theorem established previously for $\tau(S_m \square H)$ with $H = M_n$ reduces it to

$$\tau(S_m \square M_n) = \frac{\frac{n}{2}\,\Omega_n(4)}{m+1} \cdot (det(I + L(M_n)))^{m-1} \cdot det\big((m+1)I + L(M_n)\big).$$

The Möbius ladder $M_n$ has $2n$ vertices $1,2,\ldots,2n$ arranged on a cycle $C_{2n}$, with each vertex $i$ also joined to its antipode $i+n$. Every vertex has degree 3. For $n = 2$, this gives $K_4$, and for $n = 3$ it gives $K_{3,3}$.

Row $i$ of $L$ has 3 on the diagonal and $-1$ in the columns $\mathrm{i}-1$, $\mathrm{i}+1$ and $\mathrm{i}+\mathrm{n}$ (indices taken $mod\ 2n$). This makes $L$ a circulant matrix, $L = 3I - A$. Splitting the vertices into the first $n$ and the last $n$ gives the block form:

$$L(M_n) = \begin{pmatrix} 3I - A(P_n) & -(I+E) \\ -(I+E) & 3I - A(P_n) \end{pmatrix}_{2n\times 2n}, \qquad E = e_1 e_n^T + e_n e_1^T,$$

where $A(P_n)$ is the adjacency matrix of the path. The matrix $E$ adds the two "twist" edges $u_n v_1$ and $v_n u_1$, which are the only difference from the ordinary ladder.

Adding x to every diagonal entry gives

$$xI + L(M_n) = \begin{pmatrix} (x+3)I - A(P_n) & -(I+E) \\ -(I+E) & (x+3)I - A(P_n) \end{pmatrix},$$

Since $L(M_n)$ is circulant of order $2n$, its eigenvectors are $v_j = \omega^{jk}$ with $\omega = e^{2\pi i/2n} = e^{\pi i/n}$. Row $j$ of $Lv = \lambda v$ reads

$$3v_j - v_{j-1} - v_{j+1} - v_{j+n} = (3 - \omega^k - \omega^{-k} - \omega^{nk})v_j.$$

Since $\omega^n = -1$, we have $\omega^{nk} = (-1)^k$, so the eigenvalues are

$$\lambda_k = 3 - 2\cos\frac{k\pi}{n} - (-1)^k, k = 0,1, \dots, 2n-1.$$

The eigenvalues of $xI + L$ are $x + \lambda_k$, so

$$\det\big(xI + L(M_n)\big) = \prod_{k=0}^{2n-1}\left(x + 3 - 2\cos\frac{k\pi}{n} - (-1)^k\right)$$

To simplify this, we separate even and odd $k$. For $k = 2m$ the factor is $x + 2 - 2\cos\frac{2\pi m}{n}$, and for $k = 2m+1$ it is $x + 4 - 2\cos\frac{(2m+1)\pi}{n}$, so

$$\det\big(xI + L(M_n)\big) = \prod_{m=0}^{n-1}\left(x + 2 - 2\cos\frac{2\pi m}{n}\right)\prod_{m=0}^{n-1}\left(x + 4 - 2\cos\frac{(2m+1)\pi}{n}\right).$$

The first product is exactly $\det\big(xI + L(C_n)\big)$. For the second, the numbers $2\cos\frac{(2m+1)\pi}{n}$ are the roots of $2T_n(z/2) + 2$, which gives the closed form

$$\det\big(xI + L(M_n)\big) = \left[2T_n\left(\frac{x+2}{2}\right) - 2\right]\left[2T_n\left(\frac{x+4}{2}\right) + 2\right].$$

Applying the closed form above with $x = 1$ gives

$$det\big(I + L(M_n)\big) = [2T_n\left(\frac{3}{2}\right) - 2]\,[2T_n\left(\frac{5}{2}\right) + 2] = T_n(3)\,\Omega_n(5)$$

and applying it with $x = m + 1$ gives

$$\det\big((m+1)I + L(M_n)\big) = \left[2T_n\left(\frac{\mathrm{m}+3}{2}\right) - 2\right]\left[2T_n\left(\frac{\mathrm{m}+5}{2}\right) + 2\right] = T_n(m+3)\Omega_n(m+5).$$

