AIMS Mathematics, 11(11): 38170–38184.
DOI: 10.3934/math.20261474
Received: 08 September 2026
Revised: 15 October 2026
Accepted: 09 November 2026
Published: 17 November 2026

https://www.aimspress.com/journal/Math

Research article

The distribution of the largest digit in semi-regular continued fractions

A. Kazin[1,*] **and S. Kadyrov**[2]

[1] School of Information Technology and Applied Mathematics, SDU University, Kaskelen, Kazakhstan

[2] College of Humanities and Sciences, Ajman University, Ajman, UAE

* **Correspondence:** Email: aiken.kazin@sdu.edu.kz.

Abstract: Fix a sign sequence $\sigma \in \{\pm 1\}^{\mathbb{N}}$, and let $T_n^{\sigma}(x)$ be the largest of the first n partial quotients in the corresponding semi-regular continued fraction expansion of $x \in (0,1) \setminus \mathbb{Q}$. Let $\varphi : \mathbb{N} \to (0,\infty)$ be non-decreasing and satisfy

$$\varphi(n) \to \infty, \qquad \lim_{n\to\infty} \frac{\log \varphi(n)}{\log n} = b < \infty.$$

We prove that, for every $\alpha \in [0, +\infty]$, the set of points for which

$$\lim_{n\to\infty} \frac{T_n^{\sigma}(x)}{\varphi(n)} = \alpha$$

has Hausdorff dimension one. The value $\alpha = +\infty$ denotes divergence to positive infinity. The proof uses a bounded-digit core and sparse increases of digits at fixed positions. Uniform continuant and separation estimates yield Hölder continuity of the inverse map and transfer the dimension of the core to the required level sets.

Keywords: semi-regular continued fractions; Hausdorff dimension; fractal dimension
Mathematics Subject Classification: 11K50

1. Introduction

The metric and fractal properties of partial quotients in regular continued fractions have been studied extensively; see, for example, [9, 11–13, 16]. The distribution of the largest partial quotient was first investigated by Galambos [7, 8]. Philipp [22] proved an iterated logarithm law associated with a conjecture of Erdős, and Okano [20] constructed examples with prescribed limiting behaviour.

Wu and Xu [27] then proved that the corresponding level sets have full Hausdorff dimension. Their argument also gives the same conclusion for every increasing function φ satisfying $\varphi(n) \to \infty$ and $\log \varphi(n)/\log n \to b < \infty$.

Further results on the largest partial quotient in the regular continued fraction were obtained in [1,2, 6, 18, 23]. These works show that the answer depends on the growth of the normalizing function once one passes beyond the polynomial regime. Tan, Tian and Wang [25] investigated the Lebesgue measure and Hausdorff dimension of sets defined by the occurrence of at least two large partial quotients among the first n digits for infinitely many n. Related dimension questions for restricted partial quotients were studied in [26], while quantitative estimates for bounded-digit Cantor sets were obtained in [9–11].

Semi-regular continued fractions allow the numerators to have prescribed signs. Their classical theory goes back to Perron [21]; see also [3, 17]. Hausdorff-dimension questions for regular and minus continued fractions were considered in [4]. More recently, dimension results for arbitrary fixed sign sequences were proved in [14, 15, 19]. Takahasi [24] established a largest-digit result for a class of autonomous parabolic iterated function systems, including the backward continued fraction. The purpose of the present paper is to extend the Wu–Xu theorem to every fixed SRCF sign sequence, including sign sequences which are not eventually constant.

Throughout, $\mathbb{N} = \{1, 2, 3, \ldots\}$. Fix a sign sequence $\sigma = (\sigma_n)_{n\geq 1} \in \{-1, 1\}^{\mathbb{N}}$ and put $\mathcal{I} = (0, 1) \setminus \mathbb{Q}$. For $x \in \mathcal{I}$ we write its σ–SRCF expansion as

$$x = a_0(x) + \cfrac{\sigma_1}{a_1(x) + \cfrac{\sigma_2}{a_2(x) + \cfrac{\sigma_3}{\ddots}}}, \qquad a_0(x) = \frac{1 - \sigma_1}{2}.$$

Here $a_n(x) \in \mathbb{N}$ for $n \geq 1$, and we also write $a_{n,\sigma}(x) = a_n(x)$. With this indexing, the admissibility conditions are

$$a_n(x) + \sigma_{n+1} \geq 1 \quad (n \geq 1), \qquad a_n(x) + \sigma_{n+1} \geq 2 \quad \text{for infinitely many } n.$$

The second condition is equivalent to irrationality. The sign sequence in the convention of [19] is $(\sigma_2, \sigma_3, \ldots)$. If $\sigma_1 = -1$, set $\widehat{\sigma} = (1, \sigma_2, \sigma_3, \ldots)$. Then $a_{n,\sigma}(x) = a_{n,\widehat{\sigma}}(1 - x)$ for every $n \geq 1$. Thus changing the outer sign corresponds to the reflection $x \mapsto 1 - x$, which preserves Hausdorff dimension.

Let p_n/q_n be the nth convergent. The convergents satisfy

$$p_n = a_n p_{n-1} + \sigma_n p_{n-2}, \qquad q_n = a_n q_{n-1} + \sigma_n q_{n-2} \qquad (n \geq 1),$$

where $p_{-1} = 1$, $p_0 = a_0$, $q_{-1} = 0$, and $q_0 = 1$.

Define

$$T_n^{\sigma}(x) = \max\{a_{k,\sigma}(x) : 1 \leq k \leq n\}.$$

Let $\varphi : \mathbb{N} \to (0, \infty)$ be non-decreasing, assume that $\varphi(n) \to \infty$, and suppose that

$$\lim_{n\to\infty} \frac{\log \varphi(n)}{\log n} = b < \infty. \tag{1.1}$$

For $\alpha \in [0, +\infty]$, set

$$E_\varphi^\sigma(\alpha) = \left\{ x \in \mathcal{I} : \lim_{n\to\infty} \frac{T_n^\sigma(x)}{\varphi(n)} = \alpha \right\}.$$

For $\alpha = +\infty$, the limit in this definition means divergence to positive infinity.

