AIMS Mathematics, 11(11): 37976–38018.
DOI: 10.3934/math.20261468
Received: 08 May 2026
Revised: 30 July 2026
Accepted: 07 August 2026
Published: 16 November 2026

https://www.aimspress.com/journal/Math

*Research article*

# Subgradient extragradient-type iterative methods for variational inequality and convex minimization problem

**Maryam Iqbal***

Academy of Computer Science and Software Engineering, University of Johannesburg, South Africa

* **Correspondence:** Email: miqbal@uj.ac.za.

**Abstract:** In this study, I aimed to propose two inertial viscosity-type subgradient extragradient methods to approximate the solution of the variational inequality problems in real Hilbert spaces. These methods hybridize the viscosity and the inertial techniques to speed up the convergence rate. Furthermore, weak convergence results for the proposed algorithms were established under relaxed $\alpha$-monotonicity and Lipschitz continuity assumptions. I present applications of my results to approximate the solution of the convex minimization problem. Moreover, I present numerical experiments with graphical illustrations to demonstrate the efficiency of the proposed methods. My results generalize and complement several results in the literature.

**Keywords:** subgradient extragradient; variational inequality; relaxed $\alpha$-monotone; Hilbert spaces; minimization problem
**Mathematics Subject Classification:** 49J40, 47H09, 49J20, 47H10

## 1. Introduction

Several problems in nonlinear analysis can be modeled as the variational inequality problem involving certain nonlinear operators. However, it is not always possible to solve the variational inequality problem analytically. To overcome this issue, iterative algorithms are valuable tools for approximate solutions to variational inequality problems. Variational inequality problems and their solutions are significant due to their applications in signal processing, image restoration, tomography, and radiation therapy. Therefore, the solution of variational inequality and inclusion problems remains the interest of many authors; for more details, I refer to [1–3]. Several efficient extragradient-type methods have been proposed for solving variational inequality problems under different generalized monotonicity assumptions. Long et al. [30] introduced modified subgradient extragradient algorithms equipped with a new line-search rule to improve the convergence performance for variational inequality problems. Moreover, Long et al. [31] proposed a self-adaptive double inertial extragradient

algorithm for solving quasimonotone variational inequality and fixed point problems, together with applications to image restoration. Motivated by these developments, I propose two inertial viscosity-type subgradient extragradient algorithms for variational inequality problems involving relaxed $\alpha$-monotone operators.

## 2. Preliminaries

Throughout this paper, I assume that $\mathcal{Z}_1$ is a nonempty closed and convex subset of a real Hilbert space $\mathcal{Y}$. The sets of real numbers and natural numbers are denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively. Let $\mathcal{D} : \mathcal{Z}_1 \to \mathcal{Y}$ be a mapping. For any real number $a \in \mathbb{R}$, I write

$$[a]_+ := \max\{a, 0\}, \qquad [a]_- := \min\{a, 0\}.$$

The variational inequality problem ($VIP$) is defined as follows:
Find $l_1 \in \mathcal{Z}_1$ such that

$$\langle \mathcal{D}l_1, l_2 - l_1\rangle \geq 0, \text{ for all } l_2 \in \mathcal{Z}_1. \tag{2.1}$$

The $VIP$ was introduced by Stampacchia [19] in finite dimensional space and then extended to infinite dimensional spaces [21].

I recall the following definitions for the sake of completeness:
Let $\mathcal{D} : \mathcal{Y} \to \mathcal{Y}$ be a mapping. Then, for all $l_1, l_2 \in \mathcal{Y}$, $\mathcal{D}$ is known as

- $\lambda$-contraction, if I have a $\lambda \in (0, 1)$, and

$$\|\mathcal{D}l_1 - \mathcal{D}l_2\| \leq \lambda\|l_1 - l_2\|,$$

- $\sigma$-strongly monotone, if I have a $\sigma > 0$, and

$$\langle \mathcal{D}l_1 - \mathcal{D}l_2, l_1 - l_2\rangle \geq \sigma\|l_1 - l_2\|^2,$$

- monotone, if $\langle \mathcal{D}l_1 - \mathcal{D}l_2, l_1 - l_2\rangle \geq 0$,
- pseudomonotone, if $\langle \mathcal{D}l_2, l_1 - l_2\rangle \geq 0$, then $\langle \mathcal{D}l_1, l_1 - l_2\rangle \geq 0$,
- quasimonotone, if $\langle \mathcal{D}l_2, l_1 - l_2\rangle > 0$, then $\langle \mathcal{D}l_1, l_1 - l_2\rangle \geq 0$.

**Definition 1.** *[22] A mapping $\mathcal{D} : \mathcal{Z}_1 \to \mathcal{Y}$ is known as relaxed $\alpha$-monotone if there exists a map $\alpha : \mathcal{Y} \to \mathbb{R}$ satisfying $\alpha(qz) = q^t\alpha(z)$ for all $q > 0$ and $z \in \mathcal{Y}$, where $t > 1$ is a constant, such that*

$$\langle \mathcal{D}l_1 - \mathcal{D}l_2, l_1 - l_2\rangle \geq \alpha(l_1 - l_2), \quad \textit{for all } l_1, l_2 \in \mathcal{Z}_1.$$

As monotonicity has an important role in the theory of variational inequalities, I refer the reader to [22–24].

**Definition 2.** *[7] The metric projection $P_{\mathcal{Z}_1} : \mathcal{Y} \to \mathcal{Z}_1$, is a mapping with*

$$\|l_1 - P_{\mathcal{Z}_1}l_1\| = \inf\{\|l_1 - l_2\| : l_2 \in \mathcal{Z}_1\}.$$

*An important characterization of metric projection $P_{\mathcal{Z}_1}$ is:*

$$P_{\mathcal{Z}_1}l_1 = l \quad \textit{iff} \quad \langle l_1 - l, l_2 - l\rangle \leq 0, \quad \textit{for all } l_2 \in \mathcal{Z}_1. \tag{2.2}$$

For solving variational inequality problems, the following gradient-projection algorithm has been proposed:

$$\omega_{k+1} = P_{\mathcal{Z}_1}(\omega_k - \psi \mathcal{D}\omega_k), \quad k \in \mathbb{N}. \tag{2.3}$$

The sequence $\{\omega_k\}$ converges to the solution of problem (2.1) if $\mathcal{D}$ is $\sigma$-strongly monotone and $\mathcal{L}$-Lipschitzian and the parameter $\psi$ is constrained within the interval $(0, \frac{2\sigma}{\mathcal{L}^2})$.

On the other hand, if the mapping $\mathcal{D}$ is monotone and Lipschitz continuous, then the method given in (2.3) fails to converge to the solution of (2.1). For instance, if I set $\mathcal{Y} = \mathbb{R}^2$ and $\mathcal{D}$ is a rotation map with angle $\frac{\pi}{2}$, then the method given in (2.3) may not converge. To overcome this drawback, Korpelevich [8] introduced the following extragradient algorithm that converges to a solution of problem (2.1) in the case of a monotone and Lipschitzian mapping, which is given as follows:

$$\begin{aligned} \omega_1 &\in \mathcal{Z}_1, \\ v_k &= P_{\mathcal{Z}_1}(\omega_k - \psi_k \mathcal{D}\omega_k), \\ \omega_{k+1} &= P_{\mathcal{Z}_1}(\omega_k - \psi_k \mathcal{D}v_k), \end{aligned}$$

where $\psi_k \in (0, \frac{1}{\mathcal{L}})$.

Many authors have discussed the convergence behavior of this algorithm for monotone and pseudomonotone mappings [9–11]. However, one of the major drawbacks of these methods is the involvement of two projections onto the feasible set at each iteration, which becomes computationally demanding when the feasible set possesses a complicated structure. To overcome this difficulty, a subgradient extragradient algorithm was proposed and its weak convergence was established to a solution of the *VIP* for monotone and Lipschitzian continuous mappings [12].

$$\begin{aligned} \omega_1 &\in \mathcal{Y}, \\ v_k &= P_{\mathcal{Z}_1}(\omega_k - \psi_k \mathcal{D}\omega_k), \\ \mathcal{Z}_{2_k} &= \{w \in \mathcal{Y} : \langle \omega_k - \psi_k \mathcal{D}\omega_k - v_k, w - v_k \rangle \leq 0\}, \\ \omega_{k+1} &= P_{\mathcal{Z}_{2_k}}(\omega_k - \psi_k \mathcal{D}v_k), \quad k \geq 1. \end{aligned}$$

Ogwo et al. [13] introduced a new method to solve the *VIP* for quasi-monotone and Lipschitz continuous mappings and given as follows:

**Algorithm: 1.1**

Choose $\vartheta_k \in [0, 1)$ and $\tau_k \in (0, 1]$, suppose $\psi \in (0, 1/\mathcal{L})$, and let $\omega_0, \omega_1 \in \mathcal{Y}$ be arbitrary.

**Step 1:** For $\omega_{k-1}$ and $\omega_k$, find

$$\rho_k = \omega_k + \vartheta_k(\omega_k - \omega_{k-1}).$$

Find

$$v_k = P_{\mathcal{Z}_1}(\rho_k - \psi \mathcal{D}\rho_k),$$

if $\rho_k = v_k$: then **STOP**. Otherwise move next.

**Step 2:** Calculate

$$\mathcal{Z}_{2_k} = \{w \in \mathcal{Y} : \langle \rho_k - \psi \mathcal{D}\rho_k - v_k, w - v_k \rangle \leq 0\}.$$

Compute

$$t_k = P_{\mathcal{Z}_{2_k}}(\rho_k - \psi \mathcal{D}v_k).$$

Find

$$\omega_{k+1} = (1 - \tau_k)\rho_k + \tau_k t_k.$$

Subgradient extragradient methods are generally computationally cheaper than conventional extragradient methods. The convergence of an algorithm in a fewer number of steps has attracted the interest of many authors [4–6]. Motivated by [12, 13, 23], in this paper, I propose inertial viscosity-type subgradient extragradient methods for relaxed $\alpha$-monotone and Lipschitz continuous operators. My methods exhibit faster convergence to the solution of the *VIP*, and I extend the applicability of the proposed algorithms to convex minimization problems.

Yao et al. [25] obtained convergence results for proximal-type algorithms with reflections for variational inequalities, and Peng et al. [26] developed an extrapolated adaptive proximal method with a single dynamic parameter together with applications to optimization problems. A two-term inertial method for monotone variational inequality problems, with implementations in multiclass classification and portfolio optimization, was studied in [27]. Beyond linear settings, the golden ratio method for monotone variational inequality problems in Hadamard spaces was investigated in [28], while modified Halpern and viscosity methods for hierarchical variational inequalities on Hadamard manifolds were analyzed in [29]. The methods in [25–27] were designed for monotone operators in Hilbert spaces, whereas the schemes in [28, 29] were formulated in nonlinear geodesic spaces. In contrast, the algorithms proposed in this work are developed for relaxed $\alpha$-monotone operators in real Hilbert spaces and combine the inertial and viscosity techniques within the subgradient extragradient framework. Since the underlying assumptions and settings of these works are different from ours, a direct numerical comparison with them would not be meaningful; nevertheless, they indicate useful directions in which my results may be further developed.

In this paper, I denote the weak convergence of a sequence $\{\omega_k\}$ to $\omega^*$ by $\omega_k \rightharpoonup \omega^*$, and the solution set of (2.1) is denoted by $\Upsilon$. Comments about the contributions of this paper are provided below.

**Remark 1.** *If I set $\vartheta_k = 0$ and take $\varphi$ to be the identity mapping, then our proposed Algorithm 2.1 reduces to Algorithm 3.3 given in [17]. Similarly, if $\varphi$ is the identity mapping, then my proposed Algorithm 2.1 reduces to Algorithm 3.3 given in [13]. Observe that the parameters and the involvement of the contraction mapping play a key role to improve the rate of convergence of an iterative scheme, which is also verified in Section 4. Hence, my proposed scheme is a significant improvement over the schemes given in [13, 17]. Moreover, the numerical results in Section 4 show that my Algorithms 2.1 and 2.2 converge faster than Algorithms 3.3 and 3.4 given in [13]. Further, in the study of inertial methods, the parameter is often restricted or required to be non-decreasing with an upper limit. Additionally, many authors assume that the inertial parameter depends on the Lipschitz constant, which may be very difficult to find in practice. As an analogue of this condition, I use another condition that is less restrictive to guarantee the convergence of inertial approaches.*

**Definition 3.** *[13] A mapping $\mathcal{D} : \mathcal{Z}_1 \to \mathcal{Y}$ is said to be sequentially weakly continuous if, for any sequence $\{\omega_k\}$ in $\mathcal{Z}_1$, which converges weakly to $w \in \mathcal{Z}_1$, the sequence $\{\mathcal{D}\omega_k\}$ converges weakly to $\mathcal{D}w$ in $\mathcal{Y}$.*

**Remark 2.** *The proposed Algorithms 2.1 and 2.2 do not take the classical inertial form*

$$\omega_{k+1} = \omega_k + \vartheta_k(\omega_k - \omega_{k-1}).$$

*The following discussion is included only to motivate the summability condition* (2.4). *According to [14], for the classical inertial iteration*

$$\omega_{k+1} = \omega_k + \vartheta_k(\omega_k - \omega_{k-1}),$$

*one has*

$$\omega_{k+1} - \omega_k = \left(\prod_{j=1}^{k} \vartheta_j\right)(\omega_1 - \omega_0),$$

*which implies*

$$\omega_k = \omega_1 + \left(\sum_{m=1}^{k-1}\prod_{j=1}^{m} \vartheta_j\right)(\omega_1 - \omega_0).$$

*Hence, the convergence of the classical inertial sequence is ensured whenever*

$$\sum_{m=1}^{\infty}\prod_{j=1}^{m} \vartheta_j < \infty.$$

*This motivates Assumption* (2.4)*, which is required to define the auxiliary sequence* $\{h_\alpha\}$ *in* (2.5) *and is used throughout the convergence analysis of the proposed algorithms.*

$$\sum_{m=\alpha}^{\infty}\prod_{j=\alpha}^{m} \vartheta_j < \infty, \tag{2.4}$$

*which is adopted in the convergence analysis of the proposed algorithms. Based on this condition, the auxiliary sequence*

$$h_\alpha = \sum_{m=\alpha-1}^{\infty}\prod_{j=\alpha}^{m} \vartheta_j = 1 + \sum_{m=\alpha}^{\infty}\prod_{j=\alpha}^{m} \vartheta_j \tag{2.5}$$

*is well defined, where*

$$\prod_{j=\alpha}^{\alpha-1} \vartheta_j = 1.$$

**Remark 3.** *By [14], the condition* (2.4) *guarantees that the sequence* $\{h_\alpha\}$ *given by* (2.5) *is well-defined, and*

$$h_\alpha = 1 + \vartheta_\alpha h_{\alpha+1}, \text{ for all } \alpha \geq 1. \tag{2.6}$$

*The subsequent result ensures the assumption* (2.4).

**Proposition 1.** *[14] Suppose* $\{\vartheta_k\} \subset [0, 1)$. *If*

$$\lim_{k\to\infty}\left(\frac{1}{1-\vartheta_{k+1}} - \frac{1}{1-\vartheta_k}\right) = r,$$

*where* $r \in [0, 1)$. *Then I get:*

- *The* (2.4) *holds, and* $h_{k+1} \sim \frac{1}{(1-r)(1-\vartheta_k)}$ *as* $k \to \infty$,
- *as* $k \to \infty$, *then equivalence of* $1 - \vartheta_k \sim 1 - \vartheta_{k+1}$ *exists. Hence,* $h_{k+1} \sim h_{k+2}$.

**Remark 4.** *[13] By Proposition 1 , we have* $\vartheta_k = 1 - \frac{\bar{\vartheta}}{k}$, *and* $\bar{\vartheta} > 1$, *is an example that fulfils* (2.4). *Indeed, I have that*

$$\lim_{k\to\infty}\left(\frac{1}{1-\vartheta_{k+1}} - \frac{1}{1-\vartheta_k}\right) = \lim_{k\to\infty}\frac{1}{\bar{\vartheta}}(k+1) - \frac{1}{\bar{\vartheta}}(k) = \frac{1}{\bar{\vartheta}} \in [0,1),$$

*which fulfils the conditions of Proposition 1. Then, by Proposition 1(i), Condition* (2.4) *holds. It is worth noting that many choices of* $\vartheta_k$ *comply with* (2.4) *[14–16]. The related finite sum of* $\{h_\alpha\}$ *is given as: Take* $\alpha, k \geq 1$,

$$h_{\alpha,k} = \begin{cases} \sum_{m=\alpha-1}^{k-1}\left(\prod_{j=\alpha}^{m}\vartheta_j\right) = 1 + \sum_{m=\alpha}^{k-1}\left(\prod_{j=\alpha}^{m}\vartheta_j\right), & \text{if } \alpha \leq k, \\ 0, & \text{otherwise.} \end{cases} \tag{2.7}$$

*In a similar way,* $\{h_{\alpha,k}\}$ *is well-defined, and*

$$h_{\alpha,k} = 1 + \vartheta_\alpha h_{\alpha+1,k}, \quad \text{for all} \quad \alpha \geq 1, \quad k \geq \alpha + 1. \tag{2.8}$$

**Lemma 1.** *[14] Assume that* $\{l_k\}$, $\{\vartheta_k\}$, *and* $\{m_k\}$ *are sequences in* $\mathbb{R}$ *satisfying the following conditions:*

$$l_{k+1} \leq \vartheta_k l_k + m_k.$$

*Suppose* $\vartheta_k \geq 0$.
*(a) Then I get that*

$$\sum_{\alpha=1}^{k} l_\alpha \leq h_{1,k}l_1 + \sum_{\alpha=1}^{k-1} h_{\alpha+1,k}m_\alpha,$$

*where* $\{h_{\alpha,k}\}$ *is as given in* (2.7).
*(b) For* (2.4), *assume that the sequence* $\{h_\alpha\}$ *defined by* (2.7) *such that*

$$\sum_{\alpha=1}^{\infty} h_{\alpha+1}[m_\alpha]_+ < \infty.$$

*Then, the series* $\sum_{\alpha\geq 1}[l_\alpha]_+$ *is convergent. Additionally,*

$$\sum_{\alpha=1}^{\infty}[l_\alpha]_+ \leq h_1[l_1]_+ + \sum_{\alpha=1}^{\infty} h_{\alpha+1}[m_\alpha]_+,$$

*here* $[h]_+ := \max\{h, 0\}$ *and* $h$ *is real.*

**Lemma 2.** *[14] Suppose that* $\{\omega_k\} \subset \mathcal{Y}$ *and* $\{\vartheta_k\} \subset [0,1)$. *For* $t \in \mathcal{Y}$, *if* $\Upsilon_k := \frac{1}{2}\|\omega_k - t\|^2$, *then we get*

$$\begin{aligned} \Upsilon_{k+1} - \Upsilon_k - \vartheta_k(\Upsilon_k - \Upsilon_{k-1}) = \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 + \langle \omega_{k+1} - \rho_k, \rho_k - t\rangle \\ + \frac{1}{2}\|\omega_{k+1} - \rho_k\|^2, \end{aligned} \tag{2.9}$$

*where* $\rho_k = \omega_k + \vartheta_k(\omega_k - \omega_{k-1})$.

Now we recall some well-known results.

**Lemma 3.** *Suppose that* $\{\omega_k\} \subset \mathcal{Y}$ *and* $\omega_k \rightharpoonup w$, *then*

$$\liminf_{k\to\infty}\|\omega_k - w\| < \liminf_{k\to\infty}\|\omega_k - v\|, \quad \forall v \neq w.$$

**Lemma 4.** *If* $w, v \in \mathcal{Y}$, *then*

$$2\langle w, v\rangle = \|w\|^2 - \|w - v\|^2 + \|v\|^2 = \|w + v\|^2 - \|w\|^2 - \|v\|^2.$$

## 3. Convergence analysis

Here, I present an inertial viscosity-type subgradient extragradient method for *VIP*.

Now I have the following assumptions:

Before stating the assumptions, I recall that the sequence $\{h_k\}$ is defined by (2.5) and satisfies the recursive relation (2.6). Moreover, $\vartheta_k \in [0,1)$ denotes the inertial parameter, $\tau_k \in (0,1]$ is the viscosity parameter, and $\zeta_k = \frac{1-\tau_k}{2\tau_k}$ whenever $\tau_k \in (0,1)$. Throughout the paper, $\|\cdot\|$ denotes the norm induced by the inner product of the underlying real Hilbert space $\mathcal{Y}$.

**Assumption 1.** Suppose that the sequences $\{\vartheta_k\} \subset [0,1)$, $\{\tau_k\} \subset (0,1]$ and $\{\zeta_k\} \subset (0,1)$ satisfy $\liminf_{k\to\infty} \tau_k > 0$. Assume there exists $\xi \in (0,1)$ such that, for all sufficiently large $k$, the following inequality holds:

$$\begin{cases} 2\xi\zeta_{k-1}(1-\vartheta_{k-1})\|\omega_k - \varphi(\omega_{k-1})\|^2 \le 2\zeta_{k-1}(1-\vartheta_{k-1})\|\omega_k - \varphi(\omega_{k-1})\|^2 \\ \quad - \vartheta_k h_{k+1}\Big(2[\zeta_k(1-\vartheta_k)]_+\|\varphi(\omega_k - \omega_{k-1})\|^2 - 2[\zeta_{k-1}(1-\vartheta_{k-1})]_-\|\omega_k - \varphi(\omega_{k-1})\|^2 \\ \quad + (1+\vartheta_k+\lambda^2+\lambda)\|\omega_k - \omega_{k-1}\|^2 + (1+\vartheta_k)\|\omega_k - \omega_{k-1} - \varphi(\omega_k - \omega_{k-1})\|^2 \\ \quad + 2\|\omega_k - \varphi(\omega_{k-1})\|^2\Big) - h_{k+1}\|\omega_k - \varphi(\omega_k)\|^2, \text{ if } \tau_k \in (0,1), \\ \\ \xi(1-\vartheta_{k-1})\|\omega_k - \varphi(\omega_{k-1})\|^2 \le (1-\vartheta_{k-1})\|\omega_k - \varphi(\omega_{k-1})\|^2 \\ \quad -\vartheta_k h_{k+1}\Big([1-\vartheta_k]_+\|\varphi(\omega_k - \omega_{k-1})\|^2 - [1-\vartheta_{k-1}]_-\|\omega_k - \varphi(\omega_{k-1})\|^2 \\ \quad +(1+\vartheta_k+\lambda^2+\lambda)\|\omega_k - \omega_{k-1}\|^2 + (1+\vartheta_k)\|\omega_k - \omega_{k-1} - \varphi(\omega_k - \omega_{k-1})\|^2 \\ \quad +2\|\omega_k - \varphi(\omega_{k-1})\|^2\Big) - h_{k+1}\|\omega_k - \varphi(\omega_k)\|^2, \text{ if } \tau_k = 1. \end{cases} \tag{3.1}$$

Assumption 1 is a technical condition used in the convergence analysis of the proposed algorithms. It controls the additional terms generated by the inertial and viscosity components and provides the recursive estimates required to establish the boundedness and convergence of the generated sequence. It plays the role of the decrease conditions employed in the analysis of relaxed inertial methods in [14], and it involves only quantities that are generated by the algorithm, namely the iterates, the contraction $\varphi$, and parameters $\vartheta_k$, $\tau_k$, $\zeta_k$, and $h_{k+1}$. Consequently, it can be monitored along the iterations, and it holds in particular when the weighted distances $\|\omega_k - \varphi(\omega_{k-1})\|$ decay sufficiently fast in comparison with the inertial weights $\vartheta_k h_{k+1}$. This assumption is introduced for the theoretical analysis and is not intended as an additional restriction on the *VIP*.