Substituting these two evaluations, together with $\tau(M_n) = (\frac{n}{2})\,\Omega_n(4)$, into the reduced formula for $\tau(S_m \square M_n)$ obtained above, we have

$$\tau(S_m \square M_n) = \frac{n}{2(\mathrm{m}+1)}\,\Omega_n(4) \cdot (T_n(3)\,\Omega_n(5))^{m-1} \cdot T_n(m+3)\,\Omega_n(m+5)$$

Finally, substituting the closed forms of $T_n$ and $\Omega_n$, we have

$$\tau(S_m \square M_n) = \frac{n}{2^{2nm+1}(m+1)} \cdot [(2+\sqrt{3})^n + (2-\sqrt{3})^n + 2]$$
$$\cdot \left([(3+\sqrt{5})^n + (3-\sqrt{5})^n - 2^{n+1}]\right)^{m-1}$$
$$\cdot \left([(5+\sqrt{21})^n + (5-\sqrt{21})^n + 2^{n+1}]\right)^{m-1}$$
$$\cdot [(m+3+\sqrt{(m+1)(m+5)})^n + (m+3-\sqrt{(m+1)(m+5)})^n$$
$$- 2^{n+1}] \cdot [(m+5+\sqrt{(m+3)(m+7)})^n$$
$$+ (m+5-\sqrt{(m+3)(m+7)})^n + 2^{n+1}].$$

The proof is complete. ■

**Theorem10.** For $m \geq 0$ and $n \geq 2$, the number of spanning trees of the Cartesian product $S_m \square G_n$ is given by

$$\tau(S_m \square G_n) = \frac{(5n+8)^{m-1}(m^2+(n+7)m+5n+8)}{2^n \cdot 8^{m-1}(m^2+7m+8)} \cdot [(2+\sqrt{3})^n + (2-\sqrt{3})^n$$
$$- 2] \cdot \left([(5+2\sqrt{6})^n + (5-2\sqrt{6})^n - 2]\right)^{m-1}$$
$$\cdot [\left((m+2)(m+5) + \sqrt{(m+3)(m+4)(m^2+7m+8)}\right)^n$$
$$+ \left((m+2)(m+5) - \sqrt{(m+3)(m+4)(m^2+7m+8)}\right)^n - 2^{n+1}].$$

*Proof.* We first dispose of the degenerate case $m = 0$: Here, $S_0$ consists of a single vertex, so that $S_0 \square G_n$ is isomorphic to $G_n$, and we have $\tau(S_0 \square G_n) = T_n(4)$, which agrees with the stated formula as $m = 0$. We may therefore assume throughout that $m \geq 1$.

Applying the general theorem established previously for $\tau(S_m \square H)$ with $H = G_n$ reduces it to

$$\tau(S_m \square G_n) = \frac{T_n(4)}{m+1} \cdot (det(I + L(G_n)))^{m-1} \cdot det((m+1)I + L(G_n)).$$

The gear $G_n$ is the wheel $W_n$ with an extra vertex inserted on each rim edge. It has $2n + 1$ vertices: the hub h, the rim vertices $u_1, u_2, \ldots, u_n$, and the inserted vertices $v_1, v_2, \ldots, v_n$, where $v_i$ lies between $u_i$ and $v_i$ (and $v_n$ between $u_n$ and $u_1$). The hub has degree n, each rim vertex has degree 3 (the hub and two inserted vertices), and each inserted vertex has degree 2. Ordering the vertices as h, u, v, we get

$$L(G_n) = \begin{pmatrix} n & -\mathbf{1}^T & \mathbf{0}^T \\ -\mathbf{1} & 3I & -B \\ \mathbf{0} & -B^T & 2I \end{pmatrix}_{(2n+1)\times(2n+1)}, \qquad B = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 1 \end{pmatrix}_{n\times n}$$

Adding x to every diagonal entry gives

$$xI + L(G_n) = \begin{pmatrix} x+n & -\mathbf{1}^T & \mathbf{0}^T \\ -\mathbf{1} & (x+3)I & -B \\ \mathbf{0} & -B^T & (x+2)I \end{pmatrix}$$