Our main result is the following.

Theorem 1.1. *Let $\varphi : \mathbb{N} \to (0, \infty)$ be non-decreasing, let $\varphi(n) \to \infty$, and assume* (1.1)*. Then, for every $\alpha \in [0, +\infty]$,*

$$\dim_H E_\varphi^\sigma(\alpha) = 1.$$

For $\sigma_n = 1$ for all n, the finite-α part of Theorem 1.1 is the result of Wu and Xu [27, Section 3]. In particular, the choice $\varphi(n) = n/\log\log n$ for all sufficiently large n gives their original normalization, with the initial values chosen so that φ is positive and non-decreasing on $\mathbb{N}$. The new point is that the sign sequence is arbitrary and may be non-autonomous.

Theorem 1.1 is related to, but is not a direct corollary of, the dimension results in [19, 24]. Nakajima and Takahasi [19] use non-autonomous conformal iterated function systems to study restricted digit sets in which all partial quotients tend to infinity; their dimension is $\tau(B)/2$, where $\tau(B)$ is the exponent of convergence of the allowed digit set B. Their contraction and bounded-distortion estimates [19, Lemmas 2.3(b) and 2.4] assume digits at least 3, whereas the bounded cores used here retain the admissible branches with digits 1 and 2. The bounded-digit conversion from [14], recalled and proved in Lemma 3.2, is therefore supplemented by the expansion and separation estimates in Lemmas 3.3 and 3.4. Takahasi [24] proves a largest-digit theorem for autonomous parabolic iterated function systems, with normalization $n/\log\log n$, including the backward continued fraction. An arbitrary prescribed sign sequence is not directly covered by that theorem. Our argument uses the same general strategy of a large-dimensional bounded core and sparse digit modifications, but keeps every sign and coordinate position fixed and proves the estimates needed for this construction directly.

The proof follows the general strategy of [27]. We start with a regular bounded-digit Cantor set and rewrite its points as SRCFs with the prescribed signs. We then increase the partial quotients at a sparse set of positions, keeping the sign sequence and all coordinate positions fixed. A Hölder estimate for the inverse map, which subtracts these prescribed increments, shows that the construction does not decrease Hausdorff dimension. The main additional point is to obtain estimates which are uniform in the sign sequence.

Section 2 contains the required facts about signed continuants and cylinders. The proof of Theorem 1.1 is given in Section 3. Throughout the paper, log denotes the natural logarithm and $\dim_H$ denotes Hausdorff dimension.

2. Background and auxiliary results

Throughout this section we fix the sign sequence σ. When a_j is written without an argument, it denotes a prescribed positive integer in a finite or infinite word; $a_j(x) = a_{j,\sigma}(x)$ denotes the digit of the expansion of a point $x \in \mathcal{I}$. For a word $(a_1, \dots, a_n)$ and $0 \le j \le n$, write $q_j(a_1, \dots, a_j)$ for the continuant of its first j digits, with $q_{-1} = 0$, $q_0 = 1$, and the recurrence below. When the word is fixed we abbreviate this to q_j. Define $p_j(a_1, \dots, a_j)$ similarly, with $p_{-1} = 1$ and $p_0 = a_0 = (1 - \sigma_1)/2$. For the digits of a point, write $q_j^\sigma(x) = q_j(a_1(x), \dots, a_j(x))$.

2.1. Convergents, determinant identity, and cylinders

The convergents p_n/q_n satisfy, for all $n \geq 1$,

$$\begin{aligned} p_n &= a_n p_{n-1} + \sigma_n p_{n-2}, & p_{-1} &= 1,\ p_0 = a_0, \\ q_n &= a_n q_{n-1} + \sigma_n q_{n-2}, & q_{-1} &= 0,\ q_0 = 1. \end{aligned} \tag{2.1}$$

A finite word $(a_1, \ldots, a_n)$ is called *admissible* if

$$a_j + \sigma_{j+1} \geq 1 \qquad (1 \leq j \leq n).$$

Every such word has an infinite admissible continuation: append digits equal to 3, so that the inequality is strict at every new position. For an infinite word we also require strict inequality for infinitely many indices. These conditions give the unique irrational SRCF coding; see [19, Lemma 2.1], with the index shift specified in the introduction.

Lemma 2.1 (Determinant identity)**.** *For every $n \geq 1$,*

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1} \prod_{j=1}^{n} \sigma_j,$$

in particular $|p_n q_{n-1} - p_{n-1} q_n| = 1$.

Proof. Set $A_n = \left(\begin{smallmatrix} a_n & \sigma_n \\ 1 & 0 \end{smallmatrix}\right)$. Then

$$\begin{pmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{pmatrix} = A_n A_{n-1} \cdots A_1 \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Taking determinants and using $\det A_j = -\sigma_j$ and $\det\left(\begin{smallmatrix} a_0 & 1 \\ 1 & 0 \end{smallmatrix}\right) = -1$ gives

$$p_n q_{n-1} - p_{n-1} q_n = (-1) \prod_{j=1}^{n} (-\sigma_j) = (-1)^{n+1} \prod_{j=1}^{n} \sigma_j.$$

□

For an admissible word $(a_1, \ldots, a_n)$, define the nth cylinder

$$I_n^{\sigma}(a_1, \ldots, a_n) = \{x \in \mathcal{I} : a_{1,\sigma}(x) = a_1, \ldots, a_{n,\sigma}(x) = a_n\}.$$

Lemma 2.2 (Cylinder endpoints and length)**.** *Let $(a_1, \ldots, a_n)$ be an admissible word, and let p_n/q_n be the corresponding convergents. Then the closure of $I_n^{\sigma}(a_1, \ldots, a_n)$ is an interval with endpoints*

$$\frac{p_n}{q_n} \quad \text{and} \quad \frac{p_n + \sigma_{n+1} p_{n-1}}{q_n + \sigma_{n+1} q_{n-1}}.$$