**Assumption 2.** Moreover, we have the following assumptions:

(a) $\Upsilon \neq \emptyset$.

(b) $\mathcal{D}$ is relaxed $\alpha$-monotone and Lipschitzian on the set $\mathcal{Y}$, and it is sequentially weakly continuous on the subset $\mathcal{Z}_1$.

(c) The set $\{t \in \mathcal{Z}_1 : \mathcal{D}t = 0\} \setminus \Upsilon$ is finite.

(d) Let $\varphi : \mathcal{Z}_1 \to \mathcal{Y}$ be a linear and $\lambda$-contraction mapping satisfying

$$\|\rho_k - \varphi(t)\| \le \|\rho_k - t\|, \qquad \forall\, t \in \mathcal{Z}_1.$$

**Algorithm 2.1** Proposed algorithm with fixed step size.

Choose $\vartheta_k \in [0, 1)$ and $\tau_k \in (0, 1]$, assume $\psi \in (0, 1/\mathcal{L})$ and $\omega_0, \omega_1 \in \mathcal{Y}$ are arbitrary.

**Step 1:** Find

$$\rho_k = \omega_k + \vartheta_k(\omega_k - \omega_{k-1}).$$

Find

$$v_k = P_{\mathcal{Z}_1}(\varphi(\rho_k) - \psi\mathcal{D}\varphi(\rho_k)).$$

**Step 2:** Construct

$$\mathcal{Z}_{2_k} = \{w \in \mathcal{Y} : \langle \varphi(\rho_k) - \psi\mathcal{D}\varphi(\rho_k) - v_k, w - v_k\rangle \leq 0\}.$$

Compute

$$t_k = P_{\mathcal{Z}_{2_k}}(\varphi(\rho_k) - \psi\mathcal{D}v_k).$$

Compute

$$\omega_{k+1} = (1 - \tau_k)\varphi(\rho_k) + \tau_k\mathcal{D}t_k.$$

## 4. Results

**Theorem 1.** *Assume that $\{\omega_k\}$ is generated by Algorithm 2.1 and that Assumption 2(a),(b) holds. Then I get that:*

$$\begin{aligned}
&\Upsilon_{k+1} - \Upsilon_k - \vartheta_k(\Upsilon_k - \Upsilon_{k-1}) \\
&\leq \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 - \zeta_k\|\omega_{k+1} - \varphi(\rho_k)\|^2 - \frac{\tau_k}{2}(1 - \psi\mathcal{L})\Big[\|\varphi(\rho_k) - v_k\|^2 \\
&\quad + \|\mathcal{D}t_k - v_k\|^2\Big] + \frac{1}{2}\|\rho_k - \varphi(\rho_k)\|^2,
\end{aligned}$$

*where $\zeta_k = \frac{1-\tau_k}{2\tau_k}$ and $\Upsilon_k = \frac{1}{2}\|\omega_k - t\|^2$, for all $t \in \Upsilon$.*

*Proof.* Let $t \in \Upsilon$. Using Algorithm 2.1, I have

$$\omega_{k+1} - \varphi(\rho_k) = \tau_k(\mathcal{D}t_k - \varphi(\rho_k)). \tag{4.1}$$

By (4.1), I have

$$\|\omega_{k+1} - \varphi(\rho_k)\|^2 = \tau_k^2\|\mathcal{D}t_k - \varphi(\rho_k)\|^2.$$

Hence,

$$\tau_k^{-1}\|\omega_{k+1} - \varphi(\rho_k)\|^2 = \tau_k\|\mathcal{D}t_k - \varphi(\rho_k)\|^2. \tag{4.2}$$

Using the characterization (2.2) of $P_{\mathcal{Z}_{2_k}}$, I have

$$\tau_k\langle \varphi(\rho_k) - \psi\mathcal{D}v_k - \mathcal{D}t_k, t - \mathcal{D}t_k\rangle \leq 0. \tag{4.3}$$

Using (4.1), (4.2) and the contraction condition, I have

$$\langle \omega_{k+1} - \rho_k, \rho_k - t\rangle$$

$$
\begin{aligned}
&= \langle \omega_{k+1} - \varphi(\rho_k), \rho_k - t\rangle + \langle \varphi(\rho_k) - \rho_k, \rho_k - t\rangle \\
&\leq -\tau_k \langle \mathcal{D}t_k - \varphi(\rho_k), \mathcal{D}t_k - \rho_k + t - \mathcal{D}t_k\rangle + \lambda\|\rho_k - t\|^2 + \langle \varphi(t) - \rho_k, \rho_k - t\rangle \\
&= -\tau_k \langle \mathcal{D}t_k - \varphi(\rho_k), \mathcal{D}t_k - \varphi(\rho_k)\rangle - \tau_k \langle \mathcal{D}t_k - \varphi(\rho_k), \varphi(\rho_k) - \rho_k + t - \mathcal{D}t_k\rangle \\
&\quad + \lambda\|\rho_k - t\|^2 + \langle \varphi(t) - \rho_k, \rho_k - t\rangle.
\end{aligned}
$$

According to Assumption 2(d), I get

$$
\begin{aligned}
&\langle \omega_{k+1} - \rho_k, \rho_k - t\rangle \\
&= -\tau_k \|\mathcal{D}t_k - \varphi(\rho_k)\|^2 - \tau_k \langle \mathcal{D}t_k - \varphi(\rho_k), \varphi(\rho_k) - \rho_k\rangle - \tau_k \langle \mathcal{D}t_k - \varphi(\rho_k), t - \mathcal{D}t_k\rangle \\
&\quad + \lambda\|\rho_k - t\|^2 - \langle \rho_k - \varphi(t), \rho_k - t\rangle \\
&\leq \tau_k^{-1}\|\omega_{k+1} - \varphi(\rho_k)\|^2 - \tau_k \langle \mathcal{D}t_k - \varphi(\rho_k), \varphi(\rho_k) - \rho_k\rangle \\
&\quad - \tau_k \langle \mathcal{D}t_k + \psi\mathcal{D}v_k - \psi\mathcal{D}v_k - \varphi(\rho_k), t - \mathcal{D}t_k\rangle + \lambda\|\rho_k - t\|^2 - \|\rho_k - t\|^2 \\
&= \tau_k^{-1}\|\omega_{k+1} - \varphi(\rho_k)\|^2 - \tau_k \langle \mathcal{D}t_k - \varphi(\rho_k), \varphi(\rho_k) - \rho_k\rangle - \tau_k \langle -\psi\mathcal{D}v_k, t - \mathcal{D}t_k\rangle \\
&\quad - \tau_k \langle \mathcal{D}t_k + \psi\mathcal{D}v_k - \varphi(\rho_k), t - \mathcal{D}t_k\rangle + (\lambda - 1)\|\rho_k - t\|^2 \\
&\leq \tau_k^{-1}\|\omega_{k+1} - \varphi(\rho_k)\|^2 - \tau_k \langle \mathcal{D}t_k - \varphi(\rho_k), \varphi(\rho_k) - \rho_k\rangle + \psi\tau_k \langle \mathcal{D}v_k, t - \mathcal{D}t_k\rangle \\
&\quad + \tau_k \langle \varphi(\rho_k) - \psi\mathcal{D}v_k - \mathcal{D}t_k, t - \mathcal{D}t_k\rangle.
\end{aligned}
$$

Using (4.3) and $t \in \Upsilon$, I get

$$
\begin{aligned}
&\langle \omega_{k+1} - \rho_k, \rho_k - t\rangle \\
&\leq \tau_k^{-1}\|\omega_{k+1} - \varphi(\rho_k)\|^2 - \tau_k \langle \mathcal{D}t_k - \varphi(\rho_k), \varphi(\rho_k) - \rho_k\rangle + \psi\tau_k \langle \mathcal{D}v_k, t - \mathcal{D}t_k\rangle.
\end{aligned} \tag{4.4}
$$

By (4.4) and (4.2) in Lemmas 2 and 4, respectively, I get

$$
\begin{aligned}
&\Upsilon_{k+1} - \Upsilon_k - \vartheta_k(\Upsilon_k - \Upsilon_{k-1}) \\
&\leq \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 + \frac{1}{2}\|\omega_{k+1} - \rho_k\|^2 - \tau_k^{-1}\|\omega_{k+1} - \varphi(\rho_k)\|^2 \\
&\quad - \tau_k \langle \mathcal{D}t_k - \varphi(\rho_k), \varphi(\rho_k) - \rho_k\rangle + \psi\tau_k \langle \mathcal{D}v_k, t - \mathcal{D}t_k\rangle \\
&= \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 + \frac{1}{2}\|\omega_{k+1} - \rho_k\|^2 - \tau_k\|\mathcal{D}t_k - \varphi(\rho_k)\|^2 \\
&\quad + \psi\tau_k \langle \mathcal{D}v_k, t - \mathcal{D}t_k\rangle - \tau_k \langle \mathcal{D}t_k - \varphi(\rho_k), \varphi(\rho_k) - \rho_k\rangle \\
&= \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 + \frac{1}{2}\|\omega_{k+1} - \rho_k\|^2 - \frac{1}{2}\tau_k\|\mathcal{D}t_k + v_k - v_k - \varphi(\rho_k)\|^2 \\
&\quad - \frac{1}{2}\tau_k\|\mathcal{D}t_k - \varphi(\rho_k)\|^2 + \psi\tau_k \langle \mathcal{D}v_k, t - \mathcal{D}t_k\rangle - \tau_k \langle \mathcal{D}t_k - \varphi(\rho_k), \varphi(\rho_k) - \rho_k\rangle.
\end{aligned}
$$

This gives

$$
\begin{aligned}
&\Upsilon_{k+1} - \Upsilon_k - \vartheta_k(\Upsilon_k - \Upsilon_{k-1}) \\
&\leq \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 + \frac{1}{2}\|\omega_{k+1} - \rho_k\|^2 - \frac{1}{2}\tau_k\|(\varphi(\rho_k) - v_k) - (\mathcal{D}t_k - v_k)\|^2 \\
&\quad - \frac{1}{2}\tau_k\|\mathcal{D}t_k - \varphi(\rho_k)\|^2 + \psi\tau_k \langle \mathcal{D}v_k, t - \mathcal{D}t_k\rangle - \tau_k \langle \mathcal{D}t_k - \varphi(\rho_k), \varphi(\rho_k) - \rho_k\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 + \frac{1}{2}\|\omega_{k+1} - \rho_k\|^2 - \frac{1}{2}\tau_k\Big[\|\varphi(\rho_k) - v_k\|^2 + \|\mathcal{D}t_k - v_k\|^2 \\
&\quad - 2\langle\varphi(\rho_k) - v_k, \mathcal{D}t_k - v_k\rangle\Big] - \frac{1}{2}\tau_k\|\mathcal{D}t_k - \varphi(\rho_k)\|^2 + \psi\tau_k\langle\mathcal{D}v_k, t - \mathcal{D}t_k\rangle \\
&\quad - \tau_k\langle\mathcal{D}t_k - \varphi(\rho_k), \varphi(\rho_k) - \rho_k\rangle \\
&= \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 + \frac{1}{2}\|\omega_{k+1} - \rho_k\|^2 - \frac{1}{2}\tau_k\Big[\|\varphi(\rho_k) - v_k\|^2 + \|\mathcal{D}t_k - v_k\|^2\Big] \\
&\quad - \frac{1}{2}\tau_k\|\mathcal{D}t_k - \varphi(\rho_k)\|^2 - \tau_k\langle\varphi(\rho_k) - v_k, v_k - \mathcal{D}t_k\rangle + \psi\tau_k\langle\mathcal{D}v_k, t - \mathcal{D}t_k\rangle \\
&\quad - \tau_k\langle\mathcal{D}t_k - \varphi(\rho_k), \varphi(\rho_k) - \rho_k\rangle. \qquad (4.5)
\end{aligned}
$$

Since $\{v_k\} \subset \mathcal{Z}_1$ and $t \in \Upsilon$, I have

$$\langle\mathcal{D}v_k,\ v_k - t\rangle \geq 0.$$

Adding and subtracting $\mathcal{D}t_k$ inside the second argument yields

$$\langle\mathcal{D}v_k,\ (v_k - \mathcal{D}t_k) + (\mathcal{D}t_k - t)\rangle \geq 0,$$

which implies

$$\psi\langle\mathcal{D}v_k,\ t - \mathcal{D}t_k\rangle \leq \psi\langle\mathcal{D}v_k,\ v_k - \mathcal{D}t_k\rangle.$$

Thus, I get that

$$
\begin{aligned}
\psi\langle\mathcal{D}v_k, t - \mathcal{D}t_k\rangle - \langle\varphi(\rho_k) - v_k, v_k - \mathcal{D}t_k\rangle &\leq \psi\langle\mathcal{D}v_k, v_k - \mathcal{D}t_k\rangle - \langle\varphi(\rho_k) - v_k, v_k - \mathcal{D}t_k\rangle \\
&= \langle\psi\mathcal{D}v_k - \varphi(\rho_k) + v_k, v_k - \mathcal{D}t_k\rangle. \qquad (4.6)
\end{aligned}
$$

Substituting (4.6) into (4.5), I obtain that

$$
\begin{aligned}
&\Upsilon_{k+1} - \Upsilon_k - \vartheta_k(\Upsilon_k - \Upsilon_{k-1}) \\
&\leq \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 + \frac{1}{2}\|\omega_{k+1} - \rho_k\|^2 - \frac{1}{2}\tau_k\Big[\|\varphi(\rho_k) - v_k\|^2 \\
&\quad + \|\mathcal{D}t_k - v_k\|^2\Big] - \frac{1}{2}\tau_k\|\mathcal{D}t_k - \varphi(\rho_k)\|^2 - \tau_k\langle\mathcal{D}t_k - \varphi(\rho_k), \varphi(\rho_k) - \rho_k\rangle \\
&\quad + \tau_k\langle\psi\mathcal{D}v_k - \varphi(\rho_k) + v_k, v_k - \mathcal{D}t_k\rangle \\
&= \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 + \frac{1}{2}\|\omega_{k+1} - \rho_k\|^2 - \frac{1}{2}\tau_k\Big[\|\varphi(\rho_k) - v_k\|^2 \\
&\quad + \|\mathcal{D}t_k - v_k\|^2\Big] - \frac{1}{2}\tau_k\|\mathcal{D}t_k - \varphi(\rho_k)\|^2 - \tau_k\langle\mathcal{D}t_k - \varphi(\rho_k), \varphi(\rho_k) - \rho_k\rangle \\
&\quad + \tau_k\langle\psi\mathcal{D}v_k - \psi\mathcal{D}\varphi(\rho_k), v_k - \mathcal{D}t_k\rangle + \tau_k\langle\psi\mathcal{D}\varphi(\rho_k) - \varphi(\rho_k) + v_k, v_k - \mathcal{D}t_k\rangle \\
&= \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 + \frac{1}{2}\|\omega_{k+1} - \rho_k\|^2 - \frac{1}{2}\tau_k\Big[\|\varphi(\rho_k) - v_k\|^2 \\
&\quad + \|\mathcal{D}t_k - v_k\|^2\Big] - \frac{1}{2}\tau_k\|\mathcal{D}t_k - \varphi(\rho_k)\|^2 - \tau_k\langle\mathcal{D}t_k - \varphi(\rho_k), \varphi(\rho_k) - \rho_k\rangle \\
&\quad + \psi\tau_k\langle\mathcal{D}\varphi(\rho_k) - \mathcal{D}v_k, \mathcal{D}t_k - v_k\rangle + \tau_k\langle\varphi(\rho_k) - \psi\mathcal{D}\varphi(\rho_k) - v_k, \mathcal{D}t_k - v_k\rangle.
\end{aligned}
$$

By definition of $\mathcal{Z}_{2_k}$, I have

$$\begin{aligned}
&\Upsilon_{k+1} - \Upsilon_k - \vartheta_k(\Upsilon_k - \Upsilon_{k-1}) \\
&\leq \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 + \frac{1}{2}\|\omega_{k+1} - \rho_k\|^2 - \frac{1}{2}\tau_k\Big[\|\varphi(\rho_k) - v_k\|^2 \\
&\quad + \|\mathcal{D}t_k - v_k\|^2\Big] - \frac{1}{2}\tau_k\|\mathcal{D}t_k - \varphi(\rho_k)\|^2 - \tau_k\langle \mathcal{D}t_k - \varphi(\rho_k), \varphi(\rho_k) - \rho_k\rangle \\
&\quad + \psi\tau_k\langle \mathcal{D}\varphi(\rho_k) - \mathcal{D}v_k, \mathcal{D}t_k - v_k\rangle. \qquad (4.7)
\end{aligned}$$

Since $\mathcal{D}$ is Lipschitzian, by the Cauchy–Schwarz inequality and Young's inequality, I obtain

$$\begin{aligned}
&\langle \mathcal{D}\varphi(\rho_k) - \mathcal{D}v_k,\ \mathcal{D}t_k - v_k\rangle \\
&\leq \|\mathcal{D}\varphi(\rho_k) - \mathcal{D}v_k\|\,\|\mathcal{D}t_k - v_k\| \\
&\leq \mathcal{L}\|\varphi(\rho_k) - v_k\|\,\|\mathcal{D}t_k - v_k\| \\
&\leq \frac{\mathcal{L}}{2}\left(\|\varphi(\rho_k) - v_k\|^2 + \|\mathcal{D}t_k - v_k\|^2\right). \qquad (4.8)
\end{aligned}$$

Using (4.8), (4.7), (2.1), and Lemma 4, I have

$$\begin{aligned}
&\Upsilon_{k+1} - \Upsilon_k - \vartheta_k(\Upsilon_k - \Upsilon_{k-1}) \\
&\leq \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 + \frac{1}{2}\|\omega_{k+1} - \rho_k\|^2 - \frac{1}{2}\tau_k\Big[\|\varphi(\rho_k) - v_k\|^2 \\
&\quad + \|\mathcal{D}t_k - v_k\|^2\Big] - \frac{1}{2}\tau_k\|\mathcal{D}t_k - \varphi(\rho_k)\|^2 - \tau_k\langle \mathcal{D}t_k - \varphi(\rho_k), \varphi(\rho_k) - \rho_k\rangle \\
&\quad + \psi\tau_k\frac{\mathcal{L}}{2}\Big(\|\varphi(\rho_k) - v_k\|^2 + \|\mathcal{D}t_k - v_k\|^2\Big) \\
&= \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 + \frac{1}{2}\|\omega_{k+1} - \rho_k\|^2 - \frac{\tau_k}{2}(1 - \psi\mathcal{L})\Big[\|\varphi(\rho_k) - v_k\|^2 \\
&\quad + \|\mathcal{D}t_k - v_k\|^2\Big] - \frac{1}{2}\tau_k\|\mathcal{D}t_k - \varphi(\rho_k)\|^2 - \tau_k\langle \mathcal{D}t_k - \varphi(\rho_k), \varphi(\rho_k) - \rho_k\rangle \\
&= \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 + \frac{1}{2}\|\omega_{k+1} - \rho_k\|^2 - \frac{\tau_k}{2}(1 - \psi\mathcal{L})\Big[\|\varphi(\rho_k) - v_k\|^2 \\
&\quad + \|\mathcal{D}t_k - v_k\|^2\Big] - \frac{1}{2}\tau_k\|\mathcal{D}t_k - \varphi(\rho_k)\|^2 + \langle \omega_{k+1} - \varphi(\rho_k), \rho_k - \varphi(\rho_k)\rangle \\
&= \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 + \frac{1}{2}\|\omega_{k+1} - \rho_k\|^2 - \frac{\tau_k}{2}(1 - \psi\mathcal{L})\Big[\|\varphi(\rho_k) - v_k\|^2 \\
&\quad + \|\mathcal{D}t_k - v_k\|^2\Big] - \frac{1}{2}\tau_k\|\mathcal{D}t_k - \varphi(\rho_k)\|^2 + \frac{1}{2}\Big(\|\omega_{k+1} - \varphi(\rho_k)\|^2 + \|\rho_k - \varphi(\rho_k)\|^2 \\
&\quad - \|(\omega_{k+1} - \varphi(\rho_k)) - (\rho_k - \varphi(\rho_k))\|^2\Big).
\end{aligned}$$

Then I have

$$\begin{aligned}
&\Upsilon_{k+1} - \Upsilon_k - \vartheta_k(\Upsilon_k - \Upsilon_{k-1}) \\
&\leq \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 + \frac{1}{2}\|\omega_{k+1} - \rho_k\|^2 - \frac{\tau_k}{2}(1 - \psi\mathcal{L})\Big[\|\varphi(\rho_k) - v_k\|^2
\end{aligned}$$

$$
\begin{aligned}
&+ \|\mathcal{D}t_k - v_k\|^2\Big] - \frac{1}{2}\tau_k\|\mathcal{D}t_k - \varphi(\rho_k)\|^2 + \frac{1}{2}\|\omega_{k+1} - \varphi(\rho_k)\|^2 + \frac{1}{2}\|\rho_k - \varphi(\rho_k)\|^2 \\
&- \frac{1}{2}\|\omega_{k+1} - \rho_k\|^2 \\
&= \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 - \frac{\tau_k}{2}(1 - \psi\mathcal{L})\Big[\|\varphi(\rho_k) - v_k\|^2 + \|\mathcal{D}t_k - v_k\|^2\Big] \\
&- \frac{1}{2\tau_k}\|\omega_{k+1} - \varphi(\rho_k)\|^2 + \frac{1}{2}\|\omega_{k+1} - \varphi(\rho_k)\|^2 + \frac{1}{2}\|\rho_k - \varphi(\rho_k)\|^2.
\end{aligned}
$$