To compute $\boldsymbol{det}(xI + L(G_n))$, we first eliminate the last block $(x+2)I$ using the Schur complement, which is valid for $x \neq -2$:

$$\det(xI + L) = (x+2)^n \det\begin{pmatrix} x+n & -\mathbf{1}^T \\ -\mathbf{1} & C \end{pmatrix}, \qquad C = (x+3)I - \frac{BB^T}{x+2}.$$

Since $BB^T = 2I + P + P^T = 4I - L(C_n)$, where $P$ is the cyclic shift matrix, matrix $C$ is a polynomial in $L(C_n)$. Recalling that the eigenvalues of $L(C_n)$ are $\lambda_k = 2 - 2\cos\frac{2\pi k}{n}$, the eigenvalues of $C$ are

$$\mu_k = \frac{x^2 + 5x + 2 + \lambda_k}{x+2} = \frac{x^2 + 5x + 4 - 2\cos\frac{2\pi k}{n}}{x+2}, \qquad k = 0,1,\dots,n-1.$$

The vector $\mathbf{1}$ is an eigenvector of $C$ for $k = 0$, so $\mathbf{1}^T C^{-1}\mathbf{1} = \frac{n}{\mu_0}$. Applying the Schur complement once more, now with respect to $C$, we obtain

$$\det\begin{pmatrix} x+n & -\mathbf{1}^T \\ -\mathbf{1} & C \end{pmatrix} = \left(\prod_{k=1}^{n-1} \mu_k\right)(\mu_0(x+n) - n).$$

The last factor simplifies as follows:

$$\mu_0(x+n) - n = \frac{(x^2+5x+2)(x+n) - n(x+2)}{x+2} = \frac{x(x^2 + (n+5)x + 4n + 2)}{x+2}.$$

Substituting everything, the powers of $x+2$ cancel completely: $(x+2)^n \cdot (x+2)^{-(n-1)} \cdot (x+2)^{-1} = 1$. Hence,

$$\det\big(xI + L(G_n)\big) = x\,(x^2 + (n+5)x + 4n + 2)\prod_{k=1}^{n-1}\left(x^2 + 5x + 4 - 2\cos\frac{2\pi k}{n}\right).$$

Using $\prod_{k=0}^{n-1}\left(z - 2\cos\frac{2\pi k}{n}\right) = 2T_n(z/2) - 2$ with $z = x^2 + 5x + 4$, and removing the $k = 0$ factor $z - 2 = x^2 + 5x + 2$, we obtain

$$\det\big(xI + L(G_n)\big) = \frac{x\,(x^2 + (n+5)x + 4n + 2)}{x^2 + 5x + 2}\left[2T_n\left(\frac{x^2 + 5x + 4}{2}\right) - 2\right].$$

Applying the spectrum formula with $x = 1$, so that $Y = (2)(5) = 10$, gives

$$det(I + L(G_n)) = \frac{(5n + 8)\,T_n(10)}{8}$$

and applying it with $x = m + 1$, so that $Y = (m + 2)(m + 5)$, and the quadratic factor evaluates to $m^2 + (n + 7)m + (5n + 8)$ (a direct expansion of$(m + 1)^2 + (n + 5)(m + 1) + (4n + 2)$, gives

$$det((m + 1)I + L(G_n)) = \frac{(m + 1)\,T_n((m + 2)(m + 5))\,[m^2 + (n + 7)m + (5n + 8)]}{m^2 + 7m + 8}$$

Substituting these two evaluations, together with $\tau(G_n) = T_n(4)$ into the reduced formula above, the factor of $m + 1$ cancels against the denominator, and we obtain

$$\tau(S_m \square G_n) = T_n(4) \cdot \left(\frac{5n+8}{8}\,T_n(10)\right)^{m-1} \cdot \frac{m^2 + (n+7)m + 5n + 8}{m^2 + 7m + 8}\,T_n((m + 2)(m + 5))$$