Consequently, writing $|I_n^{\sigma}(a_1, \ldots, a_n)|$ for the length of this interval,

$$|I_n^{\sigma}(a_1, \ldots, a_n)| = \frac{1}{q_n \, |q_n + \sigma_{n+1} q_{n-1}|}.$$

Proof. Write x with tail parameter $t \in (0,1)$:

$$x = \frac{p_n + \sigma_{n+1} t\, p_{n-1}}{q_n + \sigma_{n+1} t\, q_{n-1}}.$$

By the irrational coding theorem, every irrational $t \in (0,1)$ has a normalized tail expansion with leading sign $+1$ and subsequent signs $\sigma_{n+2}, \sigma_{n+3}, \ldots$. Concatenating it with the admissible prefix preserves the coding conditions. Thus the possible tail parameters are dense in $[0,1]$. Induction in the denominator recurrence gives $q_j > 0$ and $q_j > q_{j-1}$ whenever $\sigma_{j+1} = -1$ (the details are also recorded in (3.3)). Hence the denominator of the tail map is positive on $[0,1]$. The map is continuous and strictly monotone, so the cylinder closure is its image of $[0,1]$, with the two stated endpoints. The length equals the absolute difference of these endpoints, i.e.

$$\left| \frac{p_n}{q_n} - \frac{p_n + \sigma_{n+1} p_{n-1}}{q_n + \sigma_{n+1} q_{n-1}} \right| = \frac{|p_n q_{n-1} - p_{n-1} q_n|}{q_n\, |q_n + \sigma_{n+1} q_{n-1}|}.$$

The numerator is equal to 1 by Lemma 2.1. □

2.2. Regular bounded-digit sets

For each integer $L \geq 1$, let

$$E_L = \{[0; b_1, b_2, \ldots] : 1 \leq b_j \leq L \text{ for every } j \geq 1\}$$

be the regular continued-fraction set. By [27, Lemma 2.3], we have

$$\dim_H E_L \geq 1 - \frac{1}{L \log 2} \qquad (L \geq 8). \tag{2.2}$$

The conversion of these same points to the prescribed SRCF coding is given in Lemma 3.2. The restriction $L \geq 8$ is needed only when applying this dimension estimate.

2.3. Matrix formalism and a signed continuant splitting identity

For $m \geq 1$ set

$$A_m := \begin{pmatrix} a_m & \sigma_m \\ 1 & 0 \end{pmatrix}.$$

Then for $n \geq 1$,

$$\begin{pmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{pmatrix} = A_n A_{n-1} \cdots A_1 \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.3}$$

Lemma 2.3 (Signed continuant identity)**.** *Let $n \geq 2$ and $1 \leq k \leq n-1$. For each $i \geq 1$, define the block continuants by*

$$K_{i,i-2} = 0, \qquad K_{i,i-1} = 1, \qquad K_{i,j} = a_j K_{i,j-1} + \sigma_j K_{i,j-2} \quad (j \geq i).$$

Every block retains the prescribed signs at their original indices. For example, $K_{i,i} = a_i$ and $K_{i,i+1} = a_{i+1} a_i + \sigma_{i+1}$; the term containing σ_i in the first step is zero. No reindexing of the sign sequence is implicit in the block notation. Since $K_{1,j} = q_j$, the splitting identity is

$$q_n = q_k K_{k+1,n} + \sigma_{k+1} q_{k-1} K_{k+2,n}. \tag{2.4}$$

Proof. Taking the second column of (2.3) gives

$$\begin{pmatrix} q_n \\ q_{n-1} \end{pmatrix} = A_n \cdots A_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} q_k \\ q_{k-1} \end{pmatrix} = A_k \cdots A_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Set $T := A_n \cdots A_{k+1}$. Then

$$\begin{pmatrix} q_n \\ q_{n-1} \end{pmatrix} = T \begin{pmatrix} q_k \\ q_{k-1} \end{pmatrix} = q_k \, T \begin{pmatrix} 1 \\ 0 \end{pmatrix} + q_{k-1} \, T \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

By definition of the block continuants with the original sign indices,

$$\left(T \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)_1 = K_{k+1,n}.$$

For the second term, $A_{k+1}\binom{0}{1} = \left(\begin{smallmatrix} a_{k+1} & \sigma_{k+1} \\ 1 & 0 \end{smallmatrix}\right)\binom{0}{1} = \sigma_{k+1}\binom{1}{0}$, so

$$T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = A_n \cdots A_{k+2} \cdot A_{k+1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sigma_{k+1} \, A_n \cdots A_{k+2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The first component of the last vector is $\sigma_{k+1} K_{k+2,n}$. Hence

$$q_n = q_k K_{k+1,n} + \sigma_{k+1} q_{k-1} K_{k+2,n},$$

which is (2.4). □

3. Proof of Theorem 1.1

Throughout this section, φ is non-decreasing, tends to infinity, and satisfies

$$\frac{\log \varphi(n)}{\log n} \longrightarrow b < \infty.$$

All cylinders below correspond to admissible finite words.

We first establish two elementary estimates for admissible continuants.

Lemma 3.1 (Admissible continuant estimates)**.** *Let $(a_1, \ldots, a_n)$ be an admissible word. Then*

$$q_{n-1} \le n q_n, \qquad \frac{1}{(n+1) q_n^2} \le |I_n^{\sigma}(a_1, \ldots, a_n)| \le \frac{n+1}{q_n^2}. \tag{3.1}$$

If only a_k is increased by a nonnegative integer ξ, the resulting denominator q_n' satisfies

$$q_n \le q_n' \le (1 + n\xi) q_n. \tag{3.2}$$

Proof. We first record two consequences of admissibility. Induction in the recursion for q_j shows that

$$q_j > 0, \qquad \sigma_{j+1} = -1 \implies q_j > q_{j-1}. \tag{3.3}$$

Indeed, in the minus case at step j, the condition at the preceding place gives $q_{j-1} > q_{j-2}$, and hence $q_j \geq q_{j-1} - q_{j-2} > 0$. If the next sign is minus, then $a_j \geq 2$, which gives the second assertion in the same way.