Hence, I get

$$
\begin{aligned}
&\Upsilon_{k+1} - \Upsilon_k - \vartheta_k(\Upsilon_k - \Upsilon_{k-1}) \\
&\leq \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 - \frac{\tau_k}{2}(1 - \psi\mathcal{L})\Big[\|\varphi(\rho_k) - v_k\|^2 + \|\mathcal{D}t_k - v_k\|^2\Big] \\
&- \Big(\frac{1 - \tau_k}{2\tau_k}\Big)\|\omega_{k+1} - \varphi(\rho_k)\|^2 + \frac{1}{2}\|\rho_k - \varphi(\rho_k)\|^2 \\
&= \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 - \zeta_k\|\omega_{k+1} - \varphi(\rho_k)\|^2 - \frac{\tau_k}{2}(1 - \psi\mathcal{L})\Big[\|\varphi(\rho_k) - v_k\|^2 \\
&+ \|\mathcal{D}t_k - v_k\|^2\Big] + \frac{1}{2}\|\rho_k - \varphi(\rho_k)\|^2,
\end{aligned}
$$

where $\zeta_k = \frac{1-\tau_k}{2\tau_k}$ and $\Upsilon_k = \frac{1}{2}\|\omega_k - t\|^2$, for all $t \in \Upsilon$. □

**Theorem 2.** *Assume that* $\{\omega_k\}$ *is generated by Algorithm 2.1 and that* (2.4) *and Assumption 2 hold. Then*

$$
\begin{aligned}
&\sum_{\alpha=1}^{k-1} h_{\alpha+1,k}\Big[2\zeta_\alpha(1 - \vartheta_\alpha)^2\|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2 - (\vartheta_\alpha + \vartheta_\alpha^2)\|\omega_\alpha - \omega_{\alpha-1}\|^2 \\
&\quad + 2\zeta_\alpha(1 - \vartheta_\alpha)(\|\omega_{\alpha+1} - \varphi(\omega_\alpha)\|^2 - \|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2) \\
&\quad - (1 + \vartheta_\alpha)\|\omega_\alpha - \varphi(\omega_\alpha)\|^2 - (\vartheta_\alpha^2 + \vartheta_\alpha)\|\omega_\alpha - \omega_{\alpha-1} - \varphi(\omega_\alpha - \omega_{\alpha-1})\|^2\Big] \\
&\leq 2h_1|\Upsilon_1 - \Upsilon_0| + 2\Upsilon_0,
\end{aligned}
$$

*here* $h_{\alpha,k}$ *as in* (2.7) *and provided that the* $\varphi$ *is linear.*

*Proof.* Using Lemma 4, I have

$$
\begin{aligned}
&\|\omega_{k+1} - \varphi(\omega_k)\|^2 \\
&= \|\omega_{k+1} + \varphi(\omega_k - \omega_{k+1}) - \varphi(\omega_k - \omega_{k+1}) - \varphi(\omega_k)\|^2 \\
&= \|\varphi(\omega_k - \omega_{k-1})\|^2 + \|\omega_{k+1} - \varphi(\omega_k) - \varphi(\omega_k - \omega_{k-1})\|^2 \\
&\quad + 2\langle\omega_{k+1} - \varphi(\omega_k) - \varphi(\omega_k - \omega_{k-1}), \varphi(\omega_k - \omega_{k-1})\rangle.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&2\langle\omega_{k+1} - \varphi(\omega_k) - \varphi(\omega_k - \omega_{k-1}), \varphi(\omega_k - \omega_{k-1})\rangle \\
&= \|\omega_{k+1} - \varphi(\omega_k)\|^2 - \|\varphi(\omega_k - \omega_{k-1})\|^2 - \|\omega_{k+1} - \varphi(\omega_k) - \varphi(\omega_k - \omega_{k-1})\|^2.
\end{aligned}
$$

Using the contraction condition along with the above equality, I have

$$\begin{aligned}
&\|\omega_{k+1} - \varphi(\rho_k)\|^2 \\
&= \|\omega_{k+1} - \varphi(\omega_k + \vartheta_k(\omega_k - \omega_{k-1}))\|^2 \\
&= \|\omega_{k+1} - \varphi(\omega_k) - \varphi(\omega_k - \omega_{k-1})\|^2 + (1-\vartheta_k)^2\|\varphi(\omega_k - \omega_{k-1})\|^2 \\
&\quad + 2(1-\vartheta_k)\langle \omega_{k+1} - \varphi(\omega_k) - \varphi(\omega_k - \omega_{k-1}), \varphi(\omega_k - \omega_{k-1})\rangle \\
&= \vartheta_k\|\omega_{k+1} - \varphi(\omega_k) - \varphi(\omega_k - \omega_{k-1})\|^2 + (1-\vartheta_k)^2\|\varphi(\omega_k - \omega_{k-1})\|^2 \\
&\quad + (1-\vartheta_k)\Big(\|\omega_{k+1} - \varphi(\omega_k)\|^2 - \|\varphi(\omega_k - \omega_{k-1})\|^2\Big) \\
&\geq (1-\vartheta_k)^2\|\varphi(\omega_k - \omega_{k-1})\|^2 + (1-\vartheta_k)\Big(\|\omega_{k+1} - \varphi(\omega_k)\|^2 - \|\varphi(\omega_k - \omega_{k-1})\|^2\Big). \qquad (4.9)
\end{aligned}$$

Using Lemma 4, I have

$$\begin{aligned}
&\|\rho_k - \varphi(\rho_k)\|^2 \\
&= \|\omega_k + \vartheta_k(\omega_k - \omega_{k-1}) - \varphi(\omega_k + \vartheta_k(\omega_k - \omega_{k-1}))\|^2 \\
&= \|\omega_k - \varphi(\omega_k)\|^2 + \vartheta_k^2\|\omega_k - \omega_{k-1} - \varphi(\omega_k - \omega_{k-1})\|^2 \\
&\quad + 2\vartheta_k\langle \omega_k - \varphi(\omega_k), \omega_k - \omega_{k-1} - \varphi(\omega_k - \omega_{k-1})\rangle \\
&= \|\omega_k - \varphi(\omega_k)\|^2 + \vartheta_k^2\|\omega_k - \omega_{k-1} - \varphi(\omega_k - \omega_{k-1})\|^2 + \vartheta_k\Big[\|\omega_k - \varphi(\omega_k)\|^2 \\
&\quad + \|\omega_k - \omega_{k-1} - \varphi(\omega_k - \omega_{k-1})\|^2 - \|\omega_k - \varphi(\omega_k) - (\omega_k - \omega_{k-1} - \varphi(\omega_k - \omega_{k-1}))\|^2\Big] \\
&\leq (1+\vartheta_k)\|\omega_k - \varphi(\omega_k)\|^2 + (\vartheta_k^2 + \vartheta_k)\|\omega_k - \omega_{k-1} - \varphi(\omega_k - \omega_{k-1})\|^2. \qquad (4.10)
\end{aligned}$$

Using (4.9) and (4.10) in Theorem 1 and $1 - \psi\mathcal{L} > 0$, I have

$$\begin{aligned}
&\Upsilon_{k+1} - \Upsilon_k - \vartheta_k(\Upsilon_k - \Upsilon_{k-1}) \\
&\leq \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 - \zeta_k\Big[(1-\vartheta_k)^2\|\varphi(\omega_k - \omega_{k-1})\|^2 \\
&\quad + (1-\vartheta_k)(\|\omega_{k+1} - \varphi(\omega_k)\|^2 - \|\varphi(\omega_k - \omega_{k-1})\|^2)\Big] \\
&\quad + \frac{1}{2}((1+\vartheta_k)\|\omega_k - \varphi(\omega_k)\|^2 + (\vartheta_k^2 + \vartheta_k)\|\omega_k - \omega_{k-1} - \varphi(\omega_k - \omega_{k-1})\|^2) \\
&= \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 - \zeta_k(1-\vartheta_k)^2\|\varphi(\omega_k - \omega_{k-1})\|^2 \\
&\quad - \zeta_k(1-\vartheta_k)(\|\omega_{k+1} - \varphi(\omega_k)\|^2 - \|\varphi(\omega_k - \omega_{k-1})\|^2) \\
&\quad + \frac{1}{2}(1+\vartheta_k)\|\omega_k - \varphi(\omega_k)\|^2 + \frac{1}{2}(\vartheta_k^2 + \vartheta_k)\|\omega_k - \omega_{k-1} - \varphi(\omega_k - \omega_{k-1})\|^2. \qquad (4.11)
\end{aligned}$$

Combining (4.11) with Lemma 1(a), I get

$$\begin{aligned}
&\Upsilon_k - \Upsilon_0 \\
&= \sum_{\alpha=1}^{k}(\Upsilon_\alpha - \Upsilon_{\alpha-1})
\end{aligned}$$

$$\leq h_{1,k}(\Upsilon_1 - \Upsilon_0) + \sum_{\alpha=1}^{k-1} h_{\alpha+1,k}\Big[\frac{1}{2}(\vartheta_\alpha + \vartheta_\alpha^2)\|\omega_\alpha - \omega_{\alpha-1}\|^2 - \zeta_\alpha(1-\vartheta_\alpha)^2\|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2$$
$$- \zeta_\alpha(1-\vartheta_\alpha)\Big(\|\omega_{\alpha+1} - \varphi(\omega_\alpha)\|^2 - \|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2\Big)$$
$$+ \frac{1}{2}(1+\vartheta_\alpha)\|\omega_\alpha - \varphi(\omega_\alpha)\|^2 + \frac{1}{2}(\vartheta_\alpha^2 + \vartheta_\alpha)\|\omega_\alpha - \omega_{\alpha-1} - \varphi(\omega_\alpha - \omega_{\alpha-1})\|^2\Big].$$

Thus, I obtain that

$$2(\Upsilon_k - \Upsilon_0)$$
$$\leq 2h_{1,k}(\Upsilon_1 - \Upsilon_0) + \sum_{\alpha=1}^{k-1} h_{\alpha+1,k}\Big[(\vartheta_\alpha + \vartheta_\alpha^2)\|\omega_\alpha - \omega_{\alpha-1}\|^2 - 2\zeta_\alpha(1-\vartheta_\alpha)^2\|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2$$
$$- 2\zeta_\alpha(1-\vartheta_\alpha)\Big(\|\omega_{\alpha+1} - \varphi(\omega_\alpha)\|^2 - \|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2\Big)$$
$$+ (1+\vartheta_\alpha)\|\omega_\alpha - \varphi(\omega_\alpha)\|^2 + (\vartheta_\alpha^2 + \vartheta_\alpha)\|\omega_\alpha - \omega_{\alpha-1} - \varphi(\omega_\alpha - \omega_{\alpha-1})\|^2\Big].$$

Since $h_{1,k} \leq h_1$, this gives

$$\sum_{\alpha=1}^{k-1} h_{\alpha+1,k}\Big[2\zeta_\alpha(1-\vartheta_\alpha)^2\|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2 - (\vartheta_\alpha + \vartheta_\alpha^2)\|\omega_\alpha - \omega_{\alpha-1}\|^2$$
$$+ 2\zeta_\alpha(1-\vartheta_\alpha)(\|\omega_{\alpha+1} - \varphi(\omega_\alpha)\|^2 - \|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2)$$
$$- (1+\vartheta_\alpha)\|\omega_\alpha - \varphi(\omega_\alpha)\|^2 - (\vartheta_\alpha^2 + \vartheta_\alpha)\|\omega_\alpha - \omega_{\alpha-1} - \varphi(\omega_\alpha - \omega_{\alpha-1})\|^2\Big]$$
$$\leq 2h_1(\Upsilon_1 - \Upsilon_0) + 2\Upsilon_0 - 2\Upsilon_k.$$

This implies that

$$\sum_{\alpha=1}^{k-1} h_{\alpha+1,k}\Big[2\zeta_\alpha(1-\vartheta_\alpha)^2\|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2 - (\vartheta_\alpha + \vartheta_\alpha^2)\|\omega_\alpha - \omega_{\alpha-1}\|^2$$
$$+ 2\zeta_\alpha(1-\vartheta_\alpha)(\|\omega_{\alpha+1} - \varphi(\omega_\alpha)\|^2 - \|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2)$$
$$- (1+\vartheta_\alpha)\|\omega_\alpha - \varphi(\omega_\alpha)\|^2 - (\vartheta_\alpha^2 + \vartheta_\alpha)\|\omega_\alpha - \omega_{\alpha-1} - \varphi(\omega_\alpha - \omega_{\alpha-1})\|^2\Big]$$
$$\leq 2h_1|\Upsilon_1 - \Upsilon_0| + 2\Upsilon_0.$$

□

**Theorem 3.** *If I use the assumptions of Theorem 2 then I get that*

$$\sum_{\alpha=1}^{k-1}\Big[2\zeta_{\alpha-1}(1-\vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 - \vartheta_\alpha h_{\alpha+1}\Big(2[\zeta_\alpha(1-\vartheta_\alpha)]_+\|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2$$
$$- 2[\zeta_{\alpha-1}(1-\vartheta_{\alpha-1})]_-\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 + (1 + \vartheta_\alpha + \lambda^2 + \lambda)\|\omega_\alpha - \omega_{\alpha-1}\|^2$$
$$+ (1+\vartheta_\alpha)\|\omega_\alpha - \omega_{\alpha-1} - \varphi(\omega_\alpha - \omega_{\alpha-1})\|^2 + 2\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2\Big)$$
$$- h_{\alpha+1}\|\omega_\alpha - \varphi(\omega_\alpha)\|^2\Big] \leq 2\Big[h_1|\Upsilon_1 - \Upsilon_0| + \Upsilon_0 + h_1\zeta_0(1-\vartheta_0)\|\omega_1 - \varphi(\omega_0)\|^2\Big].$$

*Proof.* Observe that

$$\begin{aligned}
&\sum_{\alpha=1}^{k-1} h_{\alpha+1,k}2\zeta_\alpha(1-\vartheta_\alpha)\Big(\|\omega_{\alpha+1}-\varphi(\omega_\alpha)\|^2-\|\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2\Big)\\
&=\sum_{\alpha=1}^{k-1} h_{\alpha+1,k}2\zeta_\alpha(1-\vartheta_\alpha)\|\omega_{\alpha+1}-\varphi(\omega_\alpha)\|^2-\sum_{\alpha=1}^{k-1} h_{\alpha+1,k}2\zeta_\alpha(1-\vartheta_\alpha)\|\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2\\
&=2\sum_{\alpha=1}^{k-1} h_{\alpha+1,k}\zeta_\alpha(1-\vartheta_\alpha)\|\omega_{\alpha+1}-\varphi(\omega_\alpha)\|^2+2h_{1,k}\zeta_0(1-\vartheta_0)\|\omega_1-\varphi(\omega_0)\|^2\\
&-2h_{1,k}\zeta_0(1-\vartheta_0)\|\omega_1-\varphi(\omega_0)\|^2-2\sum_{\alpha=1}^{k-1} h_{\alpha+1,k}\zeta_\alpha(1-\vartheta_\alpha)\|\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2\\
&=2\sum_{\alpha=1}^{k-2} h_{\alpha+1,k}\zeta_\alpha(1-\vartheta_\alpha)\|\omega_{\alpha+1}-\varphi(\omega_\alpha)\|^2+2h_{k,k}\zeta_{k-1}(1-\vartheta_{k-1})\|\omega_k-\varphi(\omega_{k-1})\|^2\\
&\quad+2h_{1,k}\zeta_0(1-\vartheta_0)\|\omega_1-\varphi(\omega_0)\|^2-2h_{1,k}\zeta_0(1-\vartheta_0)\|\omega_1-\varphi(\omega_0)\|^2\\
&\quad-2\sum_{\alpha=1}^{k-1} h_{\alpha+1,k}\zeta_\alpha(1-\vartheta_\alpha)\|\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2\\
&=2\sum_{\alpha=1}^{k-1} h_{\alpha,k}\zeta_{\alpha-1}(1-\vartheta_{\alpha-1})\|\omega_\alpha-\varphi(\omega_{\alpha-1})\|^2-2\sum_{\alpha=1}^{k-1} h_{\alpha+1,k}\zeta_\alpha(1-\vartheta_\alpha)\|\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2\\
&\quad-2h_{1,k}\zeta_0(1-\vartheta_0)\|\omega_1-\varphi(\omega_0)\|^2.
\end{aligned}$$

Since $h_{1,k}\le h_1$, I obtain

$$\begin{aligned}
&\sum_{\alpha=1}^{k-1} h_{\alpha+1,k}2\zeta_\alpha(1-\vartheta_\alpha)\Big(\|\omega_{\alpha+1}-\varphi(\omega_\alpha)\|^2-\|\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2\Big)\\
&\ge 2\sum_{\alpha=1}^{k-1}\Big(h_{\alpha,k}\zeta_{\alpha-1}(1-\vartheta_{\alpha-1})\|\omega_\alpha-\varphi(\omega_{\alpha-1})\|^2-h_{\alpha+1,k}\zeta_\alpha(1-\vartheta_\alpha)\|\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2\Big)\\
&\quad-2h_{1,k}\zeta_0(1-\vartheta_0)\|\omega_1-\varphi(\omega_0)\|^2. \qquad (4.12)
\end{aligned}$$

Now, using (4.12) and Theorem 2, I have

$$\begin{aligned}
&\sum_{\alpha=1}^{k-1} h_{\alpha+1,k}\Big[2\zeta_\alpha(1-\vartheta_\alpha)^2\|\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2-(\vartheta_\alpha+\vartheta_\alpha^2)\|\omega_\alpha-\omega_{\alpha-1}\|^2\\
&\quad-(1+\vartheta_\alpha)\|\omega_\alpha-\varphi(\omega_\alpha)\|^2-(\vartheta_\alpha^2+\vartheta_\alpha)\|\omega_\alpha-\omega_{\alpha-1}-\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2\Big]\\
&\quad+2\sum_{\alpha=1}^{k-1}\Big(h_{\alpha,k}\zeta_{\alpha-1}(1-\vartheta_{\alpha-1})\|\omega_\alpha-\varphi(\omega_{\alpha-1})\|^2-h_{\alpha+1,k}\zeta_\alpha(1-\vartheta_\alpha)\|\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2\Big)\\
&\quad-2h_{1,k}\zeta_0(1-\vartheta_0)\|\omega_1-\varphi(\omega_0)\|^2\le 2h_1|\Upsilon_1-\Upsilon_0|+2\Upsilon_0.
\end{aligned}$$

This means

$$\begin{aligned}
&\sum_{\alpha=1}^{k-1}\Big[h_{\alpha+1,k}\Big(2(\zeta_\alpha(1-\vartheta_\alpha)^2-\zeta_\alpha(1-\vartheta_\alpha))\|\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2-(\vartheta_\alpha+\vartheta_\alpha^2)\|\omega_\alpha-\omega_{\alpha-1}\|^2\\
&\quad-(1+\vartheta_\alpha)\|\omega_\alpha-\varphi(\omega_\alpha)\|^2-(\vartheta_\alpha^2+\vartheta_\alpha)\|\omega_\alpha-\omega_{\alpha-1}-\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2\Big)\\
&\quad+2h_{\alpha,k}\zeta_{\alpha-1}(1-\vartheta_{\alpha-1})\|\omega_\alpha-\varphi(\omega_{\alpha-1})\|^2\Big]\\
&\le 2\Big[h_1|\Upsilon_1-\Upsilon_0|+\Upsilon_0+h_1\zeta_0(1-\vartheta_0)\|\omega_1-\varphi(\omega_0)\|^2\Big].
\end{aligned}\tag{4.13}$$

As

$$(1-\vartheta_\alpha)^2-(1-\vartheta_\alpha)=-\vartheta_\alpha(1-\vartheta_\alpha).\tag{4.14}$$