Finally, substituting the closed form of $T_n$, we get

$$\begin{aligned}\tau(S_m \square G_n) = {} & \frac{(5n + 8)^{m-1}\,(m^2 + (n + 7)m + 5n + 8)}{2^n \cdot 8^{m-1}\,(m^2 + 7m + 8)} \cdot \left[\left(2 + \sqrt{3}\right)^n + \left(2 - \sqrt{3}\right)^n - 2\right] \\ & \cdot \left(\left[\left(5 + 2\sqrt{6}\right)^n + \left(5 - 2\sqrt{6}\right)^n - 2\right]\right)^{m-1} \\ & \cdot \left[\left((m + 2)(m + 5) + \sqrt{(m + 3)(m + 4)(m^2 + 7m + 8)}\right)^n \right. \\ & \left. + \left((m + 2)(m + 5) - \sqrt{(m + 3)(m + 4)(m^2 + 7m + 8)}\right)^n - 2^{n+1}\right].\end{aligned}$$

The proof is complete. ■

**Theorem 11.** For $m \geq 0$ and $n \geq 1$, the number of spanning trees of the Cartesian product $(S_m \square Q_n)$, where $Q_n$ is the $n$ -dimensional hypercube graph, is given by

$$\tau(S_m \square Q_n) = \frac{1}{2^n}\prod_{k=1}^{n}(2k)^{\binom{n}{k}} \cdot \left(\prod_{k=1}^{n}(2k + 1)^{\binom{n}{k}}\right)^{m-1} \cdot \prod_{k=1}^{n}(m + 2k + 1)^{\binom{n}{k}}$$

where $\binom{n}{k}$ is the binomial coefficient.

*Proof.* We first dispose of the degenerate case $m = 0$: Here, $S_0$ consists of a single vertex, so that $S_0 \square Q_n$ is isomorphic to $Q_n = K_2 \square K_2 \square \ldots \square K_2$ ($n$ factors), and we have $\tau(S_0 \square Q_n) = \tau(Q_n) =$

$\frac{1}{2^n}\prod_{k=1}^{n}(2k)^{\binom{n}{k}}$, which agrees with the stated formula as $m = 0$. We may therefore assume throughout that $m \geq 1$.

Applying the general theorem established previously for $\tau(S_m \square H)$ with $H = Q_n$ reduces it to

$$\tau(S_m \square Q_n) = \frac{\tau(Q_n)}{m+1} \cdot (det(I + L(Q_n)))^{m-1} \cdot det((m+1)I + L(Q_n)).$$

The vertices of $Q_n$ are the binary strings of length n, so there are $2^n$ of them. Two vertices are adjacent when they differ in exactly one bit, so every vertex has degree n. Splitting the vertices by their first bit gives two copies of $Q_{n-1}$ joined by a perfect matching. This gives the recursive block form of $L(Q_n)$ as follows:

$$L(Q_n) = \begin{pmatrix} L(Q_{n-1}) + I & -I \\ -I & L(Q_{n-1}) + I \end{pmatrix}_{2^n \times 2^n}, \quad L(Q_0) = (0).$$

Adding x to every diagonal entry gives

$$xI + L(Q_n) = \begin{pmatrix} A & -I \\ -I & A \end{pmatrix}, \quad A = (x+1)I + L(Q_{n-1}).$$

To compute $\det(xI + L(Q_n))$, write $D_n(x) = \det(xI + L(Q_n))$. The four blocks are polynomials in $L(Q_{n-1})$, so they commute, and the determinant of the block matrix reduces to a determinant of half the size:

Here, $A - I = xI + L(Q_{n-1})$ and $A + I = (x+2)I + L(Q_{n-1})$. This gives the recurrence

$$D_n(x) = D_{n-1}(x)\, D_{n-1}(x+2), D_0(x) = x.$$

Next, we solve the recurrence. We claim that $D_n(x) = \prod_{k=0}^{n}(x+2k)^{\binom{n}{k}}$. This is true for $n = 0$. If it holds for $n-1$, then

$$D_{n-1}(x)\, D_{n-1}(x+2) = \prod_{k=0}^{n-1}(x+2k)^{\binom{n-1}{k}} \prod_{k=0}^{n-1}(x+2k+2)^{\binom{n-1}{k}} = \prod_{k=0}^{n}(x+2k)^{\binom{n-1}{k}+\binom{n-1}{k-1}},$$