We shall also use

$$\sigma_{m+1} = -1 \implies q_m \leq (m+1)(q_m - q_{m-1}). \tag{3.4}$$

For $m = 1$, this is $a_1 \leq 2(a_1 - 1)$, since $a_1 \geq 2$. Assume $m \geq 2$. If $\sigma_m = 1$, then

$$(m+1)(q_m - q_{m-1}) - q_m = \big(m(a_m - 1) - 1\big)q_{m-1} + mq_{m-2} \geq 0.$$

If $\sigma_m = -1$, the induction hypothesis applied to the prefix of length $m-1$ gives $q_{m-2} \leq (1 - 1/m)q_{m-1}$. Therefore

$$\begin{aligned}(m+1)(q_m - q_{m-1}) - q_m &= \big(m(a_m - 1) - 1\big)q_{m-1} - mq_{m-2} \\ &\geq m(a_m - 2)q_{m-1} \geq 0,\end{aligned}$$

which proves (3.4).

If $q_n < q_{n-1}$, then $\sigma_n = -1$. In that case a_n cannot be at least 2, by (3.3); hence $a_n = 1$. Applying (3.4) to the prefix of length $n-1$ yields

$$q_{n-1} \leq n(q_{n-1} - q_{n-2}) = nq_n.$$

The same inequality is immediate when $q_n \geq q_{n-1}$. This proves the first estimate in (3.1).

Put $s_n = q_n + \sigma_{n+1}q_{n-1}$. By (3.3), $s_n > 0$. If $\sigma_{n+1} = 1$, then

$$q_n \leq s_n \leq (n+1)q_n;$$

if $\sigma_{n+1} = -1$, (3.4) gives

$$\frac{q_n}{n+1} \leq s_n < q_n.$$

Lemma 2.2 now gives (3.1).

It remains to prove (3.2). Let

$$U = q_{k-1}, \quad V = q_{k-2},$$

and put $W = K_{k+1,n}$, $Z = K_{k+2,n}$, using the original sign indices as in Lemma 2.3. For $k = n$, these definitions give $W = 1, Z = 0$. The splitting identity gives

$$q_n = (a_k U + \sigma_k V)W + \sigma_{k+1}UZ. \tag{3.5}$$

Writing $C_i = K_{i,n}$, the reversed recursion for the right-hand continuants is

$$C_i = a_i C_{i+1} + \sigma_{i+1}C_{i+2}.$$

Starting from $C_{n+1} = 1, C_{n+2} = 0$, admissibility shows inductively that $C_i \geq C_{i+1} > 0$ for $i \leq n$. Hence $0 \leq Z/W \leq 1$. If $\sigma_k = -1$, (3.4) applied to the prefix of length $k-1$ gives

$$0 \leq \frac{V}{U} \leq 1 - \frac{1}{k}; \tag{3.6}$$

for $k = 1$, take $V/U = 0$. Dividing (3.5) by UW and considering the four choices of (σ_k, σ_{k+1}) gives

$$\frac{q_n}{UW} = a_k + \sigma_k \frac{V}{U} + \sigma_{k+1} \frac{Z}{W} \geq \frac{1}{k}.$$

Indeed, when $\sigma_{k+1} = -1$, one has $a_k \geq 2$; when $\sigma_k = -1$, use (3.6). Therefore $UW \leq kq_n \leq nq_n$. Since increasing a_k by ξ changes the right side of (3.5) by $\xi UW \geq 0$, both inequalities in (3.2) follow. □

We next establish conversion, growth, and separation estimates for the regular bounded-digit set E_L defined in Section 2.2. For each integer $L \geq 1$, put

$$\mathcal{K}_L = \{[0; b_1, b_2, \ldots] : 1 \leq b_1 \leq L + 1, \quad 1 \leq b_j \leq L \ (j \geq 2)\},$$

$$\phi_{\varepsilon,a}(t) = \frac{1}{a + \varepsilon t}.$$

Lemma 3.2 (Conversion to bounded SRCF digits)**.** *Let $L \geq 1$ be an integer. Every $x \in E_L$ has a σ-SRCF expansion with $a_j(x) \leq L + 2$ for every $j \geq 1$. Its tails z_j, defined by*

$$x = a_0 + \sigma_1 z_1, \qquad z_j = \phi_{\sigma_{j+1}, a_j(x)}(z_{j+1}),$$

belong to $\mathcal{K}_L$. The corresponding sequence of branches contains at most L consecutive copies of $h = \phi_{-1,2}$.

Proof. For $z = [0; b_1, b_2, \ldots] \in \mathcal{K}_L$, one has

$$z = \phi_{\varepsilon,a}(z^+), \qquad (a, z^+) = \begin{cases} (b_1, [0; b_2, b_3, \ldots]), & \varepsilon = 1, \\ (b_1 + 1, 1 - [0; b_2, b_3, \ldots]), & \varepsilon = -1. \end{cases} \tag{3.7}$$

The identities

$$1 - [0; c_1, c_2, \ldots] = \begin{cases} [0; 1, c_1 - 1, c_2, \ldots], & c_1 \geq 2, \\ [0; c_2 + 1, c_3, \ldots], & c_1 = 1 \end{cases} \tag{3.8}$$

show that $z^+ \in \mathcal{K}_L$. They also show that both x and $1 - x$ belong to $\mathcal{K}_L$ whenever $x \in E_L$. Start with $z_1 = x$ if $\sigma_1 = 1$, and $z_1 = 1 - x$ otherwise. Applying (3.7) successively with $\varepsilon = \sigma_{j+1}$ gives the stated tails and digits, with $a_j \leq L + 2$ and $a_j + \sigma_{j+1} \geq 1$.