Using (2.6) and (4.14), I have

$$\begin{aligned}
&h_{\alpha+1,k}\Big[-2\zeta_\alpha\vartheta_\alpha(1-\vartheta_\alpha)\|\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2-(\vartheta_\alpha+\vartheta_\alpha^2)\|\omega_\alpha-\omega_{\alpha-1}\|^2\\
&\quad-(1+\vartheta_\alpha)\|\omega_\alpha-\varphi(\omega_\alpha)\|^2-(\vartheta_\alpha+\vartheta_\alpha^2)\|\omega_\alpha-\omega_{\alpha-1}-\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2\Big]\\
&\quad+2(1+\vartheta_\alpha h_{\alpha+1,k})\zeta_{\alpha-1}(1-\vartheta_{\alpha-1})\|\omega_\alpha-\varphi(\omega_{\alpha-1})\|^2\\
&=h_{\alpha+1,k}\Big[-2\zeta_\alpha\vartheta_\alpha(1-\vartheta_\alpha)\|\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2-(\vartheta_\alpha+\vartheta_\alpha^2)\|\omega_\alpha-\omega_{\alpha-1}\|^2\\
&\quad-(1+\vartheta_\alpha)\|\omega_\alpha-\varphi(\omega_\alpha)\|^2-(\vartheta_\alpha+\vartheta_\alpha^2)\|\omega_\alpha-\omega_{\alpha-1}-\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2\Big]\\
&\quad+2(\zeta_{\alpha-1}(1-\vartheta_{\alpha-1})+\vartheta_\alpha h_{\alpha+1,k}\zeta_{\alpha-1}(1-\vartheta_{\alpha-1}))\|\omega_\alpha-\varphi(\omega_{\alpha-1})\|^2\\
&=h_{\alpha+1,k}\Big[-2\zeta_\alpha\vartheta_\alpha(1-\vartheta_\alpha)\|\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2-(\vartheta_\alpha+\vartheta_\alpha^2)\|\omega_\alpha-\omega_{\alpha-1}\|^2\\
&\quad-(1+\vartheta_\alpha)\|\omega_\alpha-\varphi(\omega_\alpha)\|^2-(\vartheta_\alpha+\vartheta_\alpha^2)\|\omega_\alpha-\omega_{\alpha-1}-\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2\\
&\quad+2\vartheta_\alpha\zeta_{\alpha-1}(1-\vartheta_{\alpha-1})\|\omega_\alpha-\varphi(\omega_{\alpha-1})\|^2\Big]+2\zeta_{\alpha-1}(1-\vartheta_{\alpha-1})\|\omega_\alpha-\varphi(\omega_{\alpha-1})\|^2\\
&=2\zeta_{\alpha-1}(1-\vartheta_{\alpha-1})\|\omega_\alpha-\varphi(\omega_{\alpha-1})\|^2+h_{\alpha+1,k}\Big[-2\vartheta_\alpha\zeta_\alpha(1-\vartheta_\alpha)\|\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2\\
&\quad-\vartheta_\alpha(1+\vartheta_\alpha)\|\omega_\alpha-\omega_{\alpha-1}\|^2-\vartheta_\alpha(1+\vartheta_\alpha)\|\omega_\alpha-\omega_{\alpha-1}-\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2\\
&\quad+2\vartheta_\alpha\zeta_{\alpha-1}(1-\vartheta_{\alpha-1})\|\omega_\alpha-\varphi(\omega_{\alpha-1})\|^2-\vartheta_\alpha\|\omega_\alpha-\varphi(\omega_\alpha)\|^2\Big]-h_{\alpha+1,k}\|\omega_\alpha-\varphi(\omega_\alpha)\|^2\\
&=2\zeta_{\alpha-1}(1-\vartheta_{\alpha-1})\|\omega_\alpha-\varphi(\omega_{\alpha-1})\|^2-\vartheta_\alpha h_{\alpha+1,k}\Big[2\zeta_\alpha(1-\vartheta_\alpha)\|\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2\\
&\quad+(1+\vartheta_\alpha)\|\omega_\alpha-\omega_{\alpha-1}\|^2+(1+\vartheta_\alpha)\|\omega_\alpha-\omega_{\alpha-1}-\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2\\
&\quad-2\zeta_{\alpha-1}(1-\vartheta_{\alpha-1})\|\omega_\alpha-\varphi(\omega_{\alpha-1})\|^2+\|\omega_\alpha-\varphi(\omega_\alpha)\|^2\Big]-h_{\alpha+1,k}\|\omega_\alpha-\varphi(\omega_\alpha)\|^2.
\end{aligned}$$

By $h_{\alpha+1,k}\le h_{\alpha+1}$, I get that

$$h_{\alpha+1,k}\Big[-2\zeta_\alpha\vartheta_\alpha(1-\vartheta_\alpha)\|\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2-(\vartheta_\alpha+\vartheta_\alpha^2)\|\omega_\alpha-\omega_{\alpha-1}\|^2-(1+\vartheta_\alpha)\|\omega_\alpha-\varphi(\omega_\alpha)\|^2$$

$$- (\vartheta_\alpha + \vartheta_\alpha^2)\|\omega_\alpha - \omega_{\alpha-1} - \varphi(\omega_\alpha - \omega_{\alpha-1})\|^2\Big] + 2(1 + \vartheta_\alpha h_{\alpha+1,k})\zeta_{\alpha-1}(1 - \vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2$$
$$\geq 2\zeta_{\alpha-1}(1 - \vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 - \vartheta_\alpha h_{\alpha+1}\Big[2\zeta_\alpha(1 - \vartheta_\alpha)\|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2$$
$$+ (1 + \vartheta_\alpha)\|\omega_\alpha - \omega_{\alpha-1}\|^2 + (1 + \vartheta_\alpha)\|\omega_\alpha - \omega_{\alpha-1} - \varphi(\omega_\alpha - \omega_{\alpha-1})\|^2$$
$$- 2\zeta_{\alpha-1}(1 - \vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 + \|\omega_\alpha - \varphi(\omega_\alpha)\|^2\Big] - h_{\alpha+1}\|\omega_\alpha - \varphi(\omega_\alpha)\|^2. \tag{4.15}$$

Here, by (4.15) in (4.13), I get

$$\sum_{\alpha=1}^{k-1}\Big[2\zeta_{\alpha-1}(1 - \vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 - \vartheta_\alpha h_{\alpha+1}\Big(2[\zeta_\alpha(1 - \vartheta_\alpha)]_+\|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2$$
$$- 2[\zeta_{\alpha-1}(1 - \vartheta_{\alpha-1})]_-\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 + (1 + \vartheta_\alpha)\|\omega_\alpha - \omega_{\alpha-1}\|^2$$
$$+ (1 + \vartheta_\alpha)\|\omega_\alpha - \omega_{\alpha-1} - \varphi(\omega_\alpha - \omega_{\alpha-1})\|^2 + \|\omega_\alpha - \varphi(\omega_\alpha)\|^2\Big) - h_{\alpha+1}\|\omega_\alpha - \varphi(\omega_\alpha)\|^2\Big]$$
$$\leq 2\Big[h_1|\Upsilon_1 - \Upsilon_0| + \Upsilon_0 + h_1\zeta_0(1 - \vartheta_0)\|\omega_1 - \varphi(\omega_0)\|^2\Big]. \tag{4.16}$$

Now using Lemma 4, I get

$$\begin{aligned}
&\|\omega_\alpha - \varphi(\omega_\alpha)\|^2\\
&= \|\omega_\alpha - \varphi(\omega_{\alpha-1}) + \varphi(\omega_{\alpha-1}) - \varphi(\omega_\alpha)\|^2\\
&= \|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 + \|\varphi(\omega_{\alpha-1}) - \varphi(\omega_\alpha)\|^2 + 2\langle\omega_\alpha - \varphi(\omega_{\alpha-1}), \varphi(\omega_{\alpha-1}) - \varphi(\omega_\alpha)\rangle\\
&= \|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 + \lambda^2\|\omega_{\alpha-1} - \omega_\alpha\|^2 + \big[\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 + \lambda\|\omega_{\alpha-1} - \omega_\alpha\|^2\\
&\quad - \|\omega_\alpha - \varphi(\omega_{\alpha-1}) - \varphi(\omega_{\alpha-1}) + \varphi(\omega_\alpha)\|^2\\
&\leq 2\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 + (\lambda^2 + \lambda)\|\omega_\alpha - \omega_{\alpha-1}\|^2.
\end{aligned} \tag{4.17}$$

By (4.17) in (4.16),

$$\sum_{\alpha=1}^{k-1}\Big[2\zeta_{\alpha-1}(1 - \vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 - \vartheta_\alpha h_{\alpha+1}\Big(2[\zeta_\alpha(1 - \vartheta_\alpha)]_+\|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2$$
$$- 2[\zeta_{\alpha-1}(1 - \vartheta_{\alpha-1})]_-\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 + (1 + \vartheta_\alpha)\|\omega_\alpha - \omega_{\alpha-1}\|^2 + (1 + \vartheta_\alpha)\|\omega_\alpha - \omega_{\alpha-1}$$
$$- \varphi(\omega_\alpha - \omega_{\alpha-1})\|^2 + 2\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 + (\lambda^2 + \lambda)\|\omega_\alpha - \omega_{\alpha-1}\|^2\Big) - h_{\alpha+1}\|\omega_\alpha - \varphi(\omega_\alpha)\|^2\Big]$$
$$\leq 2\Big[h_1|\Upsilon_1 - \Upsilon_0| + \Upsilon_0 + h_1\zeta_0(1 - \vartheta_0)\|\omega_1 - \varphi(\omega_0)\|^2\Big],$$

I now have

$$\sum_{\alpha=1}^{k-1}\Big[2\zeta_{\alpha-1}(1 - \vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 - \vartheta_\alpha h_{\alpha+1}\Big(2[\zeta_\alpha(1 - \vartheta_\alpha)]_+\|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2$$
$$- 2[\zeta_{\alpha-1}(1 - \vartheta_{\alpha-1})]_-\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 + (1 + \vartheta_\alpha + \lambda^2 + \lambda)\|\omega_\alpha - \omega_{\alpha-1}\|^2$$
$$+ (1 + \vartheta_\alpha)\|\omega_\alpha - \omega_{\alpha-1} - \varphi(\omega_\alpha - \omega_{\alpha-1})\|^2 + 2\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2\Big)$$

$$- h_{\alpha+1}\|\omega_\alpha - \varphi(\omega_\alpha)\|^2\Big] \leq 2\Big[h_1|\Upsilon_1 - \Upsilon_0| + \Upsilon_0 + h_1\zeta_0(1-\vartheta_0)\|\omega_1 - \varphi(\omega_0)\|^2\Big].$$

□

**Theorem 4.** *Assume that $\{\omega_k\}$ is generated by Algorithm 2.1 and that* (2.4) *and Assumptions 1 and 2 hold. Then I have*

$$\sum_{\alpha=1}^{\infty} \vartheta_\alpha h_{\alpha+1}\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 < \infty.$$

**Remark 5.** *The summability condition*

$$\sum_{\alpha=1}^{\infty} \vartheta_\alpha h_{\alpha+1}\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 < \infty$$

*is a standard assumption in the analysis of inertial iterative algorithms. It guarantees that the accumulated inertial perturbations remain controlled and asymptotically vanish, which is essential for establishing the convergence of the proposed method. This condition can be ensured by an appropriate choice of the inertial parameters together with the decay behavior of the iterative sequence.*

*Proof.* Assume that inequality (3.1) holds and I split the proof in two cases.
**Case 1.** Consider $\tau_k \in (0,1)$, for all $k \geq 1$. Now, by (3.1) in Theorem 3,

$$2\sum_{\alpha=1}^{k-1} \xi\zeta_{\alpha-1}(1-\vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 \leq 2\Big[h_1|\Upsilon_1 - \Upsilon_0| + \Upsilon_0 + h_1\zeta_0(1-\vartheta_0)\|\omega_1 - \varphi(\omega_0)\|^2\Big],$$

which gives

$$\sum_{\alpha=1}^{k-1} \xi\zeta_{\alpha-1}(1-\vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 \leq h_1|\Upsilon_1 - \Upsilon_0| + \Upsilon_0 + h_1\zeta_0(1-\vartheta_0)\|\omega_1 - \varphi(\omega_0)\|^2. \tag{4.18}$$

As $k \to \infty$ in (4.18), I have

$$\sum_{\alpha=1}^{\infty} \zeta_{\alpha-1}(1-\vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 < \infty. \tag{4.19}$$

Again from (3.1), I have

$$\begin{aligned}
&2\xi\zeta_{\alpha-1}(1-\vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 \\
&\leq 2\zeta_{\alpha-1}(1-\vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 - \vartheta_\alpha h_{\alpha+1}\Big(2[\zeta_\alpha(1-\vartheta_\alpha)]_+\|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2 \\
&\quad - 2[\zeta_{\alpha-1}(1-\vartheta_{\alpha-1})]_-\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 + (1+\vartheta_\alpha+\lambda^2+\lambda)\|\omega_\alpha - \omega_{\alpha-1}\|^2 \\
&\quad + (1+\vartheta_\alpha)\|\omega_\alpha - \omega_{\alpha-1} - \varphi(\omega_\alpha - \omega_{\alpha-1})\|^2 + 2\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2\Big) - h_{\alpha+1}\|\omega_\alpha - \varphi(\omega_\alpha)\|^2.
\end{aligned}$$

This implies that

$$0 \leq 2\zeta_{\alpha-1}(1-\vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 - 2\xi\zeta_{\alpha-1}(1-\vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2$$

$$- 2\vartheta_\alpha h_{\alpha+1}\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 - \vartheta_\alpha h_{\alpha+1}\Big(2[\zeta_\alpha(1-\vartheta_\alpha)]_+\|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2$$
$$- 2[\zeta_{\alpha-1}(1-\vartheta_{\alpha-1})]_-\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 + (1+\vartheta_\alpha+\lambda^2+\lambda)\|\omega_\alpha - \omega_{\alpha-1}\|^2$$
$$+ (1+\vartheta_\alpha)\|\omega_\alpha - \omega_{\alpha-1} - \varphi(\omega_\alpha - \omega_{\alpha-1})\|^2\Big) - h_{\alpha+1}\|\omega_\alpha - \varphi(\omega_\alpha)\|^2.$$

Thus,

$$2\vartheta_\alpha h_{\alpha+1}\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 \leq 2\zeta_{\alpha-1}(1-\vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2$$
$$- 2\xi\zeta_{\alpha-1}(1-\vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 - \vartheta_\alpha h_{\alpha+1}\Big(2[\zeta_\alpha(1-\vartheta_\alpha)]_+\|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2$$
$$- 2[\zeta_{\alpha-1}(1-\vartheta_{\alpha-1})]_-\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 + (1+\vartheta_\alpha+\lambda^2+\lambda)\|\omega_\alpha - \omega_{\alpha-1}\|^2$$
$$+ (1+\vartheta_\alpha)\|\omega_\alpha - \omega_{\alpha-1} - \varphi(\omega_\alpha - \omega_{\alpha-1})\|^2\Big) - h_{\alpha+1}\|\omega_\alpha - \varphi(\omega_\alpha)\|^2.$$

Now, I have

$$\vartheta_\alpha h_{\alpha+1}\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 \leq \zeta_{\alpha-1}(1-\vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2.$$

Taking summation on both sides, I have

$$\sum_{\alpha=1}^{\infty} \vartheta_\alpha h_{\alpha+1}\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 \leq \sum_{\alpha=1}^{\infty} \zeta_{\alpha-1}(1-\vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2. \tag{4.20}$$

Using (4.20) in (4.19) I get

$$\sum_{\alpha=1}^{\infty} \vartheta_\alpha h_{\alpha+1}\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 < \infty.$$

**Case 2.** Now consider $\tau_k = 1$, for all $k \geq 1$. Then, $\omega_{k+1} = \mathcal{D}t_k$ and $\zeta_k = 0$, for all $k \geq 1$. Set $w = \varphi(\rho_k) - v_k$ and $v = \omega_{k+1} - v_k$ in Lemma 4

$$\|(\varphi(\rho_k) - v_k) + (\omega_{k+1} - v_k)\|^2 - \|\varphi(\rho_k) - v_k\|^2 - \|\omega_{k+1} - v_k\|^2 = \|\varphi(\rho_k) - v_k\|^2$$
$$+\|\omega_{k+1} - v_k\|^2 - \|(\varphi(\rho_k) - v_k) - (\omega_{k+1} - v_k)\|^2,$$

which gives

$$\|(\varphi(\rho_k) + \omega_{k+1} - 2v_k)\|^2 + \|\varphi(\rho_k) - \omega_{k+1}\|^2 = 2[\|\varphi(\rho_k) - v_k\|^2 + \|\omega_{k+1} - v_k\|^2].$$

It follows that

$$\|(\varphi(\rho_k) + \omega_{k+1} - 2v_k)\|^2 + \|\varphi(\rho_k) - \omega_{k+1}\|^2 = 2[\|\varphi(\rho_k) - v_k\|^2 + \|\mathcal{D}t_k - v_k\|^2].$$

This gives

$$\|\varphi(\rho_k) - v_k\|^2 + \|\mathcal{D}t_k - v_k\|^2 \leq \|(\varphi(\rho_k) + \omega_{k+1} - 2v_k)\|^2 + \|\varphi(\rho_k) - \omega_{k+1}\|^2. \tag{4.21}$$

By (4.21) in Theorem 1 for $1 - \psi\mathcal{L} > 0$, as $\zeta_k = 0$ for $\tau_k = 1$, I have

$$\Upsilon_{k+1} - \Upsilon_k - \vartheta_k(\Upsilon_k - \Upsilon_{k-1})$$

$$\leq \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 - \frac{1}{2}\Big[\|(\varphi(\rho_k) + \omega_{k+1} - 2v_k)\|^2 + \|\varphi(\rho_k) - \omega_{k+1}\|^2\Big]$$
$$+ \frac{1}{2}\|\rho_k - \varphi(\rho_k)\|^2$$
$$= \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 - \frac{1}{2}\|(\varphi(\rho_k) + \omega_{k+1} - 2v_k)\|^2 - \frac{1}{2}\|\varphi(\rho_k) - \omega_{k+1}\|^2$$
$$+ \frac{1}{2}\|\rho_k - \varphi(\rho_k)\|^2$$
$$\leq \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 - \frac{1}{2}\|\varphi(\rho_k) - \omega_{k+1}\|^2 + \frac{1}{2}\|\rho_k - \varphi(\rho_k)\|^2. \tag{4.22}$$

Now, using (4.9) in (4.22) and by (4.10), I have

$$\Upsilon_{k+1} - \Upsilon_k - \vartheta_k(\Upsilon_k - \Upsilon_{k-1})$$
$$\leq \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 - \frac{1}{2}(1 - \vartheta_k)^2\|\varphi(\omega_k - \omega_{k-1})\|^2$$
$$- \frac{1}{2}(1 - \vartheta_k)\Big(\|\omega_{k+1} - \varphi(\omega_k)\|^2 - \|\varphi(\omega_k - \omega_{k-1})\|^2\Big)$$
$$+ \frac{1}{2}(1 + \vartheta_k)\|\omega_k - \varphi(\omega_k)\|^2 + \frac{1}{2}(\vartheta_k^2 + \vartheta_k)\|\omega_k - \omega_{k-1} - \varphi(\omega_k - \omega_{k-1})\|^2.$$

This together with Lemma 1(a) implies that

$$\Upsilon_k - \Upsilon_0 = \sum_{\alpha=1}^{k}(\Upsilon_\alpha - \Upsilon_{\alpha-1})$$
$$\leq h_{1,k}(\Upsilon_1 - \Upsilon_0) + \sum_{\alpha=1}^{k-1} h_{\alpha+1,k}\Big[\frac{1}{2}(\vartheta_\alpha + \vartheta_\alpha^2)\|\omega_\alpha - \omega_{\alpha-1}\|^2$$
$$- \frac{1}{2}(1 - \vartheta_\alpha)^2\|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2 - \frac{1}{2}(1 - \vartheta_\alpha)\Big(\|\omega_{\alpha+1} - \varphi(\omega_\alpha)\|^2$$
$$- \|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2\Big) + \frac{1}{2}(1 + \vartheta_\alpha)\|\omega_\alpha - \varphi(\omega_\alpha)\|^2 + \frac{1}{2}(\vartheta_\alpha^2 + \vartheta_\alpha)\|\omega_\alpha - \omega_{\alpha-1} - \varphi(\omega_\alpha - \omega_{\alpha-1})\|^2\Big].$$

Hence, I get that

$$2(\Upsilon_k - \Upsilon_0) \leq 2h_{1,k}(\Upsilon_1 - \Upsilon_0) + \sum_{\alpha=1}^{k-1} h_{\alpha+1,k}\Big[(\vartheta_\alpha + \vartheta_\alpha^2)\|\omega_\alpha - \omega_{\alpha-1}\|^2$$
$$- (1 - \vartheta_\alpha)^2\|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2 - (1 - \vartheta_\alpha)\Big(\|\omega_{\alpha+1} - \varphi(\omega_\alpha)\|^2$$
$$- \|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2\Big) + (1 + \vartheta_\alpha)\|\omega_\alpha - \varphi(\omega_\alpha)\|^2 + (\vartheta_\alpha^2 + \vartheta_\alpha)\|\omega_\alpha - \omega_{\alpha-1} - \varphi(\omega_\alpha - \omega_{\alpha-1})\|^2\Big].$$

This yields that

$$\sum_{\alpha=1}^{k-1} h_{\alpha+1,k}\Big[(1 - \vartheta_\alpha)^2\|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2 - (\vartheta_\alpha + \vartheta_\alpha^2)\|\omega_\alpha - \omega_{\alpha-1}\|^2$$

$$+ (1-\vartheta_\alpha)\Big(||\omega_{\alpha+1}-\varphi(\omega_\alpha)||^2 - ||\varphi(\omega_\alpha-\omega_{\alpha-1})||^2\Big) + (1+\vartheta_\alpha)||\omega_\alpha-\varphi(\omega_\alpha)||^2$$
$$+ (\vartheta_\alpha^2+\vartheta_\alpha)||\omega_\alpha-\omega_{\alpha-1}-\varphi(\omega_\alpha-\omega_{\alpha-1})||^2\Big]$$
$$\leq 2h_{1,k}(\Upsilon_1-\Upsilon_0) - 2(\Upsilon_k-\Upsilon_0).$$

Using $h_{1,k} \leq h_1$ in above inequality

$$\sum_{\alpha=1}^{k-1} h_{\alpha+1,k}\Big[(1-\vartheta_\alpha)^2||\varphi(\omega_\alpha-\omega_{\alpha-1})||^2 - (\vartheta_\alpha+\vartheta_\alpha^2)||\omega_\alpha-\omega_{\alpha-1}||^2$$
$$+ (1-\vartheta_\alpha)\Big(||\omega_{\alpha+1}-\varphi(\omega_\alpha)||^2 - ||\varphi(\omega_\alpha-\omega_{\alpha-1})||^2\Big) + (1+\vartheta_k)||\omega_\alpha-\varphi(\omega_\alpha)||^2$$
$$+ (\vartheta_\alpha^2+\vartheta_\alpha)||\omega_\alpha-\omega_{\alpha-1}-\varphi(\omega_\alpha-\omega_{\alpha-1})||^2\Big]$$
$$\leq 2h_1(\Upsilon_1-\Upsilon_0) + 2\Upsilon_0. \tag{4.23}$$