and Pascal's rule $\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$ completes the induction. Therefore,

$$\det(xI + L(Q_n)) = \prod_{k=0}^{n}(x+2k)^{\binom{n}{k}}.$$

Applying the spectrum formula above with $x = 1$ gives

$$det(I + L(Q_n)) = \prod_{k=1}^{n}(2k+1)^{\binom{n}{k}},$$

and applying it with $x = m+1$ gives

$$det\big((m\ +\ 1)I\ +\ L(Q_n)\big)\ =\ (m\ +\ 1)\prod_{k\ =\ 1}^{n}(m\ +\ 2k\ +\ 1)^{\binom{n}{k}}.$$

Substituting these two evaluations into the reduced formula above, the factor of $m+1$ cancels exactly against the denominator, and we obtain

$$\tau(S_m\ \square\ Q_n)\ =\ \tau(Q_n)\cdot\left(\prod_{k\ =\ 1}^{n}(2k\ +\ 1)^{\binom{n}{k}}\right)^{m\ -\ 1}\ \cdot\ \prod_{k\ =\ 1}^{n}(m\ +\ 2k\ +\ 1)^{\binom{n}{k}}.$$

The proof is complete. ■

**Theorem 12.** For $m \geq 0$ and $p, q \geq 1$, the number of spanning trees of the Cartesian product $S_m \square K_{p,q}$ is given by

$$\begin{aligned}\tau\big(S_m\ \square\ K_{p,q}\big)\ =\ p^{q\ -\ 1}\,q^{p\ -\ 1}\ \cdot\ &((p\ +\ 1)^{q\ -\ 1}\,(q\ +\ 1)^{p\ -\ 1}\,(p\ +\ q\ +\ 1))^{m\ -\ 1}\\ &\cdot\ (m\ +\ p\ +\ 1)^{q\ -\ 1}\,(m\ +\ q\ +\ 1)^{p\ -\ 1}\,(m\ +\ p\ +\ q\ +\ 1)\end{aligned}$$

*Proof.* As before, we first dispose of $m = 0$ : Here, $S_0$ is a single vertex, so $S_0 \square K_{p,q}$ is isomorphic to $K_{p,q}$, and by the well-known formula for complete bipartite graphs,

$$\tau(K_{p,q})\ =\ p^{q\ -\ 1}\,q^{p\ -\ 1}$$

which agrees with the stated formula as $m = 0$, we may assume $m \geq 1$.

Applying the general theorem established previously for $\tau(S_m \square H)$ immediately reduces it to

$$\tau\big(S_m\ \square\ K_{p,q}\big)\ =\ \frac{p^{q\ -\ 1}\,q^{p\ -\ 1}}{m\ +\ 1}\ \cdot\ \big(det(I\ +\ L(K_{p,q}))\big)^{m\ -\ 1}\ \cdot\ det((m\ +\ 1)I\ +\ L(K_{p,q}))$$

where $L(K_{p,q})$ denotes the Laplacian matrix of $K_{p,q}$. It remains only to evaluate the two determinants on the right-hand side explicitly.

The vertex set splits into two parts of sizes $p$ and $q$. Every vertex in the first part is adjacent to all $q$ vertices of the second part, and vice versa. Hence, each vertex in the first part has degree $q$, each vertex in the second part has degree $p$, and there are no edges inside a part. Writing the matrix in block form gives the Laplacian matrix $L(K_{p,q})$:

$$L\big(K_{p,q}\big)=\begin{pmatrix} q & 0 & \cdots & 0 & -1 & \cdots & \cdots & -1\\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & q & -1 & \cdots & \cdots & -1\\ -1 & \cdots & \cdots & -1 & p & 0 & \cdots & 0\\ \vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0\\ -1 & \cdots & \cdots & -1 & 0 & \cdots & 0 & p \end{pmatrix}_{(p+q)\times(p+q)} = \begin{pmatrix} qI_p & -J_{p\times q}\\ -J_{q\times p} & pI_q \end{pmatrix}.$$

Here, $J$ denotes an all-ones matrix of the indicated size.