Equality in the last inequality occurs only for the branch $h = \phi_{-1,2}$. Such a branch occurs when the current regular expansion begins with $(1, c, \ldots)$ and the prescribed sign is minus. If $c \geq 2$, (3.8) changes this beginning to $(1, c - 1, \ldots)$; if $c = 1$, the next first digit is at least 2. Since $c \leq L$, at most L consecutive copies of h can occur. Thus the admissibility inequality is strict infinitely often, so the constructed sequence is the irrational σ-SRCF coding of x. □

Lemma 3.3 (Uniform denominator growth)**.** *For each integer $L \geq 1$, there are constants $C_L \geq 1$ and $\lambda_L > 1$, independent of the sign sequence, such that*

$$q_n^{\sigma}(x) \geq C_L^{-1} \lambda_L^n \qquad (x \in E_L, \ n \geq 1). \tag{3.9}$$

Proof. Put $g = \phi_{1,1}$, $h = \phi_{-1,2}$, and $N = L + 2$. Every branch arising in Lemma 3.2 maps $[0, 1]$ into itself and has absolute derivative at most 1 there. Equality can occur only for g at 0, or for h at 1. If a composition of N such branches has derivative norm 1, the chain rule therefore forces it to be h^N or $h^{N-1} \circ g$. Indeed, no branch takes a value in $[0, 1]$ to 0, so a g attaining equality must be the innermost branch; all outer branches must then be h. Both compositions contain more than L consecutive copies of h, contrary to Lemma 3.2.

There are only finitely many branch words of length N with digits at most $L + 2$ and signs in $\{\pm 1\}$. Consequently, for some $0 < \rho_L < 1$, every word of length N realized by a point of E_L, at any position and for any sign sequence, has derivative norm at most ρ_L. For the first n branches, the tail formula and the determinant identity give

$$\left| \frac{d}{dt} \frac{p_n + \sigma_{n+1} t p_{n-1}}{q_n + \sigma_{n+1} t q_{n-1}} \right|_{t=0} = q_n^{-2}.$$

Splitting the composition into blocks of length N yields $q_n^{-2} \le \rho_L^{\lfloor n/N \rfloor}$. Thus (3.9) holds with

$$\lambda_L = \rho_L^{-1/(2N)}, \qquad C_L = \rho_L^{-1/2}.$$

□

For a fixed sequence $\Delta = (\Delta_i)_{i \ge 1}$ of nonnegative integers, define the coordinate map $G_\Delta : \mathcal{I} \to \mathcal{I}$ by

$$a_{i,\sigma}(G_\Delta(x)) = a_{i,\sigma}(x) + \Delta_i \qquad (i \ge 1). \tag{3.10}$$

This map keeps the sign sequence and every coordinate position fixed. Increasing digits preserves the admissibility inequalities, including strictness at infinitely many indices, so the irrational SRCF coding makes G_Δ well defined. Uniqueness of this coding makes it injective. In particular, two expansions and their images have the same maximal common-prefix length. For a finite word $w = (a_1, \dots, a_m)$, write $w^\Delta = (a_1 + \Delta_1, \dots, a_m + \Delta_m)$.

Lemma 3.4 (Separation before and after digit changes)**.** *Let $L \ge 1$ be an integer, and let distinct $x, y \in E_L$ have maximal common σ-prefix length $n \ge 0$. Then there are $\ell \in \{1, 2\}$ and an admissible word w of length $n + \ell$ extending this common prefix such that:*

(1) The closure of $I^\sigma_{n+\ell}(w)$ lies strictly between x and y, and the last ℓ digits of w are at most $L + 4$;
(2) For every sequence Δ of nonnegative integers, the closure of $I^\sigma_{n+\ell}(w^\Delta)$ lies strictly between $G_\Delta(x)$ and $G_\Delta(y)$. Its digit at position $n + i$ is at most $L + 4 + \Delta_{n+i}$, for $1 \le i \le \ell$.

Proof. Interchanging x and y if necessary, write

$$d = a_{n+1}(x) < a_{n+1}(y) = e, \qquad \varepsilon = \sigma_{n+2}.$$

Let $t_d = z_{n+2}(x)$ and $t_e = z_{n+2}(y)$ be the tails after these first unequal digits, as in Lemma 3.2. Both belong to $\mathcal{K}_L$, every member of which is greater than $1/(L + 2)$. Before applying the common-prefix map, the two points are $\phi_{\varepsilon,d}(t_d)$ and $\phi_{\varepsilon,e}(t_e)$. The common-prefix map is monotone; for $n = 0$, it is the affine map $t \mapsto a_0 + \sigma_1 t$.

If $e \ge d + 2$, append the single digit $d + 1$ to the common prefix. For $\varepsilon = 1$, its branch interval has endpoints $1/(d + 2)$ and $1/(d + 1)$, with the e-branch point below and the d-branch point above these

endpoints. For $\varepsilon = -1$, its endpoints are $1/(d+1)$ and $1/d$, again with the two points on opposite sides. This gives a separator of depth $\ell = 1$. Adding the same Δ_{n+1} to the three digits preserves these inequalities, proving both claims in this case.

Suppose $e = d + 1$. Append $(e, L + 4)$ if $\varepsilon = 1$, and $(d, L + 4)$ if $\varepsilon = -1$. A separator tail after the first new digit has the form

$$t_s = \frac{1}{L + 4 + \sigma_{n+3} t}, \qquad 0 < t < 1,$$

where t is the tail after the second new digit. Thus $t_s < 1/(L+3) < 1/(L+2) < t_d, t_e$. For the two choices of ε, respectively, this gives

$$\phi_{1,e}(t_e) < \phi_{1,e}(t_s) < \frac{1}{e} < \phi_{1,d}(t_d), \tag{3.11}$$

and

$$\phi_{-1,e}(t_e) < \frac{1}{d} < \phi_{-1,d}(t_s) < \phi_{-1,d}(t_d). \tag{3.12}$$

The same strict separation holds at the separator cylinder's endpoints, where t is 0 or 1, since $t_s \leq 1/(L+3) < t_d, t_e$. The chosen digits are admissible and at most $L+4$, so this proves the first claim with $\ell = 2$.