Similar to (4.12), I have

$$\sum_{\alpha=1}^{k-1} h_{\alpha+1,k}(1-\vartheta_\alpha)\Big(||\omega_{\alpha+1}-\varphi(\omega_\alpha)||^2 - ||\varphi(\omega_\alpha-\omega_{\alpha-1})||^2\Big)$$
$$= \sum_{\alpha=1}^{k-1} h_{\alpha+1,k}(1-\vartheta_\alpha)||\omega_{\alpha+1}-\varphi(\omega_\alpha)||^2 + h_{1,k}(1-\vartheta_0)||\omega_1-\varphi(\omega_0)||^2$$
$$- h_{1,k}(1-\vartheta_0)||\omega_1-\varphi(\omega_0)||^2 - \sum_{\alpha=1}^{k-1} h_{\alpha+1,k}(1-\vartheta_\alpha)||\varphi(\omega_\alpha-\omega_{\alpha-1})||^2$$
$$= \sum_{\alpha=1}^{k-2} h_{\alpha+1,k}(1-\vartheta_\alpha)||\omega_{\alpha+1}-\varphi(\omega_\alpha)||^2 + h_{k,k}(1-\vartheta_{k-1})||\omega_k-\varphi(\omega_{k-1})||^2$$
$$+ h_{1,k}(1-\vartheta_0)||\omega_1-\varphi(\omega_0)||^2 - h_{1,k}(1-\vartheta_0)||\omega_1-\varphi(\omega_0)||^2$$
$$- \sum_{\alpha=1}^{k-1} h_{\alpha+1,k}(1-\vartheta_\alpha)||\varphi(\omega_\alpha-\omega_{\alpha-1})||^2$$
$$= \sum_{\alpha=1}^{k-1} h_{\alpha,k}(1-\vartheta_{\alpha-1})||\omega_\alpha-\varphi(\omega_{\alpha-1})||^2 - \sum_{\alpha=1}^{k-1} h_{\alpha+1,k}(1-\vartheta_\alpha)||\varphi(\omega_\alpha-\omega_{\alpha-1})||^2$$
$$- h_{1,k}(1-\vartheta_0)||\omega_1-\varphi(\omega_0)||^2.$$

Since $h_{1,k} \leq h_1$, I obtain

$$\sum_{\alpha=1}^{k-1} h_{\alpha+1,k}(1-\vartheta_\alpha)\Big(||\omega_{\alpha+1}-\varphi(\omega_\alpha)||^2 - ||\varphi(\omega_\alpha-\omega_{\alpha-1})||^2\Big)$$
$$\geq \sum_{\alpha=1}^{k-1}\Big(h_{\alpha,k}(1-\vartheta_{\alpha-1})||\omega_\alpha-\varphi(\omega_{\alpha-1})||^2$$

$$- h_{\alpha+1,k}(1-\vartheta_\alpha)\|\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2\Big)$$
$$- h_1(1-\vartheta_0)\|\omega_1-\varphi(\omega_0)\|^2. \tag{4.24}$$

Additionally, by (4.24) in Theorem 2, I have

$$\sum_{\alpha=1}^{k-1} h_{\alpha+1,k}\Big[(1-\vartheta_\alpha)^2\|\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2-(\vartheta_\alpha+\vartheta_\alpha^2)\|\omega_\alpha-\omega_{\alpha-1}\|^2$$
$$-(1+\vartheta_\alpha)\|\omega_\alpha-\varphi(\omega_\alpha)\|^2-(\vartheta_\alpha^2+\vartheta_\alpha)\|\omega_\alpha-\omega_{\alpha-1}-\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2\Big]$$
$$+\sum_{\alpha=1}^{k-1}\Big(h_{\alpha,k}(1-\vartheta_{\alpha-1})\|\omega_\alpha-\varphi(\omega_{\alpha-1})\|^2-h_{\alpha+1,k}(1-\vartheta_\alpha)\|\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2\Big)$$
$$-h_1(1-\vartheta_0)\|\omega_1-\varphi(\omega_0)\|^2\le 2h_1|\Upsilon_1-\Upsilon_0|+2\Upsilon_0.$$

This means

$$\sum_{\alpha=1}^{k-1}\Big[h_{\alpha+1,k}\Big(((1-\vartheta_\alpha)^2-(1-\vartheta_\alpha))\|\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2-(\vartheta_\alpha+\vartheta_\alpha^2)\|\omega_\alpha-\omega_{\alpha-1}\|^2$$
$$-(1+\vartheta_\alpha)\|\omega_\alpha-\varphi(\omega_\alpha)\|^2-(\vartheta_\alpha^2+\vartheta_\alpha)\|\omega_\alpha-\omega_{\alpha-1}-\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2\Big)$$
$$+2h_{\alpha,k}(1-\vartheta_{\alpha-1})\|\omega_\alpha-\varphi(\omega_{\alpha-1})\|^2\Big]$$
$$\le 2h_1|\Upsilon_1-\Upsilon_0|+2\Upsilon_0+h_1(1-\vartheta_0)\|\omega_1-\varphi(\omega_0)\|^2. \tag{4.25}$$

I get that

$$\sum_{\alpha=1}^{k-1}\Big[h_{\alpha+1,k}\Big(-\vartheta_\alpha(1-\vartheta_\alpha)\|\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2-(\vartheta_\alpha+\vartheta_\alpha^2)\|\omega_\alpha-\omega_{\alpha-1}\|^2$$
$$-(1+\vartheta_\alpha)\|\omega_\alpha-\varphi(\omega_\alpha)\|^2-(\vartheta_\alpha^2+\vartheta_\alpha)\|\omega_\alpha-\omega_{\alpha-1}-\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2\Big)$$
$$+2h_{\alpha,k}(1-\vartheta_{\alpha-1})\|\omega_\alpha-\varphi(\omega_{\alpha-1})\|^2\Big]$$
$$\le 2h_1|\Upsilon_1-\Upsilon_0|+2\Upsilon_0+h_1(1-\vartheta_0)\|\omega_1-\varphi(\omega_0)\|^2. \tag{4.26}$$

From (2.5) and (4.14), I have

$$h_{\alpha+1,k}\Big[-\vartheta_\alpha(1-\vartheta_\alpha)\|\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2-(\vartheta_\alpha+\vartheta_\alpha^2)\|\omega_\alpha-\omega_{\alpha-1}\|^2$$
$$-(1+\vartheta_\alpha)\|\omega_\alpha-\varphi(\omega_\alpha)\|^2-(\vartheta_\alpha+\vartheta_\alpha^2)\|\omega_\alpha-\omega_{\alpha-1}-\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2\Big]$$
$$+(1+\vartheta_\alpha h_{\alpha+1,k})(1-\vartheta_{\alpha-1})\|\omega_\alpha-\varphi(\omega_{\alpha-1})\|^2$$
$$=h_{\alpha+1,k}\Big[-\vartheta_\alpha(1-\vartheta_\alpha)\|\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2-(\vartheta_\alpha+\vartheta_\alpha^2)\|\omega_\alpha-\omega_{\alpha-1}\|^2$$
$$-(1+\vartheta_\alpha)\|\omega_\alpha-\varphi(\omega_\alpha)\|^2-(\vartheta_\alpha+\vartheta_\alpha^2)\|\omega_\alpha-\omega_{\alpha-1}-\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2$$

$$
\begin{aligned}
&+ \vartheta_\alpha(1-\vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2\Big] + (1-\vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2\\
&= (1-\vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 - \vartheta_\alpha h_{\alpha+1,k}\Big[(1-\vartheta_\alpha)\|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2\\
&\quad + (1+\vartheta_\alpha)\|\omega_\alpha - \omega_{\alpha-1}\|^2 + (1+\vartheta_\alpha)\|\omega_\alpha - \omega_{\alpha-1} - \varphi(\omega_\alpha - \omega_{\alpha-1})\|^2\\
&\quad - (1-\vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 + \|\omega_\alpha - \varphi(\omega_\alpha)\|^2\Big] - h_{\alpha+1,k}\|\omega_\alpha - \varphi(\omega_\alpha)\|^2.
\end{aligned}
$$

Since $h_{\alpha+1,k} \le h_{\alpha+1}$, I have

$$
\begin{aligned}
&h_{\alpha+1,k}\Big[ - \vartheta_\alpha(1-\vartheta_\alpha)\|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2 - (\vartheta_\alpha + \vartheta_\alpha^2)\|\omega_\alpha - \omega_{\alpha-1}\|^2 - (1+\vartheta_\alpha)\|\omega_\alpha - \varphi(\omega_\alpha)\|^2\\
&\quad - (\vartheta_\alpha + \vartheta_\alpha^2)\|\omega_\alpha - \omega_{\alpha-1} - \varphi(\omega_\alpha - \omega_{\alpha-1})\|^2\Big] + (1+\vartheta_\alpha h_{\alpha+1,k})(1-\vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2\\
&\ge (1-\vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 - \vartheta_\alpha h_{\alpha+1}\Big[(1-\vartheta_\alpha)\|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2\\
&\quad + (1+\vartheta_\alpha)\|\omega_\alpha - \omega_{\alpha-1}\|^2 + (1+\vartheta_\alpha)\|\omega_\alpha - \omega_{\alpha-1} - \varphi(\omega_\alpha - \omega_{\alpha-1})\|^2\\
&\quad - (1-\vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 + \|\omega_\alpha - \varphi(\omega_\alpha)\|^2\Big] - h_{\alpha+1}\|\omega_\alpha - \varphi(\omega_\alpha)\|^2. \qquad (4.27)
\end{aligned}
$$

Using (4.27) in (4.26), I get

$$
\begin{aligned}
&\sum_{\alpha=1}^{k-1}\Big[(1-\vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 - \vartheta_\alpha h_{\alpha+1}\Big([1-\vartheta_\alpha]_+\|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2\\
&\quad - [1-\vartheta_{\alpha-1}]_-\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 + (1+\vartheta_\alpha)\|\omega_\alpha - \omega_{\alpha-1}\|^2\\
&\quad + (1+\vartheta_\alpha)\|\omega_\alpha - \omega_{\alpha-1} - \varphi(\omega_\alpha - \omega_{\alpha-1})\|^2 + \|\omega_\alpha - \varphi(\omega_\alpha)\|^2\Big) - h_{\alpha+1}\|\omega_\alpha - \varphi(\omega_\alpha)\|^2\Big]\\
&\le 2h_1|\Upsilon_1 - \Upsilon_0| + 2\Upsilon_0 + h_1(1-\vartheta_0)\|\omega_1 - \varphi(\omega_0)\|^2.
\end{aligned}
$$

Now, using (4.17), I have

$$
\begin{aligned}
&\sum_{\alpha=1}^{k-1}\Big[(1-\vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 - \vartheta_\alpha h_{\alpha+1}\Big([1-\vartheta_\alpha]_+\|\varphi(\omega_\alpha - \omega_{\alpha-1})\|^2\\
&\quad - [1-\vartheta_{\alpha-1}]_-\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 + (1+\vartheta_\alpha + \lambda^2 + \lambda)\|\omega_\alpha - \omega_{\alpha-1}\|^2\\
&\quad + (1+\vartheta_\alpha)\|\omega_\alpha - \omega_{\alpha-1} - \varphi(\omega_\alpha - \omega_{\alpha-1})\|^2 + 2\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2\Big) - h_{\alpha+1}\|\omega_\alpha - \varphi(\omega_\alpha)\|^2\Big]\\
&\le 2h_1|\Upsilon_1 - \Upsilon_0| + 2\Upsilon_0 + h_1(1-\vartheta_0)\|\omega_1 - \varphi(\omega_0)\|^2. \qquad (4.28)
\end{aligned}
$$

Using (3.1) in (4.28), I get

$$
\begin{aligned}
&\sum_{\alpha=1}^{k-1}\xi(1-\vartheta_{\alpha-1})\|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2\\
&\le 2h_1|\Upsilon_1 - \Upsilon_0| + 2\Upsilon_0 + h_1(1-\vartheta_0)\|\omega_1 - \varphi(\omega_0)\|^2.
\end{aligned}
$$

Also similar to Case 1, I obtain that

$$\sum_{\alpha=1}^{\infty} \vartheta_\alpha h_{\alpha+1} \|\omega_\alpha - \varphi(\omega_{\alpha-1})\|^2 < \infty.$$

□

**Theorem 5.** *If I use assumptions of Theorem 4, then we have*
*(a)* $\lim_{k\to\infty} \|\omega_k - t\|$ *exists for any* $t \in \Upsilon$, *So* $\{\omega_k\}$ *is bounded.*
*(b)* $\lim_{k\to\infty} \left[\|\varphi(\rho_k) - v_k\|^2 + \|\mathcal{D}t_k - v_k\|^2\right] = 0.$
*(c)* $\lim_{k\to\infty} \|\omega_k - \rho_k\| = 0.$

*Proof.* (*a*) By Theorem 1

$$\Upsilon_{k+1} - \Upsilon_k \leq \vartheta_k(\Upsilon_k - \Upsilon_{k-1}) + \frac{1}{2}(\vartheta_k + \vartheta_k^2)\|\omega_k - \omega_{k-1}\|^2 - \frac{\tau_k}{2}(1 - \psi\mathcal{L})\left[\|\varphi(\rho_k) - v_k\|^2 + \|\mathcal{D}t_k - v_k\|^2\right] + \frac{1}{2}\|\rho_k - \varphi(\rho_k)\|^2.$$

Using $\vartheta_k^2 \leq \vartheta_k$, I have

$$\Upsilon_{k+1} - \Upsilon_k \leq \vartheta_k(\Upsilon_k - \Upsilon_{k-1}) + \vartheta_k\|\omega_k - \omega_{k-1}\|^2 - \frac{\tau_k}{2}(1 - \psi\mathcal{L})\left[\|\varphi(\rho_k) - v_k\|^2 + \|\mathcal{D}t_k - v_k\|^2\right] + \frac{1}{2}\|\rho_k - \varphi(\rho_k)\|^2. \tag{4.29}$$

Now, using $1 - \psi\mathcal{L} > 0$, I get

$$\Upsilon_{k+1} - \Upsilon_k \leq \vartheta_k(\Upsilon_k - \Upsilon_{k-1}) + \vartheta_k\|\omega_k - \omega_{k-1}\|^2 + \frac{1}{2}\|\rho_k - \varphi(\rho_k)\|^2. \tag{4.30}$$

Applying Lemma 1(b) and (4.7) in (4.30), I obtain that

$$\sum_{k=1}^{\infty}[\Upsilon_k - \Upsilon_{k-1}]_+ \leq h_1[\Upsilon_1 - \Upsilon_0]_+ + \sum_{k=1}^{\infty} h_{k+1}\left(\vartheta_k\|\omega_k - \omega_{k-1}\|^2 + \frac{1}{2}\|\rho_k - \varphi(\rho_k)\|^2\right).$$

Set $P := \sum_{k=1}^{\infty} h_{k+1}\vartheta_k\|\omega_k - \omega_{k-1}\|^2$ and $Q := \sum_{k=1}^{\infty} \frac{1}{2}h_{k+1}\|\rho_k - \varphi(\rho_k)\|^2$. I now verify that $P$ and $Q$ are finite. Since all the terms appearing in the bracket of (3.1) are nonnegative, rearranging (3.1) gives, for all sufficiently large $k$,

$$\vartheta_k h_{k+1}\left[(1 + \vartheta_k + \lambda^2 + \lambda)\|\omega_k - \omega_{k-1}\|^2 + (1 + \vartheta_k)\|\omega_k - \omega_{k-1} - \varphi(\omega_k - \omega_{k-1})\|^2\right] + h_{k+1}\|\omega_k - \varphi(\omega_k)\|^2 \leq 2(1 - \xi)\zeta_{k-1}(1 - \vartheta_{k-1})\|\omega_k - \varphi(\omega_{k-1})\|^2,$$

when $\tau_k \in (0, 1)$, while the analogous estimate without the factor $2\zeta_{k-1}$ holds when $\tau_k = 1$. Summing these inequalities over $k$ and using (4.19), together with its analogue obtained in Case 2 of the proof of Theorem 4, I deduce that

$$\sum_{k=1}^{\infty} \vartheta_k h_{k+1}\|\omega_k - \omega_{k-1}\|^2 < \infty, \qquad \sum_{k=1}^{\infty} \vartheta_k h_{k+1}\|\omega_k - \omega_{k-1} - \varphi(\omega_k - \omega_{k-1})\|^2 < \infty,$$

$$\sum_{k=1}^{\infty} h_{k+1}\|\omega_k - \varphi(\omega_k)\|^2 < \infty.$$

The first of these series dominates $P$, so $P < \infty$. Moreover, combining the last two series with (4.10) yields

$$\sum_{k=1}^{\infty} h_{k+1}\|\rho_k - \varphi(\rho_k)\|^2 \leq 2\sum_{k=1}^{\infty} h_{k+1}\|\omega_k - \varphi(\omega_k)\|^2 + 2\sum_{k=1}^{\infty} \vartheta_k h_{k+1}\|\omega_k - \omega_{k-1} - \varphi(\omega_k - \omega_{k-1})\|^2 < \infty,$$

which shows that $Q < \infty$. Therefore, I have

$$\sum_{k=1}^{\infty}[\Upsilon_k - \Upsilon_{k-1}]_+ \leq h_1[\Upsilon_1 - \Upsilon_0]_+ + P + Q < \infty.$$

Since $\Upsilon_k = \frac{1}{2}\|\omega_k - t\|^2$, this gives $\lim_{k\to\infty} \|\omega_k - t\|^2$, which implies the boundedness of $\{\omega_k\}$.
($b$) Using Lemma 1 in (4.29), I have

$$\begin{aligned}
\Upsilon_k - \Upsilon_0 &= \sum_{\alpha=1}^{k}(\Upsilon_\alpha - \Upsilon_{\alpha-1}) \\
&\leq h_{1,k}(\Upsilon_1 - \Upsilon_0) + \sum_{\alpha=1}^{k-1} h_{\alpha+1,k}\Big[\vartheta_\alpha\|\omega_\alpha - \omega_{\alpha-1}\|^2 - \frac{\tau_\alpha}{2}(1-\psi\mathcal{L})[\|\varphi(\rho_\alpha) - v_\alpha\|^2 \\
&\quad + \|\mathcal{D}t_\alpha - v_\alpha\|^2] + \frac{1}{2}\|\rho_\alpha - \varphi(\rho_\alpha)\|^2\Big].
\end{aligned}$$

By $h_{1,k} \leq h_1$, I get

$$\begin{aligned}
\Upsilon_k - \Upsilon_0 &= \sum_{\alpha=1}^{k}(\Upsilon_\alpha - \Upsilon_{\alpha-1}) \\
&\leq h_1(\Upsilon_1 - \Upsilon_0) + \sum_{\alpha=1}^{k-1} h_{\alpha+1,k}\Big[\vartheta_\alpha\|\omega_\alpha - \omega_{\alpha-1}\|^2 + \frac{1}{2}\|\rho_\alpha - \varphi(\rho_\alpha)\|^2\Big] \\
&\quad - \sum_{\alpha=1}^{k-1} h_{\alpha+1,k}\frac{\tau_\alpha}{2}(1-\psi\mathcal{L})[\|\varphi(\rho_\alpha) - v_\alpha\|^2 + \|\mathcal{D}t_\alpha - v_\alpha\|^2].
\end{aligned}$$

Thus, I have

$$\begin{aligned}
&\sum_{\alpha=1}^{k-1} h_{\alpha+1,k}\frac{\tau_\alpha}{2}(1-\psi\mathcal{L})[\|\varphi(\rho_\alpha) - v_\alpha\|^2 + \|\mathcal{D}t_\alpha - v_\alpha\|^2] \\
&\leq \Upsilon_0 - \Upsilon_k + h_1(\Upsilon_1 - \Upsilon_0) + \sum_{\alpha=1}^{k-1} h_{\alpha+1,k}\Big[\vartheta_\alpha\|\omega_\alpha - \omega_{\alpha-1}\|^2 + \frac{1}{2}\|\rho_\alpha - \varphi(\rho_\alpha)\|^2\Big] \\
&\leq \Upsilon_0 + h_1(\Upsilon_1 - \Upsilon_0) + \sum_{\alpha=1}^{k-1} h_{\alpha+1,k}\Big[\vartheta_\alpha\|\omega_\alpha - \omega_{\alpha-1}\|^2 + \frac{1}{2}\|\rho_\alpha - \varphi(\rho_\alpha)\|^2\Big] < \infty.
\end{aligned}$$

As $h_{\alpha+1,k} = 0$, for all $\alpha \geq k$, for $k \to \infty$, then by the monotone convergence result, I have

$$\sum_{\alpha=1}^{\infty} h_{\alpha+1} \frac{\tau_\alpha}{2}(1 - \psi\mathcal{L})[\|\varphi(\rho_\alpha) - v_\alpha\|^2 + \|\mathcal{D}t_\alpha - v_\alpha\|^2] < \infty. \tag{4.31}$$

Since $\liminf_{k\to\infty} \tau_k > 0$, there is a $\bar{\tau} > 0$ with $\tau_k \geq \bar{\tau}$, when $k$ is very large. Replace $\alpha$ with $k$ in (4.31), and $h_k \geq 1$ for all $k \geq 1$, I get by (4.31) that $\sum_{k=1}^{\infty} \frac{\bar{\tau}}{2}(1 - \psi\mathcal{L})\Big[\|\varphi(\rho_k) - v_k\|^2 + \|\mathcal{D}t_k - v_k\|^2\Big] < \infty$, which further gives

$$\lim_{k\to\infty}(\|\varphi(\rho_k) - v_k\|^2 + \|\mathcal{D}t_k - v_k\|^2) = 0.$$

(*c*) Using Algorithm 2.1, I have

$$\|\rho_k - \omega_k\| = \vartheta_k\|\omega_k - \omega_{k-1}\|.$$

Since $\vartheta_k^2 \leq \vartheta_k$, I obtain

$$\sum_{k=1}^{\infty} h_{k+1}\|\rho_k - \omega_k\|^2 = \sum_{k=1}^{\infty} \vartheta_k^2 h_{k+1}\|\omega_k - \omega_{k-1}\|^2 \leq \sum_{k=1}^{\infty} \vartheta_k h_{k+1}\|\omega_k - \omega_{k-1}\|^2 = P < \infty,$$

where the finiteness of $P$ was established in the proof of part (a). Since $h_k \geq 1$ for all $k \geq 1$, the terms of a convergent series tend to zero, and we conclude that $\lim_{k\to\infty} \|\rho_k - \omega_k\| = 0$. □

**Theorem 6.** *Assume that* $\{\omega_k\}$ *is generated by Algorithm 2.1 under* (2.4)*, and that Assumptions 1 and 2(a)–(d) hold. If* $\{\omega_k\} \rightharpoonup \omega^*$*, then* $\omega^* \in \Upsilon$.