Adding $x$ to every diagonal entry gives

$$xI + L(K_{p,q}) = \begin{pmatrix} x+q & 0 & \cdots & 0 & -1 & \cdots & \cdots & -1 \\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & x+q & -1 & \cdots & \cdots & -1 \\ -1 & \cdots & \cdots & -1 & x+p & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\ -1 & \cdots & \cdots & -1 & 0 & \cdots & 0 & x+p \end{pmatrix} = \begin{pmatrix} (x+q)I_p & -J_{p\times q} \\ -J_{q\times p} & (x+p)I_q \end{pmatrix}$$

Since the matrix is in block form, we can apply the Schur complement with respect to the upper-left block $(x+q)I_p$, which is invertible when $x \neq -q$:

$$\det(xI + L) = \det\Big((x+q)I_p\Big) \cdot \det\left((x+p)I_q - \frac{1}{x+q} J_{q\times p} J_{p\times q}\right).$$

The first factor equals $(x+q)^p$. For the second, the product of the two all-ones blocks is $J_{q\times p} J_{p\times q} = p J_q$, where $J_q$ is the $q \times q$ all-ones matrix, so we need the determinant of

$$(x+p)I_q - \frac{p}{x+q} J_q.$$

The matrix $J_q$ has eigenvalue q once (with eigenvector $\mathbf{1}$) and eigenvalue $0$ with multiplicity $q-1$. Consequently, the matrix above has eigenvalue $(x+p) - \frac{pq}{x+q}$ once and eigenvalue $x+p$ with multiplicity $q-1$. Multiplying everything together:

$$\begin{aligned} \det(xI + L) &= (x+q)^p\,(x+p)^{q-1}\left[(x+p) - \frac{pq}{x+q}\right] \\ &= (x+q)^{p-1}(x+p)^{q-1}[(x+p)(x+q) - pq]. \end{aligned}$$

Since $(x+p)(x+q) - pq = x^2 + (p+q)x = x(x+p+q)$, we get

$$\det\Big(xI + L\big(K_{p,q}\big)\Big) = x\,(x+p+q)\,(x+p)^{\,q-1}\,(x+q)^{\,p-1}$$

Applying this with $x = 1$ gives

$$det(I + L(K_{p,q})) = (p+1)^{q-1}\,(q+1)^{p-1}\,(p+q+1)$$

and applying it with $x = m+1$ gives

$$\begin{aligned} &det((m+1)I + L(K_{p,q})) \\ &\quad = (m+1)\,(m+p+1)^{q-1}\,(m+q+1)^{p-1}\,(m+p+q+1). \end{aligned}$$

Substituting these two evaluations into the reduced formula for $\tau\big(S_m \square K_{p,q}\big)$ obtained above, the factor of $m+1$ cancels against the denominator, and we get

$$\tau(S_m \square K_{p,q}) = p^{q-1} q^{p-1} \cdot ((p+1)^{q-1} (q+1)^{p-1} (p+q+1))^{m-1} \cdot (m+p+1)^{q-1} (m+q+1)^{p-1} (m+p+q+1).$$

The proof is complete. ■

## 4. Numerical results

To illustrate the derived results, Tables 1–11 present selected numerical values of the complexity of each product graph family, obtained by direct application of the closed-form formulas established in Theorems 2–12, respectively.

**Table 1.** Selected values of the complexity of the cartesian product $S_m \square P_n$.

| $n$ | $m$ | $\tau(S_m \square P_n)$ | $n$ | $m$ | $\tau(S_m \square P_n)$ |
|---|---|---|---|---|---|
| 2 | 1 | 4 | 2 | 2 | 15 |
| 2 | 3 | 54 | 3 | 2 | 192 |
| 2 | 4 | 189 | 4 | 2 | 2415 |
| 3 | 3 | 2240 | 3 | 4 | 24576 |
| 4 | 3 | 89964 | 4 | 4 | 3046869 |