For the second claim, let t_d^Δ, t_e^Δ denote the corresponding tails of the image points, and let t_s^Δ denote a tail of the modified separator. Then

$$t_s^\Delta = \frac{1}{L + 4 + \Delta_{n+2} + \sigma_{n+3} t} < \frac{1}{L + 3 + \Delta_{n+2}} \qquad (0 < t < 1). \tag{3.13}$$

The tail of the image point sharing the separator's first new digit begins with $b + \Delta_{n+2}$, where $b \leq L+2$. If $\sigma_{n+3} = 1$, this tail is greater than $1/(b+\Delta_{n+2}+1)$; if $\sigma_{n+3} = -1$, it is greater than $1/(b+\Delta_{n+2})$. In either case it is strictly greater than $1/(L + 3 + \Delta_{n+2})$, hence greater than t_s^Δ, including the latter's endpoint values. Replacing d, e by $d + \Delta_{n+1}, e + \Delta_{n+1}$ in (3.11) or (3.12) now proves separation after applying the modified common-prefix map. All preceding digits have also been increased at their original positions, so the resulting word is exactly w^Δ, as required. □

The next lemma shows that sufficiently sparse coordinate changes do not decrease Hausdorff dimension.

Lemma 3.5 (Sparse digit perturbation)**.** *Let $L \geq 1$ be an integer, let $S = \{m_1 < m_2 < \cdots\} \subset \mathbb{N}$, and let $c_j \in \mathbb{Z}_{\geq 0}$. Suppose that*

$$R_n := \sum_{m_j \leq n} \log((n+1)(c_j+1)) = o(n). \tag{3.14}$$

Define $\Delta_{m_j} = c_j$ and $\Delta_i = 0$ for $i \notin S$, and let $G = G_\Delta|_{E_L}$. Then

$$\dim_H G(E_L) \geq \dim_H E_L. \tag{3.15}$$

Proof. Condition (3.14) also implies $\#(S \cap [1, n]) = o(n)$, since

$$\#(S \cap [1, n]) \log(n+1) \leq R_n.$$

The inverse of G subtracts c_j at coordinate m_j. Let q_n and Q_n be the denominators before and after the changes. Repeated application of (3.2) gives

$$q_n \le Q_n \le q_n \prod_{m_j \le n} (1 + nc_j) \le q_n e^{R_n}. \tag{3.16}$$

Take distinct $u = G(x)$, $v = G(y)$, and let n be their maximal common σ-prefix length, which is also that of x, y. For $n \ge 1$, the cylinder estimate gives

$$|x - y| \le (n + 1)q_n^{-2}. \tag{3.17}$$

By Lemma 3.4, a modified separator cylinder of order $n + \ell$, with $\ell \in \{1, 2\}$, lies between u and v. Its last digits are bounded by $L + 4 + \Delta_{n+i}$, for $1 \le i \le \ell$. If $Q^*_{n+\ell}$ is its denominator, the recurrence and Lemma 3.1 give

$$Q^*_{n+\ell} \le Q_n \prod_{i=1}^{\ell} (L + 4 + \Delta_{n+i} + n + 2) \le q_n e^{\omega_n}, \tag{3.18}$$

where

$$\omega_n = R_n + R_{n+2} + 2\log(n + L + 6) = o(n).$$

Indeed, each factor in the product is at most $(n + L + 6)(\Delta_{n+i} + 1)$, and the sum of the logarithms of $\Delta_{n+i} + 1$ is bounded by R_{n+2}. The lower cylinder estimate in (3.1) now yields

$$|u - v| \ge \frac{1}{(n + \ell + 1)(Q^*_{n+\ell})^2} \ge q_n^{-2} \exp(-2\omega_n - \log(n + 3)). \tag{3.19}$$

All error terms here depend only on L and the prescribed increments, not on x, y or the separator chosen.

Fix $\varepsilon > 0$ and choose $\theta > 0$ so small that $2 - \theta \ge (2 + \theta)/(1 + \varepsilon)$. The exponential bound (3.9), together with (3.17) and (3.19), gives, uniformly for all sufficiently large n,

$$|x - y| \le q_n^{-(2-\theta)}, \qquad |u - v| \ge q_n^{-(2+\theta)}.$$

Consequently, $|x - y| \le |u - v|^{1/(1+\varepsilon)}$ for such pairs. For bounded common-prefix lengths, including $n = 0$, only finitely many modified separator words can occur: the original digits are bounded by $L + 2$, the separator depth is at most 2, and the increments at these finitely many positions are fixed. If such pairs exist, the corresponding cylinder lengths have a positive minimum. Using $|x - y| \le 1$, we can therefore choose a constant C_ε covering these remaining pairs as well. Thus

$$|G^{-1}(u) - G^{-1}(v)| \le C_\varepsilon |u - v|^{1/(1+\varepsilon)} \qquad (u, v \in G(E_L)).$$

The Hausdorff-dimension inequality for a Hölder map [5, Proposition 3.3] implies

$$\dim_H E_L \le (1 + \varepsilon) \dim_H G(E_L).$$

Letting $\varepsilon \downarrow 0$ proves (3.15). □

Proof of Theorem 1.1. Suppose first that $\alpha = 0$. By Lemma 3.2, every point of E_L has σ-SRCF digits bounded by $L + 2$. Since $\varphi(n) \to \infty$, it follows that $E_L \subset E_\varphi^\sigma(0)$ for every integer $L \geq 8$. Consequently, (2.2) gives

$$\dim_H E_\varphi^\sigma(0) \geq \dim_H E_L \geq 1 - \frac{1}{L \log 2}.$$

Letting $L \to \infty$ proves the assertion for $\alpha = 0$.