*Proof.* By Theorem 5, take a subsequence $\{\omega_{k_l}\}$ of $\{\omega_k\}$ with $\omega_{k_l} \rightharpoonup \omega^* \in \mathcal{Y}$. Additionally, from Theorem 5(b),(c), one can see $\lim_{k\to\infty} \|v_k - \omega_k\| = 0$. Thus, the subsequence $\{v_{k_l}\}$ of $\{v_k\}$ satisfies $v_{k_l} \rightharpoonup \omega^*$. Moreover, $\omega^* \in \mathcal{Z}_1$ since $\{v_{k_l}\} \subset \mathcal{Z}_1$. Here, I have the two different cases.
**Case 1.** Assume $\limsup_{k\to\infty} \|\mathcal{D}v_{k_l}\| = 0$. Thus,

$$\lim_{k\to\infty} \|\mathcal{D}v_{k_l}\| = \liminf_{k\to\infty} \|\mathcal{D}v_{k_l}\| = 0.$$

Since $\mathcal{D}$ is sequentially weakly continuous and $v_{k_l} \rightharpoonup \omega^*$, I have

$$\mathcal{D}v_{k_l} \rightharpoonup \mathcal{D}\omega^*.$$

Moreover, from Theorem 5(b), I have

$$\lim_{k\to\infty} \|\mathcal{D}v_{k_l}\| = 0.$$

Hence, $\mathcal{D}v_{k_l} \to 0$ strongly. The strong convergence implies weak convergence, and therefore the weak limit of $\{\mathcal{D}v_{k_l}\}$ is zero. By the uniqueness of weak limits, we obtain

$$\mathcal{D}\omega^* = 0.$$

Since $\omega^* \in \mathcal{Z}_1$ and $\mathcal{D}\omega^* = 0$, it follows from the definition of $\Upsilon = \mathrm{VIP}(\mathcal{Z}_1, \mathcal{D})$ that $\omega^* \in \Upsilon$.
**Case 2.** Assume $\limsup_{k\to\infty} \|\mathcal{D}v_{k_l}\| > 0$. Now, take a subsequence of $\{\mathcal{D}v_{k_l}\}$ with

$$\lim_{k\to\infty} \|\mathcal{D}v_{k_l}\| = M > 0.$$

Additionally, by the characterization (2.2) of $P_{\mathcal{Z}_1}$, for all $w \in \mathcal{Z}_1$, I have

$$\begin{aligned}0 &\geq \langle \varphi(\rho_{k_l}) - \psi\mathcal{D}\varphi(\rho_{k_l}) - v_{k_l}, w - v_{k_l}\rangle \\ &= \langle \varphi(\rho_{k_l}) - v_{k_l}, w - v_{k_l}\rangle - \psi\langle \mathcal{D}\varphi(\rho_{k_l}), w - v_{k_l}\rangle \\ &= \langle \varphi(\rho_{k_l}) - v_{k_l}, w - v_{k_l}\rangle - \psi\langle \mathcal{D}\varphi(\rho_{k_l}), w - \varphi(\rho_{k_l})\rangle - \psi\langle \mathcal{D}\varphi(\rho_{k_l}), \varphi(\rho_{k_l}) - v_{k_l}\rangle.\end{aligned}$$

This gives

$$\langle \varphi(\rho_{k_l}) - v_{k_l}, w - v_{k_l}\rangle - \psi\langle \mathcal{D}\varphi(\rho_{k_l}), \varphi(\rho_{k_l}) - v_{k_l}\rangle \leq \psi\langle \mathcal{D}\varphi(\rho_{k_l}), w - \varphi(\rho_{k_l})\rangle,$$

which implies that

$$\frac{1}{\psi}\langle \varphi(\rho_{k_l}) - v_{k_l}, w - v_{k_l}\rangle - \langle \mathcal{D}\varphi(\rho_{k_l}), \varphi(\rho_{k_l}) - v_{k_l}\rangle \leq \langle \mathcal{D}\varphi(\rho_{k_l}), w - \varphi(\rho_{k_l})\rangle. \tag{4.32}$$

As $\langle \varphi(\rho_{k_l}) - v_{k_l}, w - v_{k_l}\rangle \leq \frac{1}{2}[\|\varphi(\rho_{k_l}) - v_{k_l}\|^2 + \|w - v_{k_l}\|^2]$. Using Theorem 5(b), I have

$$0 \leq \liminf_{l\to\infty}\langle \mathcal{D}\varphi(\rho_{k_l}), w - \varphi(\rho_{k_l})\rangle \leq \limsup_{l\to\infty}\langle \mathcal{D}\varphi(\rho_{k_l}), w - \varphi(\rho_{k_l})\rangle < \infty, \quad \text{for all } w \in \mathcal{Z}_1. \tag{4.33}$$

Now,

$$\begin{aligned}&\langle \mathcal{D}v_{k_l}, w - v_{k_l}\rangle \\ &= \langle \mathcal{D}v_{k_l}, w - \varphi(\rho_{k_l})\rangle + \langle \mathcal{D}v_{k_l}, \varphi(\rho_{k_l}) - v_{k_l}\rangle \\ &= \langle \mathcal{D}v_{k_l} - \mathcal{D}\varphi(\rho_{k_l}), w - \varphi(\rho_{k_l})\rangle + \langle \mathcal{D}\varphi(\rho_{k_l}), w - \varphi(\rho_{k_l})\rangle + \langle \mathcal{D}v_{k_l}, \varphi(\rho_{k_l}) - v_{k_l}\rangle.\end{aligned} \tag{4.34}$$

Since $\mathcal{D}$ is Lipschitzian, from Theorem 5(b) I have

$$\lim_{l\to\infty}\|\mathcal{D}\varphi(\rho_{k_l}) - \mathcal{D}v_{k_l}\| = 0.$$

Additionally, from Theorem 5, (4.33) and (4.34), I get

$$0 \leq \liminf_{l\to\infty}\langle \mathcal{D}v_{k_l}, w - v_{k_l}\rangle \leq \limsup_{l\to\infty}\langle \mathcal{D}v_{k_l}, w - v_{k_l}\rangle < \infty, \quad \text{for all } w \in \mathcal{Z}_1. \tag{4.35}$$

From (4.35), I have the following cases:
**Case A.** Suppose that $\limsup_{l\to\infty}\langle \mathcal{D}v_{k_l}, w - v_{k_l}\rangle > 0$, for all $w \in \mathcal{Z}_1$.
Then, there exists a subsequence $\{v_{k_{l_j}}\}$ of $\{v_{k_l}\}$ as follows:

$$\lim_{j\to\infty}\langle \mathcal{D}v_{k_{l_j}}, w - v_{k_{l_j}}\rangle > 0.$$

Thus, there exists $j_0 \geq 1$, such that

$$\langle \mathcal{D}v_{k_{l_j}}, w - v_{k_{l_j}}\rangle > 0, \quad \text{for any} \quad j \geq j_0.$$

Thus, I have

$$0 < \langle \mathcal{D}v_{k_{l_j}}, w - v_{k_{l_j}}\rangle$$

$$= \langle \mathcal{D}w, w - v_{k_{l_j}} \rangle - \langle \mathcal{D}w - \mathcal{D}v_{k_{l_j}}, w - v_{k_{l_j}} \rangle$$
$$\leq \langle \mathcal{D}w, w - v_{k_{l_j}} \rangle - \alpha(w - v_{k_{l_j}}),$$

for all $w \in \mathcal{Z}_1$, where $\alpha$ is the map given in Definition 1.1.
As $j \to \infty$, I get $\langle \mathcal{D}w, w - \omega^* \rangle - \alpha(w - \omega^*) \geq 0$.
This implies that

$$\langle \mathcal{D}w, w - \omega^* \rangle \geq 0, \text{ for all } w \in \mathcal{Z}_1.$$

Hence, I have $\omega^* \in \Upsilon$.
**Case B.** Suppose that $\limsup_{l\to\infty} \langle \mathcal{D}v_{k_l}, w - v_{k_l} \rangle = 0$, for all $w \in \mathcal{Z}_1$. Then, by (4.35), I obtain that

$$\lim_{l\to\infty} \langle \mathcal{D}v_{k_l}, w - v_{k_l} \rangle = 0, \text{ for all } w \in \mathcal{Z}_1. \tag{4.36}$$

For each $l \geq 1$, put

$$\epsilon_l := \left| \langle \mathcal{D}v_{k_l}, w - v_{k_l} \rangle \right| + \frac{1}{l+1}. \tag{4.37}$$

Then $\epsilon_l > 0$ and $\lim_{l\to\infty} \epsilon_l = 0$ by (4.36). Moreover, the scalar inequality

$$\langle \mathcal{D}v_{k_l}, w - v_{k_l} \rangle + \epsilon_l > 0, \quad \text{for all } w \in \mathcal{Z}_1, \tag{4.38}$$

holds. As $\lim_{l\to\infty} \left\| \mathcal{D}v_{k_l} \right\| = M > 0$, there is an $l_0 \geq 1$ with $\left\| \mathcal{D}v_{k_l} \right\| > \frac{M}{2}$, for all $l \geq l_0$. Thus, I can define $q_{k_l} = \frac{\mathcal{D}v_{k_l}}{\|\mathcal{D}v_{k_l}\|^2}$. Thus, $\langle \mathcal{D}v_{k_l}, q_{k_l} \rangle = 1$. Multiplying this identity by $\epsilon_l$ and adding the resulting equality to (4.38), I obtain

$$\langle \mathcal{D}v_{k_l},\ w + \epsilon_l q_{k_l} - v_{k_l} \rangle > 0. \tag{4.39}$$

Since $\mathcal{D}$ is relaxed $\alpha$-monotone, I have

$$\langle \mathcal{D}(w + \epsilon_l q_{k_l}) - \mathcal{D}v_{k_l},\ w + \epsilon_l q_{k_l} - v_{k_l} \rangle \geq \alpha(w + \epsilon_l q_{k_l} - v_{k_l}).$$

Adding this inequality to (4.39) yields

$$\langle \mathcal{D}(w + \epsilon_l q_{k_l}),\ w + \epsilon_l q_{k_l} - v_{k_l} \rangle > \alpha(w + \epsilon_l q_{k_l} - v_{k_l}). \tag{4.40}$$

Writing $\mathcal{D}(w + \epsilon_l q_{k_l}) = \mathcal{D}w + (\mathcal{D}(w + \epsilon_l q_{k_l}) - \mathcal{D}w)$ in (4.40) and using the Lipschitz continuity of $\mathcal{D}$ together with the Cauchy–Schwarz inequality, I get

$$\begin{aligned} \langle \mathcal{D}w,\ w + \epsilon_l q_{k_l} - v_{k_l} \rangle &> \alpha(w + \epsilon_l q_{k_l} - v_{k_l}) - \langle \mathcal{D}(w + \epsilon_l q_{k_l}) - \mathcal{D}w,\ w + \epsilon_l q_{k_l} - v_{k_l} \rangle \\ &\geq \alpha(w + \epsilon_l q_{k_l} - v_{k_l}) - \mathcal{L}\,\epsilon_l \|q_{k_l}\| \left\| w + \epsilon_l q_{k_l} - v_{k_l} \right\| \\ &\geq \alpha(w + \epsilon_l q_{k_l} - v_{k_l}) - \frac{2\mathcal{L}}{M}\,\epsilon_l\, M_1, \end{aligned} \tag{4.41}$$

where $M_1 > 0$, whose existence follows from the boundedness of the sequence $\{w + \epsilon_l q_{k_l} - v_{k_l}\}$, and I use the bound $\|q_{k_l}\| = 1/\|\mathcal{D}v_{k_l}\| \leq 2/M$, which is for all $l \geq l_0$. Now, letting $l \to \infty$ in (4.41) and using $\epsilon_l \to 0$ together with $v_{k_l} \rightharpoonup \omega^*$, I get, as in Case A, $\langle \mathcal{D}w, w - \omega^* \rangle - \alpha(w - \omega^*) \geq 0$. This implies that

$$\langle \mathcal{D}w, w - \omega^* \rangle \geq 0, \text{ for all } w \in \mathcal{Z}_1.$$

Hence, $\omega^* \in \Upsilon$. □

**Theorem 7.** *Assume* $\{\omega_k\}$ *as given in Algorithm 2.1 with* (2.4)*, Assumptions 1 and 2(a)–(d) fulfilled, then* $\{\omega_k\}$ *has at most one weak limit point in* $\Upsilon$.

*Proof.* On the contrary, assume $\{\omega_k\}$ has two different weak limits in $\Upsilon$; these are $\omega^*$ and $\bar{w}$. Additionally, assume $\{\omega_{n_j}\}$ is a subsequence of $\{\omega_k\}$ with $\omega_{n_j} \rightharpoonup \bar{w}$, when $j \to \infty$. Then, using Theorem 5(a) and (2.7), Imply

$$\begin{aligned}
&\lim_{k\to\infty} \|\omega_k - \bar{w}\| = \lim_{j\to\infty} \|\omega_{n_j} - \bar{w}\| \\
&= \liminf_{j\to\infty} \|\omega_{n_j} - \bar{w}\| \\
&< \liminf_{j\to\infty} \|\omega_{n_j} - \omega^*\| \\
&= \lim_{k\to\infty} \|\omega_k - \omega^*\| \\
&= \liminf_{l\to\infty} \|\omega_{k_l} - \omega^*\| \\
&< \liminf_{l\to\infty} \|\omega_{k_l} - \bar{w}\| \\
&= \lim_{k\to\infty} \|\omega_k - \bar{w}\|,
\end{aligned}$$

This shows the contradiction. Thus, I get proof. □

**Theorem 8.** *If we use the assumptions of Theorem 7, then* $\{\omega_k\}$ *converges weakly to a solution of* VIP$(\mathcal{Z}_1, \mathcal{D})$.

*Proof.* By Theorem 6, every weak cluster point of $\{\omega_k\}$ belongs to $\Upsilon = \text{VIP}(\mathcal{Z}_1, \mathcal{D})$. Moreover, by Assumption 2(c), the set $\{\omega \in \mathcal{Z}_1 : \mathcal{D}\omega = 0\} \setminus \Upsilon$ is finite. Hence, $\{\omega_k\}$ has only finitely many weak cluster points, all belonging to $\Upsilon$. Suppose $\omega^1, \omega^2, \ldots, \omega^m$ are the weak limits of $\{\omega_k\}$, and assume that $\{\omega^\alpha_{k_l}\}$ is a subsequence of $\{\omega_k\}$ with $\omega^\alpha_{k_l} \rightharpoonup \omega^\alpha$, as $l \to \infty$. Then, we obtain

$$\lim_{l\to\infty} \langle \omega^\alpha_{k_l}, \omega^\alpha - \omega^j \rangle = \langle \omega^\alpha, \omega^\alpha - \omega^j \rangle, \quad \forall\, j \neq \alpha. \tag{4.42}$$

Here, if $j \neq \alpha$, set $\gamma^{-1}(\omega^\alpha - \omega^j) = q$, so $\gamma = \|\omega^\alpha - \omega^j\|$, I obtain that

$$\begin{aligned}
\langle \omega^\alpha, q \rangle &= \gamma^{-1} \langle \omega^\alpha, \omega^\alpha - \omega^j \rangle \\
&= \gamma^{-1} (\|\omega^\alpha\|^2 - \langle \omega^\alpha, \omega^j \rangle) \\
&= \gamma^{-1} [\|\omega^\alpha\|^2 - \frac{1}{2}\|\omega^\alpha\|^2 - \frac{1}{2}\|\omega^j\|^2 + \frac{1}{2}\|\omega^\alpha - \omega^j\|^2] \\
&= \frac{\gamma^{-1}}{2} (\|\omega^\alpha\|^2 - \|\omega^j\|^2) + \frac{\gamma^{-1}}{2}\gamma^2 \\
&= \frac{1}{2\gamma} (\|\omega^\alpha\|^2 - \|\omega^j\|^2) + \frac{\gamma}{2} \\
&> \frac{1}{2\gamma} (\|\omega^\alpha\|^2 - \|\omega^j\|^2) + \frac{1}{4}\gamma.
\end{aligned} \tag{4.43}$$

To distinguish weak cluster points, I construct mutually separated neighborhoods around each weak cluster point. These neighborhoods are used to show that, for sufficiently large $k$, the iterates cannot

move from the neighborhood of one weak cluster point to another. This leads to a contradiction by using the fact that

$$\|\omega_{k+1} - \omega_k\| \to 0.$$

As $k$ is large enough, from (4.42) and (4.43), I get

$$\omega_{k_l}^{\alpha} \in \left\{ w : \langle w, q \rangle > \frac{1}{2\gamma}(\|\omega^{\alpha}\|^2 - \|\omega^{j}\|^2) + \frac{1}{4}\gamma \right\}. \tag{4.44}$$

Thus, I have $N_1 > N$ and

$$\begin{aligned} \omega_{k_l}^{\alpha} \in \mathbf{A}^{\alpha} := \bigcap_{j=1, j\neq\alpha}^{m} \Bigg\{ & w : \langle w, \frac{\omega^{\alpha} - \omega^{j}}{\|\omega^{\alpha} - \omega^{j}\|} \rangle \\ & > \frac{1}{2\|\omega^{\alpha} - \omega^{j}\|}(\|\omega^{\alpha}\|^2 - \|\omega^{j}\|^2) + \xi_0 \Bigg\}, \text{ for all } k \geq N_1, \end{aligned} \tag{4.45}$$

where $\xi_0 = \min\left\{\frac{1}{4}\|\omega^{\alpha} - \omega^{j}\| : \alpha, j = 1, 2, \ldots, m, \alpha \neq j\right\}$.

Since there are only finitely many distinct weak cluster points, the minimum above is strictly positive. Hence, $\xi_0$ provides a uniform positive separation between the neighborhoods corresponding to different weak cluster points. Now, from Theorem 5(b), I obtain that $\lim_{k\to\infty} \|t_k - \rho_k\| = 0$. Since $\omega_{k+1} - \omega_k = \tau_k(t_k - \rho_k)$ and $\{\tau_k\}$ a bounded sequence, then $\lim_{k\to\infty} \|\omega_{k+1} - \omega_k\| = 0$. By Theorem 5(c), I get $\lim_{k\to\infty} \|\omega_{k+1} - \omega_k\| = 0$. Thus, there exists $N_2 > N_1 > N$, such that

$$\|\omega_{k+1} - \omega_k\| < \xi_0, \ \forall\ k \geq N_2.$$

I claim that $\{\omega_k\}$ has a weak limit in VIP$(\mathcal{Z}_1, \mathcal{D})$. On the contrary, assume that $\{\omega_k\}$ has two or more limits. Then, there exists $N_3 \geq N_2 > N_1 > N$ such that $\omega_{N_3} \in \mathbf{A}^{\alpha}$ and $\omega_{N_3+1} \in \mathbf{A}^{j}$, where

$$\mathbf{A}^{j} := \bigcap_{\alpha=1, \alpha\neq j}^{m} \left\{ w : \langle -w, \frac{\omega^{\alpha} - \omega^{j}}{\|\omega^{\alpha} - \omega^{j}\|} \rangle < \frac{1}{2\|\omega^{\alpha} - \omega^{j}\|}(\|\omega^{\alpha}\|^2 - \|\omega^{j}\|^2) - \xi_0 \right\}, \tag{4.46}$$

where $m \geq 2$ and

$$\|\omega_{N_3+1} - \omega_{N_3}\| < \xi_0. \tag{4.47}$$

As $\omega_{N_3} \in \mathbf{A}^{\alpha}$ and $\omega_{N_3+1} \in \mathbf{A}^{j}$, I get

$$\langle \omega_{N_3}, \frac{\omega^{\alpha} - \omega^{j}}{\|\omega^{\alpha} - \omega^{j}\|} \rangle > \frac{1}{2\|\omega^{\alpha} - \omega^{j}\|}(\|\omega^{\alpha}\|^2 - \|\omega^{j}\|^2) + \xi_0. \tag{4.48}$$

This gives

$$\langle \omega_{N_3+1}, \frac{\omega^{\alpha} - \omega^{j}}{\|\omega^{j} - \omega^{\alpha}\|} \rangle > \frac{-1}{2\|\omega^{j} - \omega^{\alpha}\|}(\|\omega^{j}\|^2 - \|\omega^{\alpha}\|^2) - \xi_0.$$

This implies that

$$\langle -\omega_{N_3+1}, \frac{\omega^{\alpha} - \omega^{j}}{\|\omega^{\alpha} - \omega^{j}\|} \rangle < \frac{1}{2\|\omega^{\alpha} - \omega^{j}\|}(\|\omega^{j}\|^2 - \|\omega^{\alpha}\|^2) + \xi_0. \tag{4.49}$$

Adding (4.48) and (4.49), I have

$$\langle \omega_{N_3} - \omega_{N_3+1}, \frac{\omega^\alpha - \omega^j}{\|\omega^\alpha - \omega^j\|} \rangle > 2\xi_0,$$

and using (4.47), I obtain that

$$2\xi_0 < \langle \omega_{N_3} - \omega_{N_3+1}, \frac{\omega^\alpha - \omega^j}{\|\omega^\alpha - \omega^j\|} \rangle \leq \|\omega_{N_3+1} - \omega_{N_3}\| < \xi_0. \tag{4.50}$$

I conclude that $\{\omega_k\}$ has at most one weak limit and this gives that $\{\omega_k\}$ converges weakly to a point of VIP($\mathcal{Z}_1, \mathcal{D}$). □

I next introduce the following algorithm incorporating a self-adaptive step size.

**Algorithm 2.2** Proposed algorithm containing the self adaptive step-size.

Choose $\vartheta_k \in [0, 1)$ and $\tau_k \in (0, 1]$, for all $k \geq 1$. Suppose that $\psi_1 > 0$, $\eta \in (0, 1)$, and $\omega_0, \omega_1 \in \mathcal{Y}$ are arbitrary. Choose a nonnegative real sequence $\{c_k\}$ such that $\sum_{k=1}^{\infty} c_k < \infty$.