**Table 2.** Selected values of the complexity of the cartesian product $S_m \square C_n$.

| $n$ | $m$ | $\tau(S_m \square C_n)$ | $n$ | $m$ | $\tau(S_m \square C_n)$ |
|---|---|---|---|---|---|
| 3 | 1 | 75 | 4 | 1 | 384 |
| 3 | 2 | 1728 | 4 | 2 | 31500 |
| 3 | 3 | 37632 | 4 | 3 | 2332800 |
| 3 | 4 | 786432 | 4 | 4 | 160744500 |
| 3 | 5 | 15925248 | 4 | 5 | 10497600000 |

**Table 3.** Selected values of the complexity of the cartesian product $S_m \square K_n$.

| $n$ | $m$ | $\tau(S_m \square K_n)$ | $n$ | $m$ | $\tau(S_m \square K_n)$ |
|---|---|---|---|---|---|
| 2 | 1 | 4 | 2 | 2 | 15 |
| 2 | 3 | 54 | 3 | 2 | 1728 |
| 2 | 4 | 189 | 4 | 2 | 686000 |
| 3 | 3 | 37632 | 3 | 4 | 786432 |
| 4 | 3 | 128000000 | 4 | 4 | 22781250000 |

**Table 4.** Selected values of the complexity of the cartesian product $S_{\mathrm{m}} \square F_n$.

| $n$ | $m$ | $\tau(S_{\mathrm{m}} \square F_n)$ | $n$ | $m$ | $\tau(S_{\mathrm{m}} \square F_n)$ |
|---|---|---|---|---|---|
| 2 | 1 | 75 | 2 | 2 | 1728 |
| 2 | 3 | 37632 | 3 | 2 | 147000 |
| 2 | 4 | 786432 | 4 | 2 | 11515392 |
| 3 | 3 | 17280000 | 3 | 4 | 1913625000 |
| 4 | 3 | 7019986176 | 4 | 4 | 3951107112960 |

**Table 5.** Selected values of the complexity of the cartesian product $S_{\mathrm{m}} \square W_n$.

| $n$ | $m$ | $\tau(S_{\mathrm{m}} \square W_n)$ | $n$ | $m$ | $\tau(S_{\mathrm{m}} \square W_n)$ |
|---|---|---|---|---|---|
| 3 | 1 | 3456 | 4 | 1 | 55125 |
| 3 | 2 | 686000 | 4 | 2 | 59179680 |
| 3 | 3 | 128000000 | 4 | 3 | 59256852480 |
| 3 | 4 | 22781250000 | 4 | 4 | 55037657088000 |
| 3 | 5 | 3906250000000 | 4 | 5 | 48548166940753920 |

**Table 6.** Selected values of the complexity of the cartesian product $S_{\mathrm{m}} \square L_n$.

| $n$ | $m$ | $\tau(S_{\mathrm{m}} \square L_n)$ | $n$ | $m$ | $\tau(S_{\mathrm{m}} \square L_n)$ |
|---|---|---|---|---|---|
| 2 | 1 | 384 | 2 | 2 | 31500 |
| 2 | 3 | 2332800 | 3 | 2 | 49766400 |
| 2 | 4 | 160744500 | 4 | 2 | 76752081000 |
| 3 | 3 | 65840947200 | 3 | 4 | 77052719923200 |
| 4 | 3 | 1784546631705600 | 4 | 4 | 34871187345215319000 |

**Table 7.** Selected values of the complexity of the cartesian product $S_{\mathrm{m}} \square R_n$.

| $n$ | $m$ | $\tau(S_{\mathrm{m}} \square R_n)$ | $n$ | $m$ | $\tau(S_{\mathrm{m}} \square R_n)$ |
|---|---|---|---|---|---|
| 3 | 1 | 367500 | 4 | 1 | 42467328 |
| 3 | 2 | 1492992000 | 4 | 2 | 3500658000000 |
| 3 | 3 | 5333116723200 | 4 | 3 | 237027409920000000 |
| 3 | 4 | 17336861982720000 | 4 | 4 | 13927110117054750000000 |
| 3 | 5 | 52432020296014233600 | 4 | 5 | 73497014841600000000000000000 |