Assume next that $0 < \alpha < \infty$. For each integer $k \geq 2$, put

$$\mathcal{D}_k = \{2^k, \ldots, 2^{k+1} - 1\}, \qquad \tau_k = k^{-1/2}.$$

Within $\mathcal{D}_k$, start with $m_{k,1} = 2^k$. Having chosen $m_{k,i}$, choose $m_{k,i+1}$ to be the least integer in $\mathcal{D}_k$ larger than $m_{k,i}$ for which

$$\varphi(m_{k,i+1}) > (1 + \tau_k)\varphi(m_{k,i}), \tag{3.20}$$

if such an integer exists; otherwise stop. Denote the number of selected positions in $\mathcal{D}_k$ by N_k, and enumerate their union as

$$S = \bigcup_{k \geq 2} \{m_{k,1}, \ldots, m_{k,N_k}\} = \{m_1 < m_2 < \cdots\}.$$

At the selected position m_j, prescribe the increment

$$c_j = \lfloor \alpha \varphi(m_j) \rfloor. \tag{3.21}$$

For $n \in \mathcal{D}_k$, let $\nu(n)$ be the last selected position in $\mathcal{D}_k$ not exceeding n. Such a position exists because 2^k is selected. It is also the last element of S not exceeding n. By the minimality in (3.20) and the stopping rule,

$$1 \leq \frac{\varphi(n)}{\varphi(\nu(n))} \leq 1 + \tau_k. \tag{3.22}$$

In particular, any jump exceeding this factor is itself a selected position, so (3.22) does not require a bound on individual jumps of φ.

Iterating (3.20) within $\mathcal{D}_k$ gives

$$(N_k - 1) \log(1 + \tau_k) \leq \log \frac{\varphi(2^{k+1})}{\varphi(2^k)} = O(k).$$

The non-strict inequality also covers $N_k = 1$, including a flat block. The last estimate follows from monotonicity, $\varphi(n) \to \infty$, and the polynomial upper bound implied by (1.1). Since $\log(1 + \tau_k) \asymp k^{-1/2}$, we obtain

$$N_k = O(k^{3/2}). \tag{3.23}$$

Summing over the dyadic blocks yields

$$\#(S \cap [1, 2^R]) = O(R^{5/2}) = o(2^R). \tag{3.24}$$

Also, (3.21) and the polynomial upper bound on φ imply that $\log(c_j + 1) = O(R)$ uniformly for $m_j \leq 2^R$. Thus,

$$\sum_{m_j \leq 2^R} \log((2^R + 1)(c_j + 1)) = O(R^{7/2}) = o(2^R). \tag{3.25}$$

For an arbitrary sufficiently large n, choose R so that $2^{R-1} < n \le 2^R$. The logarithmic sum in (3.14) is bounded by its counterpart at 2^R, and $R^{7/2} = o(n)$. Hence (3.25) verifies (3.14).

Fix an integer $L \ge 8$. Apply Lemma 3.5 to E_L, and write $F_L = G(E_L)$. Its unmodified core digits are at most $L + 2$, by Lemma 3.2. Since (c_j) is non-decreasing, for $n \ge 4$ with $\nu(n) = m_j$ we have

$$c_j + 1 \le T_n^{\sigma}(x) \le c_j + L + 2 \qquad (x \in F_L). \tag{3.26}$$

The lower bound comes from the digit at m_j; the upper bound holds because every earlier prescribed increment is at most c_j. Combining (3.22), (3.21), and (3.26) gives, for $n \in \mathcal{D}_k$,

$$\frac{\alpha}{1+\tau_k} \le \frac{T_n^{\sigma}(x)}{\varphi(n)} \le \alpha + \frac{L+2}{\varphi(n)} \qquad (x \in F_L).$$

Both bounds tend to α, uniformly for $x \in F_L$. Therefore $F_L \subset E_{\varphi}^{\sigma}(\alpha)$, and Lemma 3.5 together with (2.2) gives

$$\dim_H E_{\varphi}^{\sigma}(\alpha) \ge \dim_H F_L \ge \dim_H E_L \ge 1 - \frac{1}{L \log 2}.$$

Letting $L \to \infty$ proves the assertion for every finite $\alpha > 0$.

Finally, suppose that $\alpha = +\infty$. Use the same set S, selected using φ alone, and replace (3.21) by

$$k_j = \lfloor \log_2 m_j \rfloor, \qquad c_j = \lfloor k_j \varphi(m_j) \rfloor. \tag{3.27}$$

Both (k_j) and $(\varphi(m_j))$ are positive and non-decreasing, so (c_j) is non-decreasing. If $m_j \le 2^R$, then

$$c_j + 1 \le R\varphi(2^R) + 1,$$

and hence $\log(c_j + 1) = O(R)$ uniformly over these indices. The count (3.24) is unchanged, and the same logarithmic estimate (3.25) holds for the new increments. Thus (3.14) again holds.