**Step 1:** Compute

$$\rho_k = \omega_k + \vartheta_k(\omega_k - \omega_{k-1}).$$

Find

$$v_k = P_{\mathcal{Z}_1}(\varphi(\rho_k) - \psi_k \mathcal{D}\varphi(\rho_k)).$$

**Step 2:** Construct

$$\mathcal{Z}_{2_k} = \{w \in \mathcal{Y} : \langle \varphi(\rho_k) - \psi_k \mathcal{D}\varphi(\rho_k) - v_k, w - v_k \rangle \leq 0\}.$$

Compute

$$t_k = P_{\mathcal{Z}_{2_k}}(\varphi(\rho_k) - \psi_k \mathcal{D}v_k).$$

Compute

$$\omega_{k+1} = (1 - \tau_k)\varphi(\rho_k) + \tau_k \mathcal{D}t_k,$$

where

$$\psi_{k+1} = \begin{cases} \min\left\{\frac{\eta(\|\varphi(\rho_k) - v_k\|^2 + \|\mathcal{D}t_k - v_k\|^2)}{2\langle \mathcal{D}\varphi(\rho_k) - \mathcal{D}v_k, \mathcal{D}t_k - v_k\rangle}, \psi_k + c_k\right\}, & \text{if } \langle \mathcal{D}\varphi(\rho_k) - \mathcal{D}v_k, \mathcal{D}t_k - v_k\rangle > 0 \\ \psi_k + c_k, & \text{otherwise.} \end{cases} \tag{4.51}$$

**Remark 6.** *(i) The result of Theorem 8 remains valid if the constant step size $\psi \in (0, 1/\mathcal{L})$ in Algorithm 2.1 is replaced by a variable sequence $\{\psi_k\}$ satisfying*

$$0 < \inf_{k\geq 1} \psi_k \leq \sup_{k\geq 1} \psi_k < \frac{1}{\mathcal{L}}.$$

*However, such a choice of variable step size still requires the Lipschitz constant.*

*(ii) If the Lipschitz constant is unknown, we consider Algorithm 2.2 since the step size $\psi_k$ does not depend on it. I now show that the sequence $\{\psi_k\}$ generated by* (4.51) *is convergent. Since $\mathcal{D}$ is $\mathcal{L}$-Lipschitzian, in the case $\langle \mathcal{D}\varphi(\rho_k) - \mathcal{D}v_k, \mathcal{D}t_k - v_k\rangle > 0$, the Cauchy–Schwarz and Young inequalities give*

$$\langle \mathcal{D}\varphi(\rho_k) - \mathcal{D}v_k, \mathcal{D}t_k - v_k\rangle \leq \mathcal{L}\|\varphi(\rho_k) - v_k\| \, \|\mathcal{D}t_k - v_k\| \leq \frac{\mathcal{L}}{2}(\|\varphi(\rho_k) - v_k\|^2 + \|\mathcal{D}t_k - v_k\|^2),$$

*so that*

$$\frac{\eta(\|\varphi(\rho_k)-v_k\|^2+\|\mathcal{D}t_k-v_k\|^2)}{2\langle\mathcal{D}\varphi(\rho_k)-\mathcal{D}v_k,\mathcal{D}t_k-v_k\rangle}\geq\frac{\eta}{\mathcal{L}}.$$

*Hence, by induction,* $\psi_k\geq\min\{\frac{\eta}{\mathcal{L}},\psi_1\}$ *for all* $k\geq 1$. *On the other hand, the definition* (4.51) *gives* $\psi_{k+1}\leq\psi_k+c_k$, *so that* $\psi_k\leq\psi_1+c$ *for all* $k\geq 1$, *where* $c=\sum_{k=1}^{\infty}c_k<\infty$. *Setting* $\Psi_k:=\psi_k+\sum_{j=k}^{\infty}c_j$, *I obtain* $\Psi_{k+1}\leq\psi_k+c_k+\sum_{j=k+1}^{\infty}c_j=\Psi_k$, *so* $\{\Psi_k\}$ *is non increasing and bounded below, and hence convergent. Since* $\sum_{j=k}^{\infty}c_j\to 0$ *as* $k\to\infty$, *it follows that* $\psi_k\to\psi$ *as* $k\to\infty$, *where*

$$\psi\in\left[\min\left\{\frac{\eta}{\mathcal{L}},\psi_1\right\},\psi_1+c\right],$$

*and* $\mathcal{L}$ *denotes the Lipschitz constant of* $\mathcal{D}$ *(see also [17]).*

*(iii) If* $c_k=0$, *then the step size generated by Algorithm 2.2 reduces to that of [18]. The step size in [18] is monotone and therefore may depend on the initial choice* $\psi_1$. *In contrast, the step size defined in* (4.51) *is non-monotone and hence may be less sensitive to the initial value* $\psi_1$.

In view of the above remark, I proceed to analyze the convergence of Algorithm 2.2.

**Theorem 9.** *Suppose that* $\{\omega_k\}$ *is generated by Algorithm 2.2 and that Assumption 2(a),(b) holds. Then I get*

$$\begin{aligned}
&\Upsilon_{k+1}-\Upsilon_k-\vartheta_k(\Upsilon_k-\Upsilon_{k-1})\\
&\leq\frac{1}{2}(\vartheta_k+\vartheta_k^2)\|\omega_k-\omega_{k-1}\|^2-\zeta_k\|\omega_{k+1}-\varphi(\rho_k)\|^2-\frac{\tau_k}{2}(1-\psi_k\frac{\eta}{\psi_{k+1}})\Big[\|\varphi(\rho_k)-v_k\|^2\\
&\quad+\|\mathcal{D}t_k-v_k\|^2\Big]+\frac{1}{2}\|\rho_k-\varphi(\rho_k)\|^2,
\end{aligned}$$

*where* $\zeta_k=\frac{1-\tau_k}{2\tau_k}$ *and* $\Upsilon_k=\frac{1}{2}\|\omega_k-t\|^2$, *for all* $t\in\Upsilon$.

*Proof.* Similar to (4.7), I get

$$\begin{aligned}
&\Upsilon_{k+1}-\Upsilon_k-\vartheta_k(\Upsilon_k-\Upsilon_{k-1})\\
&\leq\frac{1}{2}(\vartheta_k+\vartheta_k^2)\|\omega_k-\omega_{k-1}\|^2+\frac{1}{2}\|\omega_{k+1}-\rho_k\|^2-\frac{1}{2}\tau_k\Big[\|\varphi(\rho_k)-v_k\|^2\\
&\quad+\|\mathcal{D}t_k-v_k\|^2\Big]-\frac{1}{2}\tau_k\|\mathcal{D}t_k-\varphi(\rho_k)\|^2-\tau_k\langle\mathcal{D}t_k-\varphi(\rho_k),\varphi(\rho_k)-\rho_k\rangle\\
&\quad+\psi_k\tau_k\langle\mathcal{D}\varphi(\rho_k)-\mathcal{D}v_k,\mathcal{D}t_k-v_k\rangle\\
&=\frac{1}{2}(\vartheta_k+\vartheta_k^2)\|\omega_k-\omega_{k-1}\|^2+\frac{1}{2}\|\omega_{k+1}-\rho_k\|^2-\frac{1}{2}\tau_k\Big[\|\varphi(\rho_k)-v_k\|^2\\
&\quad+\|\mathcal{D}t_k-v_k\|^2\Big]-\frac{1}{2}\tau_k\|\mathcal{D}t_k-\varphi(\rho_k)\|^2+\langle\omega_{k+1}-\varphi(\rho_k)),\rho_k-\varphi(\rho_k)\rangle\\
&\quad+\psi_k\tau_k\langle\mathcal{D}\varphi(\rho_k)-\mathcal{D}v_k,\mathcal{D}t_k-v_k\rangle\\
&=\frac{1}{2}(\vartheta_k+\vartheta_k^2)\|\omega_k-\omega_{k-1}\|^2+\frac{1}{2}\|\omega_{k+1}-\rho_k\|^2-\frac{1}{2}\tau_k\Big[\|\varphi(\rho_k)-v_k\|^2\\
&\quad+\|\mathcal{D}t_k-v_k\|^2\Big]-\frac{1}{2}\tau_k\|\mathcal{D}t_k-\varphi(\rho_k)\|^2+\Big(\frac{1}{2}\|\omega_{k+1}-\varphi(\rho_k)\|^2
\end{aligned}$$

$$+\frac{1}{2}\|\rho_k-\varphi(\rho_k)\|^2-\frac{1}{2}\|\omega_{k+1}-\rho_k\|^2\Big)+\psi_k\tau_k\langle\mathcal{D}\varphi(\rho_k)-\mathcal{D}v_k,\mathcal{D}t_k-v_k\rangle$$
$$=\frac{1}{2}(\vartheta_k+\vartheta_k^2)\|\omega_k-\omega_{k-1}\|^2-\zeta_k\|\omega_{k+1}-\varphi(\rho_k)\|^2+\frac{1}{2}\|\rho_k-\varphi(\rho_k)\|^2$$
$$-\frac{\tau_k}{2}\Big[\|\varphi(\rho_k)-v_k\|^2+\|\mathcal{D}t_k-v_k\|^2\Big]+\psi_k\tau_k\langle\mathcal{D}\varphi(\rho_k)-\mathcal{D}v_k,\mathcal{D}t_k-v_k\rangle, \tag{4.52}$$

where $\zeta_k=(\frac{1-\tau_k}{2\tau_k})$. If $\langle\mathcal{D}\varphi(\rho_k)-\mathcal{D}v_k,\mathcal{D}t_k-v_k\rangle\leq 0$, then the last term of (4.52) is nonpositive, and since $\psi_k$, $\tau_k$, $\eta$ and $\psi_{k+1}$ are positive, I directly obtain

$$\psi_k\tau_k\langle\mathcal{D}\varphi(\rho_k)-\mathcal{D}v_k,\mathcal{D}t_k-v_k\rangle\leq 0\leq\frac{\psi_k\tau_k\eta}{2\psi_{k+1}}(\|\varphi(\rho_k)-v_k\|^2+\|\mathcal{D}t_k-v_k\|^2),$$

which gives the required conclusion in this case.
In the case $\langle\mathcal{D}\varphi(\rho_k)-\mathcal{D}v_k,\mathcal{D}t_k-v_k\rangle>0$, by the definition of the step size in (4.51), I have

$$\psi_{k+1}=\min\left\{\frac{\eta(\|\varphi(\rho_k)-v_k\|^2+\|\mathcal{D}t_k-v_k\|^2)}{2\langle\mathcal{D}\varphi(\rho_k)-\mathcal{D}v_k,\mathcal{D}t_k-v_k\rangle},\psi_k+c_k\right\}.$$

Then, I have

$$\psi_{k+1}\leq\frac{\eta(\|\varphi(\rho_k)-v_k\|^2+\|\mathcal{D}t_k-v_k\|^2)}{2\langle\mathcal{D}\varphi(\rho_k)-\mathcal{D}v_k,\mathcal{D}t_k-v_k\rangle}.$$

This implies that

$$\langle\mathcal{D}\varphi(\rho_k)-\mathcal{D}v_k,\mathcal{D}t_k-v_k\rangle\leq\frac{\eta}{2\psi_{k+1}}(\|\varphi(\rho_k)-v_k\|^2+\|\mathcal{D}t_k-v_k\|^2). \tag{4.53}$$

Substituting (4.53) into (4.52), I obtain that

$$\Upsilon_{k+1}-\Upsilon_k-\vartheta_k(\Upsilon_k-\Upsilon_{k-1})$$
$$\leq\frac{1}{2}(\vartheta_k+\vartheta_k^2)\|\omega_k-\omega_{k-1}\|^2-\zeta_k\|\omega_{k+1}-\varphi(\rho_k)\|^2-\frac{\tau_k}{2}(1-\psi_k\frac{\eta}{\psi_{k+1}})\Big[\|\varphi(\rho_k)-v_k\|^2$$
$$+\|\mathcal{D}t_k-v_k\|^2\Big]+\frac{1}{2}\|\rho_k-\varphi(\rho_k)\|^2.$$

□

**Theorem 10.** *If we use the assumptions of Theorem 9, I have:*

$$\sum_{\alpha=1}^{k-1}h_{\alpha+1,k}\Big[2\zeta_\alpha(1-\vartheta_\alpha)^2\|\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2-(\vartheta_\alpha+\vartheta_\alpha^2)\|\omega_\alpha-\omega_{\alpha-1}\|^2$$
$$+2\zeta_\alpha(1-\vartheta_\alpha)(\|\omega_{\alpha+1}-\varphi(\omega_\alpha)\|^2-\|\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2)$$
$$-(1+\vartheta_\alpha)\|\omega_\alpha-\varphi(\omega_\alpha)\|^2-(\vartheta_\alpha^2+\vartheta_\alpha)\|\omega_\alpha-\omega_{\alpha-1}-\varphi(\omega_\alpha-\omega_{\alpha-1})\|^2\Big]$$
$$\leq 2h_1|\Upsilon_1-\Upsilon_0|+2\Upsilon_0,$$

*where $h_{\alpha,k}$ is from* (2.7)*, provided that $\varphi$ is linear.*

*Proof.* In the same way as (4.9), I get

$$\begin{aligned}&\|\omega_{k+1}-\varphi(\rho_k)\|^2\\&\geq(1-\vartheta_k)^2\|\varphi(\omega_k-\omega_{k-1})\|^2+(1-\vartheta_k)\Big(\|\omega_{k+1}-\varphi(\omega_k)\|^2-\|\varphi(\omega_k-\omega_{k-1})\|^2\Big).\end{aligned}\tag{4.54}$$

Using Remark 6(ii), one can see $\lim_{k\to\infty}\psi_k\frac{\eta}{\psi_{k+1}}\in(0,1)$. Thus, there exists $n_0\geq 1$ such that $k\geq n_0$, $0<\psi_k\frac{\eta}{\psi_{k+1}}<1$. By (4.54), (4.10), and Theorem 9, I get

$$\begin{aligned}&\Upsilon_{k+1}-\Upsilon_k-\vartheta_k(\Upsilon_k-\Upsilon_{k-1})\\&\leq\frac{1}{2}(\vartheta_k+\vartheta_k^2)\|\omega_k-\omega_{k-1}\|^2-\zeta_k\|\omega_{k+1}-\varphi(\rho_k)\|^2+\frac{1}{2}\|\rho_k-\varphi(\rho_k)\|^2\\&\leq\frac{1}{2}(\vartheta_k+\vartheta_k^2)\|\omega_k-\omega_{k-1}\|^2-\zeta_k(1-\vartheta_k)^2\|\varphi(\omega_k-\omega_{k-1})\|^2\\&\quad-\zeta_k(1-\vartheta_k)\Big(\|\omega_{k+1}-\varphi(\omega_k)\|^2-\|\varphi(\omega_k-\omega_{k-1})\|^2\Big)\\&\quad+\frac{1}{2}(1+\vartheta_k)\|\omega_k-\varphi(\omega_k)\|^2+\frac{1}{2}(\vartheta_k^2+\vartheta_k)\|\omega_k-\omega_{k-1}-\varphi(\omega_k-\omega_{k-1})\|^2.\end{aligned}$$

The rest of the proof follows as in Theorem 2. □

Similarly, I have that Theorems 3 to 7 hold for Algorithm 2.2.

**Theorem 1.** *Assume that* $\{\omega_k\}$ *is generated by Algorithm 2.2 and that* (2.4) *and Assumptions 1 and 2 hold. Then,* $\{\omega_k\}$ *converges weakly to a point of* $\mathrm{VIP}(\mathcal{Z}_1,\mathcal{D})$.

*Proof.* I verify that all the conditions used in the proof of Theorem 8 are satisfied for Algorithm 2.2. First, by Remark 6(ii), the step size sequence $\{\psi_k\}$ converges to some $\psi>0$, and therefore $\lim_{k\to\infty}(1-\psi_k\frac{\eta}{\psi_{k+1}})=1-\eta>0$. Hence, there exists $k_0\geq 1$ such that $1-\psi_k\frac{\eta}{\psi_{k+1}}>0$ for all $k\geq k_0$, and Theorem 9 provides, for all $k\geq k_0$, the same recursive estimate as Theorem 1, with the coefficient $(1-\psi\mathcal{L})$ replaced by $(1-\psi_k\frac{\eta}{\psi_{k+1}})$. Second, Theorem 10 yields the analogue of Theorem 2. Consequently, repeating the arguments of Theorems 3 to 7 for $k\geq k_0$, I obtain that $\{\omega_k\}$ is bounded, that $\lim_{k\to\infty}[\|\varphi(\rho_k)-v_k\|^2+\|\mathcal{D}t_k-v_k\|^2]=0$, that every weak cluster point of $\{\omega_k\}$ belongs to $\Upsilon$, and that $\{\omega_k\}$ has at most one weak limit point in $\Upsilon$. Since Assumptions 1 and 2 remain unchanged for Algorithm 2.2, the conclusion follows exactly as in the proof of Theorem 8. □

## 5. Application

The proposed inertial viscosity-type subgradient extragradient methods can be applied to various optimization problems by establishing their connection with variational inequality problems. In this section, I demonstrate an application of the obtained results to the convex minimization problem. This application highlights the practical significance of the proposed methods and shows that the developed theoretical framework can be used to solve important optimization models.

Convex minimization problems are fundamental in optimization theory and have numerous applications in machine learning, signal processing, operations research, and engineering. My

objective is to find a point that minimizes a convex function over a given feasible set. Let $g : \mathcal{Y} \to \mathbb{R}$ be a convex and differentiable function. The convex minimization problem is defined as

$$\min_{w \in \mathcal{Z}_1} g(w). \tag{5.1}$$

A point $w^* \in \mathcal{Z}_1$ is called a solution of problem (5.1) if

$$g(w^*) \leq g(w), \qquad \forall w \in \mathcal{Z}_1.$$

The relationship between convex minimization problems and variational inequality problems enables me to apply the proposed algorithms for solving (5.1). The following lemma provides the required optimality condition.

**Lemma 1.** *[20] Let $g : \mathcal{Y} \to \mathbb{R}$ be convex and differentiable. Then $w^*$ solves* (5.1) *if and only if $w^* \in \mathrm{VIP}(\mathcal{Z}_1, \nabla g)$.*

By Lemma 1, the convex minimization problem can be reformulated as a variational inequality problem with operator $\mathcal{D} = \nabla g$. Therefore, the convergence results obtained for the *VIP* can be directly applied to find the minimizers of convex functions. Applying Algorithm 2.2 to the operator $\nabla g$, I obtain the following method.

**Algorithm 3.1**

Choose $\vartheta_k \in [0, 1)$ and $\tau_k \in (0, 1]$, for all $k \geq 1$. Suppose that $\psi_1 > 0$, $\eta \in (0, 1)$, and $\omega_0, \omega_1 \in \mathcal{Y}$ are arbitrary. Choose a nonnegative real sequence $\{c_k\}$ such that $\sum_{k=1}^{\infty} c_k < \infty$.

**Step 1:** Compute

$$\rho_k = \omega_k + \vartheta_k(\omega_k - \omega_{k-1}).$$

Find

$$v_k = P_{\mathcal{Z}_1}(\varphi(\rho_k) - \psi_k \nabla g(\varphi(\rho_k))).$$

**Step 2:** Construct

$$\mathcal{Z}_{2_k} = \{w \in \mathcal{Y} : \langle \varphi(\rho_k) - \psi_k \nabla g(\varphi(\rho_k)) - v_k, w - v_k \rangle \leq 0\},$$

and compute

$$t_k = P_{\mathcal{Z}_{2_k}}(\varphi(\rho_k) - \psi_k \nabla g(v_k)).$$

Compute

$$\omega_{k+1} = (1 - \tau_k)\varphi(\rho_k) + \tau_k \nabla g(t_k),$$

where

$$\psi_{k+1} = \begin{cases} \min\left\{ \dfrac{\eta(\|\varphi(\rho_k) - v_k\|^2 + \|\nabla g(t_k) - v_k\|^2)}{2\langle \nabla g(\varphi(\rho_k)) - \nabla g(v_k), \nabla g(t_k) - v_k \rangle}, \psi_k + c_k \right\}, & \text{if } \langle \nabla g(\varphi(\rho_k)) - \nabla g(v_k), \nabla g(t_k) - v_k \rangle > 0, \\ \psi_k + c_k, & \text{otherwise.} \end{cases} \tag{5.2}$$

Set $k := k + 1$ and repeat.

The convergence result of the proposed algorithm for the convex minimization problem is stated as follows:

**Theorem 11.** *Let $g : \mathcal{Y} \to \mathbb{R}$ be a convex and differentiable function such that $\nabla g$ is relaxed $\alpha$-monotone and Lipschitz continuous on $\mathcal{Y}$ and sequentially weakly continuous on $\mathcal{Z}_1$. Suppose that problem* (5.1) *has a solution, that the set $\{t \in \mathcal{Z}_1 : \nabla g(t) = 0\} \setminus \Upsilon$ is finite, and that Assumption 1, Assumption 2(d), and condition* (2.4) *hold. If $\{\omega_k\}$ is generated by Algorithm 3.1, then $\{\omega_k\}$ converges weakly to a solution of* (5.1)*.*

*Proof.* I verify that operator $\mathcal{D} = \nabla g$ satisfies all the requirements of Theorem 1. By Lemma 1, the solution set of (5.1) coincides with $\Upsilon = \mathrm{VIP}(\mathcal{Z}_1, \nabla g)$; since (5.1) is solvable, Assumption 2(a) holds. The conditions imposed on $\nabla g$ guarantee that Assumption 2(b) holds, while the finiteness of the set $\{t \in \mathcal{Z}_1 : \nabla g(t) = 0\} \setminus \Upsilon$ gives Assumption 2(c). Assumption 2(d), Assumption 1, and condition (2.4) are imposed directly. Moreover, Algorithm 3.1 is exactly Algorithm 2.2 applied to the operator $\mathcal{D} = \nabla g$. Hence, all the conditions of Theorem 1 are satisfied, and $\{\omega_k\}$ converges weakly to a point of $\mathrm{VIP}(\mathcal{Z}_1, \nabla g)$. By Lemma 1, this weak limit is a solution of problem (5.1). □

## 6. Numerical experiments

In this section, I present numerical simulations to demonstrate the efficiency and applicability of the proposed methods. All computations are carried out using MATLAB (R2018a) on a personal computer with an AMD A6-6310 APU processor operating at 1.80 GHz and 16 GB of RAM.

For performance evaluation, Algorithms 2.1 and 2.2 are compared with Algorithms 3.3 and 3.4 introduced by Ogwo et al. [13]. The comparison is performed using different initial points along with various parameter selections. In the implementation, the tolerance at the $k$-th iteration is defined as

$$\mathrm{TOL}_k = \|\omega_k - \omega_{k-1}\|.$$

The stopping criterion for all iterative procedures is given by

$$\mathrm{TOL}_k < \epsilon,$$

where $\epsilon = 10^{-6}$.