**Table 8.** Selected values of the complexity of the cartesian product $S_{\mathrm{m}} \square M_n$.

| $n$ | $m$ | $\tau(S_{\mathrm{m}} \square M_n)$ | $n$ | $m$ | $\tau(S_{\mathrm{m}} \square M_n)$ |
|---|---|---|---|---|---|
| 2 | 1 | 3456 | 2 | 2 | 686000 |
| 2 | 3 | 128000000 | 3 | 2 | 1693052928 |
| 2 | 4 | 22781250000 | 4 | 2 | 3607347807000 |
| 3 | 3 | 6245298339840 | 3 | 4 | 21001565443719168 |
| 4 | 3 | 245922522801177600 | 4 | 4 | 14554034380334150289000 |

**Table 9.** Selected values of the complexity of the cartesian product $S_{\mathrm{m}} \square G_n$.

| $n$ | $m$ | $\tau(S_{\mathrm{m}} \square G_n)$ | $n$ | $m$ | $\tau(S_{\mathrm{m}} \square G_n)$ |
|---|---|---|---|---|---|
| 2 | 1 | 6720 | 2 | 2 | 3110400 |
| 2 | 3 | 1269789696 | 3 | 2 | 5500182050 |
| 2 | 4 | 474054819840 | 4 | 2 | 8193540096000 |
| 3 | 3 | 40360433287900 | 3 | 4 | 257550319881291250 |
| 4 | 3 | 1019640545280000000 | 4 | 4 | 104659168417284096000000 |

**Table 10.** Selected values of the complexity of the cartesian product $S_{\mathrm{m}} \square Q_n$.

| $n$ | $m$ | $\tau(S_{\mathrm{m}} \square Q_n)$ | $n$ | $m$ | $\tau(S_{\mathrm{m}} \square Q_n)$ |
|---|---|---|---|---|---|
| 2 | 1 | 384 | 2 | 2 | 31500 |
| 2 | 3 | 2332800 | 3 | 2 | 3500658000000 |
| 2 | 4 | 160744500 | 4 | 2 | 616347489319805117952000000000 |
| 3 | 3 | 137027409920000000 | 3 | 4 | 13927110117054750000000 |

**Table 11.** Selected values of the complexity of the cartesian product $S_{\mathrm{m}} \square K_{p,q}$.

| $p$ | $q$ | $m$ | $\tau(S_{\mathrm{m}} \square K_{p,q})$ | $p$ | $q$ | $m$ | $\tau(S_{\mathrm{m}} \square K_{p,q})$ |
|---|---|---|---|---|---|---|---|
| 2 | 2 | 1 | 384 | 2 | 2 | 2 | 31500 |
| 2 | 3 | 1 | 6720 | 3 | 2 | 2 | 3110400 |
| 2 | 4 | 1 | 98304 | 4 | 2 | 2 | 238140000 |
| 3 | 3 | 1 | 405000 | 3 | 3 | 2 | 1693052928 |
| 4 | 3 | 1 | 17496000 | 4 | 3 | 2 | 585252864000 |
| 4 | 4 | 1 | 1911029760 | 4 | 4 | 2 | 745424064000000 |

## 5. Conclusions

In this paper, we derived exact closed-form expressions for the number of spanning trees of the Cartesian product of a star graph with eleven well-known families of graphs: paths, cycles, complete graphs, fans, wheels, ladders, prisms, Möbius ladders, gear graphs, hypercubes, and complete bipartite graphs. The main contributions are twofold. First, we proved a general reduction showing that, for any connected graph, the count for its product with a star depends only on a shifted Laplacian determinant of that graph evaluated at two points. Second, using Chebyshev polynomials, block-matrix factorizations, and Schur complements, we computed this determinant for each family, obtaining explicit formulas that agree with one another wherever the families coincide, such as the four-cycle as a ladder, a hypercube, and a complete bipartite graph, and were verified against direct Matrix-Tree computations. The approach applies directly to any graph whose Laplacian spectrum is known, such as grids and tori, which is a natural direction for future work.

## Author contributions

Salama Daoud: Conceptualization, Methodology, Formal analysis, Writing–original draft, Writing–review & editing, Supervision, Validation; Ahmad Asiri: Conceptualization, Methodology, Supervision, Validation, Writing–review& editing. All authors have read and approved the final

version of the manuscript for publication,

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare no conflicts of interest.

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