For every integer $L \ge 8$, Lemma 3.5 now produces a set $F_L^{(\infty)}$ with $\dim_H F_L^{(\infty)} \ge \dim_H E_L$. If $n \in \mathcal{D}_k$ and $\nu(n) = m_j$, then $k_j = k$. The digit at m_j and (3.22) therefore give

$$\frac{T_n^{\sigma}(x)}{\varphi(n)} \ge \frac{c_j + 1}{\varphi(n)} \ge \frac{k\varphi(m_j)}{\varphi(n)} \ge \frac{k}{1+\tau_k} \qquad (x \in F_L^{(\infty)}). \tag{3.28}$$

The last expression tends to $+\infty$ as $n \to \infty$, so $F_L^{(\infty)} \subset E_{\varphi}^{\sigma}(+\infty)$. Using (2.2) once more, we obtain

$$\dim_H E_{\varphi}^{\sigma}(+\infty) \ge \dim_H F_L^{(\infty)} \ge 1 - \frac{1}{L \log 2}.$$

Letting $L \to \infty$ completes the proof. □

Author contributions

Aiken Kazin: Formal analysis, Investigation, Methodology, Validation, Visualization, and Writing–original draft; Shirali Kadyrov: Conceptualization, Project administration, Supervision, Methodology, Writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

During the preparation of this manuscript, the authors used OpenAI's GPT model (Astra), to assist with proofreading, drafting and revising text, and checking mathematical arguments and notation throughout the manuscript. The authors take full responsibility for the content of the manuscript.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. D. Barbolosi, C. Faivre, Sur les grands quotients partiels du développement en fraction continue, *J. Number Theory*, **106** (2004), 299–325. https://doi.org/10.1016/j.jnt.2003.12.008
2. J. Chang, H. Chen, Slow increasing functions and the largest partial quotients in continued fraction expansions, *Math. Proc. Camb. Philos. Soc.*, **164** (2018), 1–14. https://doi.org/10.1017/S0305004116000815
3. K. Dajani, C. Kraaikamp, The mother of all continued fractions, *Colloq. Math.*, **84/85** (2000), 109–123.
4. D. A. Dastjerdi, S. Lamei, Dimension of certain sets of regular and minus continued fractions with positive partial quotients, *Arab. J. Math.*, **1** (2012), 139–148. https://doi.org/10.1007/s40065-012-0015-4
5. K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, 3 Eds, Chichester: John Wiley & Sons, 2014.
6. L. Fang, J. Liu, On the largest partial quotients in continued fraction expansions, *Fractals*, **29** (2021), 2150099. https://doi.org/10.1142/S0218348X21500997
7. J. Galambos, The distribution of the largest coefficient in continued fraction expansions, *Q. J. Math.*, **23** (1972), 147–151. https://doi.org/10.1093/qmath/23.2.147
8. J. Galambos, An iterated logarithm type theorem for the largest coefficient in continued fractions, *Acta Arith.*, **25** (1974), 359–364.
9. I. J. Good, The fractional dimensional theory of continued fractions, *Proc. Camb. Philos. Soc.*, **37** (1941), 199–228. https://doi.org/10.1017/S030500410002171X
10. D. Hensley, The Hausdorff dimensions of some continued fraction Cantor sets, *J. Number Theory*, **33** (1989), 182–198. https://doi.org/10.1016/0022-314X(89)90005-X
11. D. Hensley, Continued fraction Cantor sets, Hausdorff dimension, and functional analysis, *J. Number Theory*, **40** (1992), 336–358. https://doi.org/10.1016/0022-314X(92)90006-B
12. M. Iosifescu, C. Kraaikamp, *Metrical Theory of Continued Fractions*, Dordrecht: Kluwer Academic Publishers, 2002.
13. V. Jarník, Zur metrischen Theorie der diophantischen Approximationen, *Prace Matematyczno-Fizyczne*, **36** (1928–1929), 91–106.

14. S. Kadyrov, A. Kazin, S. Duisen, A note on semi-regular continued fractions with bounded partial quotients, *Arab. J. Math.*, **15** (2026), 1571–1580. https://doi.org/10.1007/s40065-025-00611-2
15. S. Kadyrov, A. Kazin, F. Mashurov, Semi-regular continued fractions with fast-growing partial quotients, *Fractal Fract.*, **8** (2024), 436. https://doi.org/10.3390/fractalfract8080436
16. A. Ya. Khinchin, *Continued Fractions*, Mineola, NY: Dover Publications, 1997.
17. N. Langeveld, D. Ralston, A renormalization scheme for semi-regular continued fractions, *Monatsh. Math.*, **206** (2025), 107–142. https://doi.org/10.1007/s00605-024-02031-4
18. L. Liao, M. Rams, Subexponentially increasing sums of partial quotients in continued fraction expansions, *Math. Proc. Camb. Philos. Soc.*, **160** (2016), 401–412. https://doi.org/10.1017/S0305004115000742
19. Y. Nakajima, H. Takahasi, Hausdorff dimension of sets with restricted, slowly growing partial quotients in semi-regular continued fractions, *J. Math. Soc. Japan*, **77** (2025), 903–916. https://doi.org/10.2969/jmsj/93059305
20. T. Okano, Explicit continued fractions with expected partial quotient growth, *Proc. Amer. Math. Soc.*, **130** (2002), 1603–1605. https://doi.org/10.1090/S0002-9939-02-06337-2
21. O. Perron, *Die Lehre von den Kettenbrüchen. Band I: Elementare Kettenbrüche*, 3 Eds, Stuttgart: B. G. Teubner, 1954.
22. W. Philipp, A conjecture of Erdős on continued fractions, *Acta Arith.*, **28** (1976), 379–386.
23. L. Shang, M. Wu, Some results on the largest partial quotient in continued fractions, *Fractals*, **30** (2022), 2250095. https://doi.org/10.1142/S0218348X22500955
24. H. Takahasi, The distribution of the largest digit for parabolic iterated function systems of the interval, *J. Fractal Geom.*, **13** (2026), 151–169. https://doi.org/10.4171/jfg/172
25. B. Tan, C. Tian, B. Wang, The distribution of the large partial quotients in continued fraction expansions, *Sci. China Math.*, **66** (2023), 935–956. https://doi.org/10.1007/s11425-021-1979-3
26. B. Wang, J. Wu, Hausdorff dimension of certain sets arising in continued fraction expansions, *Adv. Math.*, **218** (2008), 1319–1339. https://doi.org/10.1016/j.aim.2008.03.006
27. J. Wu, J. Xu, The distribution of the largest digit in continued fraction expansions, *Math. Proc. Camb. Philos. Soc.*, **146** (2009), 207–212. https://doi.org/10.1017/S0305004108001771



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)