Furthermore, the parameters are chosen as $\psi_k = 0.1$, $\vartheta_k = 0.5$, $\tau_k = 0.7$, and $\varphi(p) = \frac{p}{4}$. To further validate the effectiveness of the proposed schemes, I apply them to Examples 5.1 and 5.2 reported in [13]. I point out that the recent methods developed in [25–27] are designed for monotone variational inequality problems in Hilbert spaces, while the schemes in [28,29] are formulated in Hadamard spaces and Hadamard manifolds, respectively. Since the test problems considered below involve operators that are not monotone, these methods are not included in the comparison; their relationship with the proposed algorithms is discussed in Section 1.

**Example 6.1.** *Consider the set $\mathcal{Z}_1 = [-1, 1]$ and define the mapping $\mathcal{D} : \mathbb{R} \to \mathbb{R}$ by*

$$\mathcal{D}(\omega) = \begin{cases} 2\omega, & \text{if } \omega > 1, \\ \omega^2, & \text{if } \omega \in [-1, 1], \\ -2\omega - 1, & \text{if } \omega < -1. \end{cases}$$

*It can be verified that $\mathcal{D}$ is quasimonotone and Lipschitz continuous with constant* 2*. Moreover, the solution set is given by* $\mathrm{VIP}(\mathcal{Z}_1, \mathcal{D}) = \{-1, 0\}$.

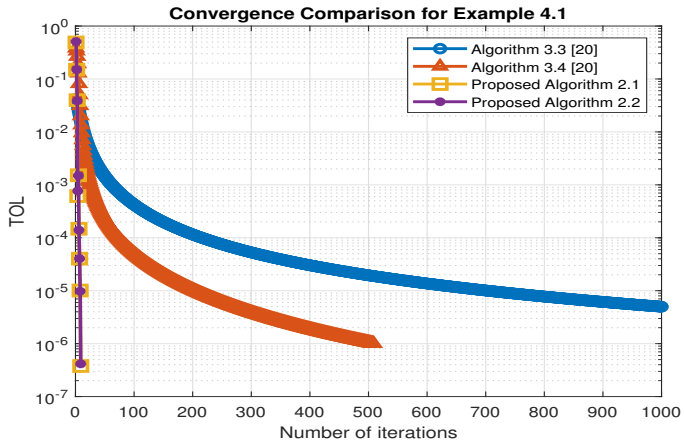



**Figure 1.** Comparison of $\mathrm{TOL}_n$ for Example 6.1.

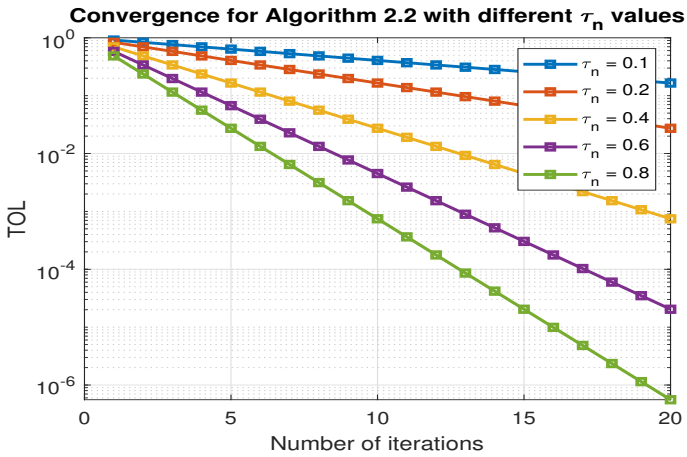



**Figure 2.** Effect of varying the relaxation parameter while keeping the inertial term fixed.

**Table 1.** Execution time comparison for Example 6.1.

| CPU time for Example 6.1 | |
|---|---|
| Algorithm | Time (Sec) |
| Algorithm 3.3 [13] | 0.11872 |
| Algorithm 3.4 [13] | 0.08981 |
| Algorithm 2.1 | 0.02323 |
| Algorithm 2.2 | 0.02619 |

**Discussion.** The CPU times reported in Table 1 indicate that the proposed algorithms are computationally more efficient than the methods in [13]. In particular, Algorithm 2.1 requires only 0.02323 seconds, which is substantially less than the execution times of Algorithm 3.3 (0.11872 seconds) and Algorithm 3.4 (0.08981 seconds). Although Algorithm 2.2 requires slightly more time than Algorithm 2.1, it outperforms both algorithms. These results demonstrate that the proposed algorithms achieve faster execution while maintaining the desired convergence properties.

**Example 6.2.** *Let* $\mathcal{Z}_1 = \{\omega \in \mathbb{R}^2 : \omega_1^2 + \omega_2^2 \leq 1,\ 0 < \omega_1\}$ *and* $\mathcal{D}(\omega_1, \omega_2) = (-\omega_1 e^{\omega_2}, \omega_2)$. *Then,* $\mathcal{D}$ *need not be quasimonotone and* $\mathrm{VIP}(\mathcal{Z}_1, \mathcal{D}) = \{(1, 0)\}$.

*In this case, other than the above parameters, I take* $\omega_0 = (0.5, 0.5), \omega_1 = (0.6, 0.5)$, *and* $\varphi(p, q) = (\frac{p}{4}, \frac{q}{4})$.

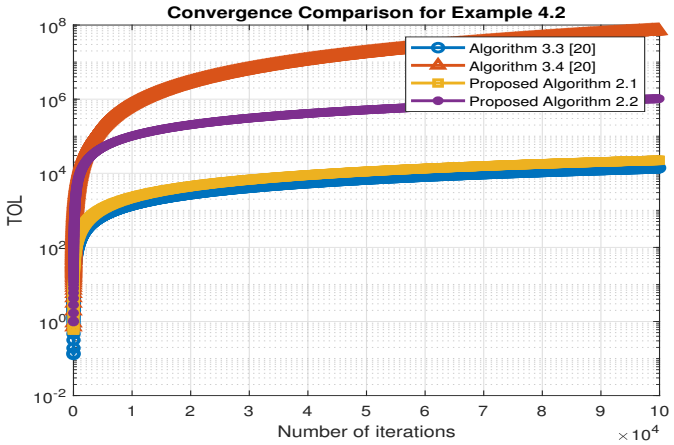



**Figure 3.** The $\mathrm{TOL}_n$ comparison for Example 6.2.

**Table 2.** Numerical results for Example 6.2.

| CPU time for Example 6.2 | |
|---|---|
| Algorithm | Time (Sec) |
| Algorithm 3.3 [13] | 9.3079 |
| Algorithm 3.4 [13] | 10.6671 |
| Algorithm 2.1 | 9.3039 |
| Algorithm 2.2 | 9.7172 |

**Discussion.** Table 2 presents the CPU times of the proposed algorithms and the methods for Example 6.2. It is observed that Algorithm 2.1 achieves the smallest execution time (9.3039 s), while Algorithm 2.2 exhibits competitive computational performance. Compared with [13, Algorithms 3.3 and 3.4], the proposed methods require comparable or less computational time, demonstrating their computational efficiency for this test problem. These results indicate that the proposed algorithms maintain competitive performance while preserving the desired convergence behavior.

Figures 1–3 illustrate the convergence behavior of the proposed algorithms and the methods under different parameter settings and initial guesses. It can be observed that all algorithms start with a similar initial error but exhibit different convergence rates. Algorithms 3.3 and 3.4 converge rapidly during the initial iterations; however, their errors stabilize at relatively higher values as the iterations progress, indicating slower long-term improvement. In contrast, the proposed algorithms continue to reduce the error throughout the iteration process. In particular, Proposed Algorithm 2.2 achieves the lowest error values over a larger number of iterations, demonstrating superior convergence performance. These numerical results confirm the effectiveness of the proposed algorithms and are consistent with the theoretical convergence analysis.

In the sequel example, we take different initial guesses and parameters for comparisons.

**Example 6.3.** *Consider the set* $\mathcal{Z}_1 = [-1, 1] \times [-1, 1]$ *and the mapping* $\mathcal{D} : \mathcal{Z}_1 \to \mathbb{R}^2$ *defined by*

$$\mathcal{D}(\omega_1, \omega_2) = (\omega_1, 2\omega_2), \quad \forall (\omega_1, \omega_2) \in [-1, 1] \times [-1, 1].$$

*Let the function* $\alpha : \mathbb{R}^2 \to \mathbb{R}$ *be given by* $\alpha(z) = \|z\|^2$. *It is easy to verify that* $\alpha(qz) = q^2\|z\|^2 = q^2\alpha(z)$, *for all* $q > 0$. *Furthermore, define* $\varphi : \mathbb{R} \to \mathbb{R}$ *by* $\varphi(p) = \frac{p}{4}$.

*Observe that* $\mathcal{D}$ *is a relaxed* $\alpha$*-monotone mapping and is Lipschitz continuous with constant* 2. *In addition,* $\mathcal{D}$ *is a bounded linear operator, and the solution set of the associated VIP is given by*

$$\mathrm{VIP}(\mathcal{Z}_1, \mathcal{D}) = \{(0, 0)\}.$$

*The convergence behavior of the proposed algorithm is illustrated below, where it can be observed that the method demonstrates improved efficiency.*

*Moreover, Figures 4 and 5 demonstrate that the proposed algorithms exhibit convergence behavior that is comparable to or better than the existing methods while maintaining stable performance for all tested initial guesses and parameter settings.*

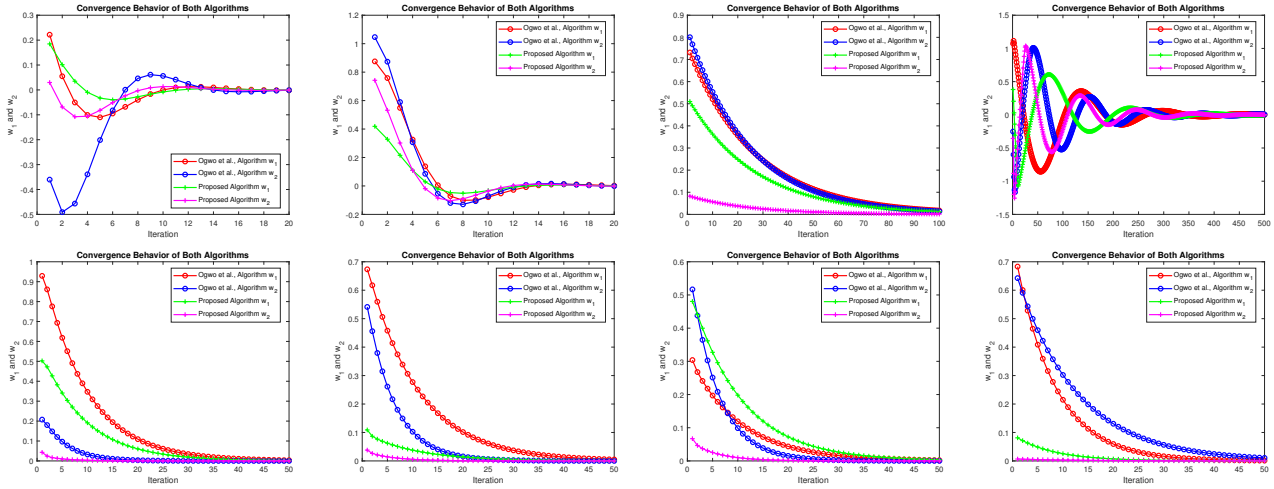



**Figure 4.** Component-wise convergence profiles of the considered algorithms.

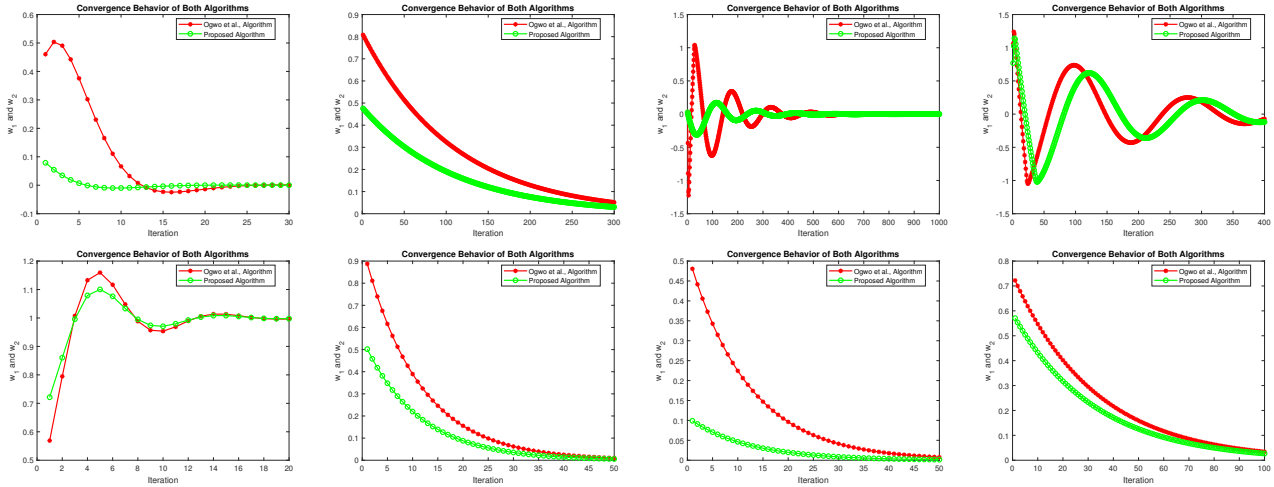



**Figure 5.** Absolute convergence behavior of the considered algorithms.

## 7. Conclusions

In this work, I introduced two novel iterative schemes based on inertial viscosity-type subgradient extragradient methods for solving VIPs and convex minimization problems in real Hilbert spaces. The proposed algorithms were developed under relaxed $\alpha$-monotonicity and Lipschitz continuity assumptions, which provide a broader framework than classical monotonicity conditions. By employing suitable inertial parameters and viscosity techniques, we established the weak convergence of the generated sequences to solutions of the considered problems. The obtained results extend and improve several convergence results in the literature by considering a more general class of operators.

Furthermore, numerical experiments were conducted to evaluate the efficiency and practical performance of the proposed methods. The computational results demonstrate that the proposed algorithms achieve competitive convergence behavior and require reasonable computational effort compared with existing approaches. These findings confirm the effectiveness and applicability of the proposed schemes for solving related optimization problems.

Despite these contributions, some limitations remain. The analysis is restricted to real Hilbert spaces and guarantees only weak convergence under the considered assumptions. Moreover, the numerical validation is based on a limited number of test examples. In future studies, researchers may focus on developing strongly convergent variants, extending the methods to broader spaces such as Banach spaces, and investigating their applications to large-scale optimization problems and real-world models.

## Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

## Acknowledgments

The author is grateful to University of Johannesburg, South Africa, for the postdoctoral fellowship support.

## Conflict of interest

The authors declare no conflict of interest.

## References

1. T. O. Alakoya, A. Taiwo, O. T. Mewomo, Y. J. Cho, An iterative algorithm for solving variational inequality, generalized mixed equilibrium, convex minimization and zeros problems for a class of nonexpansive-type mappings, *Ann. Univ. Ferrara*, **67** (2021), 1–31. https://doi.org/10.1007/s11565-020-00354-2
2. C. Izuchukwu, A. A. Mebawondu, O. T. Mewomo, A new method for solving split variational inequality problems without co-coerciveness, *J. Fix. Point Theory A.*, **22** (2020), 98. https://doi.org/10.1007/s11784-020-00834-0

3. L. O. Jolaoso, A. Taiwo, T. O. Alakoya, O. T. Mewomo, A strong convergence theorem for solving pseudo-monotone variational inequalities using projection methods, *J. Optimiz. Theory App.*, **185** (2020), 744–766. https://doi.org/10.1007/s10957-020-01672-3
4. M. W. Asghar, M. Abbas, C. D. Enyi, M. E. Omaba, Iterative approximation of fixed points of generalized $\alpha_m$-nonexpansive mappings in modular spaces, *AIMS Math.*, **8** (2023), 26922–26944. https://doi.org/10.3934/math.20231378
5. I. Beg, M. Abbas, M. W. Asghar, Convergence of AA-iterative algorithm for generalized $\alpha$-nonexpansive mappings with an application, *Mathematics*, **10** (2022), 4375. https://doi.org/10.3390/math10224375
6. M. Abbas, C. Ciobanescu, M. W. Asghar, A. Omame, Solution approximation of fractional boundary value problems and convergence analysis using AA-iterative scheme, *AIMS Math.*, **9** (2024), 13129–13158. https://doi.org/10.3934/math.2024641
7. B. V. Inh, D. S. Kim, Projection algorithms for solving nonmonotone equilibrium problems in Hilbert space, *J. Comput. Appl. Math.*, **302** (2016), 106–117. https://doi.org/10.1016/j.cam.2016.01.054
8. G. M. Korpelevich, An extragradient method for finding saddle points and other problems, *Ekon. Math. Meto.*, **12** (1976), 747–756.
9. R. Y. Apostol, A. A. Grynenko, V. V. Semenov, Iterative algorithms for monotone bilevel variational inequalities, *J. Comput. Appl. Math.*, **107** (2012), 3–14.
10. L. C. Ceng, N. Hadjisavvas, N. C. Wong, Strong convergence theorem by a hybrid extragradient-like approximation method for variational inequalities and fixed point problems, *J. Global Optim.*, **46** (2010), 635–646. https://doi.org/10.1007/s10898-009-9454-7
11. P. T. Vuong, On the weak convergence of the extragradient method for solving pseudomonotone variational inequalities, *J. Optimiz. Theory App.*, **176** (2018), 399–409. https://doi.org/10.1007/s10957-017-1214-0
12. Y. Censor, A. Gibali, S. Reich, Extensions of Korpelevich's extragradient method for the variational inequality problem in Euclidean space, *Optimization*, **61** (2012), 1119–1132. https://doi.org/10.1080/02331934.2010.539689
13. G. N. Ogwo, C. Izuchukwu, Y. Shehu, O. T. Mewomo, Convergence of relaxed inertial subgradient extragradient methods for quasimonotone variational inequality problems, *J. Sci. Comput.*, **90** (2022), 10. https://doi.org/10.1007/s10915-021-01670-1
14. H. Attouch, A. Cabot, Convergence of a relaxed inertial proximal algorithm for maximally monotone operators, *Math. Program.*, **184** (2020), 243–287. https://doi.org/10.1007/s10107-019-01412-0
15. A. Beck, M. Teboulle, A fast iterative shrinkage-thresholding algorithm for linear inverse problems, *SIAM J. Imaging Sci.*, **2** (2009), 183–202. https://doi.org/10.1137/080716542
16. A. Chambolle, C. Dossal, On the convergence of the iterates of the fast iterative shrinkage/thresholding algorithm, *J. Optimiz. Theory App.*, **166** (2015), 968–982. https://doi.org/10.1007/s10957-015-0746-4

17. H. Liu, J. Yang, Weak convergence of iterative methods for solving quasimonotone variational inequalities, *Comput. Optim. Appl.*, **77** (2020), 491–508. https://doi.org/10.1007/s10589-020-00217-8

18. J. Yang, H. Liu, Z. Li, Modified subgradient extragradient algorithms for solving monotone variational inequalities, *Optimization*, **67** (2018), 2247–2258. https://doi.org/10.1080/02331934.2018.1523404

19. G. Stampacchia, *Variational inequalities*, Italy: Istituto per le applicazioni del calcolo, 1969, 102–192.

20. M. Tian, B. N. Jiang, Inertial Haugazeau's hybrid subgradient extragradient algorithm for variational inequality problems in Banach spaces, *Optimization*, **70** (2021), 987–1007. https://doi.org/10.1080/02331934.2020.1731808

21. T. O. Alakoya, L. O. Jolaoso, O. T. Mewomo, Modified inertia subgradient extragradient method with self-adaptive stepsize for solving monotone variational inequality and fixed point problems, *Optimization*, **69** (2020), 1987–2002. https://doi.org/10.1080/02331934.2019.1711087

22. Y. P. Fang, N. J. Huang, Variational-like inequalities with generalized monotone mappings in Banach spaces, *J. Optimiz. Theory App.*, **118** (2003), 327–338. https://doi.org/10.1023/A:1025499305742

23. S. S. Chang, B. S. Lee, Y. Q. Chen, Variational inequalities for monotone operators in nonreflexive Banach spaces, *Appl. Math. Lett.*, **8** (1995), 29–34. https://doi.org/10.1016/0893-9659(95)00081-Z

24. X. P. Ding, Existence and algorithm of solutions for generalized mixed implicit quasi-variational inequalities, *Appl. Math. Comput.*, **113** (2000), 67–80. https://doi.org/10.1016/S0096-3003(99)00068-5

25. Y. Yao, S. Salisu, Y. Shehu, Convergence results of proximal-type algorithms with reflections for variational inequalities, *Optim. Eng.*, **27** (2026), 433–462. https://doi.org/10.1007/s11081-025-10010-3

26. J. W. Peng, S. Salisu, F. Mofarreh, J. C. Yao, Extrapolated adaptive proximal method with a single dynamic parameter for variational inequalities and optimization applications, *Commun. Nonlinear Sci.*, **163** (2026), 110495. https://doi.org/10.1016/j.cnsns.2026.110495

27. Z. Jia, J. W. Peng, S. Salisu, Y. Shehu, A two-term inertial method for monotone variational inequality problems: Implementation in multiclass classification and portfolio optimization, *Math. Comput. Simulat.*, **249** (2026), 510–532. https://doi.org/10.1016/j.matcom.2026.06.001

28. S. Salisu, S. Sriwongsa, P. Kumam, Golden ratio method for solving monotone variational inequality problems in Hadamard spaces, *Math. Method. Appl. Sci.*, **47** (2024), 10101–10116. https://doi.org/10.1002/mma.10112

29. K. Khammahawong, S. Salisu, Modified Halpern and viscosity methods for hierarchical variational inequalities on Hadamard manifolds, *Comput. Appl. Math.*, **43** (2024), 1. https://doi.org/10.1007/s40314-023-02503-7

30. X. J. Long, J. Yang, Y. J. Cho, Modified subgradient extragradient algorithms with a new line-search rule for variational inequalities, *B. Malays. Math. Sci. So.*, **46** (2023), 140. https://doi.org/10.1007/s40840-023-01522-1

31. X. J. Long, S. J. Yu, Y. Shehu, A self-adaptive double inertial extragradient algorithm for solving quasimonotone variational inequalities and fixed point problems with applications to image restoration, *Commun. Nonlinear Sci.*, **161** (2026), 110199. https://doi.org/10.1016/j.cnsns.2026.110199



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)